On Aumann's notion of common knowledge: an alternative approach*

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An event is common knowledge, in the intuitive sense, if everyone knows that it has ocurred, if everyone knows that everyone knows that it has occurred, and so on. We provide a Bayesian definition of common knowledge which is based on the infinite hierarchy of beliefs derived from the basic uncertainty space. Each agent's infinite stream of beliefs represents his prior on the basic uncertainty, his prior on the priors of the other agents, and so on. This framework therefore allows a direct formalization of the intuitive notion of common knowledge: an event is common knowledge in the eyes of agent i if he believes that it has occurred with probability one, if he believes that everyone else believes that it has occurred with probability one, and so on. The main theorem of the paper establishes the equivalence of the Bayesian definition and the definition of common knowledge provided by Professor Aumann in his seminal paper, Agreeing to disagree (1976).

1. Introduction

Intuitively, an event is common knowledge if everyone knows it, everyone knows that everyone knows it, ... , and so on.

Lewis (1966) was the first person to define common knowledge. However, a formal definition was not available to economists until the seminal paper of Aumann (1976). Aumann's framework is given formally in section 2. Related work familiar to economists are Milgrom (1981), Geanakoplos & Polemarcharkis (1982), and Bacharach (1985).

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The framework offered here is based on the infinite recursion of beliefs introduced by Armbruster & Böge (1979), Böge & Eisele (1979) and Mertens & Zamir (1985). The definition of common knowledge given here is of course implicit in the three earlier contributions, however, the purpose of this paper is to make it explicit and to prove an equivalence theorem between the two, mathematically quite different, approaches.

The new definition formalizes the intuitive notion of common knowledge directly. The definition which we give in section 4 translates verbally into the first paragraph of this paper. In contrast, Aumann's definition requires a moment of thought before its relation to the intuitive notion is transparent. It also allows one to easily define levels of knowledge which are lower than common knowledge. For instance, a statement like "everyone knows (everyone knows)^m that the event has ocurred" can be formalized directly, without further requiring that the event be common knowledge. (Although such statements can be made using the notion of reachability introduced in Aumann 1976, we feel it is much less awkward to make use of the infinite recursion of beliefs.)

Aumann's framework relies on information partitions of individuals. The interpretation of his definition of an event being common knowledge coincides with the intuitive notion only if the information partitions are assumed to be common knowledge to begin with. Consequently, there is an implicit self-reference which occurs precisely during that moment of thought when the reader is converting Aumann's mathematical definition into the intuitive notion of common knowledge (see the remark in section 2).

The advantage of the infinite recursion is that the mathematical definitions of knowledge up to level m as well as common knowledge coincide directly with the intuitive notions. As such, it avoids implicit self-references. Professor Aumann was of course aware of this issue in his original contribution and suggested that expanding the underlying uncertainty space would circumvent this problem. The results reported here, as well as related work by Brandenburger & Dekel (1985), demonstrate that the set of all infinite hierarchies of beliefs can be considered the universal uncertainty space in which each state of the world contains a description of both the basic uncertainty as well as the information each agent has about the world.

For the readers familiar with the Aumman definition, an important feature of our approach (and to our knowledge, a novel feature in the literature) is that in section 5 we explicitly formalize the assumption "the private information partitions and the prior are common knowledge", which is implicit in the original definition.

This alternative framework has been used to formalize "rationality is common knowledge in a game" and to derive the behavioural implications

of such a hypothesis (see Tan & Werlang, 1984 and 1988). We found that it was not always convenient to work with Aumann's framework when the events of interest were rather complex, like rationality on a class of games. In a recent paper, Aumann (1987) discusses correlated equilibrium as an expression of Bayesian rationality. In that paper, players are rational at *every* state of the world and consequently, it has to be common knowledge. Our model allows investigation of assumptions about lower levels of knowledge about rationality. Reny (1985) applies this framework for the study of games in extensive form.

The main result of this paper is the equivalence theorem (in section 5) between the definition given here (section 4) and the original definition of Aumann. We show that given an Aumann structure, with the information partitions and prior which are implicitly assumed to be common knowledge, there is an infinite recursion of beliefs structure in which those information partitions and the prior are common knowledge in the sense of section 4. Moreover, given these two structures, an event is common knowledge in the sense of Aumann if and only if it is common knowledge in the sense given here.

Thus the theorem accomplishes the suggestion in Aumann (1976) that the expansion of the uncertainty space, so that a state of the world also includes a description of the information of each agent, circumvents the implicit self-reference.

