

# The quantum of knowledge theory

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Summary: 1. Introduction; 2. The model; 3. The dynamics of the quantum of knowledge; 4. Conclusion.

In this paper, we present a new microfoundation based on knowledge level diversification. Knowledge in this model is present in both human beings and machines. In the former, knowledge is embodied in the form of schooling years plus job experience while in the latter, knowledge is present in the form of knowledge required to operate it. Firms must combine the knowledge level required by the machines with the knowledge level embodied in the workers in the final sector. Each new combination of knowledge level forms what is called *quantum of knowledge*. Our findings are that wages, profit and productivity are positively related to the *quantum of knowledge*. Therefore, this theory gives support to the applied human capital models, specially those related to education and learning by doing, developed by Ram (1990), Ramos (1991), Leal & Werlang (1991), and Feenstra, Markusen & Seile (1992). On the growth side, this theory also gives microfoundation support to the linear models of King & Rebelo (1990) and Barro (1990), where the productivity factor is exogenous, with the advantage of making the productivity factor endogenous to the economy. So, using this microfoundation, we built an economic growth model to analyze the effect of the endogenous productivity factor on the real growth rate of the economy. To finish, we analyze the effect of government waste on the economy productivity growth.

Neste artigo apresentamos um novo fundamento microeconômico baseado em níveis de conhecimentos diversificados. O conhecimento, neste modelo, está presente em seres humanos e máquinas. No primeiro, na forma de anos de escolaridade e experiência no trabalho, e no segundo, na forma de conhecimento requerido para serem operadas. A principal restrição é que as firmas do setor final devem combinar o nível de conhecimento requerido pelas máquinas com o dos trabalhadores. A cada nova combinação de nível de conhecimento forma-se o que chamamos de *quantum de conhecimento*. Os primeiros resultados teóricos indicam que salários, lucros e produtividade estão positivamente relacionados com o *quantum de conhecimento*. Portanto, esta teoria fornece suporte aos modelos de capital humano aplicado, especialmente aqueles relacionados a educação e *learning by doing*, desenvolvidos por Ram (1990), Ramos (1991), Leal & Werlang (1991) e Feenstra, Markusen & Seile (1992). Com respeito ao crescimento econômico, esta teoria fornece suporte de fundamento microeconômico para os modelos lineares desenvolvidos por King & Rebelo (1990) e Barro (1990), onde o principal problema é o fator de produtividade ser considerado exógeno, sendo que, nesta teoria, o fator de produtividade é endogeneizado. Assim, usando este fundamento microeconômico, construímos um modelo de crescimento econômico para analisar o efeito do fator de produtividade endógeno na taxa de crescimento real da economia. E, para finalizar, analisamos o efeito do desperdício governamental no crescimento da produtividade da economia.

## 1. Introduction

The objective of this paper is to develop a microfoundation for an economic growth theory that is consistent with human capital diversity or labor specialization existent in the economy. It is also our objective to incorporate an assumption on our economic growth

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model that is compatible with the continuous wide spread in the low and high knowledge levels existing in society (Ram, 1990).

More recently, human capital, also referred to as knowledge, has assumed the role of the physical capital of the early growth models (Solow, 1956). Therefore, knowledge is now treated as a commodity. By assuming this physical condition, knowledge acquires the condition of being able to be accumulated and, most important of all, measured.<sup>1</sup>

The first economist to unveil the importance of knowledge was Kaldor (1980), but Uzawa (1965) was the one that formally introduced the idea of a third sector responsible for knowledge accumulation, the educational sector. This idea, later improved by Lucas (1988) and more recently by Romer (1990), regards human capital as the engine for growth. A further refinement in the economic growth theory was done by Kim & Mohtadi (1992), which is the link between human capital diversity and firm production.

The links between human capital diversity and production will emerge naturally from our microfoundation specification. However, our model will differ in a number of ways to the previously mentioned authors. It introduces a new way of seeing human capital and goods. The following two paragraphs give a brief description of this new idea.

The new microfoundation to be developed here is based on knowledge diversity. More specifically, this new theory is based on the hypothesis that society's overall knowledge is represented by a distribution function in which each individual has a specific knowledge level. Here, we do not assume that the level of knowledge is exactly the same for every individual. At the same time, we do not restrict the knowledge embodied in everyone to be different. If two or more individuals have the same knowledge level then they belong to the same group.

This knowledge level diversity idea is then extended to intermediate goods. Intermediate goods are distributed according to the level of knowledge they require to be operated. This required knowledge is determined during their production. In this model, intermediate goods are produced by workers with specific knowledge level that is transferred to the produced good. The important aspect of this intermediate good is that it can only be used by the final sector if it is combined with workers of the same (or superior) level of knowledge as the one required by the good. The point to be stressed here is that both human beings and goods are diversified according to their knowledge level.

The utilization of both factors by the final sector, as mentioned above, refers to the combination of workers and intermediate goods at the same level of knowledge. The final sector firm then matches the knowledge embodied in the worker with the knowledge required by the intermediate good. This perfect match of knowledge levels will be referred to in this paper as *the quantum of knowledge*. In other words, quantum of knowledge refers to the same level of knowledge that is embodied in both humans and goods when combined with each other. This can be translated to reality as the capacity to use an specific machine to produce a good or service — the capacity to know how to do.

The above description of the quantum of knowledge is what makes this theory to differ on its basis from the more traditional human capital approach and from the previously mentioned authors. It requires that for an existing human knowledge level there is a counterpart required knowledge level from a good for both to be used in the production process.

