Duality and Symmetry with Time-Changed Lévy Processes*

José Fajardo**
Ernesto Mordecki***

Abstract

In this paper we review several relationships between prices of put and call options, of both the European and the American type, obtained mainly through Girsanov Theorem, when the asset price is driven by a time-changed Lévy process. This relation is called put-call duality, and includes the relation known as put-call symmetry as a particular case. Necessary and sufficient conditions for put-call symmetry to hold are shown in terms of the triplet of local characteristic of the time-changed Lévy process. This way we extend the results obtained by Fajardo and Mordecki (2006b).

Keywords: Lévy Processes, Time Change, Duality, Symmetry.

JEL Codes: G12, G13.

---

*Submitted in September 2007. Revised in June 2008. We thank Peter Carr for his comments and for pointing out this extension. We also thank an anonymous referee for his very helpful comments, Caio Ibsen de Almeida, João Amaro de Matos, conference participants at The Fourth World Congress of the Bachelier Finance Society and at the VI Meeting of the Brazilian Finance Society. The usual disclaimer applies. J. Fajardo also thanks CNPq for financial support.

**IBMEC Business School, Rio de Janeiro, Brazil. E-mail: pepe@ibmecrj.br

***Centro de Matemática, Facultad de Ciencias, Universidad de la República, Montevideo, Uruguay. E-mail: mordecki@cmat.edu.uy

1. Introduction

Since Black and Scholes (1973) seminal paper, many researchers have studied the true dynamics of the underlying asset return process. This true return differs from that seminal model assumptions in three different aspects: asset prices jump (so we do not observe normal returns), the volatility is stochastic and returns and volatility are correlated. Frequently, that correlation is negative, and this feature is called *leverage effect*.

In an effort to improve option pricing results, many models have been suggested. We can mention Merton (1976) model, where a compound Poisson process is introduced as the structure of the jumps, and the stochastic volatility model of Heston (1993), where a mean-reverting square root process is used. Of course, some of that work can be done with the affine diffusion models introduced by Duffie et al. (2000), but the use of the compound Poisson process to model jumps limits these models.

In fact, if we observe asset returns, we can see the presence of many small jumps in finite time intervals. To deal with this fact, more realistic jump structures have been suggested, such as the inverse Gaussian (IG) model of Barndorff-Nielsen (1998), the generalized hyperbolic (GH) model of Eberlein et al. (1998), the variance-gamma (VG) model of Madan et al. (1998) and the CGMY model of Carr et al. (2002). These models are called Lévy process models.

On the other hand, by a result due to Monroe (1978), we know that every semimartingale can be written as a time-changed Brownian motion. This large family of processes have allowed to show that time-changed Lévy processes can capture the best features of the above models: high jump activity and leverage effect, as was shown by Carr and Wu (2004). Nevertheless, the construction in Monroe (1978) is not direct, meaning that the problem of the specification of different models according to convenient parameter sets remains an important issue.

The stochastic time change in the Lévy process generates stochastic volatility. We can understand the original clock as a calendar time and the new random clock as a business time; more activity on a business day generates a faster business clock, and this randomness in business clock generates stochastic volatility. If we let the Lévy process be correlated with the random clock, we can capture the correlation between returns and volatility, see Black (1976) and Bekaert and Wu (2000). For an analysis of the best specification of the option pricing model with time-changed Lévy processes, see Huang and Wu (2004).

In the present paper we study the relationship between prices of put and call options, of both the European and the American type, when the underlying stock is driven by time-changed Lévy processes. This relationship is called put–call duality and it includes the relationship known as put–call symmetry as a particular case. Similar results for the cases of Lévy processes and additive processes have been obtained by Fajardo and Mordecki (2006b) and Eberlein and Papapantoleon (2005), respectively.

Duality and symmetry relationships are very useful, since they allow us to obtain American put prices from American call prices, which in some cases coincide with their respective European version, as is done by Chesney and Jeanblanc (2004). Also, we can use the symmetry relationship to price and construct static hedging for exotic options. That fact is very important, since we do not need to look for dynamic hedging, which would be more expensive and more difficult to implement. For more details, see Bowie and Carr (1994) and Carr et al. (1998). More recently, they were used to address the so-called skewness premium by Fajardo and Mordecki (2006b).