The result of Brandenburger & Dekel (1985) complements our result neatly. Given an infinite recursion in which an event is common knowledge (as in section 4), they show that there is an Aumann structure — that is, a probability space, information sub-σ-algebras for each agent, a prior and a state in the probability space — in which the same event is common knowledge in Aumann's sense. There is, however, a subtlety in their construction. The Aumann structure which they generate is not finite and therefore there are delicate technical issues which they deal with to extend Aumann's definition to this case. (Nielsen, 1984, is another approach which extends the Aumann's definition to state spaces which are not finite.)

Another substantive difference between this paper and that of Brandenburger & Dekel (1985) is that they did not investigate the explicit formulation of the common knowledge of partitions and prior. In particular, the sub-σ-algebras which they generate for the Aumann structure may not necessarily be common knowledge in the infinite recursion which they began with. (We conjecture that such a construction is possible — that is to say, a direct converse to our main theorem.)

Section 2 gives Aumann's definition. Section 3 provides an introduction to the infinite recursion of beliefs, the Bayesian interpretation of the frame-

work, and the main result in that area. Section 4 provides the alternative definition of common knowledge as well as an example of applying this definition to a problem in the foundations of game theory. Finally section 5 states and proves the main equivalence result of this paper.

Logicians have also studied knowledge but this literature is, unfortunately, not well known to economists. Aumann's model is very similar, in spirit, to Kripke (1963). Our framework is, in turn, similar to Fagin, Halpern and Vardi (1984). In fact, the logic-theoretic approach dispenses altogether with probability statements and is therefore more abstract and general than the literature which is familiar to economists. Halpern (1986) contains many contributions to models of knowledge from diverse disciplines.

2. Aumann's notion of common knowledge

Let (Ω, Σ, μ) be a probability space, where Σ is a σ-algebra on Ω (if Ω is a topological space, we assume that Σ is the Borel σ-algebra) and μ a probability measure on (Ω, Σ). Suppose that there are n agents. Let $N = \{1,...,n)$ be the set of agents. Each agent i is characterized by a sub-σ-algebra $\prod_i$ representing his private information.

2.1 *Definition* (Aumann, 1976): Let $\prod = \prod_1 \cap \prod_2 \cap ... \cap \prod_n$ ($\prod$ is the intersection of the sub-σ-algebras $\prod_i$, also called the "meet"). Suppose the true state of nature is $\omega \in \Omega$. An event $A \in \Sigma$ is said to be *common knowledge at* ω if $\exists$ $B \in \prod$ with $\omega \in B$ and $A \supset B$.

Remark: definition 2.1 is taken directly from Aumann (1976). Our point regarding the implicit self-reference is best summarized by adding the hypothesis "Suppose $\prod_1, \prod_2, ..., \prod_n$ are common knowledge" to the beginning of the definition.

During the course of the paper, it will be necessary to speak of finite levels of knowledge of an event. Here, we provide a definition which was suggested by a referee. It greatly simplifies the exposition, but it is not the same procedure as Aumann's. In the appendix we show that the definition we use is equivalent to Aumann's.

Given a set $X \in \Sigma$, let $\psi_i[X]$ denote the smallest element of $\prod_i$ which contains X. (We assume that the sub-σ-algebras and the probability space are such that $\psi_i[X]$ exist for all measurable X. This is true, for example, in case Ω is finite or the $\prod_i$'s and Σ are generated by finite partitions of Ω.)

2.2 *Definition*: Let $(\Omega, \Sigma, \prod_1, \prod_2, ..., \prod_n)$ be given as above. We say that an event A *is known by agent* i *at* ω if and only if $\omega \in K_{1i} = \{\omega' \in \Omega \,|\, A \supset \psi_i[\{\omega'\}]\}$.

In the same fashion, we define inductively, for $m \geq 1$ that *agent* i *knows that (everyone knows that)^m* A *ocurred at* ω, if and only if $\omega \in K_{m+1,i} = \{\omega' \in \Omega | K_{mk} \supset \psi_i[\{\omega'\}]$, for all $k \in N\}$.

3. The infinite recursion of beliefs

Our approach to the definition of common knowledge is Bayesian in spirit and relies on the mathematical structure of the infinite recursion of beliefs. The reader interested in the more formal aspects of the infinite recursion should refer to Armbruster & Böge (1979), Böge & Eisele (1979) and Mertens & Zamir (1985). In what follows, we shall adhere closely to the basic framework in Aumann in order to facilitate comparisons later. In particular, we assume that the basic domain of uncertainty for the agents is the same Ω as above.

