Monte Carlo test for stochastic trend in linear structural models for the location-scale family

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Abstract
In linear structural models for time series, a key point is the decision between modeling the trend of non-stationary processes through a deterministic or a stochastic term. The present paper introduces a Monte Carlo hypothesis test procedure to guide in such a decision. The method works for any time series distribution belonging to the location-scale family, where the analytical shape of the distribution must be defined by the user prior to apply the method. The proposed method provides an alpha-level test for any time series of length greater than 3 and it does not demand assumptions on the distribution of the trend term when it is actually stochastic.

Keywords: time series, structural models, Monte Carlo testing
JEL Codes: C12, C15, C18, C22

1 Introduction

One of the most comprehensive approaches for time series modeling is the class of the so called *space state models* (Harvey, 1989). The idea is to represent a time series Y_t as a linear combination of latent Markov processes and a noise term. This class of linear time series models plays a remarkable role in economic, medical, and soil science problems, and among the pioneers on illustrating its

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practical rationales we can cite the works of Harvey and Pierse (1984), Harvey and Todd (1983), and Shumway (1985).

A particular but still meaningful parametrization of a space state model is the *structural model*. With the structural model, Y_t is simply described through a level, μ_t, plus a noise, ϵ_t. One advantage of this simplified parametrization is the possibility of interpreting μ_t as a composition of practical components, such as trend, seasonality, and any other periodic structure.

The structure of particular interest in the present paper is that where μ_t is formed only by the level of the previous time and a trend term at the current time. In this case, the model is suggestively called *linear trend model*, and expressed as follows:

$$
\begin{aligned}
Y_t &= \mu_t + \epsilon_t\,, \\
\mu_t &= \mu_{t-1} + \beta_t + \eta_t\,, \\
\beta_t &= \beta_{t-1} + \xi_t, \qquad t = 1, \ldots, n\,.
\end{aligned}
\tag{1}
$$

The term β_t is interpreted as the trend term, and the independent white noise terms, ϵ_t, η_t, and ξ_t, each follows a specific probability distribution which, usually, is assumed to be a normal distribution with variances σ^2_ϵ, σ^2_η, and σ^2_ξ, respectively.

It merits note that, if $\sigma^2_\xi = 0$, then the trend term is simply given by a deterministic function of t. More precisely, from (1), for given constants μ_0 and β_0 such that $\mu_1 = \mu_0 + \eta_1$, and $\beta_1 = \beta_0 + \xi_t$, if $\sigma^2_\xi = \mathrm{Var}(\xi_t) = 0$, then $\beta_t = \beta_0$ and $\mu_t = \mu_0 + t\beta_0 + \sum_{i=1}^{t} \eta_i$. Statistical inference for this last form is far simpler than the situation where $\sigma^2_\xi > 0$. This is so because, instead of having to estimate a n-dimensional vector of hidden β_t terms, one would have to infer β_0 only. In this context, a natural question arises: how could we choose between modeling the trend as deterministic ($\sigma^2_\xi = 0$) or stochastic ($\sigma^2_\xi > 0$) for a given time series?

The question above has been attacked through many different ways. An overview on the main contributions found in the literature is offered in section 2. In general, former results require estimation of the full model, and most of the methods are constructed under the normal distribution assumption. In the present paper we introduce a new method based on a modification of the NM test statistic proposed by Nyblom and Mäkeläinen (1983). Although the NM statistic was originally developed to test deterministic against stochastic level terms, in the present work we prove that appropriate adjustments can be applied in order to construct a valid method to test the trend term too.

The proposed method presents important advantages over former proposals: (i) it holds for distributions of ϵ and η in the location-scale family, which must be defined in advance. Note that the classical normal or t-student distributions are just particular cases, and even discrete distributions can be assumed; (ii) it promotes a alpha-level test for any time series of length greater than 3; (iii) it does not demand any knowledge about the distribution of ξ_t; and (iv) it does not demand to estimate the model as a whole, that is, it is not necessary to estimate the multiple terms β_t and μ_t, $t = 1, \ldots, n$. Besides, we show through an intensive numerical study that the method presents a satisfactory statistical power even for small sample sizes.