The paper is organized as follows: in Section 2 we introduce time-changed Lévy processes. In Section 3 we describe the market model. In Section 4 we study the put–call duality relation. In the last sections we have the conclusions and an Appendix.

2. Time-changed Lévy Processes

2.1 Lévy processes

A stochastic process $X = \{X_t^1, \ldots, X_t^d \colon t \geq 0\}$ with values in $\mathbb{R}^d$, defined on the probability space $(\Omega, \mathbf{F}, \mathsf{P})$, is called a Lévy process,[1] with respect to the complete filtration $F = \{F_t, t \geq 0\}$, if it satisfies the following conditions:

- X has right continuous paths and left limits.
- $X_0 = 0$, and given $0 < t_1 < t_2 < \ldots < t_n$, the random variables

  $$X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

  are independent.
- The distribution of the increment $X_{t+h} - X_t$ is time-homogenous, that is, it does not depend on t.
- X is stochastically continuous, *i.e.* $\epsilon > 0 : \lim_{h \to 0} \mathsf{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$

Observe that the first condition implies that the sample paths can present discontinuities at random times.

[1] Named after the French mathematician Paul Lévy (1886-1971).

A key result for Lévy processes is the Lévy-Khintchine formula, which gives us the characteristic function of X_t:

$$\phi_{X_t}(z) \equiv \mathsf{E}e^{\langle z, X_t\rangle} = \exp(t\psi(z))$$

where ψ is called the characteristic exponent and is given by:

$$\psi(z) = \langle a, z\rangle + \frac{1}{2}\langle z, \Sigma z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,y\rangle} - 1 - \langle z, y\rangle \mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy) \qquad (1)$$

where $a = (a_1, \ldots, a_d)$ is a vector in $\mathbb{R}^d$, Π is a positive measure defined on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\Pi(dy)$ is finite, and Σ is a symmetric nonnegative definite matrix.

Interesting examples of Lévy processes are the multidimensional standard Brownian motion, whose triplet is $(0, I_d, 0)$ and we only have continuous component (here I_d is the identity matrix in $\mathbb{R}^d$), and the Poisson process with $d = 1$, whose triplet is $(0, 0, \lambda\delta(1))$, where $\delta(1)$ means Dirac measure at 1 and λ is the intensity parameter, this process has only a discontinuous component. For more examples and details on Lévy processes, see Cont and Tankov (2004). We also refer the reader to Fajardo and Mordecki (2006b) and Fajardo and Farias (2004) for applications of Lévy processes to Brazilian data.

2.2 Time-changed Lévy processes

A random change time is an increasing càdlàg process $\{\mathcal{T}_t \colon t \geq 0\}$, such that

(i) for each fixed t, $\mathcal{T}_t$ is a stopping time with respect to $\mathbf{F}$;

(ii) $\mathcal{T}_t$ is finite $\mathsf{P} - a.s.$, $\forall t \geq 0$; and

(iii) $\mathcal{T}_t \to \infty$ as $t \to \infty$.

Then, consider the process Y_t defined by:

$$Y_t \equiv X_{\mathcal{T}_t}, \quad t \geq 0$$

this process is called time-changed Lévy process. Using different triplets for X and different time changes $\mathcal{T}_t$, we can obtain a good candidate for the underlying asset return process. We know that if $\mathcal{T}_t$ is another Lévy process we have that Y would be another Lévy process (see Appendix). A more general situation is when $\mathcal{T}_t$ is modeled by a non-decreasing semimartingale:

$$\mathcal{T}_t = b_t + \int_0^t \int_0^\infty y\mu(dy, ds) \qquad (2)$$

where b is a drift and μ is the counting measure of jumps of the time change. Now we can obtain the characteristic function of Y_t:

$$\phi_{Y_t}(z) = \mathsf{E}(e^{zX_{\mathcal{T}_t}}) = \mathsf{E}\left(\mathsf{E}\left(e^{zX_u}/\mathcal{T}_t = u\right)\right)$$