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<sup>1</sup> Although its measurement seems far from easy, its possibility is widely assumed. The most recent attempt to measure society's stock of knowledge was done by Adams (1990).

## 2. The model

The quantum of knowledge theory combines the features of the models based on purely human capital diversity (eg. Lucas, 1988) with those models that are based purely on intermediate goods diversity (eg. Romer, 1987). The incorporation of these advanced features in an economic model will be made through the introduction of a new microeconomics approach for the final sector. To present this approach, as clearly as possible, simplifying assumptions will be made. This is because the main objective of this paper is to introduce the idea of quantum of knowledge as the *motor of economic growth*.

### *The final sector*

The final sector industries that we have in mind here are the retail stores, food stores, car sales, and so on. Although these sectors sell all kinds of merchandise, their output is service. Therefore what we are buying from them when buying their goods is service. So their service (output) is to put all the merchandise together to ease the consumers load in selecting and purchasing commodities.

According to the above paragraph each industry in the final sector produce a type of output combining workers and intermediate goods. However, each industry combines workers of different knowledge levels with intermediate goods that require also different knowledge levels to operate. This combination of knowledge levels gives the quality to the output of the final sector firm. For instance, the car sales industry produces an output that is different from the hospital industry. Their output differs mainly because of the knowledge level involved in production of each one. Thus, diversity of goods in our model is associated with this knowledge level that makes the output of the final sector firms to differ.

From the purely economic point of view, the knowledge level required by the machine prevents firm owners from combining it with a cheaper, less knowledgeable worker, while a more knowledgeable worker also prevents firm owners from combining him with a machine that require less knowledge to be operated, for this would be more expensive. Thus, to maximize profits firm owners would have to match workers and intermediate goods of the same knowledge level. To be consistent with this idea, we assume the production function in the final sector to have the following form:

$$y(i) = [\theta h(i)]^\alpha m(i)^{1-\alpha} \quad (1)$$

where  $y(i)$  is the output of the typical final sector firm in industry  $i$ ;  $h(i)$  is the amount of knowledge level  $i$ ;  $m(i)$  is the required knowledge to operate intermediate goods type  $i$ ;  $0 < \theta < 1$  is the fraction of knowledge spent producing final good; and  $0 < \alpha < 1$  is the parameter of the production function.

In equation (1) we have that  $h(i)$ , the amount of knowledge level  $i$ , is equal to the following:  $h(i) = i h_p$  where  $i$  is the knowledge level and  $h_p$  is the number of hours available of that knowledge level. On the machines side, we have that  $m(i) = i m_p$  where  $i$  is the required knowledge level and  $m_p$  the number of machines.

To capture some aspects of reality, we allow that the total knowledge to be assembled can be done by taking several workers with different knowledge levels such that their average knowledge level is  $i$ . Also, embodied in this feature is the amount of hours that can be

assembled in the same way. Mathematically, in reality, we are allowing the following:

$\int_z^{z'} xg(x)dx = i$ , where  $z, z'$  and  $x \in [1, q]^2$ ,  $g(x)$  is the knowledge weight in the production

and  $\int_z^{z'} xh_x dx = h(i)$ , where  $h_x$  is the number of hours of knowledge level  $x$ .

The most important aspect in equation (1) is the fact that the productivity and quality is driven by the combination of workers' and machines' knowledge level. According to our set up, type  $(i + \Delta)$  machine and workers, where  $\Delta$  is positive, are more productive and at the same time produce a better quality output than type  $i$ .

Now let us move our attention to the number of firms in the final sector. Let us assume that there are  $N$  identical competitive firms there. Therefore, the total production in the final sector industry  $i$  is the following:

$$Y(i) = y(i)N = [\theta i h_i N]^\alpha [i m_i N]^{1-\alpha} = i [\theta H_i]^\alpha [M_i]^{1-\alpha} \quad (2)$$

In equation (2),  $i h_i N = i H_i = H(i)$  is the total knowledge available of level  $i$ ;  $i m_i N = i M_i = M(i)$  is the total number of machines that requires knowledge level  $i$ . Thus, the knowledge level  $i$  in equation (2) refers to the combination of workers' and intermediate goods' knowledge level. Hence, we call it now on quantum of knowledge  $i$ .

As one may see in equation (2), each industry in the final sector is in charge to produce an output of a specified quality. And this quality is set by the combination of the workers' and intermediate goods' knowledge level, the quantum of knowledge.

To set a limit to the interval of the quantum of knowledge, we assume it to be continuous in the interval  $[1, q]$ , thus  $i \in [1, q]$  with 1 being the lowest quantum of knowledge while  $q$  the highest one. So, the final sector industries have a distribution function that includes firms producing output type 1, the one that has embodied the lowest quantum of knowledge, and firms producing output type  $q$ , the one that has the highest quantum of knowledge. Moreover, it is easy to see that among these industries will be the industry that produces the good with the average quantum of knowledge.

Before we move on in our analysis, we want to call attention to one important characteristic of equation (2). When there is no diversity, which implies that the quantum of knowledge is equal to 1, equation (2) becomes the usual Cobb-Douglas production function.

The objective of each firm in the final sector is to maximize profits. Recall the assumption about the final sector industry — there are  $N$  identical competitive firms. Thus, equation (1) gives us the form of their production function. Combining these two facts, we write the profit function for the firm as

$$\Pi = \{i [\theta H_i]^\alpha [M_i]^{1-\alpha} - w(i) \theta H_i - p(i) M_i\} \left(\frac{1}{N}\right) \quad (3)$$

<sup>2</sup> Here 1 represents the lowest knowledge level while  $q$  the highest one.

