The natural yield curve in Brazil
(A curva de juros natural no Brasil)

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Abstract
This paper estimates the term structure of natural interest rates for Brazil, a generalization of the concept of natural rate of interest for the yield curve. First, the Diebold-Li (2006) model is estimated with real yields. The latent factors of this model are then used in a model that includes an IS and a Phillips curve. The natural yield curve is obtained as the level, slope and curvature that closes the output gap at each point in time. This decomposition allows a broader indicator of the stance of monetary policy and a real-time measure of the natural rate. The difference between the slope of the real curve and its natural counterpart is highly correlated with the output gap.

Keywords: Yield curve; Monetary policy; Natural rate
JEL Code: E43, E52, G12.

Disclaimer: The views in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the National Treasury — STN Directive 833, November 28, 2018.

1. Introduction
This paper estimates the natural yield curve for Brazil. The natural (equivalently: r-star, neutral or equilibrium) real rate (Wicksell, 1936) is defined as the interest rate that is consistent with price stability and potential output. Usually the natural rate concept is related to policy or short-term rates, with the gap between these rates and the natural level showing the stance of monetary policy. In case a central bank keeps real rates implied by the policy rate below the natural level, the economy would be stimulated and price pressures would eventually emerge. Conversely, monetary policy can be characterized as restrictive if a central bank maintains the policy rate above the natural level, tending to slow down the economy and contain price pressures.

The natural yield curve extends the concept of natural rate to the entire yield curve, and has been recently developed by Brzoza-Brzezina & Kotłowski (2014) and Imakubo, Kojima & Nakajima (2018). It involves a two-step strategy. First, the real yield curve is estimated based on three latent factors (level, slope and curvature) using the dynamic Nelson-Siegel (DNS) model, and then these factors are used as observables in the natural yield curve model, based on the Laubach & Williams (2003) model, comprising an IS and a Phillips curve.

From a theoretical point of view, most mainstream Neo-Keynesian macroeconomic models usually neglect that monetary policy operates through the term structure of interest rates, relying instead on Taylor rules and the Euler equation, which links short-term rates to consumption growth. Information is then lost in these standard modelling assumptions, as the yield curve contains information from interest rates for several maturities, information that is likely to be valuable for policymakers.

From an empirical perspective, real rates in emerging markets are weakly correlated with potential output growth (Carrillo, Elizondo, Rodríguez-Pérez & Roldán-Peña, 2018), as specified in the Euler equations which form the theoretical basis for the description of the natural rate in the closed-economy Laubach-Williams model.1 Real rates based on Euler equations have also been shown to be negatively correlated with market rates (Canzoneri, Cumby & Diba, 2007), are sometimes associated to monetary non-neutrality (Gavin, 2018), and deliver unrealistically large effects to forward guidance (Del Negro, Giannoni & Patterson, 2015).

1In the model, the natural rate is a function of the potential output growth plus a non-growth variable, which arguably includes deleveraging, demand for safe assets, savings glut and other factors. More specifically, $r^*_t = c g_t + z_t$, where $g_t$ is the trend growth and $z_t$ captures other determinants of $r^*_t$. 
The methodology employed in this paper bears similarities to that of Atkeson & Kehoe (2008) who also criticize Euler equations and attempts of other papers to fix them. In their paper, the short-term nominal rate is governed by a secular component determined by the inflation target and a cyclical component related to exogenous shifts in real risk. Instead, in the DNS model for real yields estimated here, short-term rates are determined by a time-varying long-term real rate—the level of the curve—in addition to a business cycle term associated with the slope of the curve, and the medium-term component represented by the curvature.

Looking beyond short-term rates is consistent with the notion that the natural rate is a long-term concept, and interest rates from long maturities abstract from cyclical and short-term movements that cloud the reading about the natural rate, reflecting transitory movements that should not reflect significant changes in the neutral real rate. Therefore, one potential shortcoming of using short-term rates to assess the natural or equilibrium rate is that cuts or increases might not be sustained over time, leading to significantly different dynamics between short- and long-term rates.

Another advantage of using information from the term structure of interest rates is that, particularly after the global financial crisis of 2008, a number of central banks hit the zero lower bound of nominal rates, and attempted to affect the economy with unconventional monetary policies such as large scale asset purchases and forward guidance. With the lower bound on nominal rates, the gap between short-term real rates and the natural rate became a less reliable indicator of the stance of monetary policy (Brzoza-Brzezina & Kotlowski, 2014; Imakubo et al., 2018), with the focus shifting from short rates to the whole yield curve. Another attempt to incorporate financial conditions in measures of the natural rate is given by Hakkio & Smith (2017), who allow credit and term premiums to affect the evolution of the natural rate in the Laubach and Williams model. Most studies about the natural rate for Brazil have employed variants of the Laubach-Williams semi-structural model (Portugal & Barcellos, 2009; Ribeiro & Teles, 2013; Araújo & Gomes, 2014), with the last one including some open-economy features. A closed-economy New-Keynesian model was used to assess neutral rates for Brazil in Palma & Portugal (2017), while Barbosa, Camêlo & João (2016) use a Taylor rule, with neutral rates assumed to evolve as the sum of foreign rates and risk premiums. Several different methodologies were employed by Muinhos & Nakane (2006), Magud & Tsounta (2012), and Perrelli & Roache (2014). This paper seeks to contribute to the literature on natural rates in Brazil through the use of yield curve data, as in Bomfim (2001), Christensen & Rudebusch (2017), and Ajevskis (2018).

The content of this paper is of interest to financial market participants, policymakers and academics, having both practical and theoretical consequences.

The rest of the paper is organized as follows. Section 2 reviews the literature. Section 3 estimates the DNS model with real yields. Section 4 estimates the natural yield curve model, which includes an IS and a Phillips curve. Section 5 concludes.

2. Literature Review

The financial literature usually relies on three classes of models for the term structure of interest rates, gravitating between the empirical fit and theoretical consistency.

The first class encompasses Affine Term Structure Models (ATSM), characterized by restrictions that impose no arbitrage, understood as a riskless profit. This framework developed from Vasicek (1977) and Cox, Ingersoll & Ross (1985), with the yield curve dynamics driven by a short-term rate. Duffie & Kan (1996) introduced the affine (linear) class of models, making closed-form solutions possible. Later, in a discrete time


3This is somehow connected to the model of Rudebusch & Wu (2008), where the short-term nominal rate is also expressed as a function of the level and the slope. In their model, the level is interpreted as the underlying inflation target and the slope is related to the business cycle, a function of inflation and the output gap, as in a Taylor rule.
affine setup, Ang & Piazzesi (2003) included macroeconomic factors (inflation and real economic activity) as observed state variables in a VAR model with no-arbitrage restrictions, showing that the level and the slope of the yield curve are related to the macroeconomic factors.