If $\mathcal{T}_t$ and X_t are independent, then:

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = \int_0^\infty e^{\psi(z)u}\lambda^t(du) \tag{3}$$

where $\lambda^t(A) = (\mathcal{T}_t \in A)$, and, correspondingly, $\mathcal{L}_{\mathcal{T}_t}$ is the Laplace transform of $\mathcal{T}_t$. So if the Laplace transform of $\mathcal{T}$ and the characteristic exponent of X have closed forms, we can obtain a closed form for ϕ_{Y_t}.

This way we can obtain the distribution of Y_t for every t and then we can price some derivatives.

2.3 Examples

2.3.1 Subordinators

We say that $\mathcal{T}_t$ is a subordinator if it is a positive Lévy process. Positivity is a desired fact for a time change and the choice of a Lévy process will allow us to obtain, as a result, a very good candidate to model asset returns.

Now let $\mathcal{T}_t$ be an α-stable with zero drift and $\alpha \in (0,1)$, that is a Lévy process with Lévy measure given by:

$$v(x) = \frac{A}{x^{1+\alpha}}, \quad x > 0$$

and let X_t be a symmetric $\beta-$stable process, that is, using equation (1) we have

$$\psi(z) = -B|z|^\beta$$

where A and B are positive constants. Then we can compute the Laplace transform of $\mathcal{T}_t$:

$$\mathcal{L}_{\mathcal{T}_t}(z) = A\int_0^\infty \frac{e^{zx}-1}{x^{\alpha+1}}dx = -\frac{A\Gamma(1-\alpha)}{\alpha}(-z)^\alpha$$

Using equation (3), we have that $Y_t = X_{\mathcal{T}_t}$ has characteristic exponent given by

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = -C|z|^{\beta\alpha}$$

where $C = \frac{AB^\alpha\Gamma(1-\alpha)}{\alpha}$. That is Y_t is a $\beta\alpha-$stable symmetric process. If X_t is a Brownian motion, *i.e.* $\beta = 2$, then Y_t would be a $2\alpha-$stable symmetric process. As $\alpha < 1$, we have that Y_t will be a process with heavy tails, which is a stylized fact of the majority of the observed asset returns.

2.3.2 Interest rate models

As in Carr and Wu (2004) we can take $\mu = 0$ in (2) and just take locally deterministic time changes, so we need to specify the local intensity ν:

$$\mathcal{T}_t = \int_0^t \nu(s_-)ds \tag{4}$$

where ν is the instantaneous activity rate, observe that ν must be non-negative.

Using equation (3) we have:

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = \mathsf{E}(e^{-\psi(z)\int_0^t \nu(s_-)ds}) \tag{5}$$

From here we can understand $z\nu$ as an instantaneous interest rate, then we can search in the bond pricing literature for a closed form for ϕ_{Y_t}.

If X_t is a symmetric Lévy process, then it has ψ real and if we consider an independent time change, then Y_t has a symmetric distribution, that is ϕ_{Y_t} remains real and can be computed by (5).

For example, take $X_t = W_t$ a Brownian motion and the instantaneous activity rate as:

$$d\nu(t) = (a - \theta\nu(t))dt + \eta\sqrt{\nu(t)}dB_t$$

Where B_t is another Brownian motion independent of W_t. Then, we know that $\psi(z) = \frac{z^2}{2}$ and by Duffie et al. (2000), we know that the bond price for that affine class of activity rates is given by

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = \mathsf{E}(e^{-\frac{z^2}{2}\int_0^t \nu(s)ds}) = e^{-b(t)\nu(0)-c(t)}$$

In this particular model we have an analytic expression for $b(\cdot)$ and $c(\cdot)$, given by:

$$b(t) = \frac{z^2(1-e^{-\delta t})}{(\delta+\theta)+(\delta-\theta)e^{-\delta t}}$$

and

$$c(t) = \frac{a}{\eta^2}\left[2\ln\frac{(2\delta-(\delta-\theta)(1-e^{-\delta t}))}{2\delta} + (\delta-\theta)t\right]$$

with $\delta = \theta^2 + z^2\eta^2$.

