Title: Financial Guarantees in Brazilian Life Insurance and Pension Plans

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Financial Guarantees in Brazilian Life Insurance and Pension Plans

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Abstract

Recently regulated Brazilian life and pension products offer a benefit structure composed of minimum guaranteed annual rate, inflation adjustment according to a price index and participation on an investment fund performance. We present a valuation model for these products. We establish a fair condition relationship between minimum guarantees and participation rates, and explore its behavior over a space of maturities, interest rates, and also fund and price index volatilities and correlation. Besides consistency to reference models, we found that the effect of the fund volatility is conditioned to the price index volatility level and the correlation between them.

1 Introduction

Deregulation in the financial and insurance industry during the past decades put life insurance products in direct competition with mutual funds and pension funds, for the savings market. Faced with a new fierce competitive environment, life insurance companies started to add some financial guarantees to their traditional products in order to attract a bigger slice of this huge market. Capital guarantees, minimum annual guaranteed rates, participation on the company results or on specific equity or bond funds returns, early surrender as well as loans against the policy, are some typical examples of financial options that were attached to life insurance products and consequently transformed the liability nature of this industry. Mostly launched between the mid eighties and mid nineties, these options started to be issued at a considerably higher interest rate world. As a consequence they were very cheap, not even considered for pricing, which was ordinarily taken using quite conservative actuarial rates. From mid to late nineties on, interest rates decreased and of course those old cheap liabilities which had not even been priced, suddenly became valuable *in the money* options. The need for market value turned increasingly evident, since the valuation of embedded options became crucial to an assessment of the risk of these products. Several models have been developed since then, identifying and evaluating these embedded options, like [3], [4],[6] and many others. These models use financial economics tools to identify and evaluate the options embedded in the products. They provide a guideline to the pricing of these products since they establish, in a very simple way, the price limits that keep the contracts under fair conditions. Normally not considering actuarial risks, their method invariably consists of finding the set of pairs of minimum guarantees and participation rates, implied by a fairness condition, and to track their behavior under the change of state variables, mainly interest rates and asset volatility. The main findings provided by these models are:

- An inverse relationship between minimum guarantees and participation rates;
- Contract values increase as a result of interest rates reduction;
- In the absence of credit risk, the participation rates are inversely related to the asset volatility;
- Contract values have relatively low sensibility to the correlation between interest rates and asset returns.

In this paper, we develop a simple model applied to Brazilian products, following the same methodology. The main difference to be noticed is the inflation adjustment. In this sense we try to investigate the effects of an inflation adjustment to the minimum guarantee on the behavior of fair participation rates.

In the following section we develop the equations of the contracts and define a fair condition. In section 3, we develop the valuation model based on a change of numeraire technique. Then, Section 4 shows the main results comparing them to the above main findings and presenting the effects of new variables. The last section concludes, discusses about the limits of the model and suggest some paths for future research.

2 New Brazilian Life and Pension Products

L(P)MGP - Life (Pension) with Minimum Guarantee and Performance [1] were regulated in early 2003 [12]. They differ from traditional Brazilian life insurance products, mainly by the segregated character of the assets. In fact, the participation rate is measured on the return of a perfectly defined portfolio, called Specific Investment Fund [2]. Premiums can be paid either as unique initial sums or by periodic payments or even at arbitrary amounts and times at policyholder's whim. Benefits can be paid either as maturity lump-sums or in the form of annuities which, in their turn, can be temporary, life, and so on. Apart from this benefit structure, these products offer the possibility of early surrender as well as the transfer from life to pension plans of the same type, and vice versa. Also, the participation rate can vary in an increasing way, provided

[1] Which is called V(P)RGP - Vida (Plano) com Remuneração Garantida e Performance
[2] Which is called Fundo de Investimento Específico

it is pre-established in the contract. Adding some variations on the way management expenses can be paid, we face a highly complex product to model. In face of such complexity we chose to establish a simpler treatable generic contract, which will be our valuation object.

2.1 A generic simplified contract

Assumptions:

- Actuarial risks are not considered.
- Premium is unique and paid at the beginning of the contract.
- Benefit is paid in a lump-sum at time T, the maturity of the contract.
- This generic contract does not allow for early surrender.
- Underwriting and management expenses are not considered.
- Reversion of performance participation to the reserve account is considered to be made at maturity only.