This paper is organized as follows: Next section cites some of the main procedures related to the subject of deterministic versus stochastic trend. In the opportunity, the NM test for deterministic versus stochastic levels is described. Section 3 introduces our proposed method. Section 4 derives the analytical validity of the method with respect to the desired alpha-level. The statistical performance of the method in terms of statistical power is evaluated in section 5 by means of an appropriate simulation study. Section 6 brings an example of application to a time series of soy price in the state of Mato Grosso, Brazil. Section 7 closes the paper with some concluding remarks.

2 Overview

Tests and visual procedures to evaluate the trend component were carefully discussed in chapter 5 of Harvey (1989). Nonetheless, Franzini and Harvey (1983) is also a relevant article where the statistical framework, including the hypotheses testing, to examined if a structural model has deterministic or stochastic trend. Moreover, the trend structure has been studied with different proposals in the additive/structural time series models. Wu and Zhao (2007) presents the statistical theory as the construction of confidence bands and a test statistic for the presence of structural breaks for some relevant cases. Perron and Yabu (2009) introduced a novel approach and its theoretical formalism to perform a statistical test for the slope of a trend component. Also, a trace statistic to determine the number of common stochastic trend is proposed by Chang, Jiang, and Park (2006).

Another prominent approach is the "augmented Dickey-Fuller test" (ADF) (Gujarati, 2003). But, according to Stadnytska (2010), ADF inflates the actual significance level, and a correction for such effect leads to moderate statistical powers. Also, a method conventionally used for testing the null hypothesis of deterministic stationary trend, versus the alternative of a stochastic trend, is the test proposed by Hobijn, Franses, and Ooms (2004), and generalized by Kwiatkowski, Phillips, Schmidt, and Shin (1992), called KPSS, which has been extensively used in many fields (Chen & Pun, 2019; Montasser, 2015). Because of the prominent attention received by KPSS in this field, subsection 5.3 shows the results of a comparison study of KPSS with the method introduced in this paper.

Finally, outside of the structural models, some important contributions for testing parameter instability were done in regression analysis. For instance, King and Hillier (1985) constructed a class of invariant hypothesis testing to be implemented in cases where the covariance matrix has a specific parametrized form. Focusing on the dynamic regression models, Elliott and Müller (2006) presented a robust test that investigates the persistent time variation in the parameter coefficient.

None of the above methods work for an entire family of distributions, such as the location-scale family. Usually, the normal distribution is assumed, such as e.g. with ADF and KPSS. This is also the case of the method used for testing stochastic levels described in what follows.

2.1 The NM test for local level models

As a particular case of (1), the representation of a time series without trend or periodic components is the local level model:

$$\begin{aligned} Y_t &= \mu_t + \epsilon_t, \\ \mu_t &= \mu_{t-1} + \eta_t, \qquad t = 1, \ldots, n. \end{aligned} \tag{2}$$

The existence of stochastic level caused by the variability from η_t leads to an elegant but still elaborated estimation process based on the Kalman Filter algorithm, (Harvey, 1989). Contrariwise, if $\sigma_\eta^2 = 0$ (deterministic level), then the estimation process is far simpler since it demands to estimate a single term related to the level, μ_0. The hypotheses of interest in this scenario are of the form:

$$\begin{aligned} \mathrm{H}_0&: \quad \sigma_\eta^2 = 0, \\ \mathrm{H}_1&: \quad \sigma_\eta^2 > 0. \end{aligned} \tag{3}$$

For testing the hypotheses above, and assuming that ϵ_t follows a normal distribution, Nyblom and Mäkeläinen (1983) proposed to use the following test statistic:

$$NM = \frac{\sum_{t=1}^{n} \left[\sum_{s=t}^{n} (y_s - \bar{y}) \right]^2}{(n-1) \sum_{t=1}^{n} (y_t - \bar{y})^2}. \tag{4}$$