A way to model the correlation between X and $\mathcal{T}$ is to assume in this example that B_t and W_t are correlated. We obtain an asymmetric distribution for Y_t. Then, ϕ_{Y_t} would be a complex number and we have to treat this case with a complex change of measure and compute a generalized Laplace transform introduced by Carr and Wu (2004). Since the purpose of this work is not to address all the properties of time-changed Lévy processes, we refer the interested reader to Carr and Wu (2004) for more details and application to contingent claim valuation.

3. Market Model

Consider a *time-changed Lévy market* where we have a riskless asset that we denote by $B = \{B_t\}_{t\geq 0}$, with

$$B_t = e^{rt}, \qquad r \geq 0$$

where we take $B_0 = 1$ for simplicity, and a risky asset that we denote by $S = \{S_t\}_{t\geq 0}$,

$$S_t = S_0 e^{Y_t}, \qquad S_0 = e^y > 0 \tag{6}$$

Here Y_t is a time-changed Lévy process with independent increments.[2] Also, we assume that the stock pays dividends, with constant rate $\delta \geq 0$, and we assume that the probability measure is the chosen equivalent martingale measure. In other words, prices are computed as expectations with respect to, and the discounted and reinvested process $\{e^{-(r-\delta)t} S_t\}$ is a P–martingale.

In order for this condition be satisfied, we need that

$$E\left[e^{-(r-\delta)t} S_t\right] = S_0, \forall t \tag{7}$$

In other words, $E(e^{Y_t}) = e^{(r-\delta)t}$. That means that the characteristic exponent[3]

$$\Psi(1) = (r - \delta)t$$

To avoid arbitrage opportunities we have to restrict our attention to the time-changed Lévy process such that the exponential process e^{Y_t} is a –martingale.

Let $\Psi = (B, C, \nu)$ denote also the characteristic triplet of Y. Then, by the –martingale property, the drift characteristic B is completely determined by the other characteristics:[4]

$$B_t = (r - \delta)t - \frac{1}{2}\int_0^t c_s ds - \int_0^t \int_{\mathbb{R}} (e^x - 1 - x)\nu(ds, dx)$$

In the market model considered we introduce some derivative assets. More precisely, we consider call and put options of both European and American types.

Let us assume that τ is a stopping time with respect to the given filtration $\mathcal{F}$, that is $\tau\colon \Omega \to [0, \infty]$ belongs to $\mathcal{F}_t$ for all $t \geq 0$; and introduce the notation

[2] That is the case if $\mathcal{T}_t$ has independent increments.

[3] See Appendix.

[4] Here we assume that the moment generating function exists, which allows us to express the characteristics as integrals over time.

$$\mathcal{C}(S_0, K, r, \delta, \tau, \Psi) = \mathsf{E}e^{-r\tau}(S_\tau - K)^+ \tag{8}$$

$$\mathcal{P}(S_0, K, r, \delta, \tau, \Psi) = \mathsf{E}e^{-r\tau}(K - S_\tau)^+ \tag{9}$$

If $\tau = T$, where T is a fixed constant time, then formulas (8) and (9) give the price of the European call and put options, respectively.

3.1 Dual Martingale measure

As in Fajardo and Mordecki (2006b), we introduce the *dual martingale measure* $\tilde{\mathsf{P}}$ given by its restrictions $\tilde{\mathsf{P}}_t$ to $\mathcal{F}_t$ by

$$\frac{d\tilde{\mathsf{P}}_t}{d\mathsf{P}_t} = Z_t$$

where P_t (resp. $\widetilde{\mathsf{P}}$) is the restriction of P (resp. $\widetilde{\mathsf{P}_t}$) to $\mathcal{F}_t$ and the martingale[5] $Z = \{Z_t\}_{t\geq 0}$ is given by

$$Z_t = e^{Y_t-(r-\delta)t},\ (t \geq 0)$$

Following (ii) in Theorem 3.4 in Jacod and Shiryaev (1987), we know that the fixed time can be replaced by any stopping time in $\tau \in \mathcal{M}_T$ to give

$$\frac{d\tilde{\mathsf{P}}_\tau}{d\mathsf{P}_\tau} = Z_\tau$$

where, similarly as before, P_τ (resp. $\tilde{\mathsf{P}}_\tau$) is the restriction of P (resp. $\tilde{\mathsf{P}}$) to the σ-algebra

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \colon A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

With this dual martingale measure we can state and prove our main result.

