STABLE OUTCOMES IN DISCRETE AND CONTINUOUS MODELS OF TWO-SIDED MATCHING: A UNIFIED TREATMENT*

Alvin E. Roth**

Marilda Sotomayor***

Resumo

Apresentamos um tratamento unificado de uma classe de modelos de matching de dois lados que inclui modelos discretos (tais como o modelo do casamento de Gale e Shapley) e os modelos contínuos (tais como o modelo de designação de Shapley e Shubik e o modelo de designação generalizado de Demange e Gale). Em contraste com os tratamentos anteriores, as conclusões paralelas para os dois conjuntos de modelos são derivadas aqui do mesmo modo e a partir das mesmas hipóteses. Mostramos que os resultados em questão seguem, praticamente, da hipótese de que o núcleo coincide com o núcleo definido por dominação fraca. No modelo do casamento a hipótese de que as preferências são estritas faz com que estes dois conjuntos coincidam enquanto nos modelos contínuos os dois conjuntos coincidem porque os agentes tem preferências contínuas e os preços podem ser ajustados continuamente.

* Partially supported by J.S. Guggenheim Memorial Foundation, CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brazil, and by grants from the Alfred P. Sloan Foundation and the National Science Foundation, USA.

** Department of Economics, University of Pittsburgh, Pittsburgh, PA 15260, USA

*** Instituto de Economia Industrial, Universidade Federal do Rio de Janeiro, Av. Pasteur, 250 - Urca, Rio de Janeiro, RJ, Brasil.

Abstract

We present a unified treatment of a class of two-sided matching models that includes discrete models (such as the marriage model of Gale and Shapley) and continuous models (such as the assignment model of Shapley and Shubik and the generalized assignment model of Demange and Gale). In contrast with previous treatments, the parallel conclusions for the two sets of models are derived here in the same way from the same assumptions. We show that the results in question all follow closely from the assumption that the core coincides with the core defined by weak domination. In the marriage model, the assumption of strict preferences causes these two sets to coincide, while in the continuous models the two sets coincide because agents have continuous preferences and prices can be adjusted continuously.

Palavras-Chave: Matching, stable matching, core, lattice, optimal matching

Código JEL: C78

1. Introduction.

One of the persistent puzzles in the theory of two-sided matching is the great similarity of the results concerning the core of the game that have been obtained for the discrete models (e.g. the marriage model of Gale and Shapley (1962)) and for continuous models (e.g. the assignment model of Shapley and Shubik (1972) and the generalized assignment model of Demange and Gale (1985)), even though the two kinds of models appear to have important differences. Furthermore, the proofs by which these results have so far been obtained remain quite different, and seem to rely on different principles. As Ballinski and Gale (1990, p. 274) note:

“There is, by now, a substantial literature on these problems, and one is struck by the fact that almost all results proved for the ordinal case have analogues in the cardinal case, although the techniques of proof in the two cases are in general quite different, and there is as yet no “unified theory” which covers both.”

The most perplexing aspect of the puzzle is that the similar conclusions for the two classes of games require assumptions that are not merely different, but are in the following important sense very nearly *opposite*. To derive most of the results we will be concerned with here, in the marriage model we need to assume that players have strict preferences, i.e. that no player is indifferent between being matched to any pair of possible mates. In the continuous models, however, the assumption is that *every* player can be made indifferent between *any* pair of possible mates by appropriate adjustments of prices.

Perhaps the most dramatic conclusion common to the two models is that, in the marriage model when no agents are indifferent between alternative matches, and in the continuous models (in particular the assignment model) when all agents may be made indifferent between any two matches, there exists, for each side of the market, a point in the core which is optimal for all agents on that side of the market, in the sense that no agent on that side of the market does better at any other core outcome¹. It is already known that there may not exist one of the optimal points in the core if indifferences is allowed in the marriage market, (see Roth and Sotomayor (1990)). After presenting the formal model, we will show by example that this also happens if indifference is precluded in a model in which money is continuously divisible. This is typical of how the very similar conclusions about the two models do in fact depend on very dissimilar assumptions.

This paper presents a unified treatment of a class of two-sided matching models that includes the marriage and continuous models, such that the parallel conclusions for the two sets of models are de-

¹ For example, in a market with equal numbers of identical sellers and identical buyers of some homogeneous indivisible good, outcomes in the core are characterized by a single price, between the reservation price of the buyers and the sellers, at which all trades are made. The buyer optimal core outcomes is the one at which the price is as high as possible, i.e. at the seller reservation price, and the seller optimal core outcome is the one at which the price is as low as possible.

rived in the same way from the same assumption. The appropriate assumption turns out to be subtle but simple: the results in question all follow from the assumption that the core coincides with the core defined by weak domination. In the marriage model, the assumption of strict preferences causes the two sets to coincide, while in the continuous models the two sets coincide because agents have continuous preferences and prices can be adjusted continuously².

In our formal treatment, we develop a class of models which generalize both the marriage and continuous models. In this model, the common results for these two classes of models will follow from the assumptions that utilities are increasing in money, that the set of individually rational transfers is bounded, and that the core is non-empty and closed, together with the assumption that the core coincides with the core defined by weak domination. (All of these assumptions except for the last are satisfied by the marriage model even when preferences are not strict.)