In the Bayesian approach, each agent must have a prior on the uncertainty which he faces. Thus on the basic uncertainty Ω, each agent must have a prior which is element of:

3.1 *Definition*: $S_{1i} \equiv \Delta(\Omega)$

[If X is a compact metric space endowed with the Borel σ-algebra, $\Delta(X)$ is the set of Borel probability measures over X. Moreover, by a well known result (Billingsley, 1968), $\Delta(X)$ endowed with the topology of weak convergence of measures is compact and metrisable. We assume that Ω is a compact metric space.]

However, in addition to the basic uncertainty, the priors of the other agents are unknown to agent i as well. Since knowledge (equivalently beliefs in the Bayesian approach) and the extent of knowledge is of interest here, these priors of the other agents must also be included in the uncertainty to agent i. The Bayesian agent i must also have a prior on the first order beliefs of the other agents and this prior is an element of:

3.2 *Definition*: $S_{2i} \equiv \Delta(\Omega \; X \, X_{j \neq i} \, S_{1j})$

Remark: Ω is repeated here and in every subsequent layer of belief for a natural reason. We want to allow the possibility that there is correlation between the different layers of beliefs and events in the basic uncertainty. This is very much in the same spirit as in Aumann (1976) where an event can be common knowledge at a particular state of the world but not in another state of the world.

These second level priors are in turn unknown to agent i and again they must be included in the domain of uncertainty faced by agent i. As a result

the Bayesian agent must have a prior on that as well. Continuing this line of argument, we define successively:

3.3 *Definition*: $S_{mi} \equiv \Delta(\Omega \, X \, X_{j \neq i} \, S_{m-1,j})$

The infinite recursion of beliefs for each player or the types of each player is given by:

3.4 *Definition*: $S_i = \{(s_{mi})_{m \geq 1} \in X_{m \geq 1} S_{mi}$: the beliefs satisfy the minimum consistency requirement}

The minimum consistency requirement is that if the probability of an event can be evaluated using s_{mi} and using s_{ni}, then the two values must coincide. A formal statement of this can be found in Mertens & Zamir (1985).

Remark: the consistency requirement here is imposed so that the beliefs of each player obey simple probability calculus. That is, lower level beliefs may be recovered from higher level beliefs. Hence it formalizes the intuitive requirement that each player knows his own beliefs. Notice that S_i is an event in the set of all possible hierarchies. Consistency is therefore an explicit description of the state of the world rather than an implicit assumption. Theorem 3.5 below shows that it is possible to find an event in the universe of all possible hierarchies such that every agent is consistent, every agent 'believes' that every other agent is consistent and so on.

The approach takes as an axiom that a Bayesian agent is completely characterized by a sequence of beliefs or priors on the successive levels of uncertainty faced by him. Consequently, an agent is a point in S_i or alternatively, a point s_i in S_i is a type or a psychology for agent *i*.

The reader may wonder whether we have exhausted all the uncertainty for agent *i*. Following the line of thought which generated S_i, shouldn't agent *i* also have a prior on $S_{-i} = X_{k \neq i} \, S_k$ as well? The answer to is given by the main result in the mathematics of infinite recursions, proven in Armbruster & Böge (1979), Böge & Eisele (1979) and Mertens & Zamir (1985):

3.5 *Theorem*: $\forall i$, S_i is compact and metric in the topology induced by the product topology. Moreover, $\forall i$ there exists a canonical homeomorphism, $\Phi_i : S_i \to \Delta(\Omega \, X \, X_{k \neq i} \, S_k)$. One property of this canonical homeomorphism is that $\mathrm{marg}_{\Omega}[\Phi_i(s_i)] = s_{1i}$.

Remark: the content of this theorem is that the two are of the "same size" (homeomorphic) and although one might consider another layer of prior on S_i, it would be redundant as this information is already contained in S_i. Alternatively, a reinterpretation of agent *i*'s type or infinite layers of belief, is that it is in itself a prior on Ω as well as the infinite layers of beliefs of all the other agents. Moreover, the theorem allows us to define the universal domain of uncertainty for agent *i*, based on the basic uncertainty space Ω, to be $\Omega X S_{-i}$. It can be interpreted as saying that this space exhausts all that is uncertain, to agent *i*, about the environment when the basic space of uncertainty is Ω. As result, instead of the sequence of beliefs we generated earlier, the standard Bayesian approach can be applied by taking the space $\Omega\ X\ S_{-i}$ as the given domain of uncertainty and then a Bayesian agent must have a prior on this space.

