Review of major results of Martingale theory applied to the valuation of contingent claims¹

Cícero Augusto Vieira Neto²

Pedro L. Valls Pereira³

Abstract

This article condenses the theory for the valuation of contingent claims in complete and arbitrage-free markets by means of martingales. The main focus is centered on markets in which it is possible to negotiate at any time; that is, markets whose history takes place at a continuous time.

Resumo

Este artigo condensa a teoria de avaliação de títulos contingentes em mercados completos e livres de arbitragem por meio de martingais. Toda a atenção está voltada para mercados onde é possível negociar em qualquer instante do tempo, isto é, mercados cuja história se desenvolve em tempo contínuo.

Key Words: Martingale, Valuation of Contingent Claims .

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² Head of the Risk Administration Department, BM&F, Ph.D Universidade de São Paulo (USP) School of Economics (cicervieira@bmf.com.br)

³ Professor, Ibmecc Business School, Ph.D London School of Economics (pvalls@ibmecc.br)

1. Introduction.

This article condenses the theory for the valuation of contingent claims in complete and arbitrage-free markets by means of martingales, as proposed by Harrison and Kreps (1979) and Harrison and Pliska (1982). By using a homogenous notation throughout, the present article organizes and discusses a total of 11 definitions and 10 theorems that are the core of modern modeling of financial assets by the application of the martingale method. The main focus is centered on markets in which it is possible to negotiate at any time; that is, markets whose history takes place at a continuous time.

The article is organized as follows: section 2 characterizes continuous time financial models and discusses their four basic parts, namely: a set of dates, a filtered probability space, a standard m -dimensional Wiener process and an $n+1$ dimensional price process.

In section 3, we discuss several definitions that are pivotal in modern finance theory. Among the most important definitions we have those that characterize *a self-financing strategy, an arbitrage strategy, an attainable contingent claim, a complete market, an equivalent martingale probability measure, a reducible market and the market-price-of-risk*. After these definitions, we present a sequence of theorems, together with proofs, in some cases. These theorems are among the major results obtained in the field of finance in the last 20 years and allow “solving” almost all continuous time, complete market models. The most relevant theorems include *the probabilistic characterization of no arbitrage, reducibility and no arbitrage, hedgeability, and valuation of derivative contracts in complete and arbitrage-free markets with stochastic interest rates*.

Section 4 presents a brief conclusion of our work.

2. Characterization of continuous time financial models.

In general, the class of financial models broached here is divided into four basic parts:

- (i) initial and final dates 0 and T ($0 < T$) and set $\mathbf{T}$;
- (ii) a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{Q})$;
- (iii) a standard m -dimensional Wiener process $\{W(t); t \in \mathbf{T}\}$;
- (iv) an $n+1$ dimensional price process $\{P(t); t \in \mathbf{T}\}$;

Each of the items above will be discussed and interpreted next.

The initial date of model is $t = 0$ while the date is $t = T$. The dates on which investors can negotiate their assets belong to set $\mathbf{T}$. The terms *discrete time* and *continuous time* refer to situations: (i) $\mathbf{T}$ is a finite set $\{0 = t_1, t_2, \dots, t_k = T\} \subseteq [0, T]$, in which there is a limited number of instants at which negotiation is possible and (ii) $\mathbf{T} = [0, T]$, in which it is possible to negotiate at any time instant. The model proposed by Black and Scholes (1973) and most models for the term structure of interest rates are examples of case (ii). Other important models for the term structure were originally developed in discrete time, but later on, they received analogous versions in continuous time (for instance the models proposed by Ho and Lee (1986) and Black- Derman-Toy (1990)). The approach used in this article refers exclusively to case $\mathbf{T} = [0, T]$.

The use of the concept of continuous time is an abstraction. In real markets, there is a limited number of instants at which negotiation is possible. As the time interval between these instants decreases, approximation to reality improves. A priori, it is not possible to say which time interval is sufficiently small so that a continuous model can be a good approximation to discrete reality. Each case has to be separately analyzed.

Continuous time models are usually simpler as to their formula-

tions and richer in their results than their discrete time counterparts. This occurs because the use of continuous time opens room for the application of a powerful and mature mathematical theory: stochastic calculus.

The quadruple $(\Omega, \xi, \{\xi_t\}, Q)$ represents a filtered probability space, where Ω is the sample space, ξ is the sigma-algebra, $\{\xi_t; t \in \mathbf{T}\}$ an increasing family of sub sigma-algebras and Q is the probability measure.

The terms $\omega \in \Omega$ represent the possible states of nature. The information structure of the assets market is given by $\{\xi_t; t \in \mathbf{T}\}$, where (a) $\xi_0 = \Omega \cup \{A \subset \Omega / Q(A) = 0\}$, (b) $\xi_t = \bigcap_{s>t} \xi_s$ ($0 < t < T$) and (c) $\xi_T = \xi$. Condition (a) says that ξ_0 is formed by the sample space itself in addition to sets with null probability (the latter condition is merely technical), which means that ξ_0 represents the lowest possible level of information. Condition (b) states that ξ_t ($0 < t < T$) is “increasing”, that is, the amount of available information about the true state of nature increases over time. Finally, according to condition (c), the last evolution stage of the economy’s information structure is the proper sigma-algebra of the probability space.

We can interpret ξ_t as the set of all sets that it is possible to say, on date t , whether the true state of nature $\omega \in \Omega$ belongs to one of these sets. To put it simply, ξ_t represents the information available in t .

A standard Wiener process is a stochastic process $W : \Omega \times [0, \infty) \rightarrow R^m$ denoted by $W(t) = [W_1(t), \dots, W_m(t)]$, with the following properties:

- a) $W(0) = 0$;
- b) for dates t and $s > t$, $W(s) - W(t)$ has a multivariate normal distribution with mean zero and diagonal variance-covariance matrix, where the diagonal terms are all equal to $s - t$;

- c) for dates $0 \leq t_0 \leq t_1 \leq \dots \leq t_n \leq \infty$, increments $W(t_0), W(t_1) - W(t_0), \dots, W(t_n) - W(t_{n-1})$ are independently distributed;
- d) for each $\omega \in \Omega$, the sample trajectory $t \rightarrow W(\omega, t)$ is continuous;

An uncommon fact that we will not attempt to show is that probability space $(\Omega, \mathcal{F}, Q)$ can be constructed in such a way that the Wiener process exists. On the top of that, filtration $\{\mathcal{F}_t\}$ can also be defined so as to represent the information contained in $\{W(s); 0 \leq s \leq t\}$.