4. Put-Call Duality and Symmetry

In this section we obtain the put-call duality and symmetry relationships.

The following proposition presents a relationship that we have called put-call duality.

Proposition 4.1. *Consider a Time-changed Lévy market with driving process Y with characteristic triplet $\Psi = (B, C, \nu)$ and with a independent time change. Then, for the expectations introduced in (8) and (9) we have*

$$\mathcal{C}(S_0, K, r, \delta, \tau, \Psi) = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\Psi}) \tag{10}$$

[5]See Equation (7).

where $\tilde{\Psi}(z) = (\tilde{B}, \tilde{C}, \tilde{\nu})$ *is the characteristic triplet (of a certain semimartingale) that satisfies:*

$$\begin{cases} \tilde{B}_t & = (\delta - r)t - \frac{1}{2}\int_0^t c_s ds - \int_0^t \int_{\mathbb{R}} \left(e^x - 1 - x\mathbf{1}_{\{|x|\le 1\}}\right)\tilde{\nu}(ds, dx) \\ \tilde{C} & = C, \\ \tilde{\nu}(dy) & = e^{-y}\nu(-dy) \end{cases} \tag{11}$$

Proof. In this market the martingale $Z = \{Z_t\}_{t\ge 0}$ defined by

$$Z_t = e^{Y_t - (r-\delta)t}\,(t \ge 0) \tag{12}$$

As we have done in the latter section we introduce the *dual martingale measure* $\tilde{\mathsf{P}}$ given by its restrictions $\tilde{\mathsf{P}}_t$ to $\mathcal{F}_t$ by

$$\frac{d\tilde{\mathsf{P}}_t}{d\mathsf{P}_t} = Z_t$$

where P_t is the restriction of P to $\mathcal{F}_t$. Now

$$\begin{aligned} \mathcal{C}(S_0, K, r, \delta, \tau, \Psi) &= \mathsf{E}e^{-r\tau}(S_0 e^{Y_\tau} - K)^+ = \mathsf{E}\left[Z_\tau e^{-\delta\tau}(S_0 - Ke^{-Y_\tau})^+\right] \\ &= \tilde{\mathsf{E}}e^{-\delta\tau}(S_0 - Ke^{\tilde{Y}_\tau})^+ \end{aligned}$$

where $\tilde{\mathsf{E}}$ denotes expectation with respect to $\tilde{\mathsf{P}}$, and the process $\tilde{Y} = \{\tilde{Y}_t\}_{t\ge 0}$ given by $\tilde{Y}_t = -Y_t$ $(t \ge 0)$ is the *dual process*. In order to conclude the proof, that is, in order to verify that

$$\tilde{\mathsf{E}}e^{-\delta\tau}(S_0 - Ke^{\tilde{Y}_\tau})^+ = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\Psi})$$

we must verify that the dual process $\tilde{Y}$ is a semimartingale with characteristic triplet defined by (11). To this end, take $u = (-1, 0, 1)$ and $v = (0, 0 - 1)$ in Proposition A.1 in Appendix. This concludes the proof. ■

As in Fajardo and Mordecki (2006b), Proposition 4.1 motivates us to consider the following market model. Given a time-changed Lévy market with driving process characterized by Ψ, consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\ge 0}$, given by

$$\tilde{B}_t = e^{\delta t}, \qquad r \ge 0$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\ge 0}$, modeled by

$$\tilde{S}_t = Ke^{\tilde{Y}_t}, \qquad S_0 = e^x > 0$$

where $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$ is a semimartingale with local characteristics under $\tilde{}$ given by $\tilde{\Psi}$. This market is the *auxiliary market* in Detemple (2001), and the *dual market* in Fajardo and Mordecki (2006b) and we call the relation (10), *put-call duality* as in Fajardo and Mordecki (2006b). It must be noticed that Peskir and Shiryaev (2001) propose the same denomination for a different relation. Finally, observe that in the dual market (*i.e.* with respect to $\tilde{\mathsf{P}}$), the process $\{e^{-(\delta-r)t}\tilde{S}_t\}$ is a martingale.