Taking this simplified framework, a policyholder pays a premium X at the moment $t = 0$ to the insurance company, who invests it in a specific investment fund S. A reserve account M [3] is created in the liability side of the balance sheet and represents at every moment the *minimum* benefit owed to the policyholder if she surrendered at that given moment. The *excess result* E_i at the end of each month i is defined as:

$$E_i = M_{i-1}.\left[(1+\delta_i) - (1+g_m)(1+\theta_i)\right] \tag{1}$$

where:

- E, is the excess return of S over the minimum guarantee adjusted for inflation.
- M_{i-1}, is the reserve account at the end of the previous month.
- δ_i is the return of S during the period $(i-1, i)$
- g_m, is the monthly equivalent minimum guaranteed rate.
- θ_i is the inflation rate during the period $(i-1, i)$ measured as the variation of the price index I during that period.

When positive, a share $\alpha.E_i$ of the monthly excess return is credited in a second liability account named Provision for Excess Results which we denote by B [4]. The complement of that share, $(1-\alpha).E_i$ is credited in an equity account C

[3]This account is named Provisão Matemática de Benefícios a Conceder

[4]This account is named PTEF - Provisão Técnica para Excedentes Financeiros

of the company. When negative, it means the return of S won't be enough to pay the minimum guarantee, i.e. it is below the minimum adjusted guarantee. The difference must be covered by the balances of B and C in the proportions α and $(1-\alpha)$ respectively. In case the balance of B is not enough to cover $\alpha.E_i$, the difference will also be covered by the equity account C, regarding that it will be repaid by future positive balances of B, capitalized by the return of S. The balance of B account is periodically reverted to the reserve account M. Although not mandatory on a periodic basis, the reversion periodicity cannot exceed five consecutive years. As stated in the last assumption, we restrict our analysis to the case of maturity reversion. Meanwhile, for the sake of understanding the dynamics of the different accounts let us assume for now that reversion is made on an annual basis.

Let us now define A, the policyholder account, whose balance is the sum of the balances of M and E. Whenever reversion takes place the balance of A equals the balance of M. From that moment on, until the next reversion, M evolves according to minimum adjusted guaranteed rate and E, according to the excess results obtained every month. Immediately after every reversion, the balance of E is, of course, zero. So, assuming the reversion at the end of each year t, it is not difficult to conclude that:

$$A_t = A_{t-1}(1+g)(1+\theta_t) + \alpha.[E_t]^+ \tag{2}$$

$$\begin{aligned} C_t &= C_{t-1}(1+\delta_t) + (1-\alpha).[E_t]^+ - [E_t]^- \\ &= C_{t-1}(1+\delta_t) + [E_t]^+ - \alpha.[E_t]^+ - [E_t]^- \end{aligned} \tag{3}$$

where

- $E_t = M_{t-1}[(1+\delta_t) - (1+g)(1+\theta_t)]$.
- $[E_t]^+ = max[E_t, 0]$
- $[E_t]^- = max[-E_t, 0]$.
- δ_t is the return of S during the interval $[t-1, t]$.
- g is the minimum annual guaranteed rate.
- θ_t is the inflation rate, between $(t-1)$ and t.

The options involved are clearly shown. Considering now that reversion takes place only at maturity T, we can say that:

$$A_T = X.(1+g)^T.\frac{I_T}{I_0} + \alpha.[E_T]^+ \tag{4}$$

$$C_T = (1-\alpha).[E_T]^+ - [E_T]^- \tag{5}$$

Another way to express A_T in which we can clearly see the exchange of options between insurance company and policyholder, is:

$$A_T = X \prod_{t=1}^{T} (1 + \delta_t) - (1 - \alpha)[E_T]^+ + [E_T]^- \tag{6}$$

One sees immediately that the policyholder is long a put option on the performance. This put is indeed the minimum adjusted guarantee; short on a $(1-\alpha)$ percentage of a call option on the performance. The insurance company is short the same put option and long the same percentage on the performance. Summing up 5 and 6 we obtain:

$$A_T + C_T = X \prod_{t=1}^{T} (1 + \delta_t) = X_T \tag{7}$$

In continuous time, we can write for A:

$$A_T = X. \left\{ e^{gT}.\frac{I_T}{I_0} + \alpha \left[\frac{S_T}{S_0} - e^{gT}.\frac{I_T}{I_0} \right]^+ \right\} \tag{8}$$