In ATSM models, the evolution of the yield curve is governed by the dynamics of the short rate. This rate is expressed as a linear function of the latent factors, which are assumed to evolve as a Vector Auto-Regression (VAR). The transition of a risk-neutral measure—where there is no risk premium—to a real world measure is done by a time-varying market price of risk, which is assumed to be an affine (linear) function of the latent factors. Therefore, both the short rate and the market price of risk are linear functions of the latent factors, which then drives bond yield dynamics. Macroeconomic and financial factors are usually obtained as the principal components from a set of variables. These factors then enter the model either as additional state variables, or directly in the measurement equation affecting bond yields. Gürkaynak & Wright (2012) and Kucera (2017) provide summaries of the structure and math of no-arbitrage affine models of the term structure.

The second class uses variants of the Nelson-Siegel (NS) model (Nelson & Siegel, 1987). This model extracts unobservable factors that summarize the dynamics of the term structure, where the factors are fixed over time. Diebold & Li (2006) extended the NS model, allowing the factors to be time varying, having the interpretation of level, slope and curvature. Another strand of the literature includes macroeconomic variables in these models to assess their importance for the dynamics of the yield curve, as in Diebold, Rudebusch & Borağan Aruoba (2006). Christensen, Diebold & Rudebusch (2011) proposed a DNS model with the no-arbitrage condition, joining the DNS and the ATSM approaches. Diebold, Piazzesi & Rudebusch (2005) explain in a more accessible way the restrictions placed on the factor loadings in the no-arbitrage DNS model.

The third class of term structure models comprises models that have stronger macroeconomic foundations. Wu (2006) presents a calibrated general equilibrium model, featuring capital and price adjustment costs, in which the expectations hypothesis of the term structure holds. He associates monetary shocks with the slope of the curve, and technology shocks with the level. In Rudebusch & Wu (2008), the short-term rate is based on a reaction function, expressed as a linear function of the level and the slope of the curve, with the former factor associated with the evolution of inflation and the latter with cyclical deviations of output from its potential and inflation from its target. This approach further developed into models that impose more macroeconomic structure, but often lack a good fit to the data. Bekaert, Cho & Moreno (2010) present a New Keynesian model with a term structure component where the expectations hypothesis holds. Rudebusch & Swanson (2008) show that including habits in consumption in DSGE models and solving the model with a third-order approximation is not sufficient to match term premium moments in the data. Rudebusch & Swanson (2012) incorporate Epstein-Zin preferences and long-run real and nominal risks in a DSGE model. While these features improve the model’s fit, they come with the side effect of higher risk aversion, which is a result also found by van Binsbergen, Fernández-Villaverde, Koijen & Rubio-Ramírez (2012).

This paper intends to contribute to the macro-finance research in Brazil, which is still limited. Matsumura & Moreira (2006) use no-arbitrage models to study the interactions between the economy and the yield curve from 2000 to 2005. They use daily data, with both continuous- and discrete-time specifications, finding that inflation shocks are the main sources of long-run fluctuations in nominal variables and that no-arbitrage models perform poorer than unrestricted models in forecasting ability. Shousha (2008) finds that monetary policy expectations embedded in yield curve spreads help to predict real activity, measured by the industrial production, retail sales, and employment data. He also estimates an affine model of the term structure with three macroeconomic states (Gross Domestic Product (GDP), inflation and the nominal exchange rate) and the level and slope of the yield curve. Macroeconomic variables explain a large portion of the variance of long-term interest rates, with inflation and the exchange rate responding for a great share of the 12-month yield. Guillen & Tabak (2009) model the term premium with the Kalman filter. They reject the expectations hypothesis of the term structure, finding that term premiums are not constant between 1995-2006. They also find that global risk aversion explains a large part of term premiums.
3. Model

This paper uses the DNS model of Diebold & Li (2006), due to its simplicity and tractability. While some papers used the DNS approach for Brazilian data, their focus was on forecasting, not on interactions of macroeconomics and finance. The methodology follows Brzoza-Brzezina & Kotłowski (2014) and Imakubo et al. (2018), who developed the concept of the natural yield curve. In the DNS model, the yield curve is a function of three unobservable factors (level, curvature and slope). Given some restrictions on the factor loadings, these can be interpreted as long-, medium- and short-term factors, respectively. The model is given by the following function:

\[ r_{\tau,t} = L_t + S_t \left( \frac{1 - e^{-\lambda \tau}}{\lambda \tau} \right) + C_t \left( \frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right), \]

where \( r_{\tau,t} \) is the annualized zero-coupon interest rate of maturity \( \tau \). For the sake of standardization, bond maturities will be expressed in months (\( \tau = 12, 24, 36, 48, 60, 120 \) and \( 180 \)), not years. This follows Diebold & Li (2006), Diebold et al. (2006), and Caldeira, Moura & Portugal (2010), where the maturity of the bonds is expressed in months.

\( L_t, S_t \) and \( C_t \) are the level, slope and curvature factors, respectively, and \( \lambda \) is a constant that governs the exponential rate of decay of the loadings. The greater this parameter, the faster is the decay of long-term maturities. The loading on the first factor is constant, affecting all interest rates in the same way, so it determines the level of the curve. The loading on the slope factor affects short rates more than long rates, and the loading of the curvature factor exerts more impact on medium-term rates.

The model is expressed in state space form and estimated by maximum likelihood using the Kalman Filter, following Diebold et al. (2006). Caldeira et al. (2010) defend the estimation of the Diebold-Li model using only one step and making use of the Kalman filter as opposed to the two-step method, since it leads to efficient parameter estimates, as the measurement and state equations are estimated jointly, and also due to the fact that in the one-step estimation there is no need to assume a particular value for the decay parameter \( \lambda \).

Following Brzoza-Brzezina & Kotłowski (2014) and Imakubo et al. (2018), and departing from Equation (1) for \( \tau = 12, 24, 36, 48, 60, 120 \) and \( 180 \) months, the measurement equations are given by:

\[ r_t = \Lambda f_t + \varepsilon_t, \]