Equation (3) is in terms of the final good  $y(i)$ ;  $w(i)$  is the wage of human capital with knowledge  $i$ , in terms of the final good  $y(i)$ ; and  $p(i)$  is the price of the intermediate good  $i$ , also in terms of the final good  $y(i)$ .

Each firm's objective is to maximize equation (3). In order to maximize equation (3), firm chooses the quantity of hours and number of machines. Thus, the first-order conditions for this problem are

$$\frac{\partial \Pi}{\partial \theta H(i)} = 0, \quad w(i) = \alpha i [\theta H_i]^{\alpha-1} M_i^{1-\alpha} \quad \text{and} \quad (4)$$

$$\frac{\partial \Pi}{\partial M(i)} = 0, \quad p(i) = (1-\alpha) i [\theta H_i]^\alpha M_i^{-\alpha} \quad (5)$$

There will be a continuum of equations (4) and (5) in the interval  $i \in [1, q]$ . These two equations represent the inverse demand functions for workers and intermediate sector goods by final sector industry  $i$ . Equations (4) and (5) are the real wage and price of workers and goods with quantum of knowledge  $i$ . They are positively related to the quantum of knowledge and negatively related to their available quantity. Thus, holding everything else constant, real wage increases with the quantum of knowledge. The same holds for the intermediate goods real price.<sup>3</sup> This means that each worker and intermediate good will receive a real wage and price compensation that takes into account their respective knowledge levels.<sup>4</sup> Equation (4) and (5) also shows that the real wage and price will differ from one category to the other by three factors: the quantum of knowledge, the total amount of hours of labor,  $H_i$ ; and the total number of intermediate goods,  $M_i$ .

The implicit demand functions for both human capital and intermediate goods with knowledge  $i$  can be derived from equations (4) and (5). The desired result is easily obtained by the rearranging of equations (4) and (5) and by using  $M(i)$  and  $H(i)$  definitions. We should recall that  $h_i$  and  $m_i$  are given at any point in time.

$$v(i) |_N = \frac{\theta h_i}{m_i} = \left( \frac{\alpha \cdot i}{w(i)} \right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad (6)$$

$$v(i) |_N = \frac{\theta h_i}{m_i} = \left( \frac{p(i)}{(1-\alpha) \cdot i} \right)^{\frac{1}{\alpha}} \quad (7)$$

As we can see in the above equations, given values for  $h_i$ ,  $m_i$  and  $\theta$ , the real wage and price of the intermediate good will adjust so that both equations will be equal. Both equations are influenced by the level of knowledge. Interpreting these as implicit demands, an increase in the number of hours,  $h_i$ , would be followed by an equal increase in the number of intermediate goods,  $m_i$ , holding  $\theta$  constant. If the time devoted to the production of the final

<sup>3</sup> These results are very similar to the ones obtained by Romer (1987) and Goodfriend & McDermott (1992).

<sup>4</sup> It can be regarded as the quality factor in the price determination.

sector good,  $\theta$ , increases, so will the number of intermediate goods, holding  $m_i$  constant. Also these equations show that any improvement in quality would increase  $w(i)$  and  $p(i)$ .

The two exogenous variables in this factor market clearance are  $\theta$  and  $i$ . The first,  $\theta$ , is the percentage of hours of labor used by the final sector; the second,  $i$ , is the quantum of knowledge itself. As we shall see, the first variable is determined by the intermediate sector when maximizing its profit while the second is controlled by the consumers when deciding how much to consume.

### *The intermediate sector*

The intermediate sector is formed by a continuum of firms where each firm produces a unique type of intermediate good, let's say  $M(i)$ . Their monopoly condition arises from the following three conditions: first, a firm willing to enter in the market would have to spend time and money learning about which goods are being produced and what type of workers are available in the final sector; second, once it has learned about its supply and demand conditions, it would choose to produce an intermediate good not existent in the market; third, given the imposed learning cost, let's call it  $C$ , we assume that the intermediate firm would produce the intermediate good such that the stream of future profits, the expected return on the investment ( $E$ ), is equal to  $C$ .

The last assumption is necessarily to avoid the introduction and analysis of a market that sells information on what type of intermediate good to produce. Certainly, this market would emerge in the case that  $C < E$ . However, competition there would lead to the condition that  $C = E$ . Thus, a firm buying the information in this market would be as good as the imposed assumption. Recall, however, that the learning cost,  $C$ , takes place before the production starts, therefore this is a fixed cost.<sup>5</sup>

On the basis of the above conditions any new entrant would be better off using this fixed cost to search for a type of good not yet produced, otherwise it would incur in the same cost ( $C$ ) and share the returns with a competitor. If this happens, the resulting Bertrand competition would lead the two firms to earn zero profits (Tirole, 1988). Thus, these conditions guaranty the monopoly condition to the firm producing an intermediate good. The above rational process forms the basis on which monopolies always will be formed to access the new knowledge. For simplicity, we assume that firms being created are owned by the workers. This will avoid the introduction of an asset market. Moreover, we also assume that to produce the intermediate good, the intermediate sector firm uses only workers with the specified knowledge. More precisely, we assume the production function to be a constant return to scale production function of the following type:

$$M_i = [1 - \theta] H_i \quad (8)$$

Equation (8) tells us that the complementary time,  $(1 - \theta)$ , devoted by the workers with knowledge  $i$  in the production of good  $y(i)$  is employed in the production of the intermediate good  $M(i)$ . If we let  $\phi$  to be the time devoted to the intermediate sector, then  $\theta + \phi = 1$ .