3 Fair Contracts

The so called Contribution Principle, states that benefits must be shared between the policyholder and the insurance company in the proportion of the contributions made. According to this principle, a contract is said to be fair when the present value of the premium equals the present value of the expected benefits. It must be stressed that this present value calculation implies an expectation calculated under an equivalent martingale measure. Following [9] we can write that:

$$V_0(A_T) = V_0(X) \tag{9}$$

$$V_0(C_T) = 0 \tag{10}$$

Where V_0 is merely a present value operator. Thus,

$$\frac{V_0(A_T)}{V_0(X)} = 1 \tag{11}$$

Considering that the initial premium X is evaluated at $t = 0$, we can write $V_0(X) = X$:

$$V_0(\frac{A_T}{X}) = 1 \tag{12}$$

From 2.1

$$V_0(\frac{A_T}{X}) = 1 = e^{-rT} E^Q \left\{ e^{gT}.\frac{I_T}{I_0} + \alpha \left[\frac{S(T)}{S(0)} - e^{gT}.\frac{I_T}{I_0} \right]^+ \right\} \tag{13}$$

where E^Q is an expectation under the equivalent martingale measure, and r is the risk free short rate of interest.

4 A Simple Valuation Model

A_T/X can then be seen as the contract function for an European derivative that, at $t = 0$, has value 1. This means we face an inverted problem, like the well known case of implied volatilities. In fact we want to find pairs (g, α) that satisfy that fairness condition. We use the notation Υ for this contract function. Thus, we have:

$$\Upsilon(T) = \left\{ e^{gT}.\frac{I_T}{I_0} + \alpha \left[\frac{S(T)}{S(0)} - e^{gT}.\frac{I_T}{I_0} \right]^+ \right\} \tag{14}$$

and then, we designate $\Psi(t, \Upsilon)$ the price process of the contract, which has value 1 at $t = 0$. To obtain $\Psi(t)$ we follow closely [1], using a change of numeraire technique to evaluate options which strike price is a function of a price index. To start with, we consider an economy with the following assumptions:

- A temporal horizon $[0, T]$.
- There are no transaction costs and every security is considered perfectly divisible.
- Short selling is allowed.
- Trade is continuous.
- Real interest rate R is assumed to be constant.
- There are no opportunities of arbitrage.
- Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of all possible states of natures, $\mathcal{F}$ is the σ-algebra of the subsets of Ω and P is a probability measure. Information is revealed through time by the filtration $F = \{\mathcal{F}_t, t \in [0, T]\}$, consisting in an increasing sequence of σ-algebras, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for $t \geq s$. At $t = 0$ there is no information available and at moment T uncertainty ceases, since all information relative to the temporal horizon is already available [11].

At these conditions the value of an european derivative can be expressed by:

$$\Psi(t, \Upsilon(T)) = U(t) E_t^{Q_U} \left[\frac{\Upsilon(T)}{U(T)} \right] \tag{15}$$

where Q_U is the probability measure induced by the numeraire U. Numeraire is any tradeable asset that pays no dividends, at least during the temporal horizon considered. It is the asset in terms of which other asset values are expressed. In these more general terms, we can express 13 as:

$$U(0)E_t^{Q_U}\left\{\frac{1}{U(T)}\left[e^{gT}.\frac{I_T}{I_0}+\alpha\left[\frac{S(T)}{S(0)}-e^{gT}.\frac{I_T}{I_0}\right]^+\right]\right\}=1 \tag{16}$$