Two-sided matching models are not merely of theoretical interest; they are also powerful descriptive models. Discrete models have proved to be appropriate tools with which to analyse markets in which wages can be treated as part of the job description, such as the markets for medical interns in the U.S. - Roth (1984 a, 1986) and the U.K. - Roth (1990, 1991), or non-wage matching processes, see Mongell and Roth (1991) and Sotomayor (1996 c). It seems likely

² An analogous unification was accomplished in Roth and Sotomayor (1989) concerning the marriage model (which involves one-to-one matching) and the college admissions model (which involves many-to-one matching). That paper showed that the core of the college admissions game shares the properties found in the marriage model even when colleges may be indifferent between different entering classes, so long as they have strict preferences over individual students. The reason is that when colleges have strict preferences over students, they will not be indifferent between any two entering classes they can be assigned in the core, even though they may be indifferent between some pairs of entering classes. And the results for the marriage model follow from the weaker assumption that no agent is indifferent between any two mates who can be assigned to him in the core.

that continuous models may be more useful for analysing markets in which wages are determined endogenously. The class of models introduced here allows wages to be treated either way, i.e. either as part of the job description, or as an endogenously determined part of the market outcome.

2. The Mathematical Model.

There are two finite, disjoint sets of players $P = \{1, \dots, i, \dots, m\}$ and $Q = \{1, \dots, j, \dots, n\}$. The players in P and Q will be called P -players and Q -players respectively. The set of all permissible transfers between two players is a set of numbers which will be denoted by S . For mathematical convenience we will allow a player to make a transfer of $s = 0$ to himself. Furthermore if i transfers s to j then we say that j transfers $-s$ to i .

That is, we will be assuming that

$$I_1- 0 \in S;$$

$$I_2- s \in S \iff -s \in S.$$

The structure of preferences is given by the utility functions: $U_{ij} : S \rightarrow R$ and $V_{ij} : S \rightarrow R$. Thus $U_{ij}(s)$ is the utility to i in P of being matched with j and receiving a monetary payment of s from j , if $s \geq 0$, or transferring s to j , if $s \leq 0$. Analogously $V_{ij}(s)$ is the utility of j in Q for being matched with i .

The reservation utilities for the players i and j are given by $r_i = U_{ii}(0)$ and $s_j = V_{jj}(0)$, respectively. The market game is then given by (P, Q, S, U, V) . A transfer s from i to j is *individually rational* if $U_{ij}(-s) \geq r_i$ and $V_{ij}(s) \geq s_j$. We will be assuming that for all i in P and all j in Q :

A_1- U_{ij} and V_{ij} are strictly increasing in the variable s ;

A_2- the set of all individually rational transfers that can be made between i and j is bounded. (It may be empty).

Lacking this assumption, it would be possible for a player to regard some match as “infinitely good”, in the sense that a player would be prepared to pay any finite amount to achieve it. Specifically, any individually rational match between i and j could remain possible no matter how large the reservation utility of j .

Definition 1. A *matching* μ is a bijection from $P \cup Q$ onto $P \cup Q$ of order two (that is, $\mu^2(x) = x$), such that if $\mu(i) \notin Q$ then $\mu(i) = i$ and if $\mu(j) \notin P$ then $\mu(j) = j$.

That is, $\mu(i) = j$ means that at the matching μ i is matched to j , whereas $\mu(i) = i$ means i is single.

Definition 2. An *outcome* is a matching μ and a pair of vectors (u, v) called the *payoff*, with $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. An outcome will be denoted by $(u, v; \mu)$.

Definition 3. The outcome $(u, v; \mu)$ is *feasible* if there is a set of numbers $s_{ij} \in S$, defined only if $\mu(i) = j$ such that

- i) $u_i = U_{ij}(-s_{ij})$ and $v_j = V_{ij}(s_{ij})$, if $\mu(i) = j$
- and
- ii) $u_i = r_i$ if $\mu(i) = i$; $v_j = s_j$ if $\mu(j) = j$;
- iii) $u_i \geq r_i$, $v_j \geq s_j$ (individual rationality).

The payoff (u, v) is *feasible* if $(u, v; \mu)$ is a feasible outcome for some matching μ . In this case we say that μ is *compatible* with the feasible payoff (u, v) .

Definition 4. The feasible outcome $(u, v; \mu)$ is *stable* if there is no pair $(i, j) \in P \times Q$ with $\mu(i) \notin j$ and $s \in S$ such that $U_{ij}(-s) > u_i$ and $V_{ij}(s) > v_j$.

This is equivalent to requiring that no feasible outcome $(u', v'; \mu')$ exist such that $\mu'(i) = j$ and

$$u_i < u'_i \quad v_j < v'_j.$$

Similarly we define a *stable payoff*.