For any stochastic process $x(t, \omega)$, whenever there is no ambiguity, we will use either of the following notations: $x(t, \omega) = x(t) = x_t = x$.

The present study draws on Ito's processes only. Such processes are sufficiently general for most of the widely known applications. Let $P(t, \omega) = (P_0(t, \omega), P_1(t, \omega), \dots, P_n(t, \omega))$ be the economy's price process, where $P_i(t, \omega)$ denotes the price of the i th asset on date t , state of nature ω . In several studies, $P(t, \omega)$ is also called *market*. We say that $P_i(t, \omega)$ follows an Ito's process if

$$\begin{aligned} P_i(t, \omega) &= P_i(0) + \int_0^t \mu_i(s, \omega) \cdot ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s, \omega) \cdot dW_j(s, \omega) \\ &= P_i(0) + \int_0^t \mu_i(s, \omega) \cdot ds + \int_0^t \sigma_i(s, \omega) \cdot dW(s, \omega) \end{aligned} \quad (1)$$

with $P_i(0)$ a real number, $\mu_i(s, \omega) \in L^1$ and $\sigma_i(s, \omega) \in L^2$, where

$L^1 = \left\{ x(t, \omega) \text{ adapted such that } \int_0^T |x(s, \omega)| \cdot ds < \infty \right\}$ and

$L^2 = \left\{ x(t, \omega) \text{ adapted such that } \int_0^T |x(s, \omega)|^2 ds < \infty \right\}$;

where the symbol $|\cdot|$ denotes the norm of Euclidean space.

The second integral on the right-side of expression (1) is defined in Ito's sense (see, for example, Arnold (1973) or Oksendal (1998)). Price process $P_i(\cdot)$ is *adapted*, which simply means that price of asset i , on date t , is also part of the information available in t . The sets L^1 and L^2 represent technical integrability conditions. The terms $\mu_i(t, \omega)$ and $\sigma_i(t, \omega)$ can be interpreted from the following result (Duffie (1996)):

$$\frac{d}{d\tau} E(P_i(\tau)/\xi_t) |_{\tau=t} = \mu_i(t) \quad (2)$$

$$\frac{d}{d\tau} \text{var}(P_i(\tau)/\xi_t) |_{\tau=t} = |\sigma_i(t)|^2 \quad (3)$$

Expressions (2) and (3) mean that $\mu_i(t)$ represents the variation rate of the conditional expectation of $P_i(t)$, while $|\sigma_i(t)|^2$ expressions " $E(dP_i(t)/\xi_t) = \mu_i(t).dt$ " and " $\text{var}(dP_i(t)/\xi_t) = |\sigma_i(t)|^2.dt$ ", which are only justified because of heuristics.

In several models, it is common to suppose that asset $P_0(t, \omega)$ obeys the following process:

$$P_0(t, \omega) = 1 + \int_0^t \mu_0(s, \omega).ds = 1 + \int_0^t P_0(s, \omega).ds \quad (4)$$

As the diffusion coefficient is equal to zero ($\sigma_0(t) = 0$), we associate asset $P_0(t)$ with an instantaneous risk-free investment, which has initial value of 1 and instantaneous yield rate equal to $r(t)$. the interpretation of $r(t)$ is the very short-term interest rate.

In a summarized notation, the price process given by (1) and (4) is written as follows:

$$dP_0(t) = P_0(t).r(t).dt \quad (5)$$

$$dP_i(t) = \mu_i(t).dt + \sigma_i(t).dW(t), \quad 1 \leq i \leq n \quad (6)$$

It is worth mentioning that there is “stochastic differentiation” theory, but only of stochastic integration. Therefore, expressions (5) and (6) are just abbreviated forms that express the meaning of (1) and (4).

Along with the results discussed in the next section, we will show a thorough interpretation of expressions (5) and (6), trying to justify and explain their use for the modeling of continuous time financial assets.

3. Valuation of Contingent Claims in Complete and Arbitrage-free Markets by Means of Martingales.

In this section, we will discuss a wide and unified set of definitions and theorems that are the basis of the theory for valuation of contingent claims proposed by Harrison and Kreps (1979) and Harrison and Pliska (1981).

We decided to begin with three results obtained through stochastic calculus which are among the major work horses of the whole continuous time finance theory and are widely used throughout the present article.

Initially, let H^2 be $= \left\{ x(t, \omega) \in L^2; E \left\{ \int_0^T x_s^2 ds \right\} < \infty \right\}$.

Theorem 1 - (*Ito's Integral Martingale Property*) - Let $I(t, \omega)$ be an Ito's integral $I(t, \omega) = \int_0^t g(s, \omega) . dW(s, \omega)$ with $g(s, \omega) \in H^2$ and $(\Omega, \xi, \{\xi_t\}, Q)$ the corresponding filtered probability space. Therefore $I(t, \omega)$ is a martingale in relation to $\{\xi_t\}$, that is,

$$E^Q \left(\int_s^T g(u, \omega) dW(u, \omega) / \xi_t \right) = \int_s^t g(u, \omega) dW(u, \omega) \text{ with } s \leq t \leq T$$

where $E^Q(.|\xi_t)$ denotes the conditional expectation operator under conditional measure Q , at ξ_t .

Proof: Oksendal (1998), corollary 3.2.6. $\square$

As previously mentioned, the shape of the price processes is $P_i(T, \omega) = P_i(0) + \int_0^T \mu_i(s, \omega) . ds + \int_0^T \sigma_i(s, \omega) . dW(s, \omega)$. Supposing that $\sigma_i(s, \omega) \in H^2$ and using theorem 1, the price process can be written as:

$$P_i(T, \omega) = E^Q(P_i(T, \omega) | \xi_t) + \int_t^T \sigma_i(s, \omega) . dW(s, \omega) \quad (7)$$

In (7) we observe that $P_i(T, \omega)$ consists of an “expected” and an “unexpected” component. Thus, since Ito’s integral is a martingale in $(\Omega, \xi, \{\xi_t\}, Q)$, it plays the role of “unpredictable component”, affecting the price of assets⁴.