4.1 Symmetric markets

It is interesting to note that in a market with no jumps, the distribution (or laws) of the discounted (and reinvested) stocks in both the given and dual Lévy markets coincide. It is then natural to define a market to be *symmetric* when this relation holds, *i.e.* when

$$\mathcal{L}\big(e^{-(r-\delta)t+Y_t} \mid \mathsf{P}\big) = \mathcal{L}\big(e^{-(\delta-r)t-Y_t} \mid \tilde{\mathsf{P}}\big) \tag{13}$$

meaning equality in law. In view of (11), and to the fact that the characteristic triplet determines the law of a time-changed Lévy process, a necessary and sufficient condition for (13) to hold is

$$\nu(dy) = e^{-y}\nu(-dy) \tag{14}$$

This ensures $\tilde{\nu} = \nu$, and it follows that $b - (r-\delta) = \tilde{b} - (\delta - r)$, giving (13), as we always have $\tilde{C} = C$.

When this symmetry condition is satisfied, our put-call duality is called put-call symmetry, which has been used to price and obtain static hedging for exotic options, as the path-dependent ones. This way we can use a very simple instrument, as European options, to analyze sophisticated instruments, as Barrier options. For more details, see Bowie and Carr (1994) and Carr et al. (1998).

4.2 Bates' x% rule

An important fact from option prices is that relative prices of out-of-the-money calls and puts can be used as a measure of symmetry or skewness of the risk-neutral distribution. Bates (1991) called this diagnosis skewness premium. He obtained a relationship called Bates'*x%-rule* that, under a symmetry condition, can perfectly explain that premium. As is shown in the following corollary.

Corollary 4.1. *Take $r = \delta$ and assume (14) holds, we have*

$$\mathcal{C}(F_0, K_c, r, \tau, \Psi) = x\ \mathcal{P}(F_0, K_p, r, \tau, \Psi)$$

where $K_c = xF_0$ and $K_p = F_0/x$, with $x > 0$.

Proof. Follows directly from Proposition 4.1. Since $r = \delta$ and $\Psi = \tilde{\Psi}$. ■

From here, calls and puts at-the-money ($x = 1$) should have the same price. This $x\%-$rule, in the context of Merton's model was obtained by Bates (1997). That is, if the call and put options have strike prices $x\%$ out-of-the money relative to the forward price, then the call should be priced $x\%$ higher than the put. Also, he studied the empirical evidence of that rule in Bates (1996).

4.3 Example

Consider that $r = \delta = 0$ and the following business clock

$$\mathcal{T}_t = \int_0^t \nu(s)ds \tag{15}$$

where ν is the CIR process, *i.e.* a positive process, given by

$$d\nu(t) = (a - \theta\nu(t))dt + \eta\sqrt{\nu(t)}dB_t$$

where B_t is a Brownian motion.

Now consider X_t to be a pure jump Lévy process, independent of $\mathcal{T}$, with Lévy measure ρ. Then, we know that the pure jump process $X_{\mathcal{T}_t}$ has characteristic triplet $(B, 0, \kappa)$, where:

$$\kappa(ds, dx) = \nu(s)\rho(dx) \quad \text{and} \quad B_t = -\int_0^t \int (e^x - 1 - \mathbf{1}_{\{|x|\leq 1\}}(x))\kappa(ds, dx)$$

Then, by applying Proposition 1, we have that in a market where $S_t = S_0 e^{X_{\mathcal{T}_t}}$, call and put prices are related by

$$\mathcal{C}(S_0, K, r, \delta, \tau, \Psi) = \mathcal{P}(K, S_0, \delta, r, \tau, \tilde{\Psi})$$

where $\Psi = (\tilde{B}, 0, \tilde{\kappa})$, with

$$\tilde{\kappa}(ds, dx) = e^{-x}\nu(s)\rho(-dx)\tilde{B}_t = -\int_0^t \int (e^x - 1 - \mathbf{1}_{\{|x|\leq 1\}}(x))\tilde{\kappa}(ds, dx)$$

If furthermore

$$\kappa = \tilde{\kappa}$$

the symmetry condition is satisfied, we obtain the Bates' $x\%$ rule.