Then definition 4 says that (u, v) is stable if it is not dominated by any feasible payoff via some pair (i, j) . (Such a pair (i, j) is called a *blocking pair*). We take the rules of the game to be that any pair (i, j) in $P \times Q$ is free to match to one another at any price s if they both agree, and any individual is free to remain single. Then the pairs (i, j) are the only essential coalitions, and a payoff is stable if and only if it is in the core, which will be denoted by C . (This coincidence between the set of stable payoffs and the core does not survive the generalization from one-to-one matching to many-to-many matching: see Roth and Sotomayor (1990), Sotomayor (1992 and 1996 a,b)).

Definition 5. The stable outcome $(u, v; \mu)$ is *strictly stable* if there is no pair $(i, j) \in P \times Q$ with $\mu(i) \notin j$ and $s \in S$ such that $U_{ij}(-s) = u_i$ and $V_{ij}(s) > v_j$ or $U_{ij}(-s) > u_i$ and $V_{ij}(s) = v_j$.

Similarly we define a *strictly stable payoff*.

Definition 5 says that (u, v) is *strictly stable* if it is stable and it is not *weakly* dominated by any feasible payoff via some coalition (i, j) . The pair (i, j) is called a *weak blocking pair*.

The set of strictly stable payoffs coincides with the core defined by weak domination, which is a subset of C and will be denoted by $\bar{C}$.

The following models are well known special cases of the model we are treating here:

The Marriage Model

Here $S = \{0\}$. Since S is finite, the players can list their acceptable partners, in order, in a finite list of preferences. Let $L(i)$ denote the list of preferences of player i . Thus $L(i) = (j, 0), (k, 0), \dots, (i, 0), \dots$ means that $U_{i,j}(0) > U_{i,k}(0) \geq r_i$, etc. The fact that s does not play any role in this game implies that any outcome $((u, v); \mu)$ is completely specified by the matching μ . Such a matching is called stable if the corresponding outcome is stable. That is, μ is a stable matching if there is no pair (i, j) in $P \times Q$ with $\mu(i) \neq j$ such that $U_{i,j}(0) > U_{i,\mu(i)}(0)$ and $V_{i,j}(0) > V_{\mu(j),j}(0)$.

Thus, μ is a stable matching if there is no pair (i, j) in $P \times Q$, with $\mu(i) \neq j$, such that i and j prefer each other to their respective mates under μ .

When the preferences are strict the resulting model is the well known marriage model, introduced by Gale and Shapley (1962).

The Assignment Game

In this case $S = R$ and there are numbers a_{ij} and b_{ij} , for all pairs (i, j) in $P \times Q$, such that

$$U_{i,j}(s) = a_{ij} + s \quad V_{i,j}(s) = b_{ij} + s.$$

If we define $\alpha_{ij} = \max\{a_{ij} + b_{ij}, 0\}$, then a feasible outcome $((u, v); \mu)$ is stable if and only if there is no pair (i, j) in $P \times Q$, with $\mu(i) \neq j$, such that $u_i + v_j < \alpha_{ij}$.

This model is equivalent to the assignment game of Shapley and Shubik (1972) and it is a special case of the model below.

The Generalized Assignment Game

In this model $S = R$ and the utility functions are strictly increasing and onto R (and consequently continuous). It is the additional assumption that the utility functions are onto R that has the consequence that every player can be made indifferent between any pair

of mates by an appropriate adjustment of prices. The fact that for every pair (i, j) there is some transfer s that makes i indifferent to being unmatched (and some other transfer s' that makes j indifferent) implies that the set of individually rational transfers that can be made between i and j is bounded.

If we define $f_{ij} = U_{ij}^{-1}$ and $g_{ij} = V_{ij}^{-1}$, then a feasible outcome $((u, v); \mu)$ is stable if and only if there is no pair (i, j) in $P \times Q$, with $\mu(i) \neq j$, such that $f_{ij}(u_i) + g_{ij}(v_j) < 0$.

This model is equivalent to the model of Demange and Gale (1985). For Demange and Gale the continuity of the utility functions is a requirement of the model.

We will concentrate on markets for either S is any set of integer numbers satisfying $(I_1 \text{ or } I_2)$ or $S = R$. We will refer to a market which satisfies assumptions A_1 and A_2 as discrete or continuous, according to whether S is a set of integer numbers or $S = R$.

The results of the next section mostly depend on the assumption that $C = \overline{C}$. This condition is satisfied if S is discrete and all preferences are strict. It is also satisfied in the continuous case of Demange and Gale, for them the utilities U_{ij} and V_{ij} are continuous.

Without the opposite assumptions that produce the common results for discrete and continuous models those results may fail to hold, as the existence of a Q -optimal stable outcome, which is at least weakly preferred by all Q -players to any other stable outcome. Such an outcome exists in the model with finite number of payments when preferences are strict, and in the model with continuously divisible payments when the utility functions are onto R so that any agent can be made indifferent between any two matches.

If the preferences are not strict in the marriage model, the existence of a man optimal matching (or a woman optimal matching) is not guaranteed, as it can be seen in Roth and Sotomayor (1990, Ex: 2.15, p. 34). On the other hand, if there is no salary s that makes a worker indifferent between being employed and being unemployed

there may be no stable outcome that is optimal for the workers. The example below illustrates the role played by indifference in the discrete and continuous models.