where

\[
\begin{bmatrix}
    r_{12,t} \\
    r_{24,t} \\
    r_{36,t} \\
    r_{48,t} \\
    r_{60,t} \\
    r_{120,t} \\
    r_{180,t}
\end{bmatrix} =
\begin{bmatrix}
    1 & 1 - e^{-12\lambda} & 1 - e^{-24\lambda} & 1 - e^{-36\lambda} & 1 - e^{-48\lambda} & 1 - e^{-60\lambda} & 1 - e^{-120\lambda} & 1 - e^{-180\lambda} \\
    1 & 1 - e^{-24\lambda} & 1 - e^{-48\lambda} & 1 - e^{-72\lambda} & 1 - e^{-96\lambda} & 1 - e^{-120\lambda} & 1 - e^{-240\lambda} & 1 - e^{-360\lambda} \\
    1 & 1 - e^{-36\lambda} & 1 - e^{-72\lambda} & 1 - e^{-108\lambda} & 1 - e^{-144\lambda} & 1 - e^{-180\lambda} & 1 - e^{-360\lambda} & 1 - e^{-480\lambda} \\
    1 & 1 - e^{-48\lambda} & 1 - e^{-96\lambda} & 1 - e^{-144\lambda} & 1 - e^{-192\lambda} & 1 - e^{-240\lambda} & 1 - e^{-480\lambda} & 1 - e^{-600\lambda} \\
    1 & 1 - e^{-60\lambda} & 1 - e^{-120\lambda} & 1 - e^{-180\lambda} & 1 - e^{-240\lambda} & 1 - e^{-360\lambda} & 1 - e^{-720\lambda} & 1 - e^{-900\lambda} \\
    1 & 1 - e^{-120\lambda} & 1 - e^{-240\lambda} & 1 - e^{-360\lambda} & 1 - e^{-480\lambda} & 1 - e^{-720\lambda} & 1 - e^{-1200\lambda} & 1 - e^{-1800\lambda} \\
    1 & 1 - e^{-180\lambda} & 1 - e^{-360\lambda} & 1 - e^{-540\lambda} & 1 - e^{-720\lambda} & 1 - e^{-1200\lambda} & 1 - e^{-2400\lambda} & 1 - e^{-3600\lambda}
\end{bmatrix}
\begin{bmatrix}
    L_t \\
    S_t \\
    C_t
\end{bmatrix},
\]

\[ f_t = \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{12,t} \\ \varepsilon_{24,t} \\ \varepsilon_{36,t} \\ \varepsilon_{48,t} \\ \varepsilon_{60,t} \\ \varepsilon_{120,t} \\ \varepsilon_{180,t} \end{bmatrix}. \]

The state equations form a VAR(1):

\[ (f_t - \mu) = \Lambda (f_{t-1} - \mu) + \xi_t, \]

where \( \mu_L, \mu_s \) and \( \mu_c \) denote the means of the factors and

\[
\begin{bmatrix}
    L_t \\
    S_t \\
    C_t
\end{bmatrix}, \quad \Lambda = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_L \\ \mu_S \\ \mu_C \end{bmatrix}.
\]
so that Equation 3 can be specified in more detail as:

\[
\begin{bmatrix}
L_t - \mu_L \\
S_t - \mu_s \\
C_t - \mu_c \\
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\end{bmatrix}
\begin{bmatrix}
L_{t-1} - \mu_L \\
S_{t-1} - \mu_s \\
C_{t-1} - \mu_c \\
\end{bmatrix}
+ \begin{bmatrix}
\xi_{1t} \\
\xi_{2t} \\
\xi_{3t} \\
\end{bmatrix}.
\]

(4)

For the errors, the assumption is that the terms in the measurement equations are not cross-correlated, but can be correlated in the state equations. Therefore, it is allowed that the covariance between the level, slope and curvature factors may be different from zero. In other words, it is assumed that the \( \xi_j^t \) are mutually correlated, while the \( \varepsilon_{\tau,j} \) are mutually independent and uncorrelated with \( \xi_j^t \), for \( j = L, S, C \).

### 3.1 Data

In order to estimate the DNS model, nominal interest rates for maturities up to 60 months were used. Pre-DI swap rates for 12, 24, 36, 48, and 60 months were collected from Bloomberg.\(^4\) I then deflated these nominal interest rates by inflation expectations for 12, 24, and 36 months ahead. These constant maturity series were constructed based on fixed-horizon expectations, constructed from the SGS\(^5\) database of the Central Bank of Brazil.\(^6\)

To take into account the inflation risk premium, constant maturity standard deviation series of inflation expectations for 12 to 36 months ahead were built, taking the standard deviation of inflation expectations (disagreement) as a proxy for the inflation risk premium. Therefore, the nominal swap series were deflated by the product of inflation expectations and the standard deviation of inflation expectations, taken as a synthetic measure of the breakeven inflation rate.\(^7\) Brzoza-Brzezina & Kotłowski (2014) and Imakubo et al. (2018) deflate nominal yields only by inflation expectations, neglecting the inflation risk premium. Since in Brazil the inflation risk premium is likely to be sizeable, due to the inflation history of the country, as a conservative option I used disagreement in order to take the inflation risk premium in account when obtaining real yields from 12 to 60 months ahead.

While the association of the disagreement of inflation expectations to the inflation risk premium is not standard in the literature,\(^8\) some papers use inflation disagreement as a proxy for inflation uncertainty. Söderlind (2011) shows that inflation disagreement shares some dynamics with other measures of inflation risk and enters positively in a regression with breakeven inflation rates as a dependent variable, although not strongly statistically significant. Wright (2011) considers inflation disagreement as a measure of inflation uncertainty and finds that it has a positive and statistically significant impact on term premiums. Abrahams, Adrian, Crump, Moench & Yu (2016) show that the inflation risk premium from their estimated model strongly correlates with forecaster disagreement in order to take the inflation risk premium in account when obtaining real yields from 12 to 60 months ahead.

\(^4\)Tickers: PREDI360 Index, PREDI720 Index, PREDI1080 Index, PREDI1440 Index and PREDI1800 Index.

\(^5\)SGS stands for Sistema Gerenciador de Séries Temporais, in Portuguese.

\(^6\)Using Central Bank of Brazil survey data for subsequent years, I construct constant maturity standard deviation series of inflation expectations for 12 to 36 months ahead.

\(^7\)More specifically,

\[
\text{Real rate} (\tau) = \left( \frac{1 + \text{nominal yield} (\tau)}{100} \right) - 1 \times 100
\]

(5)

\[
\text{BE} (\tau) = \left( \left( 1 + \frac{\text{inf exp} (\tau)}{100} \right) \times \left( 1 + \frac{\text{sd inf exp} (\tau)}{100} \right) \right) - 1 \times 100,
\]

(6)

where \( \tau = 12, 24, 36, 48, 60 \) months; \( \text{BE} \) stands for the breakeven inflation rate; \( \text{inf exp} \) is the inflation expectation for \( \tau = 12, 24, 36 \) months ahead (equivalently, \( \tau = 1, 2, 3 \) years ahead); \( \text{sd inf exp} \) is the standard deviation of inflation expectations for \( \tau = 12, 24, 36 \) months ahead (equivalently, \( \tau = 1, 2, 3 \) years ahead); \( \text{nominal yield} \) comes from Bloomberg, in annualized rates, tickers specified above. For \( \text{BE} (48) \) and \( \text{BE} (60) \), \( \text{inf exp} (36) \) and \( \text{sd inf exp} (36) \) were used, since 36 months ahead (equivalently, 3 years ahead) is the maximum maturity one can build for inflation expectations and standard deviation of inflation expectations from the Focus/SGS database of the Central Bank of Brazil.