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<sup>5</sup> Romer (1990)'s model is based on the existence of a market as the one described here which he calls design market.

Hence, as one can see, leisure time is not taken into consideration by this model. The working time is given, the decision is the allocation between how much to work in the intermediate or in the final sector.

The objective of the monopolist is to maximize profit. The monopolistic firm profit function is as thus:

$$\pi = p(i) M_i - w(i) [1 - \theta] H_i \quad (9)$$

When maximizing profit each monopolist in the intermediate sector takes as given the final sector demand for intermediate goods and human capital. These two demands are expressed by equations (4) and (5). We substitute these two equations into equation (9). We also substitute into it the production function given by equation (8). The aftermath equation of these procedures is

$$\pi = ((1-\alpha) \theta^\alpha [1-\theta]^{1-\alpha} - \alpha \theta^{\alpha-1} [1-\theta]^{2-\alpha}) i H_i \quad (10)$$

As shown in the above equation, the only variable that influences the profit of the monopolist is  $\theta$ . Recall equation (5): by choosing an optimum  $\theta$ , the monopolist is, in reality, manipulating the demand function for its good. Hence, the profit maximization problem of the monopolist is just a matter of choosing the correct parameter for the demand for its good. By maximizing his profit function, the monopolist is also setting up the parameter for the demand for workers.<sup>6</sup> Note that labor has two uses in this model: (1) to produce  $M_i$ ; (2) to cooperate with  $M_i$  in producing  $y(i)$ . So, under our assumption, the monopolist has first claim on labor, and picks  $[1 - \theta]$  to produce the intermediate good  $M_i$ . What is left goes to final good production where it contributes to the demand for  $M_i$ . Thus, the monopolist balances its need for labor and its desire for plentiful demand for  $M_i$ .

Now carrying out the maximization profit problem, we find that the value of  $\theta$  that satisfies the first order condition is

$$\theta^* = \frac{\alpha}{2} + \frac{\sqrt{\alpha} \sqrt{4 - 3\alpha}}{2} \quad (11)$$

Note in the above equation that  $\theta^*$  is real for values of  $0 < \alpha < 1$ . Now substitute  $\theta^*$  and (8) into equations (4) and (5). Following simplifications, the following result appears:

$$w(i) = A i, \text{ where } A = \alpha (\theta^{*\alpha-1} [1-\theta^*]^{1-\alpha}) \text{ and} \quad (12)$$

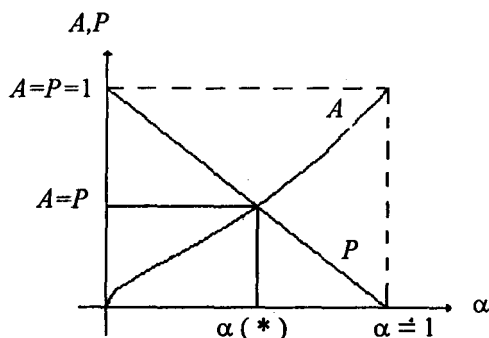
$$p(i) = P i, \text{ where } P = (1-\alpha) (\theta^{*\alpha-1} [1-\theta^*]^\alpha) \quad (13)$$

Consider equation (12). The real wage depends in last instance only upon the parameter of the production function,  $\alpha$ , and the quantum of knowledge variable,  $i$ . The number of workers with quantum of knowledge  $i$  does not affect the real wage. These results are intu-

<sup>6</sup>A similar condition appears in Goodfriend & McDermott (1992), but unlike their model intermediate firms are also buyers of the particular labor that they need.

itively pleasing because growth in the primary factor, labor, does not reduce compensation to that factor. Moreover, it links real wage solely to the quantum of knowledge. The linear relationship tells us that the workers' real wage differs from one category level to the other by the productivity factor and by their knowledge level. Moreover, the productivity parameter lies between zero and one,  $0 < A < 1$ , thus workers receive compensation proportional to their knowledge level. Equation (13) exhibits similar results for the intermediate goods. The real price of the intermediate good depends only upon its quality — the quantum of knowledge. Its price will increase with its quality. Given that the parameter  $P$  lies between  $0 < P < 1$ , then it does not receive full compensation for its quality improvement. Figure 1 shows the behavior of the parameter  $A$  and  $P$  for values of  $0 \leq \alpha \leq 1$ .

Figure 1



As we can see in the above figure, there is a  $\alpha(*)$  that makes  $A = P$ . This result is important because it gives us the point where the intermediate sector profit is zero — zero profit condition. To see this, recall equation (9) and substitute in there the definitions of  $M_i$ ,  $w(i)$ , and  $p(i)$  — equations (8), (12), and (13) respectively. The resulting equation is the following:

$$\pi = [P - A] i [1 - \theta^*] H_i \quad (14)$$

The above equation is in terms of the optimum  $\theta^*$  given by equation (11). It is easy to see that in equation (14) for  $\alpha > \alpha(*)$  the intermediate sector firm makes no profit since  $P < A$ . It is only profitable for the range  $0 < \alpha < \alpha(*)$  and makes no profit at the point where  $P = A$ : Since  $\alpha$  is the output elasticity for knowledge in the final sector,<sup>7</sup> thus the profit margin in the intermediate sector of the economy depends upon this elasticity.