The probability distribution of NM, derived by Nyblom and Mäkeläinen (1983), is formed by a ratio of linear combinations of Chi-squared random variables, that is,

$$NM \sim \frac{\sum_{k=1}^{n-1} \lambda_{kn}(1+\rho\lambda_{kn}) Z_k^2}{\sum_{k=1}^{n-1} (1+\rho\lambda_{kn}) Z_k^2}, \tag{5}$$

where $Z_1, Z_2, \ldots$ are independent standard normal random variables; $\rho = \sigma_\eta^2/\sigma_\epsilon^2$; and $\lambda_{kn}^{-1} = 2[1 - \cos(\pi k/n)]$, $k = 1, \ldots, n$. The asymptotic distribution of NM under H_0 was also derived by Nyblom and Mäkeläinen (1983), where:

$$NM/n \xrightarrow{d} \sum_{k=1}^{\infty} \left(k^{-2} Z_k^2 \right), \tag{6}$$

where $\xrightarrow{d}$ means convergence in distribution. More recently, this test was fully explored through a bootstrap approach by Franco and Souza (2002). Their results indicate that, for small to moderate sample sizes, usage of bootstrap can promote better performances in terms of statistical power than those obtained with the asymptotic approach.

3 Method

Considering the linear trend model in (1), with $n \geq 4$, we want to test

$$\begin{aligned} &\mathrm{H}_0\colon \quad \sigma_\xi^2 = 0, \qquad \text{(deterministic trend)} \\ &\mathrm{H}_1\colon \quad \sigma_\xi^2 > 0. \qquad \text{(stochastic trend)} \end{aligned} \tag{7}$$

Define the transformation $X_t = Y_t - Y_{t-1}$, with expectation μ_t and variance σ_t^2. Our method requires that one actually knows, for $\sigma_\xi^2 = 0$, to which family of the location-scale class belongs the distribution $F_Z(z)$ of the transformation:

$$Z_t = \frac{X_t - g_1(\mu_t)}{g_2(\sigma_t^2)}, \qquad t = 2, \ldots, n. \tag{8}$$

That is, under, under $\sigma_\xi^2 = 0$, the distribution of X_t belongs to a known location-scale family, $F_Z(z)$, indexed by the unknown parameters μ_t and σ_t^2 that defines the location and scale parameters through the functions $g_1(\cdot)$ and $g_2(\cdot)$, respectively. Note that even discrete distributions can be used to model Y_t if the condition above holds. For the continuous case, one example is the classical assumption of a normal distribution for Y_t, which leads to the standard normal distribution for $F_Z(z)$.

The proposed test statistic is given by:

$$U(\vec{W}) = \frac{\sum_{l=1}^{N}\left[\sum_{s=l}^{N}\left(W_s - \bar{W}\right)\right]^2}{(N-1)\sum_{l=1}^{N}\left(W_l - \bar{W}\right)^2}, \tag{9}$$

where $\vec{W} = (W_1, \ldots, W_N)$, and

$$W_l = \begin{cases} Y_{2l} - Y_{2l-1}, & \text{if } n/2 \text{ is integer,} \\ Y_{2l+1} - Y_{2l}, & \text{otherwise.} \end{cases} \tag{10}$$

Hence, if n is an even number, then $N = n/2$; otherwise, we have $N = (n-1)/2$.

Note that $U(\vec{W})$ is equivalent to applying the NM statistic to the transformation W_l, $l = 1, \ldots, N$. But, instead of calculating critical values using the asymptotic distribution derived by Nyblom and Mäkeläinen (1983), in this paper we introduce a valid Monte Carlo solution for testing the hypotheses in (7).

3.1 Monte Carlo Testing Procedure

Let u_0 denote a realized value of $U(\vec{W})$. Also, let $\vec{Z}_1, \ldots, \vec{Z}_{m-1}$ denote a sequence of $(m-1)$ independent and N-dimensional random vectors, where $Z_{i,j}$ denotes the ith entry of the jth vector, with $i = 1, \ldots, N$, $j = 1, \ldots, (m-1)$, and $Z_{i,j} \sim F_Z(z)$. In addition, let $U_1, \ldots, U_{m-1}$ denote the sequence of Monte Carlo test statistics obtained by applying the statistic $U(\cdot)$ to each of the vectors $\vec{Z}_j$, that is,

$$U_j = U(\vec{Z}_j).$$

With this, define the Monte Carlo measure of evidence:

$$G = \sum_{j=1}^{m-1} I(U_j \geq u_0). \tag{11}$$

For an arbitrary significance level of interest, say $\alpha \in (0,1)$, let $\psi(G,\alpha,m)$ denote the binary decision function based on the statistic G such that, if $\psi(G,\alpha,m) = 1$, then H_0 in (7) is rejected (stochastic trend), but, if $\psi(G,\alpha,m) = 0$, then H_0 is taken as true (deterministic trend), where:

$$\psi(G,\alpha,m) = \begin{cases} 1, & \text{if } G \leq \alpha m - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Application of the decision criterion above is equivalent to use the Monte Carlo p-value:

$$P_{mc} = (G+1)/m, \tag{13}$$

where H_0 is reject for $P_{mc} \leq \alpha$.