\(^8\)Kupfer (2018) provides a survey of several methods to estimate the inflation risk premium.
disagreement about future inflation, and is statistically significant as an explanatory variable for the inflation risk premium in a regression analysis. Ehling, Gallmeyer, Heyerdahl-Larsen & Illeditsch (2018) present empirical evidence that disagreement about inflation impacts real and nominal yields, even after controlling for several variables. In the theoretical model presented by the authors, the relationship between inflation disagreement and the inflation risk premium depends crucially on the coefficient of risk aversion. In most cases considered in their model, higher disagreement leads to a higher inflation risk premium. Importantly for Brazil, Nunes, Doi & Fernandes (2017) find that a shock on inflation disagreement, measured by the standard deviation of 12-month-ahead inflation expectations, leads to an increase in the level of the term structure of inflation risk premium, so there is evidence to use disagreement as a proxy for the inflation risk premium.

Data on swap rates, inflation expectations, and standard deviation of inflation expectations are originally on the daily frequency. Monthly averages were taken, and the nominal rates were then deflated by the synthetic breakeven rates. While swap rates for 12 and 24 months were deflated by 12 and 24-month-ahead synthetic breakevens, swaps from 36 to 60 months were deflated by the 36-month-ahead synthetic breakeven rate, the longest that can be built from the SGS database.

For long-term real yields, I use 120 and 180-month yields. These were obtained by linearly interpolating data from zero-coupon bonds available for sale to the broad public. The database therefore encompasses data from 7 maturities (12, 24, 36, 48, 60, 120 and 180-month bond yields) in annualized rates. For the estimation of the dynamic Nelson-Siegel model, the sample period runs from August 2005 to December 2018, with 161 observations. Even though I have data on nominal Pre-DI swaps—and then, from synthetic real yields—before August 2005, this month was taken as the starting month of the sample period due to the availability of information on long-term real yields (120 and 180-month yields). Bond yields used in the estimation are shown in Figure 1.

I obtain daily data, beginning in 2005, from the link http://www.tesouro.gov.br/tesouro-direto-balance-e-estatisticas, “Histórico de preços e taxas” for zero-coupon inflation-linked bonds (NTN-B principal) maturing on May 15, 2015, May 15, 2019, August 15, 2024, May 15, 2035, and May 15, 2045. I then interpolate bond yields to get constant maturity yields for 120 months (2520 business days) and 180 months (3780 business days). Finally, I transform the data to the monthly frequency, taking monthly averages.
Table 1
Estimates of parameters of the Dynamic Nelson-Siegel model

<table>
<thead>
<tr>
<th>parameter</th>
<th>initial value</th>
<th>estimate</th>
<th>std. err</th>
<th>z-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{11}$</td>
<td>0.99</td>
<td>0.97***</td>
<td>0.020</td>
<td>48.63</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>0.03</td>
<td>0.006</td>
<td>0.008</td>
<td>0.77</td>
<td>0.43</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>−0.02</td>
<td>0.009**</td>
<td>0.005</td>
<td>1.88</td>
<td>0.05</td>
</tr>
<tr>
<td>$a_{21}$</td>
<td>−0.03</td>
<td>0.03</td>
<td>0.090</td>
<td>0.42</td>
<td>0.67</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>0.94</td>
<td>0.92***</td>
<td>0.030</td>
<td>28.85</td>
<td>0.00</td>
</tr>
<tr>
<td>$a_{23}$</td>
<td>0.04</td>
<td>−0.03**</td>
<td>0.010</td>
<td>−2.31</td>
<td>0.02</td>
</tr>
<tr>
<td>$a_{31}$</td>
<td>0.03</td>
<td>−0.19</td>
<td>0.330</td>
<td>−0.56</td>
<td>0.57</td>
</tr>
<tr>
<td>$a_{32}$</td>
<td>0.02</td>
<td>0.18**</td>
<td>0.100</td>
<td>1.64</td>
<td>0.09</td>
</tr>
<tr>
<td>$a_{33}$</td>
<td>0.84</td>
<td>0.88***</td>
<td>0.040</td>
<td>17.75</td>
<td>0.00</td>
</tr>
<tr>
<td>$\mu_L$</td>
<td>8.00</td>
<td>5.84***</td>
<td>0.570</td>
<td>10.22</td>
<td>0.00</td>
</tr>
<tr>
<td>$\mu_S$</td>
<td>−1.50</td>
<td>−1.17</td>
<td>0.860</td>
<td>−1.35</td>
<td>0.17</td>
</tr>
<tr>
<td>$\mu_C$</td>
<td>−0.40</td>
<td>6.57***</td>
<td>1.220</td>
<td>−0.56</td>
<td>0.57</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.08</td>
<td>0.038***</td>
<td>0.001</td>
<td>10.94</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>parameter</th>
<th>initial value</th>
<th>estimate</th>
<th>std. err</th>
<th>z-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{sd}(\varepsilon_{12, t})$</td>
<td>1.00</td>
<td>0.60***</td>
<td>0.05</td>
<td>11.29</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd}(\varepsilon_{24, t})$</td>
<td>1.00</td>
<td>0.09***</td>
<td>0.02</td>
<td>4.80</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd}(\varepsilon_{36, t})$</td>
<td>1.00</td>
<td>0.07***</td>
<td>0.01</td>
<td>8.76</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd}(\varepsilon_{48, t})$</td>
<td>1.00</td>
<td>0.00</td>
<td>15.16</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$\text{sd}(\varepsilon_{60, t})$</td>
<td>1.00</td>
<td>0.09*</td>
<td>0.01</td>
<td>7.63</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd}(\varepsilon_{120, t})$</td>
<td>1.00</td>
<td>0.39***</td>
<td>0.04</td>
<td>9.05</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd} (L_t)$</td>
<td>1.00</td>
<td>0.16***</td>
<td>0.01</td>
<td>11.27</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd} (S_t)$</td>
<td>1.00</td>
<td>0.16***</td>
<td>0.02</td>
<td>7.86</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd} (C_t)$</td>
<td>1.00</td>
<td>0.63***</td>
<td>0.06</td>
<td>9.50</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{sd} (L_t) S_t$</td>
<td>0.20</td>
<td>−0.03*</td>
<td>0.02</td>
<td>−1.78</td>
<td>0.07</td>
</tr>
<tr>
<td>$\text{sd} (L_t) C_t$</td>
<td>0.20</td>
<td>−0.04</td>
<td>0.07</td>
<td>−0.60</td>
<td>0.54</td>
</tr>
<tr>
<td>$\text{sd} (S_t) C_t$</td>
<td>0.20</td>
<td>−0.53**</td>
<td>0.26</td>
<td>−2.02</td>
<td>0.04</td>
</tr>
</tbody>
</table>

| log likelihood | 72.16 | Akaike criterion | −0.57 |
| Schwarz criterion | −0.07 | HQ criterion | −0.37 |

***, **, *: Statistically significant at 1, 5 and 10%, respectively.

3.2 Estimation

Table 1 shows the estimated coefficients for the DNS model described in the beginning of Section 3. Starting with the state equations, the constants for the level $\mu_L = 5.84$ and the curvature $\mu_C = 6.57$ are positive and statistically significant, while the constant for the slope $\mu_S = −1.17$ is not. Current deviations of the level of the curve are impacted positively by previous deviations of the the level and of the curvature from their means, with $a_{11}$ and $a_{13}$ being significant. Deviations of the slope of the real term structure are affected by previous deviations of the slope and the curvature from its means, but not by deviations of the level, with the same dynamics happening for deviations of the curvature from its average level ($a_{22}$, $a_{23}$, $a_{32}$ and $a_{33}$ are statistically significant). Estimated standard deviations for the level, slope and curvature are all significant. Covariances between the factors are negative, but the covariance between the level and the curvature is not significant.