Theorem 2 - (Multidimensional Ito’s Formula) - Let $P(t, \omega)$ be an $n + 1$ dimensional Ito’s process, according to equation (1), and $f : R^{n+1} \times [0, \infty) \rightarrow R$ a twice continuously differentiable function in the first set of arguments and once in the second set. Therefore $\{f(P(t, \omega), t); t \geq 0\}$ is also an Ito’s process and obeys equation:

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial P_i(t)} dP_i(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial P_i(t) \partial P_j(t)} dP_i(t) dP_j(t)$$

where $dW_i(t) . dW_j(t) = dt$ if $i = j$ and 0 otherwise and if $dW_i(t) . dt = dt . dW_i(t) = 0$.

⁴ It’s important to mention that some authors, such as Oksendal (1998), define an Ito’s integral directly with $g(s, \omega) \in H^2$. For this reason, in the book written by these authors Ito’s integral is always a martingale.

Proof: Oksendal (1998), theorem 4.2.1. $\square$

The application of Ito's formula⁵ to $\ln(P_0(t))$ (see expression (5)) allows obtaining $P_0(t) = \exp\left(\int_0^t r(s)ds\right)$, where the interpretation of asset $P_0(t)$ is confirmed as an investment in which the very short-term interest rate is continuously capitalized based on initial value $P_0(0) = 1$. As shown in this simple example, Ito's formula is a powerful resource for the resolution of several differential stochastic equations.

Theorem 3 - (Ito's isometry) - Let $\int_0^t g(s, \omega) dW_i(s, \omega)$ be an Ito's integral with $g(t, \omega) \in H^2$. Therefore, we have the following formula:

$$E^Q \left(\left(\int_0^t g(s, \omega) dW_i(s, \omega) \right)^2 \right) = E^Q \left(\int_0^t g(s, \omega)^2 ds \right) \quad i = 1, \dots, m$$

Proof: Oksendal (1998), corollary 3.1.7. $\square$

In continuous time financial models, Ito's isometry is especially useful for the derivation of the conditional variance of several stochastic processes. By writing the variance as $var(x) = E(x^2) - E(x)^2$, it is possible to apply the result of the theorem to the first term on the right-side of the equation, when x is an Ito's process. The usefulness and the beauty of the result certainly reside in the elimination of the stochastic integral.

Next, we will present a set of definitions that are conventional for most continuous time financial models. Each of these definitions is interpreted and briefly discussed. All of these definitions will be necessary for the enunciations of subsequent theorems.

⁵ Also called Ito's formula

Definition 1 - The $n + 1$ dimensional stochastic process defined by

$$\begin{aligned}\bar{P}(t, \omega) &= (1, \bar{P}_1(t, \omega), \dots, \bar{P}_n(t, \omega)) \\ &= \left(1, P_1(t, \omega) \cdot P_0(t, \omega)^{-1}, \dots, P_n(t, \omega) \cdot P_0(t, \omega)^{-1}\right) \quad (8)\end{aligned}$$

is called **normalized market**.

The normalized market is simply the relative price process of the economy when asset $i = 0$ is chosen as *numeraire*.

Definition 2 - A portfolio in market $P(t, \omega)$ (or in market $\bar{P}(t, \omega)$) is an $n + 1$ dimensional process, ξ_t -adapted, denoted $\theta(t, \omega) = (\theta_0(t, \omega), \dots, \theta_n(t, \omega))$.

Term $\theta_i(t, \omega)$ represents the amount invested in the i th asset, which can be positive or negative. This amount can vary over time and with the state of nature. when we say that $\theta(t, \omega)$ is ξ -adapted, we rule out the possibility of the portfolio being created from the information that is not available yet. for instance, we rule out the absurd situation in which the investor “buys the stock on date t only if its price goes up on date $t + 1$ ”.

In some studies, portfolio $\theta(t, \omega)$ is called **strategy**.

Definition 3 - The value of a portfolio $\theta(t, \omega)$ in market $P(t, \omega)$ is the one-dimensional stochastic process.

$$V^\theta(t, \omega) = \theta(t, \omega) \cdot P(t, \omega) = \sum_{i=0}^n \theta_i(t, \omega) \cdot P_i(t, \omega).$$

To motivate the following definition, we consider the following example: we suppose that the price process is given by $P(t, \omega) = W(t, \omega)$, where $W(\cdot)$ is a multidimensional Wiener process. We also suppose that investors can negotiate only at a finite number of instants $\{0 = t_1, t_2, \dots, t_k = T\} \subseteq [0, T]$. Let z be the initial

wealth of a given investor, which is totally invested in initial instant t_1 , that is, this investor acquires a portfolio $\theta(t_1)$ such that $V^\theta(t_1) = \theta(t_1) \cdot W(t_1) = z^6$. The gain obtained by the investor between instants t_1 and t_2 will be given by $V^\theta(t_2) - V^\theta(t_1) = \theta(t_1) \cdot (W(t_2) - W(t_1))$, which allows writing $V^\theta(t_2) = V^\theta(t_1) + \theta(t_1) \cdot (W(t_2) - W(t_1))$. If the role gain is invested again in instant t_2 , then we will have $\theta(t_2)W(t_2) = V^\theta(t_2)$ and the gain obtained between t_2 and t_3 will be $V^\theta(t_3) - V^\theta(t_2) = \theta(t_2) \cdot (W(t_3) - W(t_2))$, which allows writing $V^\theta(t_3) = V^\theta(t_2) + \theta(t_2) \cdot (W(t_3) - W(t_2))$ or $V^\theta(t_3) = z + \theta(t_1) \cdot (W(t_2) - W(t_1)) + \theta(t_2) \cdot (W(t_3) - W(t_2))$. In general, we have the following expressions.

$$V^\theta(t_j) = V^\theta(t_{j-1}) + \theta(t_{j-1}) \cdot (W(t_j) - W(t_{j-1})) \quad 1 < j \leq k \quad (9)$$

$$V^\theta(t_k) = z + \sum_{i=1}^{k-1} \theta(t_i) \cdot (W(t_{i+1}) - W(t_i)) \quad k > 1 \quad (10)$$

Negotiation strategies that (i) do not receive additional financial resources over time and (ii) reinvest all the gains that were obtained are called *self-financing strategies* (or portfolios). As in expression (9), the value, on date t_j , of a self-financing strategy is equal to its value on previous date t_{j-1} plus the gain (or the loss) of portfolio value between t_{j-1} and t_j . Initial wealth z plus the accrued gains (or losses) over time result in the final value of the strategy, according to equation (10).