Now we turn our attention to the GNP of this economy. More precisely, we want to know the output per hour being produced in this economy. To get that, we must first find the output of each industry in terms of the final input. This is equivalent to substitute equation (8) into equation (2).

<sup>7</sup> The definition of the output elasticity for knowledge is:  $\zeta = \partial \ln Y / \partial \ln L$ . Applying it to the final sector firm production function, we have  $\zeta = \alpha$ .

$$Y(i) = i \theta^{*\alpha} (1 - \theta^*)^{1-\alpha} H_i \text{ or } y = \frac{Y(i)}{H_i} = i \theta^{*\alpha} (1 - \theta^*)^{1-\alpha} \quad (15)$$

Equation (15) represents the output per hour worked in industry  $i$ , where  $i \in [1, q]$ . Recall that in this model  $i$  also stands for any industry. Therefore, the industry that produces the average quality output per hour, the one that uses workers and machines with the average knowledge level, is the one in the mean of the interval  $[1, q]$  or  $\mu$ . Therefore, equation (15) for this industry is the following:

$$y(\mu) = B \mu, \text{ where } B = \theta^{*\alpha} (1 - \theta^*)^{1-\alpha} \quad (16)$$

As we may see, the average quantum of knowledge of the economy,  $\mu$ , is the single most important indicator of the economy average productivity and quality. Indeed, as the average quantum of knowledge of the society increases, so does the quality and quantity of its average output per hour worked. This theoretical result is extremely important because there are empirical support in the research done by Mankiw, Romer & Weil (1992) to this result. At the same time this result gives support to the linearity hypothesis used by models like King & Rebelo (1990) and Barro (1990), where the coefficient  $B$  in equation (16) is taken as exogenous. Thus, our microfoundation presents a way of endogenize the exogenous factor of the linear models.

Moreover, we know that the linearity, convex technologies, enable us to obtain continuous growth for the economy. Most important of all, equation (16) gives us precisely which variable must be accumulated. It is the quantum of knowledge variable.<sup>8</sup>

We want to call attention to the fact that by renaming the intermediate sector here to be the design sector — technological sector — with the final sector playing the role of the remaining firms, we have what is called a Research and Development model on the same line of Aghion & Howitt (1992) and Grossman & Helpman (1991).

In the next section, we will explain precisely the way knowledge accumulation leads to increases in the average quantum of knowledge of the economy, thereby increasing workers' productivity. So, our next section is dedicated to the dynamics of the accumulation process.

### 3. The dynamics of the quantum of knowledge

#### *Introduction*

The main hypothesis of our model is that every society is formed by individuals with a specific knowledge level, school years plus job experience. Once we know the number of individuals in each knowledge level, we have learned society's knowledge level distribution function. Therefore, underlying this hypothesis there is the assumption that every society's labor force can be classified according to their knowledge level. If we use the same point of

<sup>8</sup> It is possible to capture Romer's (1986) increasing returns hypothesis in this new microfoundation simply by writing equation (8) in the following way:  $M_i = [1 - \theta] i H_i$ . This would make equation (16) to present increasing returns to quantum of knowledge or  $y(\mu) = B \mu^{2-\alpha}$ .

view when looking at the goods, in the sense of the knowledge they require to be operated, we also can classify them according to their required knowledge. Thus, there would also exist a distribution function for the combination between workers' and goods' knowledge level, the quantum of knowledge.

We now make a special assumption. The quantum of knowledge distribution function has the shape of the Gamma function. We are making this assumption to capture the results of empirical works done in applied human field. According to this research the difference between the low and the high knowledge level seems to increase with the accumulation of knowledge (Ram, 1990). However, we want to emphasize that this assumption does not influence our two main analyses, the influence of the productivity factor and the government waste on the growth rate of the economy. As a matter fact, these analyses could be made without this assumption.

The Gamma distribution function was assumed because of three important factors: first of all, it implies that the majority of workers have near the average quantum of knowledge; second, it allows us to set the initial point (truncate) of the variable; and third, because the same parameters form both the mean and the variance. The first property is the most desirable one when we do not know the precise distribution function of the variable. The second property is very important because the variable we are dealing with requires it. And the third property captures the effect that, as the overall distribution of the quantum of knowledge moves to the right, the mean of the quantum of knowledge grows causing the widespread between its lower and the higher limit. Thus, the Gamma distribution function gives us the standards required by our problem. Nonetheless, we must mention that any other distribution function that delivers the two initial properties could be used. The precise distribution can be obtained only through empirical studies.

The Gamma distribution function assumption enable us to model the ever increasing knowledge and goods diversity, income inequality and product cycle improvement, besides the effect of the productivity factor and the government waste on the growth rate.

As we will see, the use of the distribution function will bring enhancing features to our model. For instance, it will allow us to better understand the link between knowledge accumulation and production process. Another important aspect is the economic interpretation of the parameters of the Gamma function. Since they play a very important role in our analysis, we will start by rewriting this function,

$$g(i) = \left( \frac{1}{\delta^\kappa \Gamma(\kappa)} \right) [i-1]^{\kappa-1} e^{-\left(\frac{i-1}{\delta}\right)} \quad (17)$$

where  $g(i) = 0$  for  $i < 1$ ;  $i = 1$  is the minimum knowledge level of the workers;  $\kappa$  is the average knowledge level of the workers; and  $\delta$  is a constant.

As it can be seen in the above equation, we have attached interpretations from the applied human capital literature to the parameters of this distribution.<sup>9</sup> Note that we have chosen the value of 1 as the initial condition — the minimum knowledge level. This permits us to compare this model with those of the endogenous growth literature. For instance, if all the workers have  $i = 1$ , then the final sector production function of our model becomes the traditional Cobb-Douglas function.