For the sake of clearness we can divide the contract function in two parts: The guarantee term $e^{gT}.\frac{I_T}{I_0}$; And the option term, $\left[\frac{S(T)}{S(0)}-e^{gT}.\frac{I_T}{I_0}\right]^+$. We take first the later, and denote its contract function by Φ:

$$\Phi = max[\frac{S(T)}{S(0)} - e^{gT}I(T), 0] \tag{17}$$

without any loss of generality, we can assume that:

$$S(0) = 1$$

$$I(0) = 1$$

Following [1], the underlying asset S, is assumed to evolve according to a geometric brownian motion like:

$$dS(t) = S(t)\mu dt + S(t)\sigma_S' dW \tag{18}$$

or,

$$dS(t) = S(t)rdt + S(t)\sigma_S' dW^Q \tag{19}$$

under a risk neutral measure Q [2]. On its turn, based on [8] and [7], we consider the following process for the price index, I:

$$dI(t) = I(t)(r-R)dt + I(t)\sigma_I' dW^Q \tag{20}$$

also under the measure Q. In these processes, r represents the nominal risk free short term interest rate, R is real interest rate and dW^Q a vector of two standard Wiener processes defined on the probability space $(\Omega, \mathcal{F}, P)$, constituting the two sources of uncertainty of this economy:

$$\begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix}$$

σ_S and σ_I are two column vectors:

$$\begin{pmatrix} \sigma_{S1(I1)} \\ \sigma_{S2(I2)} \end{pmatrix}$$

where $\sigma^2_{Sj(Ij)}$ represents the variance of S (or I), per unit of time, originated by the source of uncertainty $j = 1, 2$. As we can see, the strike price of the option is a function of the price index I. At every moment t, this exercise price can be expressed by:

$$K(t) = e^{gT} I(t)$$

To evaluate this option we use a change of numeraire technique. A natural candidate for numeraire would be the price index I, itself. However, I is not a tradeable asset and, as such, it cannot play that role. Notwithstanding, considering a constant real interest rate R, we can define an asset U:

$$U(t) = e^{Rt}.I(t) \tag{21}$$

which stands for a deposit that pays a constant real interest rate plus inflation, as measured by the price index I. By direct application of Ito's Lemma, the process $U(t)$ can be expressed by:

$$dU(t) = U(t).rdt + U(t)\sigma_I dW^Q \tag{22}$$

$U(t)$ is then a tradeable asset that pays no dividends and, this being so, it can be chosen for numeraire. The process value of the option can then be written as:

$$\Pi(t, \Phi(T)) = U(t) E_t^{Q_U}[max Z(T) - K(T), 0] \tag{23}$$

where:

- $\Pi(t, \Phi)$ is the process value for the option.
- $\Phi(T)$ is the option's contract function.
- Q_U is the equivalent martingale measure induced by the numeraire U.
- $Z(T) = \frac{S(T)}{U(T)}$.
- $K(T) = \frac{e^{gT} I(T)}{U(T)} = e^{(g-R)T}$, considering that $I(T) = U(T)e^{-RT}$.

By Ito's Lemma and the Main Theorem on [1], we see that Z has null drift under Q_U:

$$dZ(t) = Z(t)(\sigma_S - \sigma_I) dW^U \tag{24}$$

$\Pi(t, \Phi)$ can then be calculated by direct application of Black-Scholes for a call option with strike price K on an asset with scalar volatility expressed by[5]:

[5]Notice that $|\sigma_i| = \delta_i$, for $i = S, I$.

$$\delta_Z = \|\sigma'_S - \sigma'_I\| = \sqrt{\|\sigma_S\|^2 + \|\sigma_I\|^2 - 2\sigma_S\sigma'_I} = \sqrt{\delta_S^2 + \delta_I^2 - 2\rho\delta_S\delta_I}$$

in a normalized economy with a zero risk free rate of interest, since $\frac{U(t)}{U(t)} = 1$. In these terms, the value of this option can be written as:

$$\Pi(t,\Phi) = U(t)\left\{Z(t)N[d_1] - KN[d_2]\right\} \tag{25}$$

with:

$$d_1 = \frac{1}{\delta_Z.\sqrt{T-t}}\left\{ln\left(\frac{Z(t)}{K}\right) + \frac{1}{2}\delta_Z^2(T-t)\right\}$$

$$d_2 = d_1 - \delta_Z\sqrt{T-t}$$

substituting the expressions for $Z(t)$ and K, and taking $S(0) = I(0) = 1$, we obtain:

$$\Pi(t,\Phi) = S(t)N[d_1] - I(t)e^{-R(T-t)}e^{gT}\frac{S(0)}{I(0)}N[d_2] \tag{26}$$