We can generate a greater number of negotiation instants by dividing the intervals between dates $\{0 = t_1, t_2, \dots, t_k = T\}$ into m subintervals, allowing the negotiation of assets at the end of each instant. In such case, formula (10) becomes:

⁶In this example, we exceptionally suppose $W(0) > 0$.

$$V^\theta(t_k) = z + \sum_{i=1}^{m(k-1)} \theta(t_i/m) \cdot (W(t_{i+1}/m) - W(t_i/m)) \quad (11)$$

By supposing that $\theta(\cdot)$ fulfills the necessary technical integrability conditions, the limit of (11), in Ito's sense, as m tends to infinite, is given by stochastic integral $V^{\theta,z}(T) = z + \int_0^T \theta(s) dW(s)$. The latter is the continuous time condition for portfolio $\theta(t, \omega)$ to be self-financing in the market proposed in the example.

Definition 4 - A portfolio $\theta(t, \omega)$ is self-financing in market $P(t, \omega)$ if its value $V^\theta(t, \omega)$ fulfills condition $dV^\theta(t, \omega) = \theta(t, \omega) \cdot dP(t, \omega)$. Moreover, so that the last expression can be well defined, we should have $\sum_{i=1}^n \mu_i(t) \cdot \theta_i(t), P_0(t) \cdot r(t) \cdot \theta_0(t) \in L^1$ and $\sum_{i=1}^n \theta_i(t) \sigma_i(t) \in L^2$.

Definition 5 - An arbitrage strategy in market $P(t, \omega)$ is a self-financing portfolio $\theta(t, \omega)$ such that (i) $V^\theta(0) = 0, V^\theta(T, \omega) \geq 0, Q(V^\theta(T, \omega) > 0) > 0$ and (ii) $V^\theta(t, \omega) \in H^2$ or adapted; $\exists k \in \mathbb{R}^+$ such that $x(t, \omega) > -k$.

The first part of the definition of an arbitrage means that it is a self-financing portfolio that has initial value equal to zero and final value that is never negative and is strictly positive with probability greater than zero. The arbitrage strategy represents the possibility of gaining an unlimited amount of money at no cost and no risk. In an economy populated by investors that prefer more wealth to less wealth, the price combinations that allow for arbitrage strategies tend to fade fast.

The second part of the definition has a technical nature but plays a crucial role. Condition $V^\theta(t, \omega) \in H^2$ is an additional integrability restriction, whereas $V^\theta(t, \omega) \in K$ represents an existing credit restriction in the real world (there always exists a limit to the level

of indebtedness of an investor, which means that $V^\theta(t, \omega)$ has to be limited to a lower level).

If definition 5 included only the first part, it would be impossible to prevent arbitrage in most markets of type $P(t, \omega)$. For clarity, we can make an analogy to a dice game: let us suppose that a gambler aims at winning \$1000 per night. The gambler bets \$1000 on dice; if the result of the dice roll is an even number, he wins and if it is odd, he loses. If he wins, he takes the money and goes away after the first round, but if he loses, he doubles the bet in the second round; now if he throws the dice and gets even numbers, he wins \$2000, but if the result is otherwise, he loses. Just like the first round, he would take the money and leave if he won. If he lost, he would double the bet again in the third round, winning \$4000 with an even number result. The gambler repeats the same strategy until he wins. In fair play, the probability of not having even numbers after several rounds is $(1/2) \cdot (1/2) \dots (1/2) \cong 0$. As a consequence, if the gambler is able to always double his bets, **without suffering credit restrictions**, he can win \$1000 or any multiple amount of this value every night.

In essence, the strategy described in the example has to do with an increasingly larger bet on the risky asset. In continuous time models, it is possible to show that self-financing portfolios with such characteristic violate both the integrability condition $V^\theta(t, \omega) \in H^2$ and credit restriction $V^\theta(t, \omega) \in K$. This naturally occurs due to the explosive amounts in these portfolios.

These strategies involving an increasingly larger bet on the risky asset were called *doubling strategies* by Harrison and Kreps (1979). Duffie (1996) gives an example of *doubling strategy* in a market of type $P(t, \omega)$, with $P_0(t, \omega) = 1$ and $P_1(t, \omega) = W(t, \omega)$. Even in a market where the risky asset price follows such an erratic process as that found in the Brownian process, if $\theta(t, \omega)$ can be freely chosen, then it is possible to define a self-financing strategy with initial value

0 and with a positive and arbitrarily chosen final value. However, the amount invested in the risky asset is explosive and violates condition (ii) of definition (5).

To conclude our comments about the last definition, we would like to mention that the theorem that establishes the equivalence between an equivalent martingale probability measure and the principle of no-arbitrage depends essentially on restrictions $V^\theta(t, \omega) \in H^2$ or $V^\theta(t, \omega) \in K$. This issue will be discussed further ahead.

Definition 6 - European-style derivative contract, *also known as contingent claim*, with maturity date T , is a ξ_T -measurable random variable denoted as $c(\omega)$.

The following are examples of European-style derivative contracts: (a) A call option for asset $P_i(t, \omega) = S(t, \omega)$, with maturity date T and exercise price E . In this case, $c(\omega)$ is given by $c(\omega) = \max(0, S(T, \omega) - E)$. (b) A term contract on asset $P_i(t, \omega) = S(t, \omega)$, with maturity date T and delivery price E . In this case, $c(\omega)$ is given by $c(\omega) = S(T, \omega) - E$.

Definition 7 - A European-style derivative contract $c(\omega)$ is attainable in market $P(t, \omega)$ if there is a self-financing portfolio $\theta(t, \omega)$ with $V^\theta(t, \omega) \in K$ or $V^\theta(t, \omega) \in H^2$ and a real number z such that

$$c(\omega) = V^{\theta, z}(T) = z + \int_0^T \theta(s, \omega) dP(s, \omega) \quad (12)$$

According to definition (7), a derivative contract is attainable if there is a “well-behaved” self-financing strategy that “replicates” its pay-off value. In this case, the value of the derivative contract and of the self-financing strategy that replicates it should be the same so as to prevent arbitrage. Consequently, if it is possible to determine the value of the strategy, it is also possible to determine the value of the derivative.

Definition 8 - *Market $P(t, \omega)$ is complete if all European-style derivative contract is attainable.*

In a complete market, it is possible to have a hedge against any state of nature. In addition, all derivative contracts are redundant: there is a self-financing strategy for each of them; this strategy consists only of primary assets and behaves exactly as the derivative. Redundancy, combined with the hypothesis of no arbitrage, allows obtaining the price of any derivative contract⁷.