The canonical homeomorphism allows us to speak of S_i, the type of the agent *i*, and his prior on $\Omega\ X\ S_{-i}$ interchangeably. Hence, a type of agent *i* uniquely determines the prior of agent *i* on the state of the basic uncertainty and the types (or states) of the other agents. If an agent is of type s_i, his prior on $\Omega\ X\ S_{-i}$ is given by $\Phi_i(s_i)$. Consequently, his prior on Ω can be recovered using the canonical homeomorphism and taking the marginal of $\Phi_i(s_i)$ on Ω. Anoter feature of $\Phi_i(s_i)$ which we use of extensively below is the marginal of $\Phi_i(s_i)$ on S_k. This is agent *i*'s prior on the possible types of agent *k*; equivalently, it is agent's *i*'s prior on the infinite recursion of beliefs held by agent *k*. In particular, the set supp $\text{marg}_{Sk}\ \Phi_i(s_i)$ (i.e., the support of the marginal on S_k of agent *i*'s prior) represents the set of types which agent *i* believes agent *k* can be.

4. The alternative definition of common knowledge

Suppose that the basic uncertainty space Ω arises in a problem of interest. Suppose also that the event *A* is of interest for the purpose of the analysis. Then it is quite straightforward to formalize statements such as "everyone knows (everyone knows)^m that *A* ocurred" and "*A* is common knowledge" given the infinite recursion based on Ω defined in section 3.

4.1 *Definition:* agent *i* knows (believes) that A occurred $\Leftrightarrow$

$$s_i \in A_{1i} \equiv \{s_i \in s_i \mid \text{marg}_{\Omega}\ [\Phi_i\ (s_i)](A) = 1\}.$$

Remark: a literal translation of the definition is: agent *i*'s infinite recursion of beliefs are such that his prior on Ω (represented by the marginal of $\Phi_i(s_i)$

on Ω; see the discussion at the end of the previous section) assigns probability one to the event A.

4.2 *Definition:* agent i believes that everyone believes that A occurred $\Leftrightarrow$

$$s_i \in A_{2i} \equiv \{s_i \in A_{1i} \mid \forall k \neq i : s_k \in \text{supp marg}_{sk} [\Phi_i(s_i)] \Rightarrow s_k \in A_{1k}\}$$

Remark: literal translation is: agent i's beliefs are such that (1) he believes that A occurred; (2) if he believes that agent k can have beliefs s_k, then s_k must belong to A_{1k}. Thus, agent i believes that agent k believes that A occurred.

4.3 *Definition:* agent i believes that (everyone believes)$^{m-1}$ that A occurred $\Leftrightarrow$

$$s_i \in A_{mi} \equiv \{s_i \in A_{m-1,i} \mid \forall k \neq i : s_k \in \text{supp marg}_{Sk} [\Phi_i (s_i)] \Rightarrow s_k \in A_{m-1,k}\}$$

4.4 *Definition*: agent i believes that A is common knowledge or equivalently, in the eyes of agent i A *is* common knowledge, if: $s_i \in \cap_{m \geq i} A_{mi}$.

Remarks:

1. Note that since the definition of common knowledge here is a direct formalization of the intuitive notion (as long as one equates "knows" vith "believes with probability one"), and common knowledge of an e.ent is itself an event in the universal uncertainty space, the problem of implicit self-reference does not exist here.

2. The infinite recursion exists entirely in the mind of agent i and hence the sense of common knowledge is in the eyes of agent i. In this framework, therefore, it is possible even to define and analyse what would happen if agent i believed that A were common knowledge and agent j thought that B were common knowledge and yet A and B are mutually exclusive events.

3. No reference is made to the private information partition of agents or to a particular ω in the basic uncertainty space. (One can in fact define common knowledge of an event which did not occur in this framework.) The only implicit requirement is that A be measurable with respect to the Borel σ-algebra on Ω. Of course common knowledge of an event is now a state of the world in the universal uncertainty space.

Sometimes it is convenient to consider common knowledge from the point of view of all agents:

4.5 *Definition:* $A \in \Sigma$ is common knowledge if $\forall i : s_i \in \cap_{m \geq 1} A_{mi}$.

5. Common knowledge of the partitions and the equivalence theorem

Since the Aumann approach to common knowledge is the standard in the literature, it is of course important that the approach of section 4 be related to the tested framework. We provide here an equivalence result between the two versions of common knowledge.

The theorem we prove demonstrates that given an Aumann structure, that is, the probability space (Ω, Σ, μ) and the private information partitions $\prod_i$, there exists an infinite recursion structure such that an event A is common knowledge in the sense of Aumann if and only if it is common knowledge in the infinite recursion.

We define the infinite recursion based on Ω as in section 4. Furthermore, we restrict the types of the agents in that structure so that their priors and posteriors are consistent with "the partitions $\prod_i$ and the prior μ are common knowledge". That is, at every state in Ω, the associated beliefs are the conditional probability measures given the information partitions. Moreover, this is defined to be common knowledge in the sense of section 4. Hence, the infinite recursion allows us to explicitly formalize the implicit assumption in the mental process mentioned in section 2. Notice that the implicit assumption is made without self-reference in the infinite recursion. Next, we restrict the type of agent i further so that his beliefs are consistent with the occurrence of ω. That is, his beliefs are conditional probabilities given $\prod_i$ evaluated at ω.