with

$$d_1 = \frac{1}{\delta_Z\sqrt{(T-t)}}\left\{ln\left(\frac{S(t)I(0)}{I(t)S(0)e^{gT}}\right) + \left(R + \frac{1}{2}\delta_Z^2\right)(T-t)\right\} \tag{27}$$

$$d_2 = d_1 - \delta_Z\sqrt{T-t} \tag{28}$$

where, as already seen,

$$\delta_Z = \sqrt{\delta_S^2 + \delta_I^2 - 2\rho\delta_S\delta_I} \tag{29}$$

But the contract does not resume to the call itself. In fact, taking 16, we have:

$$\Psi(t,\Upsilon(T)) = U(t)E_t^{Q_U}\left\{e^{(g-R)T}\frac{U(T)}{U(T)} + \alpha.max[Z(T) - K(T),0]\right\} \tag{30}$$

or,

$$\begin{aligned}\Psi(0,\Upsilon(T)) &= U(0)E_t^{Q_U}\left\{e^{(g-R)T}\right\} + \\ &+ U(0)E_t^{Q_U}\left\{\alpha.max\left[Z(T) - K(T),0\right]\right\} = 1\end{aligned} \tag{31}$$

substituting $\Pi(0,\Phi(T))$ for the option term:

$$\Psi(0,\Upsilon(T)) = U(0)e^{(g-R)T} + \alpha.\Pi(0,\Phi(T)) = 1 \tag{32}$$

where, $U(0) = e^{R.0}.I(0) = 1$. Using 26 for $t = 0$:

$$e^{(g-R)T} + \alpha. \left[N(d_1) - e^{g-R} N(d_2) \right] = 1 \qquad (33)$$

with d_1 e d_2 as expressed in 27 e 28 respectively. That's precisely under this closed solution that we will find pairs of g and α which satisfy the fair condition of the contract, $\Psi(0, \Upsilon(T)) = 1$. Before we proceed, three observations must be made. First, the process we used for the price index is based on an analogy established by [8] and [7] from the Garman and Kholhagen (1983) model, where the process of an exchange rate is of the same nature of an equity that pays a constant dividend yield. Second, the equivalent martingale measures were not explicit because we simply do not need them in the Black-Scholes framework. Third, notice the market is complete, since the deposit U does not exclude the existence of the bank account B that pays nominal risk free interest rate, r. In fact, we have two sources of uncertainty, two risky assets, S and U, and the risk free asset, B.

5 Main results

In quite general terms, we want to find pairs (g, α) that satisfy:

$$F(g, \alpha, T; \delta_S, \rho, \delta_I, R) = 0 \qquad (34)$$

where the functional F is a simplified representation for 33.

5.1 Defining a Universe

Through 33 equation above we find (g, α) keeping the remaining variables constant. First, we distinguish the contract parameters, g, α and T, over which the company has some control, from the external variables, δ_S, δ_I and ρ. Of course, in a way, the company has some control over the asset volatility since it can choose the asset allocation of S, and consequently the same for ρ. Second, based either in regulation statements or in historical series, we establish a universe for each parameter and external variable.

- g has a regulated cap of 6%. For our purposes, we chose

  $$\{0\%, 1\%, 2\%, 3\%, 4\%, 5\%, 6\%\}$$

  .

- α, is considered in the interval $\{0, 1\}$.

- T is, for a unique maturity reversion, limited to 5 years, by regulation. Anyway, we assume higher maturities in order to evaluate the effects of T on the behaviour of (g, α):$T = \{1, 3, 5, 7, 9, 11, 13, 15\}$ measured in years.

- δ_S is made to variate in the set $\Delta_S = \{20\%, 30\%, 40\%, 50\%, 60\%, 70\%\}$ which is based on a series of annualized IBOVESPA daily volatilities over five year windows [6] between 1989 and 1998.
- δ_I, was observed over monthly variations of the price index $IGP - M$, between June 1989 and November 2003, also on five year windows. Given the difference in values for the periods before and after Real Plan, we chose two separate universes, so that we can evaluate the effects of changing δ_I on both regimes: pre-Real Plan $\Delta_I 1 = \{60\%, 50\%, 20\%\}$ and post-Real Plan $\Delta_I 2 = \{2\%, 3\%, 4\%\}$.
- The universe for ρ was established taking parallel series of IBOVESPA and IGP-M monthly returns from June 1989 to November 2003, taking their maximum, minimum and average values, over pre and post Real Plan:

$$\rho = \{10\%, 30\%, 60\%, 80\%\}$$

- Finally, real interest rates were taken from [10] and [5]. The universe is $\Psi = \{3\%, 6\%, 10\%, 15\%\}$.