The present study deals only with complete markets. Theorem 8 establishes the conditions that should be fulfilled by the parameters of $P(t, \omega)$ so that the markets can be complete. For further information on incomplete markets, see Karatzas and Shreve (1998).

Definition 9 - *An equivalent martingale probability measure Q^* is a probability measure in $(\Omega, \mathcal{F})$, equivalent to Q , such that the relative price process $\bar{P}(t, \omega)$ is a martingale under Q^* .*

Definition (9) will be widely used throughout our study. However, for the time being, we will just describe it briefly. Saying that measure Q^* is equivalent to Q means that $Q(A) > 0 \Leftrightarrow Q^*(A) > 0$. This condition is essential for the proof of the subsequent theorem, which establishes the connection between the existence of Q^* and the principle of no-arbitrage.

Theorem 4 - *(Probabilistic Characterization of the Principle of No-Arbitrage) - If there is an equivalent martingale probability measure Q^* in market $P(t, \omega)$, then, there is no arbitrage.*

Proof: Let's absurdly suppose there is an arbitrage $\theta(t, \omega)$ in addition to an equivalent martingale probability measure Q^* in market $P(t, \omega)$. therefore, by the definition of arbitrage $V^\theta(0) = 0$ and, by the definition of Q^* , $E^{Q^*}(V^\theta(T)P_0(T)^{-1}) =$

⁷It does not obviously mean that we should obtain closed form formulas for the prices. Quite often, it is necessary to use numerical methods.

$V^\theta(0)P_0(0)^{-1} = V^\theta(0) = 0$. As Q and Q^* are equivalent, we have $Q(V^\theta(T) > 0) > 0 \Leftrightarrow Q^*(V^\theta(T) > 0) > 0$ and $Q(V^\theta(T) \geq 0) = 1 \Leftrightarrow Q^*(V^\theta(T) \geq 0) = 1$. The last two conditions and the fact that $P_0(t)$ is positive imply $E^{Q^*}(V^\theta(T)P_0(T)^{-1}) > 0$, which is a contradiction when compared to $E^{Q^*}(V^\theta(T)P_0(T)^{-1}) = V^\theta(0) = 0$. Consequently, if Q^* exists, there could not be arbitrage. $\square$

It should be observed that the presented proof implicitly uses condition (ii) of the definition of arbitrage. The fact that $\bar{P}(t, \omega)$ is a martingale under Q^* does not warrant per se that the discounted value of a self-financing strategy $V^\theta(t).P_0(t)^{-1}$ will also be a martingale. As a matter of fact, expression $E^{Q^*}(V^\theta(t)P_0(t)^{-1})$ cannot even be well defined. when $V^\theta(t, \omega) \in H^2$, however, $E^{Q^*}(V^\theta(t)P_0(t)^{-1})$ will not only be defined as well as $V^\theta(t)P_0(t)^{-1}$ will also be a martingale under Q^* whenever $\bar{P}(t, \omega)$ is.

Theorem 5 - (Numeraire invariance) - *Initially let $\theta(t, \omega)$ be a portfolio that fulfills integrability condition $\theta(t, \omega) \in H^2$ and $c(\omega)$ a European-style derivative contract with maturity date T . Therefore:*

(5.1) $\theta(t, \omega)$ is self-financing in market $P(t, \omega)$ if and only if it is also self-financing in normalized market $\bar{P}(t, \omega)$;

(5.2) $\theta(t, \omega)$ is an arbitrage in market $P(t, \omega)$ if and only if it is also an arbitrage in normalized market $\bar{P}(t, \omega)$;

(5.3) $c(\omega)$ is attainable in market $P(t, \omega)$ if and only if $c(\omega)P_0(T)^{-1}$ is attainable in normalized market $\bar{P}(t, \omega)$;

(5.4) Market $P(t, \omega)$ is complete if and only if normalized market $\bar{P}(t, \omega)$ is also complete;

Proof: Since the proof of all items in theorem (5) is quite simple, we decided to omit it (see, for example, Duffie (1996) or Oksendal (1998)). $\square$

Theorem 5 means that the selection of a given asset as numeraire does not change the fundamental properties of a market. Independen-

dently of any mathematical proof, this result seems to be natural.

Next, we will present one definition and two theorems that will allow establishing a relation between the parameters of price process $P(t, \omega)$ and the existence of an equivalent martingale measure Q^* . According to theorem 4, this is the same as establishing a relation between the parameters of $P(t, \omega)$ and the principle of no-arbitrage.

Definition 10 - Let $dP_i(t) = \mu_i(t)dt + \sigma_i(t)dW(t)$ ($0 \leq i \leq n$) be the stochastic process followed by the price of the i th asset, where $\sigma_i(t)$ and $dW(t)$ are vectors of dimension m as already defined. We say that market $P(t, \omega)$ is reducible if there is an adapted stochastic process $\lambda(t, \omega) : [0, T] \times \Omega \rightarrow R^m$ that is the solution to the linear system $\sigma(t)\lambda(t) = \mu(t) - r(t)(P_1(t), \dots, P_n(t))$, where $\sigma(t)$ is a matrix that groups vectors $\sigma_i(t)$ ($i = 1, \dots, n$) together and $\mu(t)$ is a vector that groups scalars $\mu_i(t)$ ($i = 1, \dots, n$). In addition, $\lambda(t)$ must belong to L^2 and fulfill Novikov's condition:

$$E^Q \left(\exp \left(\frac{1}{2} \int_0^T \lambda(s, \omega)^2 ds \right) \right) < \infty.$$

The existence of a process $\lambda(t)$ such that $\sigma_i(t)\lambda(t) = \mu_i(t) - r(t)P_i(t)$ for each $i = 1, \dots, n$ means that there is a proportionality ratio between "risk" (σ_i) and "excess return over risk-free rate" ($\mu_i - rP_i$) which is the same for all assets. In the one-dimensional case ($m = 1$), the reducibility condition means that $\lambda = (\mu_i - rP_i)/\sigma_i$ for each $i = 1, \dots, n$, that is, $(\mu_i - rP_i)/\sigma_i = (\mu_j - rP_j)/\sigma_j$ for each $i, j = 1, \dots, n$. The latter expression means that "excess return per unit of risk" is the same for all economic assets. The interpretation of λ in the one-dimensional case has given it the name *market-price-of-risk*.

As shown in theorem 7 ahead, the inexistence of a solution to system $\sigma(t)\lambda(t) = \mu(t) - r(t)(P_1(t), \dots, P_n(t))$ (or of a proportionality ratio between risk and return) is a sign that "mispricing" has occurred.