All these restrictions are required to impose the structural framework in Aumann's approach into the infinite recursion. Given these restrictions on the infinite recursion, an event A is common knowledge at ω in the sense of Aumann if and only if it is common knowledge in the sense of section 4.

5.1 *Definition:* we say that the information sub-σ-algebras $(\prod_i)_{i\in N}$ and the prior μ are common knowledge in the eyes of agent i if $s_i \in \bigcap_{m\geq 1} P_{mi}$, where P_{mi} is inductively defined by:

$$1.\ \forall i: P_{1i} = \{s_i \in S_i \mid \forall k \neq i: (\omega, s_k) \in \text{supp}\ \text{marg}_{\Omega X S k}[\Phi_i(s_i)]$$

$$\Rightarrow A \in \Sigma: \text{marg}_\Omega[\Phi_k(s_k)](A) = \mu(A|\textstyle\prod_k)(\omega)\}$$

$$2.\ m \geq 2: P_{mi} = \{s_i \in P_{m-1,i} \mid \forall k \neq i: s_k \in \text{supp}\ \text{marg}_{Sk}[\Phi_i(s_i)]$$

$$\Rightarrow s_k \in P_{m-1,k}\}$$

The intuition behind the definition is quite straightforward. Recall that the state of the world, in the eyes of agent i, is fully described by an element

of $\Omega X S_{-i}$. P_{1i} is the set of types of agent i who are aware that agent k conditions his belief using the information partition $\prod_k$. In particular, if i thinks that a state Ω of the basic uncertainty is thought jointly possible with the type s_k of agent k, it must be the case that s_k has beliefs over Ω given by the conditional distribution of the prior μ given $\prod_k$ at the state Ω. Thus, P_{1i} is an event in the universal domain of uncertainty $\Omega X S_{-i}$ for agent i. As before, we wish to make this a common knowledge event. P_{mi} corresponds to the statement that agent i knows that the sub-σ-algebras are known up to level m. Finally to say that s_i is in the intersection of all P_{mi}'s is the same as requiring the statements of the form P_{1i} to be common knowledge in the eyes of agent i. Notice that in the definition of a partition being common knowledge, the event is not a simple event contained in Ω. Instead, it is a complex event relating the basic uncertainty as well as the types of the other agents. (See also the notion of Beliefs Subspace in Mertens & Zamir, 1985.)

5.2 *Definition:* agent i's beliefs are consistent with μ and $\prod_i$ at ω if $\forall X \in \Sigma$:

$$\text{marg}_\Omega[\Phi_i(s_i)](X) = \mu(X|\textstyle\prod_i)(\omega)$$

This definition requires that agent i's belief be as though ω occurred. That is, his belief is the conditional distribution of μ given $\prod_i$ evaluated at ω. Observe that agent i does not know whether ω actually occurred. He observes only $\psi_i[\{\omega\}]$. We are now ready for the main result: the equivalence of the two common knowledge definitions, under the hypothesis of common knowledge of the sub-σ-algebras and μ, the consistency of agent i with $\prod_i$ and μ at ω, when Ω is finite and when $\mu(\omega) > 0$ for every ω in Ω. We will prove the main result by proving a finite counterpart of it and then taking intersections. In words, we prove that if in the eyes of agent i the sub-σ-algebras and μ are known up to level m-1 and agent i is consistent with $\prod_i$ and μ at ω, then i knows $A \in \Sigma$ up to level m in the sense of definition 4.3 if and only if i knows A up to level m at ω in the sense of Aumann. The main result appeared before in Werlang (1986).

5.3 *Theorem*: assume that Ω is finite and $\forall \omega \in \Omega$, $\mu(\omega) > 0$. Suppose that $s_i \in P_{m-1,i}$ and agent i's beliefs are consistent with $\prod_i$ and μ at ω. In this case $s_i \in A_{mi}$ if, and only if, $\omega \in K_{mi}$.