To see the role played by indifference when $S = R$, consider the following two examples in which there may be no wage which makes a worker precisely indifferent between two different employment opportunities. In the first example the utilities are not continuous; in the second example the utilities are continuous.

Counterexample 1. Consider the case of one firm and two workers, i.e. $P = \{f\}$ and $Q = \{w_1, w_2\}$. Let $S = R$ and let the utility function of the firm be

$$\begin{aligned} U_{fw}(s) &= 1 + s && \text{for } w = w_1, w_2 \\ U_{ff}(0) &= 0 \end{aligned}$$

and let the utility function for each worker w ($w = w_1, w_2$) be

$$\begin{aligned} V_{fw}(s) &= 1 + s && \text{for } s \geq 0 \\ &= s && \text{for } s < 0, \text{ and} \\ V_{ww}(0) &= 0. \end{aligned}$$

There is no salary s that makes a worker indifferent between being employed and being unemployed. And there is no stable outcome that is optimal for the workers. There are exactly two stable outcomes,

$$\begin{aligned} \mu(f) &= w_1; \quad u_f = 1, \quad v_1 = 1, \quad v_2 = 0, \quad \text{and} \\ \mu'(f) &= w_2; \quad u'_f = 1, \quad v'_1 = 0, \quad v'_2 = 1, \end{aligned}$$

and w_1 prefers μ to μ' while w_2 has the opposite preference.

In this example the core is not a lattice under the preferences of the workers because $(u \wedge u', v \vee v')$ is not stable. Contrary to

the results for the continuous model of Demange and Gale, w_2 is unemployed at μ but he does not get his reservation utility at μ' .

Counterexample 2. Consider $P = \{p\}$ and $Q = \{q\}$. Let $S = R$, and $U_{pq}(s) = V_{pq}(s) = e^s$. The reservation utilities of both players are zero. Here again there is no monetary transfer s that makes p and q indifferent between being matched and being unmatched. The core is the lattice of all vectors (u, v) where $u = e^{-s}$ and $v = e^s$, for all $s \in \mathbb{R}$. So, the core is unbounded. Consequently there is no stable outcome that is optimal for p or q .

The literature contains a rich collection of results establishing that the core is non-empty in the general model considered here when utilities are continuous or S is finite (see, Gale and Shapley (1962), Crawford and Knoer (1981), Kaneko (1982), Roth (1982, 1984 b) and Alkan and Gale (1990), for a set of models of one-to-one matching, and see Quinzii (1984) for some related existence results). However Counterexample 1 shows that the same cannot be said for the core defined by weak domination.

Proposition 1. The core defined by weak domination, $\overline{C}$, may be empty.

Proof. In Counterexample 1, there were only two stable outcomes, with matchings μ and μ' such that $\mu(f) = w_1$ and $\mu'(f) = w_2$, corresponding to payoff vectors $(u_f, v_1, v_2) = (1, 1, 0)$ and $(u', v'_1, v'_2) = (1, 0, 1)$ respectively. But neither of these outcomes is in the core defined by weak domination, since the first weakly dominates the second via the weak blocking pair $\{f, w_2\}$, and the second weakly dominates the first via $\{f, w_1\}$.

When an outcome is in the core, but not in the core defined by weak domination, it means that there is a pair of agents $\{i, j\}$ who can be matched in a way that benefits one of them without hurting

the other, even though there is no way that they can be matched that will benefit them both. We will see that it is these weak blocking pairs that lie at the root of the different assumptions needed to obtain optimal stable matchings for each side of the market, and related results, in the discrete and continuous cases.

3. The Main Results.

The existence of optimal stable outcomes for each side of the market in the discrete and continuous cases is known to be related to the structure of the entire set of stable outcomes. (For example the set of stable outcomes forms a lattice both in the marriage model with strict preferences, Knuth (1976) and in the assignment game, Shapley and Shubik (1972). The compactness of the core in both markets implies that the lattice is complete, so it has a maximal and a minimal elements. For a comprehensive discussion, see Roth and Sotomayor (1990). In this section we show that the underlying assumptions sufficient to produce these structural results in both classes of models are that the core is non empty, compact and coincide with the core defined by weak domination. Recall that Counterexample 1 showed that no optimal stable payoff may exist if $C \neq \overline{C}$. Counterexample 2 showed that the existence of optimal stable payoffs may fail to hold if C is not compact. In that example $C = \overline{C}$.

Each of the results presented below have been separately established for the marriage market with strict preferences and for the continuous model of Demange and Gale, using incompatible assumptions. They will be established here for the general class of discrete (S finite and infinite) and continuous markets. By replacing the lists of preferences in the discrete markets by utility functions, the common arguments can now be expressed in the same language. It turns out that the proofs given by Demange and Gale for their continuous model are compatible, up to some small changes, to all models treated here under the assumptions that for all i in P and all j in Q :

- A_1 - the core is non-empty;
- A_2 - the core is closed and
- A_3 - the core coincides with the core defined by weak domination.

The small changes lie essentially in the fact that the assumption that the utility functions are continuous and onto R is replaced by A_2, A_3, A_4 and A_5 .