If $\lambda(t)$ exists, it will be used to define an equivalent probability measure Q . After that, Novikov's condition and the integrability conditions in L^2 will be necessary so that Girsanov's theorem can be used to show that such probability measure is actually an equivalent martingale measure Q^* . The final conclusion is that $\lambda(t)$ exists if and only if the market is arbitrage-free.

Theorem 6 - (Girsanov's theorem) - Let $dP_i(t, \omega) = \mu_i(t, \omega) dt + \sigma_i(t, \omega) dW(t, \omega)$ ($1 \leq i \leq n$) be an Ito's process and suppose there is an m -dimensional process $\lambda(t, \omega)$ adapted in L^2 and which fulfills Novikov's condition such that $\sigma_i(t, \omega) \lambda(t, \omega) = \mu_i(t, \omega) - r(t, \omega) P_i(t, \omega)$ for $i = 1, \dots, n$. Let process $\rho(t, \omega)$ be defined by

$$\rho(t, \omega) = \exp \left(- \int_0^t \lambda(s, \omega) dW(s, \omega) - \frac{1}{2} \int_0^t |\lambda(s, \omega)|^2 ds \right) \quad (13)$$

and let probability measure be Q^* in $(\Omega, \mathcal{F})$ such that $dQ^*(\omega) = \rho(T, \omega) dQ(\omega)$. Therefore, the process defined by $W^{Q^*}(t, \omega) \equiv W(t, \omega) + \int_0^t \lambda(s, \omega) ds$ will be an m -dimensional Wiener process in probability space $(\Omega, \mathcal{F}, Q^*)$. In addition, in this space, process $P_i(t, \omega)$ ($0 \leq i \leq n$) will be represented as $dP_i(t, \omega) = r(t, \omega) dt + \sigma_i(t, \omega) dW^{Q^*}(t, \omega)$.

Proof: The proof of the first part of the theorem can be found in Oksendal (1998), theorem 8.6.4. As to the second part, first we replace $W(t) = dW^{Q^*}(t) - \lambda(t)dt$ in $dP_i(t) = \mu_i(t)dt + \sigma_i(t)dW(t)$ and then we obtain

$$dP_i(t) = (\mu_i(t) - \sigma_i(t)\lambda(t))dt + \sigma_i(t)dW^{Q^*}(t)$$

After that, we replace $\sigma_i(t)\lambda(t) = \mu_i(t) - r(t)P_i(t)$ in the expression above and we obtain $dP_i(t) = r(t)P_i(t)dt + \sigma_i(t)dW^{Q^*}(t)$, according to the theorem. $\square$

In theorem 6, term $\rho(T, \omega)$ is the Radon-Nikodým derivative, a function of $\omega \in \Omega$ that, based on measure $\mathbb{Q}$, redistributes the probability mass of the events of ξ so as to create measure $\mathbb{Q}^*$. Due to its exponential form, $\rho(T, \omega)$ is positive and therefore $\mathbb{Q}$ and $\mathbb{Q}^*$ are equivalent.

Theorem 7 - (*Reducibility and Principle of No-Arbitrage*) - Market $P(t, \omega)$ is arbitrage-free if and only if it is reducible.

Proof: Let us first assume that market $P(t, \omega)$ is reducible. Therefore, process $\lambda(t)$ fulfills definition (10) and Girsanov's theorem can be applied; as a result, it is possible to obtain the representation of $P(t, \omega)$ in $(\Omega, \xi, \mathbb{Q}^*)$:

$$dP_i(t) = r(t)P_i(t)dt + \sigma_i(t)dW^{\mathbb{Q}^*}(t) \quad i = 0, \dots, n \quad (14)$$

By using the expression above and applying Ito's lemma to $\bar{P}_i(t) = P_i(t)P_0(t)^{-1}$, we obtain:

$$d\bar{P}_i(t) = P_0(t)^{-1}\sigma_i(t)dW^{\mathbb{Q}^*}(t) \quad i = 0, \dots, n \quad (15)$$

As Ito's integral is a martingale, we observe in (15) that the processes of normalized prices are martingales under $\mathbb{Q}^{*8}$. Consequently, $\mathbb{Q}^*$ is an equivalent martingale measure and, according to theorem 4, there is no arbitrage.

As the proof of the second part of the theorem is rather technical, we decided to omit it. For further details, see Duffie (1996) or Oksendal (1998). $\square$

⁸There is an implicit supposition that $\sigma \in H^2$, which allows applying theorem 1. In fact, some authors, such as Oksendal (1998), define Ito's integral with σ in H^2 . The definition used in equation (1) follows Duffie's approach (1996), which is more general but requires theorem 1 in order to establish the martingale property.

Harrison and Pliska (1981) have a stronger version of theorem 7, in which the market is arbitrage-free if and only if there is an equivalent martingale measure. However, the version presented in the present study is sufficient for the development of the models we will deal with.

The subsequent theorem determines the conditions that must be satisfied by the parameters of the price process $P(t, \omega)$ so that all the European derivative can be attained.

Theorem 8 - (Hedgeability) - *Let $\sigma(t, \omega)$ be the matrix with vectors σ_i ($i = 1, \dots, n$) on its lines and suppose there is an equivalent martingale measure Q^* . Therefore, market $P(t, \omega)$ is complete if and only if $\text{rank}(\sigma) = m$.*

Proof: Both Duffie (1996) and Oksendal (1998) prove this theorem. The fourth chapter in Vieira (1999) presents a proof adapted for the context of the chapter. $\square$

Assets prices are affected by a total of m random shocks, each of which is represented by a " $dW_i(t)$ ". More specifically, the price of the i th asset linearly depends on term " $\sigma_i dW(t)$ ". The value of a portfolio is a linear combination of assets prices. If $\text{rank}(\sigma) = m$, then the assets that have diffusion coefficient different from zero ($\sigma_i \neq 0$) can be linearly combined in a portfolio $\theta(t, \omega)$ so as to create any ξ_T -measurable random variable $c(\omega)$, so that we have $V^\theta(T, \omega) = c(\omega)$. Intuitively, as ξ_T is generated by $\{W(s); 0 \leq s \leq T\}$ and as $c(\omega)$ is ξ_T -measurable, all the information necessary to know $c(\omega)$ is present in the "history" of $W(\cdot)$ from $t = 0$ up to $t = T$. The rank $(\sigma) = m$ condition means that the linear combinations of prices (portfolios) are able to "capture" this history and "generate" $c(\omega)$. The demonstration of these affirmations makes use of Martingale representation theorem and requires the existence of Q^* . The instantaneously risk-free asset (with $\sigma_i = 0$) can be used to make portfolio $\theta(t, \omega)$ be self-financing. This condition together with $V^\theta(T, \omega) = c(\omega)$ make

$c(\omega)$ attainable. Since this variable was arbitrarily chosen, the market is complete.