For the measurement equations, with the exception of the 48-month bond yield, all other estimated standard deviations are significant. The decay factor is estimated at $\lambda = 0.038$. This figure is lower than what is usually obtained when the DNS model is estimated for nominal yields, pointing to a lower speed of adjustment of real relative to nominal yields. For instance, Diebold & Li (2006) use $\lambda = 0.0609$, Diebold et al. (2006) estimate $\lambda = 0.077$, and Caldeira et al. (2010) estimate $\lambda = 0.1047$. For estimations using real yields, Brzoza-Brzozina & Kotlowski (2014) use $\lambda = 0.075$ and Imakubo et al. (2018) find $\lambda = 0.143$ using quarterly data, which would translate to $\lambda = 0.0476$ with monthly frequency.
Figure 2 shows the latent factors obtained from the DNS model and their empirical counterparts. The empirical counterpart of the level of the real yield curve is the longest available yield, the 180-month yield. The correlation between both series is 0.83. The level of the real yield curve from the model shows an almost steady decline, departing from around 8% p.a. in 2005 to a low of 4% p.a. before the Taper Tantrum in 2013, when the Chairman of the Federal Reserve System signaled the beginning of the monetary policy normalization. From this episode onwards, the level of the real curve rose again, reaching more than 6% p.a. in 2014, at the onset of the recession of 2014-2016. The level of the curve has since declined again, with an average value of 4.54% p.a. from July 2017 to December 2018 and 4.67% p.a. for the 2018 calendar year. Therefore, the estimated average level of the real yield curve ($\mu_L = 5.84$) masks substantial variability over time.

Brzoza-Brzezina & Kotlowski (2014, p.6) assume that the level of the real yield curve is equal to its natural level, which can be interpreted as monetary neutrality in the long run. If indeed the level of the curve is a proxy of its natural level, and a good estimate of the short-term real rate far in the future, the difference between the 180-month bond yield and its natural level reflects real-term premia. The left panel at the top of Figure 2 shows that the level of the real yield curve tracks the evolution of the long-term bond from below, consistent with the presence of positive real-term premia over time (Christensen & Rudebusch, 2017, p.3). This difference averages 64 basis-points over the sample period. Exceptions to this dynamic happened from February to July of 2007, from May 2012 to May 2013, and from June to October of 2014. In these episodes the real curve was unnaturally low, with negative real-term premia. The second period was marked by unconventional monetary policies and ended with the Taper Tantrum episode. The other episodes are not clear enough to identify.

To gain further insight into the nature of low real-term premia in these episodes, I compare the term premium obtained from the DNS model using Brazilian data with the term premium from ACM (Adrian, Crump & Moench, 2013) for the United States. The intention was to see if there is some transmission or spillover from US term premium to Brazilian term premium, even considering different methods (DNS and ACM).

---

10The Federal Reserve Bank of New York releases the series at this link: https://www.newyorkfed.org/research/data_indicators/term_premia.html ACM term premium is on a nominal basis, but Abrahams et al. (2016, p.20) show that US nominal term premia is driven mainly by real term premia, rather than inflation risk premia. Their model shares some similarities with ACM, being also based on yield return regressions.
As mentioned before, Brzoza-Brzezina & Kotłowski (2014) argue that the level of the real yield curve can be interpreted as the spot rate at infinity, since the monetary authority cannot infinitely influence long-term real interest rates. So the difference between the actual real yield (the empirical level of the curve) and the model-implied level can be interpreted as a proxy measure of term premium. The term premium for Brazil from the DNS model was then computed as the difference between the actual 180-month zero-coupon real yield and the model-implied level:

$$DNS \text{ term premium} = 180 - \text{month actual real yield} - \text{model implied level}$$  

Figure 3 displays the series. Both are reasonably correlated until 2014, suggesting that in normal times, or in the absence of significant domestic shocks, the US term premium can be useful to understand the Brazilian term premium. Overall, there is a weak correlation between the DNS model with real yields estimated for Brazil and the nominal term premium from ACM, 0.21 throughout the sample. It is worth noting that the correlation was much higher until the end of 2014 (0.60), becoming weaker in recent years, when several shocks hit the Brazilian economy and affected risk premia. But this is only an initial exploration into the transmission channels of risk premium; a complete analysis is beyond the scope of this paper.

Back to Figure 2, the slope factor has a correlation of 0.93 with its empirical counterpart, given by the difference of the 180-month and the 12-month yield. Likewise, the curvature factor is also highly correlated (0.95) with its empirical counterpart, given by twice the 60-month yield minus the sum of the 12-month and the 180-month yield. Finally, the last panel in Figure 2 shows the combined dynamics of the smoothed latent factors from the Diebold-Li model.

11 Considering the DNS model: $$r_{t,\tau} = L_t + S_t \left( \frac{1-e^{-\lambda \tau}}{\lambda \tau} \right) + C_t \left( \frac{1-e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right)$$, taking the limit $$\lim_{\tau \to \infty} r_{t,\tau} = L_t$$, so the level of the curve corresponds to the longest available real yield.

12 An analysis in this spirit is made by Iskrev (2018). He shows a large coefficient of correlation for term premiums in long-term bond yields of the Euro Area and the United States, but rules out causation between them, suggesting common global factors as a possible explanation for the comovement.
4. Natural Yield Curve

4.1 Model

The natural yield curve model (NYC) follows the logic of Laubach and Williams’ model, commonly used to estimate the natural interest rate. The relationship between the interest gap and the output gap is given by the IS curve:

\[ y_t - y^*_t = b(r_t - r^*_t), \]  

(8)

where \( y_t \) is the log of actual output, \( y^*_t \) is the log of potential output, \( r_t \) is the actual interest rate, and \( r^*_t \) is the natural interest rate. When the former equals the latter, the output gap is closed.