This proposed Monte Carlo procedure based on the test statistic in (9) is called MCt herein.

4 Validity of MCt

In this section we show the validity of the proposed method.

Theorem 1. *The procedure in (12) is a α-level test.*

Proof. Without loss of generality, the reasoning is developed for $(n/2)$ integer. We want to prove that

$$\Pr[\psi(G,\alpha,m) = 1 \mid H_0] \leq \alpha. \tag{14}$$

To demonstrate the above, it is worth to decompose W_l as follows:

$$\begin{aligned} W_l &= Y_{2l} - Y_{2l-1} \\ &= \mu_{2l} + \epsilon_{2l} - \mu_{2l-1} - \epsilon_{2l-1} \\ &= \mu_{2l-1} + \beta_{2l} + \eta_{2l} + \epsilon_{2l} - \mu_{2l-1} - \epsilon_{2l-1} \\ &= \beta_0 + \eta_{2l} + \epsilon_{2l} - \epsilon_{2l-1}. \qquad \text{(under } H_0\text{)} \end{aligned} \tag{15}$$

From (15),

$$\begin{aligned}
\text{Cov}(W_l, W_k) &= 0, && \text{for } l \neq k; \\
\text{Var}(W_l) &= \sigma_\eta^2 + 2\sigma_\epsilon^2, && \text{for } l = 1, \ldots, N; \\
\mathbb{E}(W_l) &= \beta_0, && \text{for } l = 1, \ldots, N.
\end{aligned} \tag{16}$$

By simplicity, denote $\text{Var}(W_l)$ by σ_w^2. With this, the test statistic can be rewritten as follows:

$$\begin{aligned}
U(\vec{W}) &= \frac{\left[g_2\left(\sigma_w^2\right)\right]^2 \sum_{t=1}^{N} \left[\sum_{s=t}^{N} \left(W_s - g_1(\beta_0) + g_1(\beta_0) - \bar{W}\right)\right]^2}{\left[g_2(\sigma_w^2)\right]^2 (N-1) \sum_{t=1}^{N} \left(W_t - g_1(\beta_0) + g_1(\beta_0) - \bar{W}\right)^2} \\
&= \frac{\sum_{t=1}^{N} \left\{\sum_{s=t}^{N} \left[(W_s - g_1(\beta_0)) - \left(\bar{W} - g_1(\beta_0)\right)\right] / g_2\left(\sigma_w^2\right)\right\}^2}{(N-1) \sum_{t=1}^{N} \left\{\left[(W_t - g_1(\beta_0)) - \left(\bar{W} - g_1(\beta_0)\right)\right] / g_2(\sigma_w^2)\right\}^2} \\
&= \frac{\sum_{l=1}^{N} \left[\sum_{s=l}^{N} \left(Z_s - \bar{Z}\right)\right]^2}{(N-1) \sum_{l=1}^{N} \left(Z_l - \bar{Z}\right)^2},
\end{aligned} \tag{17}$$

where:

$$Z_l = \frac{W_l - g_1(\beta_0)}{g_2(\sigma_w^2)}. \tag{18}$$

From (16), β_0 and σ_w^2 are the unknown mean and variance of W_l, respectively. Theref2ore, under H_0, the statistic $U(\vec{W})$ is ancillary with respect to the unknown parameters μ_0, β_0, σ_η^2 and σ_ϵ^2 if the distribution of W_l is assumed to belong to a location-scale family with known standard distribution $F_Z(z)$.