The key result is Lemma 1, which will be proved here using Assumption A_5 . The remaining results follow from it. The proofs of Corollaries 1 and 2, Theorem 1, Lemma 2 and Theorem 4, which follow from Lemma 1, can be seen, in their original version, in Demange and Gale (1985) or in a little more detailed way in Roth and Sotomayor (1990). So they will not be proved here. Theorem 3 makes use of the two assumptions of the model and of the three assumptions above; for Demange and Gale's model the argument is based on the continuity of the utility functions and on the assumption that they are onto R . The proof of Theorem 5 follows the lines of the proof given by Demange and Gale, but uses A_2 instead of the assumption that the utility functions are continuous and onto.

We will use the notation $M(r, s)$ for the market in which the vectors of reservation utilities $r = (r_1, \dots, r_m)$ and $s = (s_1, \dots, s_n)$ may vary but P, Q, S, U_i restricted to $Q \times S$ and V_j restricted to $P \times S$ are fixed for all $i \in P$ and $j \in Q$. We will denote by $C(r, s)$ and $\overline{C}(r, s)$ the core and the core defined by weak domination, respectively, of $M(r, s)$.

Lemma 1. (Decomposition Lemma):

Let $(u^1, v^1) \in C(r^1, s^1)$ and $(u^2, v^2) \in C(r^2, s^2)$, where $r^2 \leq r^1 \leq u^2$ and $s^2 \leq s^1$. Let $P^1 = \{i \in P; u_i^1 > u_i^2\}$, $P^2 = \{i \in P; u_i^2 > u_i^1\}$ and $P^0 = \{i \in P; u_i^1 = u_i^2\}$. Analogously define Q^1, Q^2 and Q^0 . If $C(r^1, s^1) = \overline{C}(r^1, s^1)$ and $C(r^2, s^2) = \overline{C}(r^2, s^2)$ then $P^1 \neq \emptyset$ if and only if $Q^2 \neq \emptyset$ and $P^2 \neq \emptyset$ if and only if $Q^1 \neq \emptyset$. Furthermore, if μ^1 and μ^2 are compatible with (u^1, v^1) and (u^2, v^2) ,

respectively, $\mu^1(P^1) = \mu^2(P^1) = Q^2$, $\mu^1(P^2) = \mu^2(P^2) = Q^1$ and $\mu^1(P^0 \cup Q^0) = \mu^2(P^0 \cup Q^0) = P^0 \cup Q^0$.

Lemma 1 is called a decomposition lemma because it allows us to create a new matching μ by decomposing the stable matchings μ^1 and μ^2 . For example, the new matching could match P^1 and Q^2 by μ^1 and match $P - P^1$ and $Q - Q^2$ by μ^2 .

Proof of Lemma 1. Suppose $P^1 \neq \emptyset$ and let $i \in P^1$. Then $u_i^1 > u_i^2 \geq r_i^1$. So $\mu^1(i) = j$ for some $j \in Q$. So we cannot have $v_j^1 \geq v_j^2$, because if so (u^2, v^2) would be weakly blocked by i and j , which contradicts the assumption that $(u^2, v^2) \in \overline{C}(r^2, s^2)$. So $v_j^2 > v_j^1$ and $j \in Q^2$. Then $Q^2 \neq \emptyset$ and $\mu^1(P^1)$ is contained in Q^2 .

Now suppose that $Q^2 \neq \emptyset$. Let $j \in Q^2$. Then $v_j^2 > v_j^1 \geq s_j^1 \geq s_j^2$, so $\mu^2(j) = i$ for some $i \in P$. We cannot have $u_i^2 \geq u_i^1$, because if so (u^1, v^1) would be weakly blocked by i and j , which contradicts the assumption that $(u^1, v^1) \in \overline{C}(r^1, s^1)$. Hence $u_i^1 > u_i^2$ and $i \in P^1$. Then $P^1 \neq \emptyset$ and $\mu^2(Q^2)$ is contained in P^1 . Now, $|P^1| = |\mu^1(P^1)| \leq |Q^2| = |\mu^2(Q^2)| \leq |P^1|$ implies that $\mu^1(P^1) = Q^2$. It also implies that $\mu^2(Q^2) = P^1$, and so $\mu^2(P^1) = Q^2$.

An analogous argument shows that $\mu^1(P^2) = \mu^2(P^2) = Q^1$. The last assertion follows from the fact that $P^0 = P - (P^1 \cup P^2)$ and $Q^0 = Q - (Q^1 \cup Q^2)$ and the first two assertions.

When we are comparing two outcomes of the same market, the following simpler statement will be convenient.

Corollary 1. (Decomposition Lemma when $r = r^1 = r^2$ and $s = s^1 = s^2$).