Proof of ($\rightarrow$): the proof goes by induction on m. For $m=1$, we have that $s_i \in A_{1i}$. As agent i's beliefs are consistent with $\prod_i$ and μ at ω, given that Ω is finite and $\forall\omega \in \Omega$, $\mu(\omega) > 0$, supp $\text{marg}_\Omega[\Phi_i(s_i)] = \psi_i[\{\omega\}]$. But, since $s_i \in A_{1i}$, $\text{marg}_\Omega[\Phi_i(s_i)](A) = 1$, which implies $A \supset \psi_i[\{\omega\}]$, which is the

same as $\omega \in K_{1i}$. Suppose now that the statement is true for $m > 1$, we will show it is true for $m+1$. That is to say, given $s_i \in A_{m+1,i}$, we have to show that $\omega \in K_{m+1,i}$. By the definition of $K_{m+1,i}$, this is the same as to show that $K_{mk} \supset \psi_i[\{\omega\}]$ for all $k=1,...,n$. The case $k=i$ follows because $s_i \in A_{m+1,i}$, so that $s_i \in A_{mi}$, and, therefore, by the induction hipothesis $\omega \in K_{mi}$, and this implies $K_{mi} \supset \psi_i[\{\omega\}]$, since for all $\omega' \varepsilon\ \psi_i[\{\omega\}]$ we have $\psi_i\ [\{\omega'\}] = \psi_i[\{\omega\}]$. The case $k \neq i$ is different. Let $\omega' \in \psi_i[\{\omega\}]$. As supp $\text{marg}_\Omega[\Phi_i(s_i)] = \psi_i[\{\omega\}]$, for all $k \neq i$ there exists $s_k\ \varepsilon$ supp $\text{marg}_{Sk}[\Phi_i(s_i)]$ such that $\text{marg}_\Omega[\Phi_k(s_k)](X) = \mu(X|\prod_k)(\omega')$, for every $X \in \sum$. Given $s_i \in P_{mi}$, it follows that $s_k \in P_{m-1,k}$. As $s_i \in A_{m+1,i}$, one has $s_k \in A_{mk}$. Thus, by the induction hypothesis, $\omega' \in K_{mk}$, and the result follows.

Q.E.D (of($\Rightarrow$))

Proof of ($\Leftarrow$): again it goes by induction. For $m=1$, $\omega \in K_{1i} \Rightarrow A \supset \psi_i[\{\omega\}]$. As agent i's beliefs are consistent with $\prod_i$ and μ at ω, we have that $\text{marg}_\Omega[\Phi_i(s_i)](X) = \mu(X|\prod_i)(\omega)$, for every $X \in \Sigma$, so that $\text{marg}_\Omega[\Phi_i(s_i)](A) = 1$, which is the same as $s_i \in A_{1i}$. Suppose now the statement is valid for $m > 1$. We will show that its true for $m+1$. If $\omega \in K_{m+1,i}$, then for all $k=1,...,n$ $K_{mk} \supset \psi_i[\{\omega\}]$. Let $s_k \in$ supp $\text{marg}_{Sk}[\Phi_i(s_i)]$. Then there exists $\omega' \in$ supp $\text{marg}_\Omega[\Phi_i(s_i)] = \psi_i[\{\omega\}]$ such that $\text{marg}_\Omega[\Phi_k(s_k)](X) = \mu(X|\prod_k)(\omega')$, for every $X \in \Sigma$. Thus we have that for all $k \neq$i and $s_k \in$ supp $\text{marg}_{Sk}[\Phi_i(s_i)]$:

(1) $s_k \in P_{m-1,k}$; (2) s_k consistent with $\prod_k$ and μ at ω'; and (3) $\omega' \in K_{mk}$, this last one because $\omega' \in \psi_i[\{\omega\}]$ and $K_{mk} \supset \psi_i[\{\omega\}]$. As a result, by the induction hypothesis, $s_k \in A_{mk}$, and hence, $s_i \in A_{m+1,i}$.

Q.E.D (of($\Leftarrow$))

5.4 *Corollary*: assume that Ω is finite and $\forall\ \omega \in \Omega$, $\mu(\omega) > 0$. Suppose that in the eyes of agent i the information sub-σ-algebras $(\prod_j)_{j \in N}$ and the prior μ are common knowledge and that the beliefs of agent i are consistent with $\prod_i$ and μ at ω. Then $A \in \sum$ is common knowledge at ω in the sense of Aumann if, and only if, A is common knowledge.

Proof: by the equivalence shown in the appendix, just take infinite intersections on m of both sides of the statement of the theorem above, and the corrolary follows.

Q.E.D.

Appendix

The purpose of this appendix is to show that the iterative procedure of defining finite levels of knowledge of an event, which is described in

section 2 of text, is equivalent to Aumann's (1976) argument, that of reachability.