Let $F_U(u|\sigma_\xi^2)$ denote the probability distribution of $U(\vec{W})$. Note the dependence of this distribution on σ_ξ^2. Under the null hypothesis, i.e. $\sigma_\xi^2 = 0$, the distribution of the transformation $P = 1 - F_U(U|\sigma_\xi^2 = 0)$ is continuous uniform in the $(0,1)$ support if $F_U(u|\sigma_\xi^2)$ is continuous in u, and it is discrete uniform in the $(0,1)$ support if $F_U(u|\sigma_\xi^2)$ is discrete. This transformation is the classical p-value of the tail-type. Thus, for a given constant $\alpha \in (0,1)$, it holds that:

$$\Pr[P \leq \alpha \,|\, H_0] \leq \alpha, \qquad \alpha \in (0,1). \tag{19}$$

It is crucial to note that the Bernoulli probability behind the event $I(U_j \geq u_0)$ in (11) is a realization $P = p$ for a fixed u_0. Thus, let $F_P(p)$ denote the probability distribution of P. Under the null hypothesis, the probability of rejecting the null hypothesis with the decision rule in (12) is:

$$\begin{aligned}
\Pr[\psi(G,\alpha,m) = 1|H_0] &= \int_0^1 \sum_{g=0}^{\lfloor \alpha m \rfloor - 1} \binom{m-1}{g} p^g (1-p)^{m-1-g} dF_P(p) \\
&\leq \int_0^1 \sum_{g=0}^{\lfloor \alpha m \rfloor - 1} \binom{m-1}{g} p^g (1-p)^{m-1-g} dp \qquad \text{(from (19))} \\
&= \sum_{g=0}^{\lfloor \alpha m \rfloor - 1} \frac{\binom{m-1}{x}}{\Gamma(g+1)\Gamma(m-g)/\Gamma(m+1)} \\
&= \sum_{g=0}^{\lfloor \alpha m \rfloor - 1} \frac{\binom{m-1}{x}}{m\binom{m-1}{x}} \\
&= \sum_{g=0}^{\lfloor \alpha m \rfloor - 1} \frac{1}{m} = \frac{\lfloor \alpha m \rfloor}{m} \leq \frac{\alpha m}{m} = \alpha,
\end{aligned}$$

with the strict equality holding for rational α, m multiple of $1/\alpha$, and $F_P(p)$ continuous. Therefore, the procedure in (12) is a α-level test. □

The functions $g_1(\cdot)$ and $g_2(\cdot)$ depend on each specific $F_Z(z)$. For example, if W_t is normally distributed, then the associated standard density function is $f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$, and the location and scale parameters are $g_1(\beta_0) = \beta_0$ and $g_2(\sigma_w^2) = \sqrt{\sigma_w^2}$, respectively. If W_t follows a Logistic distribution, then $f_Z(z) = e^{-z}/(1+e^{-z})^2$, $g_1(\beta_0) = \beta_0$ and $g_2(\sigma_w^2) = \sqrt{3\sigma_w^2/\pi}$. For these or any other member of the location-scale family, the point is that the standard distribution $F_Z(z)$ does not depend on unknown parameters, so then Monte Carlo samples can be generated from $F_Z(z)$ in practice.

Important: note that the validity shown with Theorem 1 holds for any choice $m \geq 1/\alpha$ and for any sample size n. Regarding the statistical power of Monte Carlo tests in general, Silva and Assunção (2013) shows that any choice $m \geq 2000$ leads to statistical power very similar to the exact test power in most of the problems.

5 Performance of MCt: a simulation study

This section presents the results of a simulation study directed to estimate the statistical power of the proposed method, MCt, for selected tuning parameters. In the opportunity, MCt is compared with the conventional KPSS method.

This study assumes the classical normal distribution for ϵ_t and η_t, that is,

$$\epsilon_t \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right), \quad \eta_t \sim \mathcal{N}\left(0, \sigma_\eta^2\right)$$

which leads to the standard probability density function $f_Z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ to be used in the Monte Carlo testing procedure described in subsection 3.1.

5.1 Tuning Parameters Settings

There are two different groups of tuning parameters to specify. The first group relates to the operation of the test for a given times series. As the method is quite simple, there are only two parameters to specify, which are α and m. The meaningful choices for α are 0.01 and 0.05 as these are conventionally used in hypothesis testing practice. For m, the rule of thumb offered by Silva and Assunção (2013) says that any choice greater than 2,000 ensures satisfactory performances, then, aiming parsimony, we set $m = 2{,}000$ in all scenarios.