The yield curve gap – the difference between the actual real yield curve and the natural yield curve – is given by:

\[ r_{t,t} - r^*_{t,t} = (L_t - L^*_t) + (S_t - S^*_t) \left( \frac{1 - e^{\lambda \tau}}{\lambda \tau} \right) + (C_t - C^*_t) \left( \frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right). \]  

(9)

Substituting the yield curve gap in the IS curve, I arrive at:

\[ y_t - y^*_t = b_L(L_t - L^*_t) + b_S(S_t - S^*_t) + b_C(C_t - C^*_t), \]  

(10)

where \( b_L, b_S \) and \( b_C \) are the coefficients that measure the sensitivity of the output gap to each component of the curve.

Therefore, the yield curve gap can be decomposed into the level, slope and curvature gaps. Besides the IS curve, where the output gap is a function of the level, slope and curvature gaps, the model also includes a hybrid Phillips curve, that depends on lagged and expected inflation.

The state space model is given by:

\[
\begin{bmatrix}
y_{t-1} - y^*_{t-1} \\
y_t \\
\pi_t \\
L_t \\
S_t \\
C_t \\
\end{bmatrix} = \begin{bmatrix}
\alpha_y & 0 & 0 & 0 & 0 & b_L & b_S & b_C \\
\gamma & \theta & (1 - \theta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_L & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_S & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_C & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
y_{t-1} - y^*_{t-1} \\
\pi_t \\
L_t \\
S_t \\
C_t \\
\end{bmatrix} + \begin{bmatrix}
\varepsilon^Y_t \\
\varepsilon^\pi_t \\
\varepsilon^L_t \\
\varepsilon^S_t \\
\varepsilon^C_t \\
\end{bmatrix}
\]

(11)

\[
\begin{bmatrix}
L_t \\
S_t \\
C_t \\
\end{bmatrix} = \begin{bmatrix}
L^*_t \\
S^*_t \\
C^*_t \\
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \varepsilon'^Y_t \\
0 & 1 & 0 & 0 & 0 & \varepsilon'^\pi_t \\
0 & 0 & 0 & 1 & 0 & \varepsilon'^L_t \\
0 & 0 & 0 & 0 & 1 & \varepsilon'^S_t \\
0 & 0 & 0 & 0 & 1 & \varepsilon'^C_t \\
\end{bmatrix}
\]

(12)

The IS equation described above (Equation 10) had to be modified on empirical grounds. First, a lagged output gap term was included to account for the autocorrelation of this variable and improve the fit of the model. Second, the IS curve was estimated only with the second lag of the gap of the latent factors relative to their natural levels, also due to empirical reasons. An estimated version of Equation 10, which contained only the first lag, produced positive estimated coefficients, implying that real yield curve components (level, slope and curvature) above their natural counterparts lead to output above potential, which is not a reasonable economic
result. Only with the second lag was it possible to obtain negative coefficients, implying that real yield curves above their natural level slow down the economy, with output below potential. In the estimations, it was also found that the fit of the model improved when the second lag of the latent factors was included in the IS curve, managing to achieve a higher likelihood and with the expected negative sign, consistent with the evidence that monetary policy impacts real activity with a lag of around 6 months, considering the quarterly frequency of the data. This is the reason behind the different specification of the IS curve in Equation 10 and the IS equation implied by the equations in 11. Brzoza-Brzezina & Kotłowski (2014) also specify the number of lags in their measurement equations on an empirical basis.

4.2 Data

In order to estimate the NYC model, the following series were used: output gap as calculated by Instituto de Pesquisa Econômica Aplicada (IPEA),\(^\text{13}\) market (non-regulated) inflation, available from the Central Bank database (code 11428), latent factors (level, slope and curvature) from the estimated Diebold-Li model estimated in Section 3, and one-quarter-ahead inflation expectations from the Focus survey.\(^\text{14}\) The NYC was estimated at quarterly frequency, so as to align with the frequency of real GDP. Data from the output gap is originally at the quarterly frequency, so no transformation was required. For the other series, data was originally monthly, so quarterly averages were taken to arrive at quarterly data. The data used is shown in Figure 4. The sample period runs from 2006Q1 to 2018Q4, for a total of 52 observations.

\(^{13}\)Available at http://www.ipea.gov.br/cartadeconjuntura/index.php/tag/hiato-do-produto/.

\(^{14}\)This variable was constructed in the following way: the SGS database of the Central Bank of Brazil contains daily data with expectations for the monthly expected headline inflation for several months ahead. For each day, I get the expected inflation for the subsequent three quarters, i.e., on a given day in January, I get what the market was expecting IPCA monthly inflation to be in February, March and April, and compute the geometric average of these three expected figures. Next, for each month, I take the average of the daily figures within the month to get the expected inflation one quarter ahead of a given month. Finally, I take the 3-month moving average of this series as the measure of expected inflation one quarter ahead. Therefore, the expected inflation one quarter ahead of the first quarter, for instance, is the average of the expected inflation for February, March and April (quarter ahead in January), March, April and May (quarter ahead in February) and April, March and June (quarter ahead in March).
Table 2

<table>
<thead>
<tr>
<th>parameter</th>
<th>initial value</th>
<th>estimate</th>
<th>std. err</th>
<th>z-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_y)</td>
<td>0.50</td>
<td>0.70***</td>
<td>0.17</td>
<td>4.14</td>
<td>0.00</td>
</tr>
<tr>
<td>(b_L)</td>
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<td>0.17</td>
<td>0.37</td>
<td>0.70</td>
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<td>(b_S)</td>
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<td>0.11</td>
<td>-2.30</td>
<td>0.02</td>
</tr>
<tr>
<td>(b_C)</td>
<td>0.50</td>
<td>-0.03</td>
<td>0.05</td>
<td>-0.51</td>
<td>0.60</td>
</tr>
<tr>
<td>(\gamma)</td>
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<td>0.0006</td>
<td>0.06</td>
<td>0.09</td>
<td>0.92</td>
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<tr>
<td>(\theta)</td>
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<td>(\alpha_L)</td>
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<tr>
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<td>0.81***</td>
<td>0.12</td>
<td>6.69</td>
<td>0.00</td>
</tr>
<tr>
<td>(\alpha_C)</td>
<td>0.75</td>
<td>0.72***</td>
<td>0.10</td>
<td>6.99</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>parameter</th>
<th>initial value</th>
<th>estimate</th>
<th>std. err</th>
<th>z-statistic</th>
<th>p-value</th>
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<tr>
<td>(sd(\varepsilon_y^t))</td>
<td>2.37</td>
<td>0.66***</td>
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<td>3.17</td>
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<td>(sd(\varepsilon_L^t))</td>
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<td>0.80***</td>
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<td>8.58</td>
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<td>(sd(\varepsilon_S^t))</td>
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<td>0.00</td>
<td>6742.0</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>(sd(\varepsilon_C^t))</td>
<td>2.00</td>
<td>1.03***</td>
<td>0.14</td>
<td>7.93</td>
<td>0.00</td>
</tr>
<tr>
<td>(sd(\varepsilon_L^*^-t))</td>
<td>4.65</td>
<td>2.80***</td>
<td>0.38</td>
<td>7.36</td>
<td>0.00</td>
</tr>
<tr>
<td>(sd(\varepsilon_S^*^-t))</td>
<td>2.00</td>
<td>0.61***</td>
<td>0.26</td>
<td>-2.34</td>
<td>0.01</td>
</tr>
<tr>
<td>(sd(\varepsilon_C^*^-t))</td>
<td>2.00</td>
<td>2.14*</td>
<td>1.29</td>
<td>1.64</td>
<td>0.09</td>
</tr>
<tr>
<td>(sd(\varepsilon_C^*^-t))</td>
<td>4.00</td>
<td>0.00</td>
<td>20590.0</td>
<td>0.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\[**\text{log likelihood} = -364.0\]
\[\text{Akaike criterion} = 14.65\]
\[\text{Schwarz criterion} = 15.29\]
\[\text{HQ criterion} = 14.90\]