<sup>9</sup> See for example Mankiw, Romer & Weil (1992).

The mean and the variance of the Gamma distribution function are formed by the two parameters,  $\kappa$  and  $\delta$ . More precisely, their definitions are

$$\mu = \delta \kappa, \text{ and} \quad (18)$$

$$v(g) = \delta^2 \kappa \quad (19)$$

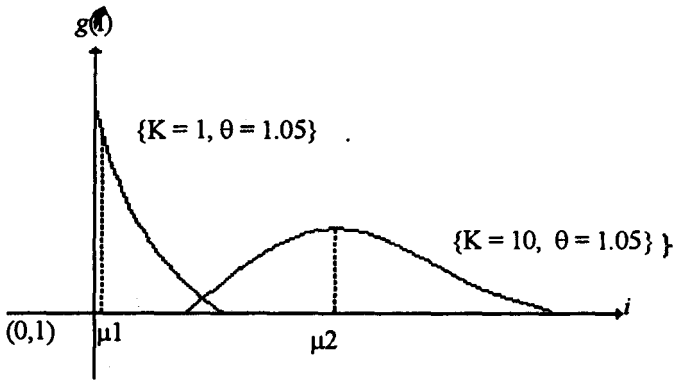
As one may notice, equation (18) expresses the average quantum of knowledge as a function of the workers' knowledge level. The explanation for this requires that we return our attention temporally to the intermediate and final sector of the economy. Recall that the intermediate goods production and quality depend upon the knowledge level of the workers. Moreover, these goods will be combined with workers in the final sector to produce the final good, service. In fact, the intermediate and final sectors output production and quality are ultimately given by the workers' knowledge level. Thus, the knowledge accumulated by the workers is the source factor of the quantum of knowledge. In simple words, workers accumulate knowledge in school and working in such way that enable them to produce goods and services of the same quality as of their knowledge level.

Equation (19), in turn, shows that the variance of the quantum of knowledge depends upon the average school years. The explanation for that evolves a digression on knowledge accumulation. The competition to sell skills leads individuals to learn more. This learning takes place at school and at the job. At the job, individuals become increasingly specialized in producing a particular good or service. This ever increasing aspect of knowledge accumulation, specialization, leads to an increase in knowledge levels and hence to an increase in the diversity of goods and services. So, equation (19) is designed to capture this ever increasing knowledge level diversity of reality.

To clarify how this quantum of knowledge diversity is captured in our model, we will make use of a numerical example. In our first case, the average knowledge level of the workers is  $\kappa = 1$  while in the second case it is  $\kappa = 10$ . We suppose that  $\delta = 1.05$ , this constant can be interpreted as the capacity that each worker has embodied in himself to boost his own knowledge level by 5%. This constant may take any value. The main aspect of it is that if  $\delta = 1$ , the worker's knowledge level is the same as the received one and if  $\delta < 1$ , the worker has the capacity to absorb less than he receives. According to the value we gave to the constant  $\delta$ , the average quantum of knowledge of the first sample is  $\mu_1 = 1.05$  while the second is  $\mu_2 = 10.5$ . These allow us to build figure 2.

In figure 2, as quantum of knowledge is accumulated, the diversity, measured by the variance, increases. This spread around the average means that workers, goods and services of different quality are being added to the economic process. Thus, this process characterizes the well-known fact of the ever increasing labor, goods and services diversity. According to our model, the factor that causes this ever increasing diversity is the continuous knowledge accumulation by the society.

Figure 2



### The accumulation process

We assume that society accumulates quantum of knowledge by forgoing units of consumption good. The amount not consumed is invested in acquiring quantum of knowledge. More precisely, the average quantum of knowledge,  $\mu$ , is accumulated through time according to the following equation:

$$\dot{\mu} = y(\mu) - c \quad (20)$$

where  $y(\mu)$  is the average income per hour worked;  $c$  is the average consumption per hour worked; and  $\mu$  is the average quantum of knowledge.

Moving ahead in our analysis, we now turn our attention to the term  $y(\mu)$  in equation (20). It represents the average income per hour worked. By plugging equation (16) into (20) we have

$$\dot{\mu} = B \cdot \mu - c \quad (21)$$

Equation (21) presents constant returns to scale in the accumulation of knowledge. This result is very much in accordance with the popular linear growth models. Therefore, the derivation process will follow the same pattern.

We now return our attention to the definition of average quantum of knowledge given by equation (18). By substituting this equation into the above equation, we have

$$\dot{K} = B \delta K - c \quad (22)$$

Recall here that  $\delta$  is assumed to be constant, therefore the only variable to be accumulated is the average knowledge level of the workers,  $\kappa$ . Thus, this equation represents the link between schooling, production and goods diversity.

The objective of our infinitely lived society's elected representative is to maximize the integral of the total future welfare discounted at time  $t = 0$ .<sup>10</sup> Thus, society's decision is to determine how much income will be consumed and/or saved. In spite of all the amenities in the real world, the saved income is assumed to be totally invested into education and job training.

The welfare function is assumed to be the following:

$$W = \int_0^{\infty} \left[ \frac{c^{1-\sigma} - 1}{1-\sigma} \right] e^{-\rho t} dt \quad \text{for } \sigma \neq 1, \text{ and} \quad (23)$$

$$W = \int_0^{\infty} \ln(c) e^{-\rho t} dt \quad \text{for } \sigma = 1.$$

where  $c$  is the average consumption per worker;  $\sigma$  is the consumption's parameter  $\sigma > 0$ ;  $\rho$  is the subjective rate of discount.