Remark 4.1. *It is important to notice that the time-changed Lévy process allows for a huge class of combinations of Levy processes with random clocks; this way, Lévy processes are a particular case, since we can take only a fixed clock. So we expect that the fit with real data we improved when we use time-changed Lévy processes.*

Remark 4.2. *From an option pricing point of view, there are a lot of improvements, since we allow for stochastic volatility and leverage effect, a work that can not be done with Lévy processes, since they do not have memory and are homogeneous in time. For an empirical analysis and comparison of many possible combinations of Lévy processes and random clocks we refer the reader to Huang and Wu (2004).*

5. Conclusions

In a time-changed Lévy market we have derived a put-call relation that we call *put-call duality*, which allowed us to obtain the put–call symmetry relation as a particular case and to obtain the Bates' x% rule.

An interesting extension is to analyze the duality relationship for Asian and Lookback options. Another important problem is to price bidimensional derivatives in a time-changed Lévy process context as is done by Fajardo and Mordecki (2006a) for the case of Lévy processes.

References

Barndorff-Nielsen, O. (1998). Non-Gaussian Orstein-Uhlenbeck based models and some of their uses in financial economics. *Journal of the Royal Statistical Society*, 63:167–241. Series B.

Bates, D. (1991). The crash of '87 – Was it expected? The evidence from options markets. *Journal of Finance*, 46(3):1009–1044.

Bates, D. (1996). Dollar jump fears, 1984-1992: Distributional abnormalities implicit in foreign currency futures options. *Journal of International Money and Finance*, 15(1):65–93.

Bates, D. (1997). The skewness premium: Option pricing under asymmetric processes. *Advances in Futures and Options Research*, 9:51–82.

Bekaert, G. & Wu, G. (2000). Asymmetric volatilities and risk in equity markets. *Review of Financial Studies*, 13:1–42.

Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659.

Black, R. (1976). Studies of stock price volatility changes. In *Proceedings of the 1976 Statistical Association*, pages 177–181. Business and Economics Statistical Association, American Statistical Association, Alexandria, VA.

Bowie, J. & Carr, P. (1994). Static simplicity. *Risk*, 7(8):45–50.

Carr, P., Ellis, K., & Gupta, V. (1998). Static hedging of exotic options. *Journal of Business*, 53(3):1165–1190.

Carr, P., Geman, H., Madan, D., & Yor, M. (2002). The fine structure of assets returns: An empirical investigation. *Journal of Business*, 75(2):305–332.

Carr, P. & Wu, L. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71(1):113–141.

Chesney, M. & Jeanblanc, M. (2004). Pricing American currency options in an exponential Lévy model. *Applied Mathematical Finance*, 11(3):207–225.

Cont, R. & Tankov, P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall. CRC Financial Mathematics Series.

Detemple, J. (2001). American options: Symmetry property. In Jouini, M. M. E. & Cvitanic, J., editors, *Option Pricing, Interest Rates and Risk Management*. Cambridge University Press.

Duffie, D., Pan, J., & Singlenton, K. (2000). Transform analysis and asset pricing for affine jump difusions. *Econometrica*, 68:1343–1376.

Eberlein, E., Keller, U., & Prause, K. (1998). New insights into smile, mispricing and value at risk: The hyperbolic model. *Journal of Business*, 71:371–405.

Eberlein, E. & Papapantoleon, A. (2005). Symmetries and pricing of exotic options in Lévy models. In Kyprianou, P. W. A. & Schoutens, W., editors, *Exotic Option Pricing and Advanced Lévy Models*. Wiley.