Let (u^1, v^1) and (u^2, v^2) be in $C(r, s)$. Let $P^1 = \{i \in P; u_i^1 > u_i^2\}$, $P^2 = \{i \in P; u_i^2 > u_i^1\}$ and $P^0 = \{i \in P; u_i^1 = u_i^2\}$. Analogously define Q^1 , Q^2 and Q^0 . If $C(r, s) = \overline{C}(r, s)$ then $P^1 \neq \emptyset$

if and only if $Q^2 \neq \emptyset$ and $P^2 \neq \emptyset$ if and only if $Q^1 \neq \emptyset$. Furthermore if μ^1 and μ^2 are compatible with (u^1, v^1) and (u^2, v^2) , respectively, $\mu^1(P^1) = \mu^2(P^1) = Q^2$, $\mu^1(P^2) = \mu^2(P^2) = Q^1$ and $\mu^1(P^0 \cup Q^0) = \mu^2(P^0 \cup Q^0) = P^0 \cup Q^0$.

The next result establishes that, if i is unmatched under some stable outcome then he will get precisely his reservation utility in all stable outcomes. Consequently, i will be unmatched at any stable outcome when S is finite and the preferences are strict.

Theorem 1. Let $x = (u^1, v^1)$ and $y = (u^2, v^2)$ be in $C(r, s)$. Let μ^1 and μ^2 be compatible matchings with x and y , respectively. Suppose $C(r, s) = \overline{C}(r, s)$. If $i \in P$ (resp. $j \in Q$) is unmatched under μ^1 then $u_i^2 = r_i$ (resp. $v_j^2 = s_j$).

The next results prepare the statement (in Theorem 2) of conditions under which the set of stable matchings is a lattice. For any vectors x^1 and x^2 of the same dimension, define $(x^1 \vee x^2) \equiv x^+$ to be the pointwise maximum of x^1 and x^2 , and $x^1 \wedge x^2 \equiv x^-$ to be their pointwise minimum.

Lemma 2. Let (u^1, v^1) be in $C(r^1, s^1)$ and let (u^2, v^2) be in $C(r^2, s^2)$, where $r^2 \leq r^1 \leq u^2$ and $s^2 \leq s^1$. If $C(r^1, s^1) = \overline{C}(r^1, s^1)$ and $C(r^2, s^2) = \overline{C}(r^2, s^2)$, then

- a) $(u^1 \vee u^2, v^1 \wedge v^2) \equiv (u^+, v^-) \in C(r^2, s^2)$;
- b) $(u^1 \wedge u^2, v^1 \vee v^2) \equiv (u^-, v^+) \in C(r^1, s^1)$.

[Equivalently, interchanging P and Q , if $r^2 \leq r^1$ and $s^2 \leq s^1 \leq v^2$, then

- a') $(u^1 \wedge u^2, v^1 \vee v^2) \in C(r^2, s^2)$;
- b') $(u^1 \vee u^2, v^1 \wedge v^2) \in C(r^1, s^1)$.]

Corollary 2. (Lemma 2 when $r^1 = r^2 = r$ and $s^1 = s^2 = s$).

Let (u^1, v^1) and (u^2, v^2) be in $C(r, s)$. If $C(r, s) = \overline{C}(r, s)$ then:

- a) $(u^1 \vee u^2, v^1 \wedge v^2) \equiv (u^+, v^-) \in C(r, s)$;
- b) $(u^1 \wedge u^2, v^1 \vee v^2) \equiv (u^-, v^+) \in C(r, s)$.

Proof. Immediate from Lemma 2.

From Corollary 2 it follows that if (u, v) and (u', v') are in $C(r, s)$ and $\overline{C}(r, s) = C(r, s)$, then $u_i \geq u'_i$ for all $i \in P$ if and only if $v_j \leq v'_j$ for all $j \in Q$. Therefore we can define two binary relations on $C(r, s)$ (one is the dual of the other):

- i) $(u, v) \geq_P (u', v')$ if and only if $u_i \geq u'_i$ and $v_j \leq v'_j$, for all $i \in P$ and $j \in Q$.
- ii) $(u, v) \geq_Q (u', v')$ if and only if $u_i \leq u'_i$ and $v_j \geq v'_j$, for all $i \in P$ and $j \in Q$.

It is clear that $\geq_P$ and $\geq_Q$ are partial orders on the set of stable payoffs.

Theorem 2. If $C(r, s) = \overline{C}(r, s) \neq \emptyset$, then $C(r, s)$ is a lattice under $\geq_P$ and $\geq_Q$.

Proof. Immediate from Corollary 2 and the definition of a lattice.

When $S = \{0\}$, Theorem 2 establishes that the core is a lattice. Recall that we have defined the core as a set of payoffs, not a set of matchings. The analogous binary relations defined on the set of stable matchings:

$$\mu \geq_P \mu' \quad \text{and} \quad \mu \geq_Q \mu'$$

may fail the anti-symmetric property when preferences are *not* strict, even when $C = \overline{C}$. Hence in this case these relations are *not*, in general, partial orders and the conclusion of Theorem 2 does not

necessarily hold for the set of stable matchings. That is, in the case $S = \{0\}$, the set of stable payoffs is a lattice when $C = \overline{C}$, although the set of stable matchings may not be. When preferences *are* strict it can be easily seen that there is a one-to-one correspondence between the set of stable payoffs and the set of stable matchings which preserves these binary relations. That is, if $(u, v; \mu)$ and $(u', v'; \mu')$ are stable outcomes then $(u, v) >_P (u', v')$ (resp. $(u, v) >_Q (u', v')$) if and only if $\mu >_P \mu'$ (resp. $\mu' >_Q \mu$). So these binary relations on the set of stable matchings *are* partial orders and Theorem 2 implies that the set of stable matchings is a complete lattice under both partial orders.