The fact that a market is complete means “hedgeability” or the possibility of getting insurance against any possible state of nature. Suppose that, for instance, $c(\omega)$ represents the future value of a liability (with maturity date T) assumed by a certain company. In a complete and arbitrage-free market, if the company wants to eliminate uncertainty over the future value of this liability, it can acquire a self-financing portfolio $\theta(t, \omega)$ such that $V^\theta(T, \omega) = -c(\omega)$. On maturity date T , the company’s cash flow will be $V^\theta(T, \omega) + c(\omega) = 0$. Portfolio $\theta(t, \omega)$ acts as an insurance against $c(\omega)$ and its initial cost will be $V^\theta(0) = \theta(0) \cdot P(0)$.

Note that $\text{rank}(\omega) = m$ condition implies that system $\sigma(t)\lambda(t) = \mu(t) - r(t)(P_1(t), \dots, P_n(t))$ has a unique $\lambda(t)$ solution. The relation of this fact with theorem 6 suggests that, in a complete market, measure Q^* should also be unique. Actually, Harrison and Pliska (1981) showed that *a market is complete if and only if there is only one equivalent martingale measure Q^** .

Theorem 9 - (Valuation of Derivative Contracts in Complete and Arbitrage-free Markets) - *Let $P(t, \omega)$ be a reducible and complete market. Therefore, the arbitrage-free price on date t , of any European-style derivative contract $c(\omega)$ with maturity date T , denoted as $\pi_t(c)$, is given by $\pi_t(c) = P_0(t) \cdot E^{Q^*}(c(\omega) \cdot P_0(T)^{-1} / \xi_t)$.*

Proof: The fact that the market is reducible implies the existence of measure Q^* , in which the discounted price processes are martingales. The fact that the market is complete implies the existence of a self-financing strategy $\theta(t, \omega)$ such that $V^\theta(T, \omega) = c(\omega)$. This, along with Q^* and $\theta(t, \omega) \in H^2$ (see definition 7) implies that $V^\theta(t) \cdot P_0(t)^{-1}$ is a martingale under Q^* , that is, $V^\theta(t) = P_0(t) E^{Q^*}(V^\theta(T) P_0(T)^{-1} / \xi_t)$. As θ and c have the same value in T , the absence of arbitrage implies that they also have the same

value in $t < T$. Consequently,

$$\pi(c) = P_0(t)E^{Q^*} (V^\theta(T)P_0(T)^{-1}/\xi_t) = P_0(t)E^{Q^*} (c(\omega)P_0(T)^{-1}/\xi_t) \quad (16)\square$$

The definition of derivative contract can be generalized, thus allowing not only a final pay-off $c(\omega)$, but also a continuous flow of dividends at rate $h(t, \omega)$. In this case, it is possible to show that the arbitrage-free price of the contract, on date t , is given by

$$E^{Q^*} \left(\int_t^T \exp \left(- \int_t^s r(v)dv \right) h(s)ds + P_0(t)P_0(T)^{-1}c(\omega)/\xi_t \right), \text{ which}$$

is clearly a generalization of (16) (see Duffie (1996), chapter.7).

We will use the theory outlined in this article and, especially theorem 9, to obtain the price of a European call option in the Generalized Black and Scholes Model. Be $P(t, \omega) = (P_0(t, \omega), P_1(t, \omega))$, with

$$dP_0(t) = r(t)P_0(t)dt$$

$$dP_1(t, \omega) = \mu(t)P_1(t, \omega)dt + \sigma(t)P_1(t, \omega)dW(t, \omega)$$

where $W(t)$ is one-dimensional while $\mu(t)$, $r(t)$ and $\sigma(t) \neq 0$ are deterministic functions.

The market is reducible (and arbitrage-free) since system $\sigma(t)P_1(t)\lambda(t) = P_1(t)\mu(t) - r(t)P_1(t)$ has solution $\lambda(t) = (\mu(t) - r(t))/\sigma(t)$. On top of that, the market is complete, since $rank(\sigma) = m = 1$.

Let $\rho(t)$ be defined as in (13) (in function of $\lambda(t)$) and let Q^* be the equivalent martingale probability defined by $dQ^*(\omega) = \rho(T)dQ(\omega)$. Therefore, $P_1(t)$ has the following representation in space (Ω, ξ, Q^*) :

$$dP_1(t, \omega) = r(t)P_1(t, \omega)dt + \sigma(t)P_1(t, \omega)dW^{Q^*}(t, \omega)$$

By applying Ito's lemma (theorem 2) to $\ln P_1(t)$, we obtain:

$$\ln P_1(T) = \ln P_1(t) + \int_t^T \left(r(s) - \frac{1}{2}\sigma(s)^2 \right) ds + \int_t^T \sigma(s)dW^{Q^*}(s)$$

By using the martingale property of Ito's integral (theorem 1), Ito's isometry (theorem 3) and the fact that $\sigma(s)$ is deterministic, we obtain the condition distribution of $\ln P_1(T)$ under Q^* :

$$\ln P_1(T)/\xi_t^{Q^*} \sim N \left(\ln P_1(t) + \int_t^T \left(r(s) - \frac{1}{2}\sigma(s)^2 \right) ds, \int_t^T \sigma(s)^2 ds \right) \quad (17)$$

Let $c(\omega)$ be a European call option for $P_1(t, \omega)$ with exercise price E and maturity date T , therefore, on maturity date T , this option is worth $c(\omega) = \max(0, P_1(T, \omega) - E)$. By using theorem 9, its arbitrage-free price on date $t < T$ is given by:

$$\begin{aligned} \pi(c) &= P_0(t)E^{Q^*} \left(\max(0, P_1(T) - E) P_0(T)^{-1} / \xi_t \right) \\ &= \exp \left(- \int_t^T r(s)ds \right) E^{Q^*} \left(\max(0, P_1(T) - E) / \xi_t \right) \end{aligned} \quad (18)$$