The second group of tuning parameters relates to the different scenarios at which the power is to be estimated. Firstly, we need to establish the number of time series replications, here denoted by M. For this we settled the value $M = 10{,}000$. The length of the time series (sample size) is determinant for the power magnitude, for which the scenarios are $n = 30$ (small), $n = 50$ (intermediate), and $n = 100,\ 150$ (large). The other parameters are related to the theoretical model. For this, we set $\mu_0 = \beta_0 = \sigma_\eta^2 = \sigma_\epsilon^2 = 1$. Finally, for the target parameter of interest, σ_ξ^2, we considered values in a grid from $\sigma_\xi^2 = 0$ (H_0 true) to $\sigma_\xi^2 = 2$ by 0.2 (H_0 false).

Besides tuning parameters, another component that can influence the performance of the method is the probability distribution, F_ξ, of ξ_t. It merits to stress that MCt is applicable for any distribution F_ξ in the location-scale family, not only for the normal distribution. To illustrate this we considered two scenarios, one where F_ξ is the normal, and another where F_ξ is discrete and asymmetric. The last was produced by defining:

$$\xi_t = Q - \lambda, \tag{20}$$

where $Q \sim Poisson(\lambda)$.

5.2 Simulation Study Results

As expected, the results in Table 1, and in Table 2, confirm the property already proved in section 4 about the validity of the procedure, that is, the test size is under the desired significance level, that is, if $\sigma_\xi^2 = 0$, then the Type I error probability equal to α for continuous $F_Z(z)$.

Note that the maximum σ_ξ^2 value used in this simulation study, $\sigma_\xi^2 = 2$, represents the situation where the variability from the trend term can influence the movements in the observations Y_t as much as the sum of variances from ϵ_t and η_t, case where the method should present elevated power. That is exactly what we observe for all scenarios of moderate to high sample sizes since the power estimates are, in most of the cases, greater than 0.6. The power in situations of small sample sizes are of intermediate magnitude (greater than 0.5) in most of the scenarios under $\alpha = 0.05$, but the performance is not so good under combinations of small samples and small α levels, such as e.g. 0.01.

5.3 Comparing MCt with KPSS

As already mentioned, for the best of our knowledge, there is no comparable method available in the literature for testing stochastic trends for the entire location-scale family. But, specifically when F_ξ is assumed to belong to the normal distribution family, then KPSS is the conventional method used to test stochastic trend. Table 3 shows estimated statistical powers of KPSS under significance levels of $\alpha = 0.01$ and $\alpha = 0.05$ with selected σ_ξ values. We note that nominal level of KPSS is not under control. Actually, the Type I error probability is greater than the nominal level in all scenarios. This is a well known fact. As remarked by Hadri and Rao (2009), in cases where one models a deterministic level when a deterministic trend is true, then KPSS leads to over-rejection of the null hypothesis. Moreover, even with larger exposition to the Type I error probability, KPSS presents a statistical power uniformly smaller than that from MCt in each of the evaluated scenarios.

6 Example for Real Data

In this section we apply the proposed test to a real time series of $n = 143$ soy prices (60 kg sack), practiced in Mato Grosso, Brazil, observed in the period of Jan/2008 to Nov/2015 under monthly measurements. Mato Grosso presents a rich biodiversity besides its population's culture diversity. This is one of the reasons that could explain its proficiency in agriculture, which represents about

Table 1. Power estimates for the MCt method under the normal distribution for ξ_t, for $\mu_0 = \beta_0 = \sigma^2_\eta = \sigma^2_\epsilon = 1$, and $\sigma^2_\xi = 0,\ 0.2, \cdots, 2$. The parametrization was $\alpha = 0.01,\ 0.05$, and $m = 2000$. The estimation was based on $M = 10000$ time series generated for each configuration of σ^2_ξ, n and α.

		$\alpha = 0.01$				$\alpha = 0.05$			
σ^2_ξ	$n =$	30	50	100	150	30	50	100	150
0		0.01	0.01	0.01	0.01	0.05	0.05	0.05	0.05
0.2		0.14	0.34	0.64	0.80	0.28	0.49	0.77	0.87
0.4		0.23	0.46	0.74	0.87	0.39	0.62	0.85	0.93
0.6		0.29	0.52	0.79	0.91	0.45	0.66	0.88	0.95
0.8		0.33	0.56	0.81	0.91	0.45	0.70	0.90	0.95
1.0		0.36	0.58	0.82	0.91	0.53	0.72	0.91	0.97
1.2		0.39	0.61	0.84	0.92	0.54	0.73	0.92	0.97
1.4		0.40	0.61	0.84	0.93	0.56	0.75	0.93	0.98
1.6		0.42	0.62	0.86	0.93	0.58	0.76	0.93	0.98
1.8		0.43	0.63	0.86	0.93	0.58	0.76	0.94	0.98
2.0		0.44	0.65	0.86	0.94	0.60	0.77	0.94	0.99