For each parameter or variable universe, a reference value was defined, as the value to be kept constant when any other parameter or variable variation is at stake. So, we have:

Reference Set	
T (years)	5
Volatility S	40%
Volatility I	3%
Correlation	30%
R	6%
g	3%

Table 1: Reference set

Using the Excel Goal-Seek numerical search algorithm within a macro for a table of numbers, we simply constructed the matrices:

$$\alpha_{g,Q} = \alpha(g, Q)_{\overline{Q_r}}$$

which is F explicit in α. Each of these matrices shows fair α for each g and a second argument Q, keeping the rest of the arguments $\overline{Q}$ at their reference values.

[6] The choice of five year windows is somewhat arbitrary and it is based simply on the reference of 5 year maturity.

5.2 Main Findings

- The minimum guarantee g is bounded by the real interest rate. In fact, if $g > R$, the guarantee will be above 1, keeping the participation rate negative, meaning that the insurance company will have a $(1+\alpha)$ over the excess result. In this case, the company would promise an above risk free rate of interest on a no risk product and then should be compensated, by a long position in a share of a call. Of course one needs to notice that this boundary is made in terms of the real rate and that constitutes a limitation of the model.

- An inverse relationship between minimum guaranteed rate and participation rate is clear from the model. Figure 1 illustrates this relationship, for different maturities. This kind of graph is meant to be a pricing guide, since it relates the three contract parameters. Higher minimum guarantee implies a lower participation. In fact, as the insured gets a higher guaranteed rate, she will accept a lower participation. Guarantees are normally issued below risk free interest rates, the participation rate making up the difference to get a fair return, i.e., a return that compensates the policyholder for the risk she takes. When no credit risk is considered as is the case, fair return means risk free interest rate.

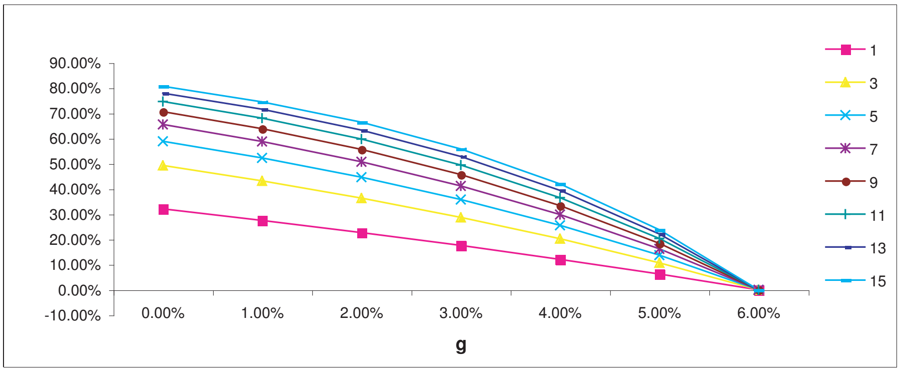

Figure 1: Participation α (alpha) as function of g, varying T. R=6%, $\delta_S = 40\%$, $\delta_I = 3\%$, $\rho = 30\%$.

- Higher maturities allow for higher levels of participation. In fact, higher T implies a lower value guarantee term and, to get the fair condition, a higher participation is then required. Notice that the option value increases with T, but this effect is clearly dominated by the effect on the guarantee term. Figure 2 shows fair contract curves for each maturity, considering reference values for external variables.