The Hamiltonian function for this problem is the following:

$$H = \left( \frac{[c-1]^{1-\sigma}}{1-\sigma} + \psi \left[ B \cdot \kappa - \frac{c}{\theta} \right] \right) e^{-\rho t} \quad (24)$$

Now we carry out the maximization process. The necessary and sufficient conditions for this problem are

$$\frac{\partial H}{\partial c} = 0, \quad c^{-\sigma} = \frac{\psi}{\theta}; \quad (25)$$

$$\frac{d\psi}{dt} - \rho\psi = -\frac{\partial H}{\partial \kappa}, \quad \dot{\psi} = -[B - \rho]\psi \quad (26)$$

$$\lim_{t \rightarrow \infty} \kappa \psi e^{-\rho t} = 0 \quad (27)$$

Equation (27) is the transversality condition. It is satisfied with the usual assumption  $B > \rho > B(1 - \sigma)$ . Differentiating equation (25) with respect to time, we get the following:

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma} \frac{\dot{\psi}}{\psi} \quad (28)$$

Equation (28) is the consumption growth rate. It can be written in another way by substituting equation (26) in there:

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<sup>10</sup> Without loss of generality, this idea of infinitely lived society's elected representative can be replaced by the family like in Blanchard & Fisher (1989).

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} [B - \rho] = \frac{\dot{v}(g)}{v(g)} = \frac{\dot{\kappa}}{\kappa} = \gamma \quad (29)$$

As we can see in the above equation, consumption, the quantum of knowledge and goods' diversity all grow at the same rate. The growth rate is endogenous to the economy. This growth rate depends upon the endogenous productivity parameter  $B$ , the subjective rate of discount  $\rho$  and the parameter  $\sigma$ . Two economies with identical parameters  $\rho$  and  $\sigma$  could have different growth rates if their endogenous productivity parameter  $B$  is different. The same would be true if, for identical  $B$ , either  $\rho$  or  $\sigma$  is different.<sup>11</sup>

Let us move our attention now to the growth rate of the economy. As we may recall, it is formed by the three parameters  $B$ ,  $\rho$  and  $\sigma$ . However, the constant  $B$  itself is formed by the coefficient of the production function,  $\alpha$  — the final sector output elasticity for knowledge. Thus the economy growth rate depends upon the following parameters:  $\sigma$ ,  $\alpha$ , and  $\rho$ . More precisely it is

$$\gamma = \frac{1}{\sigma} \left( \left[ \frac{\alpha + \sqrt{\alpha} \sqrt{(4 - 3\alpha)}}{2} \right]^\alpha \left[ \frac{2 - \alpha - \sqrt{\alpha} \sqrt{(4 - 3\alpha)}}{2} \right]^{1-\alpha} - \rho \right) \quad (30)$$

Here again this result stands in contrast to the ones found in economic growth models that take productivity of the human capital as exogenous. There, the growth rate depends upon the exogenous productivity parameter,  $B$ , which in our case is entirely formed by the final sector output elasticity for knowledge,  $\alpha$ . Thus, our microfoundation shows us that the most important factor that affects the growth rate of the economy is the output elasticity for knowledge in the final sector. Therefore, the objective of any economic policy should be directed to this parameter. Nonetheless, caution must be taken, the behavior of this parameter could decrease the growth rate instead of increasing it. To visualize this, we plot the growth rate against the parameter of the production function,  $\alpha$ .

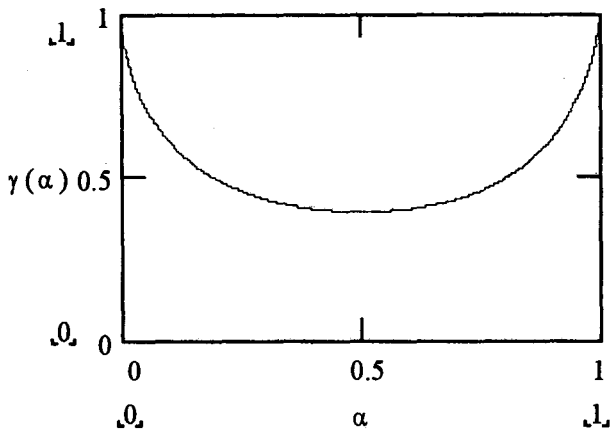
Figure 3 shows to us the behavior of the growth rate and the output elasticity for knowledge. There we have that the elasticity of substitution between consumption at any two points in time,  $1/\sigma$ , is equal to 2 and the subjective discount rate is equal to 0.01. Under these conditions, future consumption is preferred to present consumption. Which gives us a growth rate between 0.4% and 1%, approximately. Where the minimum growth rate is reached when the output elasticity for knowledge is 0.5. It is important to remember, however, that this growth rate is real growth rate of the productivity. Thus, a growth rate in the range above is extremely good for the economy. Now, we consider the case in which consumers are almost indifferent between consumption at any two points in time,  $1/\sigma = 1.01$ . This gives us a growth rate in the range 0.2%-0.5%. Essentially what these results say is that any economic policy that might affect this parameter may also affect the growth rate of the economy.<sup>12</sup> Moreover, any two economies would have different growth rates if they have different preferences for future consumption. As an extension of this model, the next sec-

<sup>11</sup> These results would be the same as the one where the assumption about the form of the distributions is not present. As a matter of fact, the same result is obtained if we use equation (21) in place of (22) when doing the consumer maximization problem.