Table 2. Power estimates under $\xi_t = Q - \lambda$, where $Q \sim Poisson(\lambda)$. The other parameters are $\mu_0 = \beta_0 = \sigma^2_\eta = \sigma^2_\epsilon = 2$, and $\sigma^2_\xi = 0,\ 0.2, \cdots, 1$. The parametrization was $\alpha = 0.01,\ 0.05$, and $m = 2000$. The estimation was based on $M = 10000$ time series generated for each configuration of σ^2_ξ, n and α.

		$\alpha = 0.01$				$\alpha = 0.05$			
$\sigma^2_\xi = \lambda$	$n =$	30	50	100	150	30	50	100	150
0		0.01	0.01	0.01	0.01	0.05	0.05	0.05	0.05
0.2		0.13	0.34	0.65	0.80	0.27	0.5	0.77	0.88
0.4		0.22	0.46	0.75	0.87	0.39	0.61	0.84	0.94
0.6		0.28	0.51	0.77	0.88	0.45	0.66	0.88	0.96
0.8		0.32	0.57	0.80	0.92	0.51	0.70	0.90	0.96
1.0		0.37	0.58	0.82	0.92	0.52	0.72	0.91	0.96
1.2		0.30	0.59	0.84	0.93	0.55	0.74	0.92	0.97
1.4		0.42	0.62	0.85	0.93	0.57	0.75	0.92	0.98
1.6		0.42	0.63	0.85	0.94	0.59	0.76	0.93	0.98
1.8		0.44	0.64	0.86	0.94	0.59	0.77	0.93	0.98
2.0		0.47	0.65	0.86	0.95	0.61	0.78	0.94	0.99

Table 3. Power estimates of the KPSS method under the normal distribution for ξ_t, for $\mu_0 = \beta_0 = \sigma_\eta^2 = \sigma_\epsilon^2 = 1$, and $\sigma_\xi^2 = 0,\ 0.2, \cdots, 2$. The significance levels used were $\alpha = 0.01,\ 0.05$. This estimation was based on $M = 10000$ time series generated for each configuration of σ_ξ^2, n and α.

		$\alpha = 0.01$				$\alpha = 0.05$			
σ_ξ^2	$n =$	30	50	100	150	30	50	100	150
0		0.017	0.035	0.193	0.328	0.200	0.313	0.580	0.674
0.2		0.023	0.106	0.376	0.612	0.210	0.321	0.610	0.776
0.4		0.050	0.158	0.448	0.681	0.210	0.391	0.703	0.853
0.6		0.051	0.197	0.489	0.710	0.233	0.418	0.704	0.887
0.8		0.071	0.238	0.520	0.73	0.235	0.495	0.748	0.873
1.0		0.070	0.248	0.543	0.771	0.233	0.520	0.736	0.904
1.2		0.083	0.271	0.544	0.765	0.279	0.529	0.763	0.890
1.4		0.080	0.305	0.552	0.777	0.265	0.531	0.779	0.904
1.6		0.084	0.292	0.562	0.764	0.278	0.564	0.794	0.915
1.8		0.093	0.312	0.555	0.789	0.259	0.603	0.774	0.914
2.0		0.098	0.309	0.572	0.781	0.286	0.585	0.801	0.916

50.46% of the whole economic activity of the state, and soy production plays an important role for such positive results.

The observed time series of soy prices is represented in Figure 1. The conventional Cox–Stuart test (Cox & Stuart, 1955), suggests existence of a trend term in this series (p-value < 0.0001). But, that test is not appropriate to decide whether the hypothetical trend is deterministic or stochastic. After running the method of section 3 assuming a normal distribution for ϵ_t and for η_t, $\alpha = 0.05$, and $m = 2000$, the observed Monte Carlo p-value, calculated according to (13), was equal to 0.079. Hence, under $\alpha = 0.05$, we conclude that there are no strong evidences against the hypothesis that, in this case, the trend is deterministic.