Definition 7. We say that the stable payoffs $(\overline{u}, \underline{v})$ is *P-optimal* if $(\overline{u}, \underline{v}) \geq_P (u, v)$ for all $(u, v) \in C(r, s)$; similarly we say that the stable payoff $(\underline{u}, \overline{v})$ is *Q-optimal* if $(\underline{u}, \overline{v}) \geq_Q (u, v)$ for all $(u, v) \in C(r, s)$.

Theorem 3. If $C(r, s)$ is a non-empty and closed set and $C(r, s) = \overline{C}(r, s)$, then there exists a unique *P-optimal* stable payoff and a unique *Q-optimal* stable payoff.

Proof. By Theorem 2, $C(r, s)$ is a lattice under both partial orders: $\geq_P$ and $\geq_Q$. $C(r, s)$ is compact, since it is closed and bounded, by assumption A_2 and the monotonicity of the utility functions. Now, for each one of the partial orders, use the fact that a compact lattice has a unique maximal element.

The existence of *P-optimal* and *Q-optimal* stable payoffs for the marriage market with strict preferences and for the continuous model of Demange and Gale is guaranteed by the fact that the core of both models is a non-empty and compact lattice.

That the core of the marriage market is compact it is a consequence from the fact that it has only a finite number of points. The continuity of the inverse of the utility functions, in the other case,

implies that C is a closed subset of $\mathbb{R}^m \times \mathbb{R}^n$. On the other hand, the fact that the utility functions are strictly increasing and onto $\mathbb{R}$ and that $u_i \geq r_i$, $v_j \geq s_j$ for all $(i, j) \in P \times Q$, imply that C is bounded and so it is compact. The lattice property follows from the fact that $C = \overline{C}$. The core and the core defined by weak domination coincide if and only if no weak blocking pairs exist. In the discrete models with strict preferences, it can never happen that one agent can benefit by being matched to another agent who remains indifferent, since no agent is indifferent between different mates, and so no weak blocking pairs exist. In continuous models with continuous utilities no weak blocking pairs exist because whenever there exists a pair of agents $\{i, j\}$ who can be matched in a way that benefits one of them without hurting the other, it is also possible to match them in a way that benefits both of them (since the price that one pays to the other can be continuously adjusted, and both agents have continuous utility functions). So under the standard assumptions for both kinds of models, the core coincides with the core defined by weak domination.

From now on we will denote the P -optimal stable payoff for $M(r, s)$, if it exists, by $(\overline{u}(r, s), \underline{v}(r, s))$ and the Q -optimal stable payoff for $M(r, s)$, if it exists, by $(\underline{u}(r, s), \overline{v}(r, s))$. The next theorem shows that $\overline{u}(r, s)$ and $\underline{u}(r, s)$ are increasing in r and decreasing in s , while $\overline{v}(r, s)$ and $\underline{v}(r, s)$ are decreasing in r and increasing in s .

Theorem 4. Let $r^1 \geq r^2$ and $s^1 \geq s^2$. Suppose that $C(r^i, s) = \overline{C}(r^i, s)$ and $C(r, s^i) = \overline{C}(r, s^i)$ for all $i = 1, 2$. Suppose there is the P -optimal and the Q -optimal stable payoffs for the markets $M(r^i, s)$ and $M(r, s^i)$, for all $i = 1, 2$. Then $\overline{u}(r^1, s) \geq \overline{u}(r^2, s)$, $\underline{v}(r^1, s) \leq \underline{v}(r^2, s)$ and $\overline{u}(r, s^1) \leq \overline{u}(r, s^2)$, $\underline{v}(r, s^1) \geq \underline{v}(r, s^2)$. Symmetrically, $\underline{u}(r^1, s) \geq \underline{u}(r^2, s)$, $\overline{v}(r^1, s) \leq \overline{v}(r^2, s)$ and $\underline{u}(r, s^1) \leq \underline{u}(r, s^2)$, $\overline{v}(r, s^1) \geq \overline{v}(r, s^2)$.

A decrease in an agent's reservation utility corresponds to an increase in the set of potential individually rational matches, so in

the marriage model, for example, it corresponds to an extension of the agent's list of acceptable partners. In general, in the finite case with strict preferences the conditions of Theorem 4 hold when the reservation utilities are changed so as not to make any individual indifferent between matched and unmatched. The theorem then states that: *When S is finite and the preferences are strict, whether we use the P -optimal or the Q -optimal stable payoffs, it will always be the case that if some players on one of the sides extend their lists of acceptable partners, without introducing indifference, no player on the same side will be made better off and no player on the opposite side will be made worse off under the original preferences.* Theorem 5 is about what happens to the P - and Q -optimal stable payoffs when new players enter one side of the market. If this does not disrupt the coincidence of the core with the core defined by weak domination (e.g. in the finite case when preferences are strict) then this can never make any of the players on the same side better off and any of the players on the opposite side worse off, no matter which of the two optimal stable payoffs are being considered.