***, **, *: Statistically significant at 1, 5 and 10%, respectively.

4.3 Estimation

Estimated results of equations in 11 and 12 are shown in Table 2. The output gap is affected by its own lag, represented by the coefficient \(\alpha_y = 0.70\). The level and the curvature gaps \((b_L \text{ and } b_C)\) are not statistically significant in the equation of the output gap, only the slope gap \((b_S = -0.25)\), with a negative coefficient, meaning that a slope of the real yield curve above its natural level tends to slow down the economy. The standard deviation of the IS equation was also significant \((sd(\varepsilon_y^t) = 0.66)\).

Inflation is expressed as a function of the output gap and a combination of past and expected inflation. The slope of the Phillips curve \((\gamma = 0.006)\) is not statistically significant, consistent with the narrative of the flattening of the curve worldwide, with inflation depending less on the slack of the economy over the last decades. In the model, inflation is mainly a function of its lags \((\theta = 0.72)\), and to a lesser extent of expected inflation. The standard deviation of the Phillips curve is statistically significant \((sd(\varepsilon_\pi^t) = 0.80)\).

The latent variables are a function of their lagged and natural values. All estimated coefficients are significant in the model \((\alpha_L = 0.59, \alpha_S = 0.81, \alpha_C = 0.72)\), with an overall high degree of persistence. As for the standard deviations, the one for the level \(sd(\varepsilon_L^t)\) is not significant, while those for the slope \(sd(\varepsilon_S^t) = 1.03\) and the curvature \(sd(\varepsilon_C^t) = 2.80)\) are.

Finally, the standard deviations of the state variables were significantly estimated for the level and the slope \((sd(\varepsilon_L^*^-t) = 0.61, sd(\varepsilon_S^*^-t) = 2.14)\).

Figure 5 shows the evolution of the real yields and their natural counterparts through the lens of the estimated model, while Figure 6 displays the estimated factors and their natural levels. Overall, natural yields show a downward trend over time. This trend was more pronounced from 2006 to 2012, against the backdrop of lower real yields in the US and stable risk premiums. This trend came to an end in 2013, with the Taper Tantrum and shocks to the risk premium. After some volatility caused by risk premium shocks during the 2014-

\[\text{To obtain the natural, “star” yields, I used } \lambda = 0.12 \text{ at the quarterly frequency, corresponding to estimated monthly figure in the DNS model (} \lambda \text{ in Table 1) times 3.}\]
2016 recession, the downward trend in real rates resumed after the “Trump Tantrum” following the election of Donald Trump in late 2016. Markets began to price in higher US rates in anticipation of the expected economic policies of Donald Trump, as the widespread view was that protectionist policies coupled with expansionist fiscal policies would fuel inflation, ultimately requiring higher interest rates by the Federal Reserve System. Bond yields then rose, negatively impacting emerging markets, with magnitudes comparable to that of the Taper Tantrum in 2013.\textsuperscript{16}

The last panel in Figure 5 shows the sum of the difference between real yields and their natural counterparts.\textsuperscript{17} Deviations from zero show periods of time when the real yield curve was unnaturally low or high. The curve was unnaturally low from 2012 to 2014, in the run-up to the Taper Tantrum, and unnaturally high in 2015-2016, with risk premiums all across the board during the recession.

The models estimated here also help to understand whether monetary policy has been sufficiently stimulating recently by comparing the short-term real yields and their natural counterparts. From 2006 to 2009, short-term real rates were aligned with their natural counterparts. From 2009 to 2014, short-term real rates were below their natural levels, with different degrees of magnitude. The largest negative gap was observed in the first quarter of 2013. Short-term rates were then set too high relative to long-run short-term natural real rates between 2015-2016, in a context of unanchored inflation expectations and a shock on regulated prices. From 2017Q3 to the end of the sample, short-term real rates were again close to their natural levels, showing policy rates close to neutral.

Figure 7 compares the slope gap of the real yield curve and the output gap. The series are highly correlated (0.80), and a formal test points to Granger causality from the slope gap to the IPEA output gap.\textsuperscript{18} Other components of the curve show milder correlations with the output gap. The curvature gap is negatively correlated (-0.46), while the level gap is almost uncorrelated with the output gap (-0.05). Since the slope is interpreted as the short-term factor of the yield curve, directly under control of monetary policy and driven by cyclical


\textsuperscript{17} $\sum_{\tau=12,24,36,48,60,120,180\text{ months}}(\text{real yield}(\tau) - \text{natural yield}(\tau))$

\textsuperscript{18} Based on a VAR with 3 lags from 2005Q4 to 2018Q4, p-value = 0.04. This result is not robust to the IFI output gap
Figure 6
Natural and actual factors

Real and natural levels
Real level
Natural level

Real and natural slopes
Real slope
Natural slope

Real and natural curvatures
Real curvature
Natural curvature

Level and output gaps
Level gap
Output gap

Slope and output gaps
Slope gap
Output gap

Curvature and output gaps
Curvature gap
Output gap
The natural yield curve in Brazil

Figure 8
Natural rate in real time

180-month real yield, level and natural level

<table>
<thead>
<tr>
<th>Year</th>
<th>Real Yield</th>
<th>Level</th>
<th>Natural Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>2008</td>
<td>7.5%</td>
<td>5%</td>
<td>6%</td>
</tr>
<tr>
<td>2010</td>
<td>5.5%</td>
<td>4.5%</td>
<td>4%</td>
</tr>
<tr>
<td>2012</td>
<td>7%</td>
<td>6%</td>
<td>5%</td>
</tr>
<tr>
<td>2014</td>
<td>6%</td>
<td>5.5%</td>
<td>5%</td>
</tr>
<tr>
<td>2016</td>
<td>5.5%</td>
<td>5%</td>
<td>4%</td>
</tr>
<tr>
<td>2018</td>
<td>4.5%</td>
<td>4.25%</td>
<td>4%</td>
</tr>
</tbody>
</table>

Figure 8 shows the evolution of the long-term real yield, the level of the curve from the DNS model, and the natural level from the NYC model. Following the taxonomy of Roberts (2018), the long-run, long-term natural real rate declined from 7.5% p.a. in the beginning of 2006 to 4.75% p.a. in the last quarter of 2018. This downward trend hides substantial variability over time. The global financial crisis of 2008 and the aftermath of the Taper Tantrum in 2013 led to substantial increases in equilibrium real rates, due to the combination of higher US real rates and spikes in risk premiums. In both episodes, long-term natural rates reached more than 6.5% per annum. The lowest value for equilibrium long-run, long-term real rates happened in late 2012, as a consequence of negative US real rates and well-behaved risk premiums, with long-term natural rates dropping to 4% per annum.