7 Concluding Remarks

The method derived in this paper works for any application where the data distribution belongs to the location-scale family, and the power performance is evidently robust over different trend distributions, such as asymmetric and discrete distributions.

The assumption that $F_Z(z)$ is known may be a limitation in some applications. However, there is a number of problems where the distribution of the data is known due to axiomatic construction and/or derived from the properties behind the sampling scheme. For example, with instrumental variables regression,

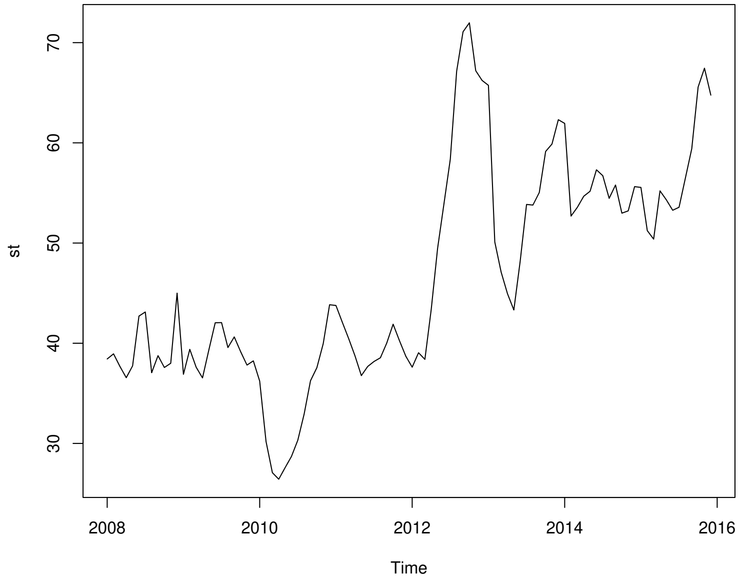

Figure 1. Average soy price (60 kg sack) is Brazilian currencies, in Mato Grosso state, monthly observed during the period of Jan/2008 to Nov/2015.

the endogenous covariates are written as a linear combination of instrumental variables and a white noise following a normal distribution for each fixed time (Nascimento, Abanto-Valle, & Mendonça, 2018); In clinical trials based on a self-control design, the binomial distribution represents the number of patients presenting adverse reactions to a new treatment over time (Silva, Kulldorff, & Yih, 2020); In post-market drug and vaccine safety surveillance, the Poisson distribution counts the number of adverse events from an exposure period window when the vaccine/drug was administered in the population (Kulldorff et al., 2011); The Erlang distribution has been used to model the time series of COVID-19 cases (Arino & Portet, 2020); In spatial statistic, the number of individuals with a certain disease in a region, conditioned on the total number of cases in the map, follows a Poisson distribution (Duczmal & Assunção, 2004); In a mark-recapture design for estimating the size of a population, the distribution of the tagged recaptured items is negative binomial under a with-replacement design (Mukhopadhyay & Bhattacharjee, 2018); In survival analysis, the time between two consecutive failures is well-fitted through a Weibull distribution (Zhang, 2016). Besides, one of the most prominent fields of the statistical inference that assumes knowledge about the distribution of the data is the class of the so-called 'generalized linear models', where the user selects one of distributions in the exponential family to model the data(Nelder & Wedderburn, 1972). In addition, one can always apply the non-parametric Chi-Square goodness-of-fit test to confirm the distribution for the data, which is valid for any sample size.

The proposed method is very simple to apply. As proved in section 4, there is no need for estimating the nuisance parameters μ_0, β_0, σ_ϵ^2 and σ_η^2. Also, the validity of the test is guaranteed irrespectively to: (i) the status of the level as deterministic or stochastic, and (ii) existence of a trend term ($\beta_0 > 0$) or not ($\beta_0 = 0$) in the true structure of the time series. But, regarding item (ii), the correct interpretation differs depending on the test decision:

1. If H_0 is rejected, then, by construction, the conclusion is that there is a trend term behinds the time series structure and that such trend is stochastic.

2. If H_0 is not rejected, then the better we can say is that, if there exists a trend, that is deterministic. Thus, further analysis would be needed to decide if the final model is to be designed with or without a trend term, which if so it is to be considered as a deterministic component.

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