Theorem 5. Suppose Q is contained in Q' . Let $C(r, s)$ be the set of stable payoffs for $M = (P, Q, S, U, V)$. Let $C(r, s')$ be the set of stable payoffs for $M' = (P, Q', S, U', V')$ where U' and V' agree with U and V , respectively, on P and Q . If $C(r, s) = \overline{C}(r, s)$ and $\overline{C}(r, s') = C(r, s')$ and the P -optimal and the Q -optimal stable payoffs exist for the markets M and M' then

$$\overline{u}_i(r, s') \geq \overline{u}_i(r, s) \quad \text{and} \quad \underline{v}_j(r, s) \geq \underline{v}_j(r, s')$$

and

$$\overline{v}_j(r, s) \geq \overline{v}_j(r, s') \quad \text{and} \quad \underline{u}_i(r, s') \geq \underline{u}_i(r, s)$$

for all $(i, j) \in P \times Q$.

The symmetrical result is obtained if P is contained in P' .

Proof. Construct a new market $M'' = (P, Q', S, U', V'')$ by choosing $s''_j = V''_{ij}(0)$ such that $s''_j = s'_j = s_j$ for all $j \in Q$. If $j \in Q' - Q$,

since the set of individually rational transfers between any pair i and j is bounded, (assumption A_2), we can rise s'_j up to some number s''_j so that $s''_j > s'_j$ and the new set of individually rational transfers between i and j is empty for all $i \in P$. The utility functions V'' differ from V' only in the reservation utilities. Therefore we can always choose s''_j so that if $j \in Q' - Q$, j will never be matched by any matching compatible with some stable payoff for M'' . Then $\bar{u}_i(r, s) = \bar{u}_i(r, s'')$ and $\underline{u}_i(r, s) = \underline{u}_i(r, s'')$ for all $i \in P$, and $\underline{v}_j(r, s) = \underline{v}_j(r, s'')$ and $\bar{v}_j(r, s) = \bar{v}_j(r, s'')$ for all $j \in Q$. Furthermore, $C(r, s'') = \bar{C}(r, s'')$ since $C(r, s) = \bar{C}(r, s)$. From Theorem 4, since $s'' > s'$, it follows that $\bar{u}_i(r, s) = \bar{u}_i(r, s'') \leq \bar{u}_i(r, s')$ and $\underline{u}_i(r, s) = \underline{u}_i(r, s'') \leq \underline{u}_i(r, s')$, for all $i \in P$, and $\underline{v}_j(r, s) = \underline{v}_j(r, s'') \geq \underline{v}_j(r, s')$ and $\bar{v}_j(r, s) = \bar{v}_j(r, s'') \geq \bar{v}_j(r, s')$, for all $j \in Q$.

4. Concluding Remarks.

This paper was motivated by a longstanding puzzle in the matching literature. Although opposite assumptions about the strictness of preferences were required, nearly identical results had been obtained for the marriage and assignment models using quite different arguments in their proofs.

Another, quite different approach to providing parallel proofs of the common results for the marriage and assignment models derives from the demonstration in Vande Vate (1989) that the set of stable matchings in the marriage model can be formulated, as in the assignment model, as the solutions to a linear programming problem (see also Rothblum, 1991). Although the linear programming formulations for these two problems are quite different from one another, Roth, Rothblum, and Vande Vate (1991) showed that they could be used to provide similar proofs of some of the common results, using the duality theorem of linear programming. However in that treatment, the central part of the puzzle addressed in this paper remained: the common results required opposite assumptions, of strict preferences in the marriage model and continuous preferences

in the assignment model. This paper resolves that paradox. We present a unified treatment of the discrete and the continuous case by providing a class of models which generalize the discrete and continuous special cases, and by identifying the common assumptions from which the common results follow.

The marriage model with strict preferences and the continuous model of Demange and Gale satisfy the assumptions A_1 and A_2 . So they are included in the class of games we treated here. Furthermore, the core of both models are non-empty, compact, and coincide with the core defined by weak domination for all vectors of reservation utilities r and s . We showed that all these common assumptions, together, yield the conclusions of Theorem 1, 2, 3, 4 and 5. So we have explained why these results always hold for these models. Furthermore our results permit us identify the reasons why these similar conclusions depend on the assumption that no agents are indifferent between any mates in the marriage model, and on the assumption that every agent may be made indifferent between any two mates in the assignment model. In each of those two models, these different assumptions have some specific role. The role played by the strictness of the preferences in the marriage market is to guarantee assumption A_5 : the core coincides with the core defined by weak domination, the only assumption which is not naturally satisfied by the model. On the other hand, when the salaries vary continuously on the set of real numbers, only A_1 is naturally satisfied, and all the other four assumptions can be violated if some player cannot be made indifferent between being unmatched or being matched to some other player, as it was illustrated by Counterexamples 2 and 3.

Submetido em Março de 1996. Revisado em Agosto de 1996.

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