The integral implied in the expectation operator in the last equation can be easily solved because $P_1(T)$ has a lognormal distribution,

as shown in expression (17) (Chapter 3 in Vieira (1999) presents all the stages for the solution of the integral equation). The call option price in Generalized Black and Scholes Model is given by:

$$\pi(c) = P_1(t)N(d_1) - E.\exp\left(-\int_t^T r(s)ds\right)N(d_2) \quad \text{where}$$

$$d_1 + \frac{1n(P_1(t)/E) + \int_t^T (r(s) + \sigma(s)^2/2) ds}{\sqrt{\int_t^T \sigma(s)^2 ds}} \quad d_2 = d_1 - \sqrt{\int_t^T \sigma(s)^2 ds}$$

The option price formula can also be written as:

$$\pi_t(c) = P_1(t).[N(d_1)] - P_0(t).[(1/P_0(T)).E.N(d_2)]$$

By looking at this formula, it is clear that the option ownership is equivalent to the ownership of a portfolio containing amount $N(d_1)$ of asset $P_1(t)$ and amount $[(1/P_0(T)).E.N(d_2)]$ of asset $P_0(t)$. It is possible to check that this portfolio is self-financing and that its value on date T is equal to $c(\omega)$.

Several financial models have the analysis of the term structure of interest rates and the valuation of derivative contracts associated with these rates as their central objective. This presupposes that the short-term rate is a stochastic process in opposition to the Generalized Black and Scholes Model, where such rate is a deterministic function of time.

When $r(t)$ is stochastic and not constant or deterministic, the solution of expression $\pi(c) = P_0(t)E^{Q^*}(c(\omega)P_0(T)^{-1}/\xi_t)$ requires that the conditional joint probability function of $c(\omega)$ and $P_0(T)^{-1}$

under Q^* be known (in expression (18), term $P_0(T)^{-1}$ “came” out of the expectation operator because $r(t)$ is deterministic). Deriving this distribution and later obtaining a formula for $\pi_t(c)$ is sometimes a hard task.

The following theorem, especially designed for an environment with stochastic interest rates, allows solving $\pi_t(c) = P_0(t)E^{Q^*}(c(\omega)P_0(T)^{-1}/\xi_t)$ without the need to know the joint probability distribution of $c(\omega)$ and $P_0(T)^{-1}$ under Q^* . Before presenting this theorem, we have to take a look at definition 11.

Definition 11 - *We say that the asset whose price is $P_i(t)$ ($t < U$) is a fixed income security with maturity date $U \leq T$ if $P_i(U) = 1$ regardless of the state of nature.*

Whenever $P_i(t)$ corresponds to a fixed income security with maturity date U , we will use notation $P_i(t) = P(t, U)$, which is more elucidative.

Theorem 10 - *(Valuation of Derivative Contracts in Complete and Arbitrage-free Markets with Stochastic Interest Rates) - Let $P(t, \omega)$ be a reducible and complete market with an equivalent martingale measure Q^* . Also let $c(\omega)$ be a European-style derivative contract and $P(t, U)$ a fixed income security, both with maturity date $U \leq T$. To conclude, let probability measure F in $(\Omega, \mathcal{F}_U)$ be equivalent to Q^* , such that $dF = \eta(U)dQ^*$, where $\eta(t)$ is given by:*

$$\eta(t) = \frac{P(t, U)}{P_0(t)P(0, U)} \quad t \leq U$$

Therefore, the arbitrage-free price of $c(\omega)$ on date $t \leq U$ is

$$\pi_t(c) = P_0(t)E^{Q^*}(c(\omega)P_0(U)^{-1}/\xi_t) = P(t, U)E^F(c(\omega)/\xi_t)$$

Proof: The theorem is based on Musiela and Rutkowski (1998). The fourth chapter in Vieira (1999) presents this proof and discusses its stages in detail. $\square$

Probability measure F is called term martingale measure. It was first used by Jamshidian (1987); however, it was formally defined by Geman in 1989.

As we can observe, the solution to expression $\pi(c) = P(t, U) E^F(c(\omega)/\xi_t)$ does not require that the joint distribution of $c(\omega)$ and $P_0(U)^{-1}$ be known. Instead, it is necessary to know the probability distribution of $c(\omega)$ under F . As shown by Vieira (1999), this can be achieved, in many cases, by means of Girsanov's theorem.

Before we conclude our review, it is important to mention that, in many applications, it is not possible to obtain analytical solutions to expressions $\pi_t(c) = P_0(t)E^{Q^*}(c(\omega)P_0(U)^{-1}/\xi_t)$ or $\pi_t(c) = P(t, U) E^F(c(\omega)/\xi_t)$. In such cases, to obtain the prices for derivative contracts, we rely on numerical methods, such as Monte Carlo simulation and the Finite Difference Method.

In our study, we did not discuss this issue. Since this specific field is sufficiently vast, we think it deserves to be dealt within a doctoral dissertation. Some noteworthy references include Duffie (1996), Munnik (1992), Boyle (1977) and Johnson and Shanno (1987). The last two authors refer exclusively to the application of Monte Carlo simulations to finance.

4. Conclusion.

As pointed out in the introduction, this article condenses the theory for the valuation of contingent claims in complete and arbitrage-free markets by means of martingales, as proposed by Harrison and Kreps (1979) and Harrison and Pliska (1981). Throughout the text, we used homogeneous notation and language. Moreover, the theo-

rems were presented in a logical sequence, that is, the first theorems were used as background for the subsequent and more complex ones.

Chapter 2 in Vieira (1999) analyzes and “resolves” 8 of the most traditional and popular models for the term structure of interest rates⁹. As shown in that chapter, all of the eight models, which were developed at different times and with different techniques, can be regarded as special cases of the general theory outlined herein. Chapters 3 and 4, also in Vieira (1999)¹⁰, are concerned with original problems and were mostly developed from the definitions and results discussed herein. Therefore, our article not only attempts to shed some light on the modeling of continuous time financial assets, but it is also a reference paper for other researchers.

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⁹The “solved” models are: Merton (1973), Vasicek (1977), Cox-Ingersoll-Ross (1985), Ho-Lee (1986), Hull-White (1990), Heath-Jarrow-Morton (1992), Ball-Touros (1983) and Munnik (1992).

¹⁰Chapter 3 in Vieira (1999) presents a formula that allows calculating the arbitrage-free price of the options for the One Day Interfinancial Deposits Index, IDI, negotiated at BM&F. Chapter 4 proposes a new multifactorial model on the Term Structure of Interest Rates.

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