Fajardo, J. & Farias, A. (2004). Generalized hyperbolic distributions and Brazilian data. *Brazilian Review of Econometrics*, 24(2):249–271.

Fajardo, J. & Mordecki, E. (2006a). Pricing derivatives on two-dimensional Lévy processes. *International Journal of Theoretical and Applied Finance*, 9(2):185–197.

Fajardo, J. & Mordecki, E. (2006b). Skewness premium with Lévy processes. Working Paper IBMEC.

Fajardo, J. & Mordecki, E. (2006c). Symmetry and duality in Lévy markets. *Quantitative Finance*, 6(3):219–227.

Heston, S. (1993). Closed-form solution for options with stochastic volatility, with application to bond and currency options. *Review of Financial Studies*, 6:327–343.

Huang, J. & Wu, L. (2004). Specification analysis of option pricing models based on time-changed Lévy processes. *Journal of Finance*, 59(3):1405–1440.

Jacod, J. & Shiryaev, A. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin, Heidelberg.

Madan, D., Carr, P., & Chang, E. (1998). The variance gamma process and option pricing. *European Finance Review*, 2:79–105.

Merton, R. (1976). Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144.

Monroe, I. (1978). Process that can be embedded in Brownian motion. *Annals of Probability*, 6:42–56.

Peskir, G. & Shiryaev, A. N. (2001). A note on the call-put parity and a call-put duality. *Theory of Probability and its Applications*, 46:181–183.

Appendix

A.1 Girsanov theorem for semimartingales

Let $Y = (Y^1, \cdots, Y^d)$ be an additive process with finite variation, that is, a semimartingale, the Law of Y is described by its characteristic function:

$$\mathsf{E}\left[e^{\langle z, Y_t\rangle}\right] = e^{\Psi(z)}$$

where

$$\Psi(z) = \int_0^t \left[\langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \langle z, x\rangle\right)\lambda_s(dx)\right] ds$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative definite $d \times d$ matrix and λ_t is a Lévy measure on $\mathbb{R}^d$, *i.e.* it satisfies $\lambda(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min\{1, |x|^2\}\lambda_t(dx) < \infty$, for all $t \leq T$. Under some technical conditions we know that the local characteristics of the semimartingale are given by:

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0,t] \times A) = \int_0^t \int_A \lambda_s(dx) ds$$

where $A \in \mathbb{R}^d$, the triplet (B, C, ν) completely characterizes the distribution of Y.

Now a Girsanov type theorem for semimartingales (Jacod and Shiryaev (1987) Ch. 3 Theorem 3.24).

Proposition A.1. *Let Y be a d-dimensional additive process with finite variation with triplet (B, C, ν) under P, let u, v be vectors in $\mathbb{R}^d$.*

Moreover let $\tilde{\mathsf{P}} \sim \mathsf{P}$, with density

$$\frac{d\tilde{\mathsf{P}}}{d\mathsf{P}} = \frac{e^{\langle v, Y_T\rangle}}{\mathsf{E}[e^{\langle v, Y_T\rangle}]}$$

Then the process $Y^* := \langle u, Y\rangle$ is a $\tilde{\mathsf{P}}-$ semimartingale with characteristic triplet (B^*, C^*, ν^*) with:

$$
\begin{aligned}
b_s^* &= \langle u, b_s \rangle + \frac{1}{2}(\langle u, c_s v \rangle + \langle v, c_s u \rangle) + \int_{\mathbb{R}^d} \langle u, x \rangle (e^{\langle v, x \rangle} - 1) \lambda_s(dx) \\
c_s^* &= \langle u, c_s u \rangle \\
\lambda_s^* &= \Lambda(\kappa_s)
\end{aligned}
$$

where Λ *is a mapping* $\lambda : \mathbb{R}^d \to \mathbb{R}$ *such that* $x \mapsto \Lambda(x) = \langle u, x \rangle$ *and* κ_s *is a measure defined by;*

$$
\kappa(A) = \int_A e^{\langle v, x \rangle} \lambda_s(dx)
$$