Recall the definitions of Ω, Σ and $\prod_1, \prod_2, ..., \prod_n$. Here, we are going to have Ω finite, $\Sigma = \wp(\Omega)$ and $\prod_1, \prod_2, ..., \prod_n$ viewed as partitions of Ω, as well as the σ-algebras they generate. For $\omega \in \Omega$, $\psi_i[\{\omega\}] \in \prod_i$ is the element of the partition $\prod_i$ with $\omega \in \psi_i[\{\omega\}]$. Also, recall the definition of the sets K_{mi}, which represent the set of states of the world where there are m levels of knowledge about event A for the i^{th} agent:

$$K_{1i} = \{\omega' \in \Omega \mid A \supset \psi_i[\{\omega'\}]\}, \text{ and for}$$
$$m \geq 2, \text{ for all } i: K_{mi} = \{\omega' \in \Omega \mid K_{m-1,k} \supset \psi_i[\{\omega'\}], \text{ for all } k \in N\}$$

Aumann (1976) defines a different procedure to represent knowledge up to finite level. One says that ω' is *reachable* from ω if for some i, and some $m \geq 1$, $\omega' \in R_{mi}(\omega)$, where the sets $R_{mi}(\omega)$ are defined inductively as follows: $R_{1i}(\omega) = \psi_i[\{\omega\}]$; and for all $m \geq 2$, for all i:

$$R_{mi}(\omega) = \cup_k (\cup P_k) \text{ where } P_k \in \prod_k \text{ and } P_k \cap R_{m-1,i}(\omega) \neq \emptyset$$

Aumann formalizes the statement "i knows that (everyone knows that)$^{m-1}$ the event A has occurred" when the state of the world is ω as $A \supset R_{mi}(\omega)$.

We will prove that the two definitions of knowledge coincide. Furthermore, that the definition of common knowledge which follows by taking m to be infinite in both cases, coincide with Aumann's definition of common knowledge. Before doing that, we need a lemma.

A.1 *Lemma*: $\forall \omega \in \Omega, \forall m \geq 1: R_{m+1,i}(\omega) = \cup_{\omega' \in \psi_i[\omega]} (\cup_k R_{mk}(\omega'))$.

Proof: We will prove first that $\cup_{\omega' \in \psi i[\{\omega\}]} (\cup_k R_{mk}(\omega')) \supset R_{m+1,i}(\omega')$. By induction on m. For $m=1$, $R_{2i}(\omega) = \{\omega'' \in \Omega \mid \exists k \text{ with } \omega'' \in P_k \in \prod_k$ and $P_k \cap \psi_i[\{\omega\}] \neq \emptyset \}$. Obviously P_k must be the same as $\psi_k[\{\omega'\}]$, where $\omega' \in P_k \cap \psi_i[\{\omega\}]$. Also, $R_{1k}(\omega') = \psi_k[\{\omega'\}]$, so that the result holds for $m=1$. Suppose now it is true for $m \geq 1$. We will show it holds for $m+1$. In fact, if $\omega'' \in R_{m+2,i}(\omega)$, then there exists k with $\omega'' \in P_k \in \prod_k$ and $P_k \cap R_{m+1,i}(\omega) \neq \emptyset$. Let $\varpi \in P_k \cap R_{m+1,i}(\omega)$. Then, by the induction hypothesis, there exists $\omega' \in \psi_i[\{\omega\}]$ and k with $\varpi \in P_k \cap R_{mk}(\omega')$. Hence, $\omega'' \in R_{m+1,k}(\omega')$ for some k, with $\omega' \in \psi_i[\{\omega\}]$ and the result follows.

We now prove the other side of the inclusion, that is to say, for all $\omega' \in \psi_i[\{\omega\}]$, and for all $k=1,\dots,n$, $R_{m+1,i}(\omega) \supset R_{mk}(\omega')$. Again, the proof goes by induction on m. The case $m=1$: $R_{1k}(\omega') = \psi_k[\{\omega'\}] \in \prod_k$. As $\omega' \in \psi_i[\{\omega\}]$, it follows that $\omega' \in \psi_i[\{\omega\}] \cap \psi_k[\{\omega'\}] \neq \emptyset$. Thus, by the definition of the set $R_{2i}(\omega)$, we have the result for $m=1$. Assume it is true for m. We will show it holds for $m+1$. Let $\omega' \in \psi_i[\{\omega\}]$, and $\omega'' \in R_{m+1,k}(\omega')$. Then there exists j with $\omega'' \in P_j \in \prod_j$, and $P_j \cap R_{mk}(\omega') \neq \emptyset$. But this means that there exists j with $\omega'' \in P_j \in \prod_j$ and $P_j \cap R_{m+1,j}(\omega) \neq \emptyset$, by the induction hypothesis. Hence $\omega'' \in R_{m+2,i}(\omega)$.

Q.E.D

The next proposition shows that the two definitions of knowledge up to a finite level are equivalent.