- For a given g, the effects of δ_S, δ_I and ρ on the level of participation are intimately related. We can use implicit derivation to explore the behaviour of α as a result of changes in each of these variables. In fact:

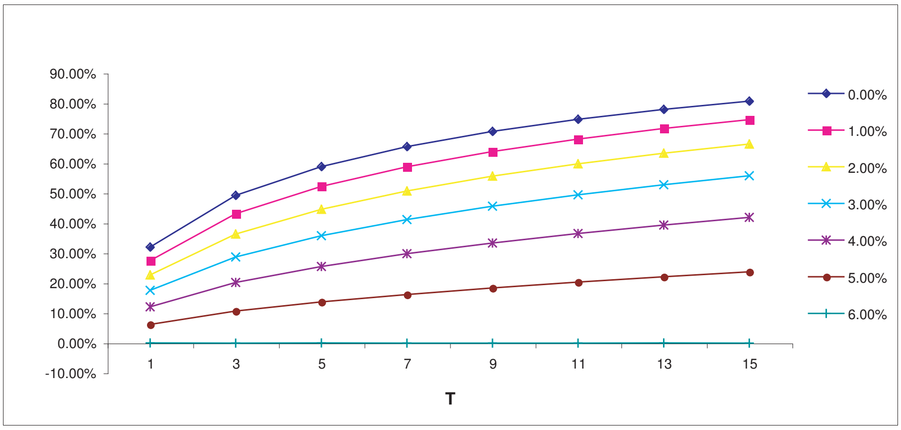

Figure 2: Participation α (alpha) as function of T, varying g. $R = 6\%$, $\delta_S = 40\%$, $\delta_I = 3\%$, $\rho = 30\%$.

$$\frac{d\alpha}{d\delta_S} = -\frac{\frac{\partial F}{\partial \delta_S}}{\frac{\partial F}{\partial \alpha}} \tag{35}$$

where

$$\frac{\partial F}{\partial \delta_S} = \frac{\partial F}{\partial \delta_Z} . \frac{\partial \delta_Z}{\partial \delta_S}$$

Substituting Eq.33 for F, it is immediate to see that $\frac{\partial F}{\partial \alpha}$ is the value of an option and, as such, is non negative. Also, $\frac{\partial F}{\partial \delta_Z}$ has positive sign, since the option value increases with its underlying asset's volatility. Then, the sign of Eq.35 will be negative when:

$$\frac{\partial \delta_Z}{\partial \delta_S} > 0$$

It is easy to conclude that, for $\rho > 0$:

$$\frac{\partial \delta_Z}{\partial \delta_S} > 0 \Leftrightarrow \frac{\delta_S}{\delta_I} > \rho$$

Then, for $\rho > 0$, the participation rate decreases with δ_S if $\frac{\delta_S}{\delta_I} > \rho$. Using the same reasoning for δ_I, we obtain:

$$\frac{\partial \delta_Z}{\partial \delta_I} > 0 \Leftrightarrow \frac{\delta_I}{\delta_S} > \rho$$

Then, for $\rho > 0$, the participation rate decreases if $\frac{\delta_I}{\delta_S} > \rho$. Of course, if $\rho < 0$, α always increases with δ_S and δ_I. Figure 3 shows the behaviour of α with δ_S for all considered levels of δ_I keeping the correlation at its reference level, 30%. It is easy to see the lower influence of δ_S on α, for higher levels of δ_I. Also, Figure 4 shows a decreasing $\alpha(\delta_S)$, for $\delta_I = 3\%$, not being significantly affected across all considered levels of ρ. Figure 5 illustrates precisely the same relationship, but now keeping δ_I at its Pre Real reference level of 50%. We can see α increasing to the point where $\delta_S = \rho.\delta_I$, when $\rho = 80\%$ or $\rho = 60\%$. For lower levels of δ_I, as is the case for Post Real Plan, α shows a decreasing behaviour with δ_S.