<sup>12</sup> Barro (1990) analyzed the case of the income tax rate.

tion covers the analysis of the specialization, product cycle improvement, and income inequality.

Figure 3



### *The specialization, product cycle improvement and income inequality*

The specialization, product cycle improvements, and income inequality in this model were captured by the assumption about the distribution function. As we notice, the variance increases with the knowledge accumulation. Note, however, that the quantum of knowledge refers to a combination of intermediate goods' and workers' knowledge level. More important yet it refers to a new industry that is formed in the final and intermediate sectors. This industry produces a good and a service of determined quality. Now if we look at figure 2, we can see that the knowledge level is moving forward, therefore goods and services that were produced by workers of a certain knowledge level after some time will not be produced anymore. This is because these workers improved their knowledge level and consequently their product's quality. So, new and better products emerge while old ones disappear, product cycle improvement. Recall that, this is equivalent to state that the spread between the lower and the upper limits of the knowledge level is ever increasing. Consequently, it also means that the income per hour being generated by workers with the lowest knowledge level becomes farther away from the income per hour generated by workers with the highest knowledge level. Thus, leading us to the possible explanation of the ever increasing income inequality in society or the same as saying that the overall society distribution function is given by a Gamma function.

### *Some extensions*

Some simplifying assumptions of this model were replaced by more realistic ones in Dias (1993).<sup>13</sup> More specifically, the following assumptions were implemented: the inter-

<sup>13</sup> This paper is still being improved, therefore its results are not definitive.

mediate sector uses physical capital to produce its intermediate good; and the representative consumer has to share its time between working in the final sector, the intermediate sector, accumulating knowledge and leisure time. The final result is that the ultimate source of growth is the quantum of knowledge with the difference that the economy average productivity presents increasing returns to it.

Equation (29) presents a result that is the same as the models of King & Rebelo (1990) and Barro (1990) used to analyze the long run government policy. Here, we plan to make a similar analysis, but focusing on the effect of the government waste on the long run growth rate. This is of extreme importance because what will be captured in this analysis is, in last instance, the government efficiency. To see this, consider the following example in which the government is taxing in one side and giving to consumers on the other side in the form of service (health care, schooling, and others). Let us assume that pure efficiency means that the government gives back everything it takes to as taxes from the same consumers. Thus, a less efficient government would waste some of the taxes between the process of taking and giving it back. So, let us represent this government waste by the variable  $\tau$ . Hence, equation (29) would be written in the following way:

$$\gamma = \frac{1}{\sigma} [\tau B - \rho] \quad (31)$$

As we may see in the above equation, a more inefficient government lowers its growth rate. Thus, any improvement in government efficiency would bring about an increase in the growth rate of the economy. So, we do expect that economies in process of reorganization would have a potencial to increase its growth rate in the long run. Moreover, similar economies can have different growth rate depending on their government efficiency.

#### 4. Conclusion

The model developed here, besides contributing to a better understanding of the links between knowledge level and production, presents the microfoundation to the popular linear models (King & Rebelo, 1990, and Barro, 1990). This model also shows that the growth rate of the economy is the same as the average growth rate of the quantum of knowledge which is endogenous to the economy. Moreover, the most important factor that influences this growth rate is the output elasticity for knowledge in the final sector. Therefore, any exogenous factor that may affect this parameter would also affect the growth rate of the economy.<sup>14</sup>

In this model the knowledge level is the ultimate source of productivity. Thus, a large human capital stock does not necessarily mean higher average knowledge level, but, as expected, a higher average knowledge level of this human capital stock is what brings higher productivity, as we may observe in reality.<sup>15</sup> This result also complies with the models that explain the links between production and human capital.<sup>16</sup>

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<sup>14</sup> This result improves considerably the exogenous hypothesis of the linear models.

<sup>15</sup> According to Romer (1990)'s model, the most important factor is the human capital size, not its knowledge level.

<sup>16</sup> For example Kim & Mohtadi (1992).

On the applied side, this model complies with the findings of Feenstra, Markusen & Zeile (1992) that supports economic growth through continuous creation of new inputs; with the findings of Ram (1990), where knowledge diversity, measured by the variance, increases with increases in the average school years of education; with the findings of the Mankiw, Romer & Weil (1992), where the average knowledge level is the factor that increases productivity; and, with the findings of Scully (1988) that a more efficient government has a higher growth rate of productivity.

On the income inequality issue, equation (12) in this model predicts that wages' distribution will follow closely the knowledge level distribution of the workers.<sup>17</sup> In sum, wages are positively related to the level of education.<sup>18</sup>

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<sup>17</sup> Reis & Barros (1990) tested this hypothesis for the Brazilian data. More specifically, the authors tested regional metropolitan wage inequality due to school years of education of the workers. Some of their important results are as follows: first, the intertemporal inequality during 1976-86 was not affected; and second, 50% of the wage inequality among regions are explained by school years of education. These results seem contradictory at first glance. Theoretically, our model would explain that, by predicting that the overall knowledge level distribution function has not changed in this period, therefore, workers' knowledge level and by extension their real income remained the same. By looking at the authors' data, we can see there that this seems to be the case. The data shows that the average knowledge level of the workers has not changed during this period while the regional knowledge level differential is quite strong and remained the same.

<sup>18</sup> Ramos (1991), Leal & Werlang (1991) found to be the case when testing this hypothesis for Brazil.

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