A.2 *Proposition*: For any event A, $\forall i$, $\forall \omega \in \Omega$, $\forall m \geq 1$: $\omega \in K_{mi} \Leftrightarrow A \supset R_{mi}(\omega)$.

Proof: ($\Rightarrow$) by induction on m. for $m=1$, $\omega \in K_{1i} \Rightarrow A \supset \psi_i[\{\omega\}]$. But $\psi_i[\{\omega\}] = R_{1i}(\omega)$, so that $A \supset R_{1i}(\omega)$. Suppose now that the result is true for all k and $m \geq 1$. We will show it holds for $m+1$. Let $\omega \in K_{m+1,i}$. Then, for all k, $K_{mk} \supset \psi_i[\{\omega\}] \Rightarrow \forall\, \omega' \in \psi_i[\{\omega\}]$, $\omega' \in K_{mk}$. Thus, by the induction hypothesis, $A \supset R_{mk}(\omega')$. By lemma A.1,
$R_{m+1,i}(\omega) = \cup_{\omega' \in \psi i[\{\omega\}]} (\cup_k R_{mk}(\omega'))$. However, the set on the right hand side is contained in the set A, so that the implication follows.

($\Leftarrow$) Once more, by induction on m. For $m=1$, $R_{1i}(\omega) = \psi_i[\{\omega\}]$. Hence, $A \supset R_{1i}(\omega) \Rightarrow \omega \in K_{1i}$, by definition. Assume the induction hypothesis, *i.e.*, the result is true for m. By lemma A.1, if $\omega' \in \psi_i[\{\omega\}]$, for all k it follows that $R_{m+1,i}(\omega) \supset R_{mk}(\omega')$. As $A \supset R_{m+1,i}(\omega)$, we have that, by the induction hypothesis for any k, $\omega' \in K_{mk}$. Thus, for all k: $K_{mk} \supset \psi_i[\{\omega\}] \Rightarrow \omega \in K_{m+1,i}$.

Q.E.D

The common knowledge equivalence is derived by taking infinite intersections and unions. The $\prod_i$'s will be interpreted as σ-algebras. Before proving that, we state without proof a useful lemma.

A.3 *Lemma*: The sets $R(\omega) = \bigcup_{m\geq 1} R_{mi}(\omega)$ and $K_A = \bigcap_{m\geq 1} K_{mi}$ independ on i. Both $R(\omega)$ and K_A are elements of the meet, *i.e.*, of $\prod_1 \cap \prod_2 \cap ... \cap \prod_n$. Furthemore, $\omega \in K_A \leftrightarrow A \supset R(\omega)$.

The set K_A is the same as the homonimous in Milgrom (1981). One should notice that the sets K_A and $R(\omega)$ need not coincide, even if $\omega \in K_A$. In the case $\omega \in A$, it is easy to prove that $K_{mi} \supset R_{mi}(\omega)$ for all m and i, so that $K_A \supset R(\omega)$. Finally, the proposition below establishes the equivalence of both definitions of common knowledge given in this appendix with Aumann's definition, as well as summarizes the results. We also state this without proof as it is an easy corollary of A.2 and A.3 together with the observations made above.

A.4 *Proposition*: The three statements below are equivalent:

(1) $\exists B \in \prod_1 \cap \prod_2 \cap ... \cap \prod_n$ with $\omega \in B$ and $A \supset B$;

(2) $\omega \in K_A$;

(3) $A \supset R(\omega)$.

Moreover, any B which satisfies (1) is such that $B \supset R(\omega)$.

Resumo

Intuitivamente, um evento é de conhecimento comum (ou domínio público, ou saber comum), se todos os indivíduos sabem que ele ocorreu, se todos sabem que todos sabem que ele ocorreu, e assim por diante. Este artigo dá uma definição bayesiana da noção de conhecimento comum, que é baseada numa hierarquia infinita de crenças, a partir de um conjunto que contém a incerteza básica. A hierarquia infinita de crenças de cada agente representa sua distribuição *a priori* sobre a incerteza básica, sua distribuição *a priori* sobre as distribuições *a priori* dos outros agentes, e assim por diante. Este arcabouço permite uma formalização direta da noção intuitiva de conhecimento comum: um evento é de conhecimento comum do ponto de vista do agente i, se ele acredita que o evento ocorreu com probabilidade 1, se ele acredita com probabilidade 1 que os outros acreditam que o evento ocorreu com probabilidade 1, etc. O principal teorema do artigo estabelece a equivalência da definição bayesiana e a definição de conhecimento comum dada por Robert Aumann em seu artigo Agreeing to disagree (1976).

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