The dynamics of the long-term natural rate can be divided in periods. From 2006 to 2010, the natural rate oscillated between 5.5 and 7.5 p.a. From 2010 to 2013, the natural rate declined to the range 4-5.5 p.a.. From 2014 to 2016, the natural rate rose to somewhere between 4.5 and 7% p.a.. From late 2016 to 2018, the natural rate dropped to the range 4.5 - 5% p.a..

4.4 Robustness

In this section, the NYC model is estimated using output gap from Instituição Fiscal Independente (IFI),\textsuperscript{19} to check the robustness of the results. In this case, the sample period again runs from 2006Q1 to 2018Q4, with a total of 52 observations.

Estimated results of Equations 11 and 12 are shown in Table 3. The output gap is again affected by its own lag, represented by the coefficient $\alpha_y = 0.72$ and the slope gap ($b_S = -0.27$). The level and the curvature gaps ($b_L$ and $b_C$) are again not statistically significant in the equation of the output gap. The standard deviation of the IS equation is statistically significant $sd(\varepsilon_t^y) = 0.56$.

\textsuperscript{19}IFI corresponds to the Brazilian Fiscal Council.

\textsuperscript{20}Link: https://www12.senado.leg.br/ifi/dados/arquivos/estimativas-do-hiato-do-produto-ifi/view
### Table 3

**Estimation of the NYC model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Initial Value</th>
<th>Estimate</th>
<th>Std. Err</th>
<th>z-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_y$</td>
<td>0.50</td>
<td>0.72***</td>
<td>0.18</td>
<td>3.97</td>
<td>0.00</td>
</tr>
<tr>
<td>$b_L$</td>
<td>0.80</td>
<td>0.17</td>
<td>0.16</td>
<td>1.04</td>
<td>0.29</td>
</tr>
<tr>
<td>$b_S$</td>
<td>0.50</td>
<td>-0.27***</td>
<td>0.11</td>
<td>-2.41</td>
<td>0.01</td>
</tr>
<tr>
<td>$b_C$</td>
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| Log likelihood | -362.0 | Akaike criterion | 14.58 |
| Schwarz criterion | 15.22 | HQ criterion | 14.82 |

***, **, *: Statistically significant at 1, 5 and 10%, respectively.

For inflation, the same results emerge. The slope of the Phillips curve ($\gamma$) is again not statistically significant. Inflation depends on a combination of its lagged value ($\theta = 0.71$) and expected inflation. The standard deviation of the Phillips curve is again statistically significant $sd(\varepsilon^\pi_t) = 0.80$.

The latent variables are expressed as a linear combination between their lagged and natural values. All estimated coefficients are statistically significant in the model ($\alpha_L = 0.59$, $\alpha_S = 0.91$, $\alpha_C = 0.99$). The estimated standard deviation for the level $sd(\varepsilon^L_t)$ is not significant, while those for the slope and the curvature are ($sd(\varepsilon^S_t) = 1.16$, $sd(\varepsilon^C_t) = 3.04$).

Lastly, the standard deviations of the state “star” variables are significantly estimated only for the level.

All in all, results are quite close to those obtained using the output gap from IPEA. The estimated coefficients from both models are quite similar to each other.

Figure 9 plots the natural level, slope and curvature from both models. The main difference is observed for the natural slopes of each model. But even in this case, both series are highly correlated (0.92) and follow a similar pattern over time. Levels from both models are almost perfectly correlated, meaning that both specifications of the NYC model give basically the same information about the evolution of the natural real rate over time. The curvature factors also differ, but this is probably due to the fact that the natural curvature in the model with IFI output gap is not statistically significant.
5. Conclusion

This paper estimated the natural yield curve for Brazil, a concept introduced by Brzoza-Brzezina & Kotłowski (2014) and Imakubo et al. (2018) that extends the concept of natural rate to the yield curve. This follows the strand of the literature that tries to obtain measures of natural rates from financial market data (Bomfim, 2001; Christensen & Rudebusch, 2017; Ajevskis, 2018).

The methodology allows the decomposition of the yield curve into dynamic latent factors (level, slope and curvature). The level of the curve is considered the long-run factor, unaffected by monetary policy, so its dynamics can be interpreted as a proxy for the natural (r-star, equilibrium, neutral) rate in real time, which can be useful to policymakers, academics, and financial market participants who try to gauge the natural rate of the economy. Since this method departs from observed real yields, which are essentially forward-looking, they attenuate the problems related to model specification and end-of-sample filtering that usually affect macroeconomic estimates of the natural rate (Clark & Kozicki, 2005; Christensen & Rudebusch, 2017).

The models show that the long-run, long-term natural real rate dropped from around 7% p.a. in 2006 to 4.75% p.a. by the end of 2018. This downward trend was not smooth. After skyrocketing during the global financial crisis in 2008, natural rates began to descend, reaching a trough in late 2012. Then natural long-term real rates climbed again to a peak of 6.5-7% p.a. in the aftermath of the Taper Tantrum in 2013 and the onset of the recession of 2014-2016. Once markets stabilized after the Trump election in late 2016, coupled with lower risk premiums, natural real rates resumed their downward trend in 2017.

It was also found that the slope gap—the difference between the slope implied by the DNS model and its natural level—is highly correlated with the output gap. This correlation is striking, given that many shocks are at play for the dynamics of the output gap. This finding is consistent with the interpretation that the slope of the curve is a short-term cyclical factor affected by monetary policy. Since short-term real rates are under the control of the central bank, this indicates that the real slope gap can be considered a reliable indicator of the stance of monetary policy.

The natural yield curve model can be extended in many ways, with the inclusion of external, fiscal and credit variables, along with richer dynamics of their interactions. Macro-finance models such as the one used in this paper are a promising and realistic way to integrate macroeconomics and finance. The methodology in
this paper is also useful for pricing long-term assets and liabilities.

References


The natural yield curve in Brazil