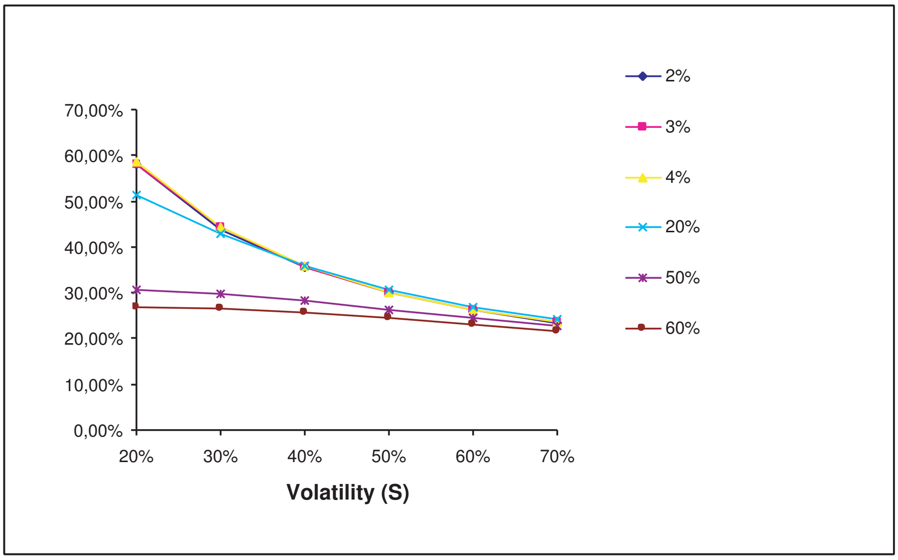

Figure 3: Participation α (alpha) as function of δ_S, varying δ_I. $T = 5$ years, $g = 3\%$, $R = 6\%$, $\rho = 30\%$.

The analysis for δ_I is entirely analogous, although one must pay attention to the universes of δ_S and δ_I. For ρ, we have:

$$\frac{d\alpha}{d\rho} = -\frac{\frac{\partial F}{\partial \rho}}{\frac{\partial F}{\partial \alpha}}$$

and

$$\frac{\partial \delta_Z}{\partial \rho} = \frac{-\delta_S \delta_I}{\delta_Z}$$

which is negative for $\rho > 0$, meaning that α increases with ρ. For $\rho < 0$, the opposite is true: α is a decreasing function of correlation. The relative effect of ρ on the participation, compared to the volatilities' effects depends also on its sign. It is not difficult to obtain:

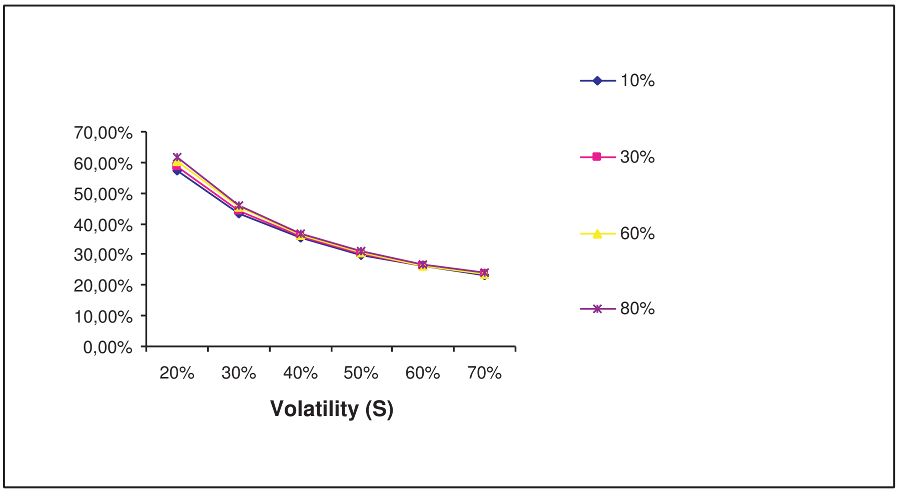

Figure 4: Participation α (alpha) as function of δ_S, varying ρ. $T = 5$ years, $g = 3\%$, $R = 6\%$, $\delta_I = 3\%$.

$$\frac{|\frac{\partial\alpha}{\partial\rho}|}{|\frac{\partial\alpha}{\partial\delta_S}|} = \frac{\delta_S\delta_I}{|\delta_S - \rho\delta_I|}$$

and analogously,

$$\frac{|\frac{\partial\alpha}{\partial\rho}|}{|\frac{\partial\alpha}{\partial\delta_S}|} = \frac{\delta_S\delta_I}{|\delta_S - \rho\delta_I|}$$

Then, it is immediate: when negative, ρ has a smaller influence on α, compared to δ_S and δ_I. However, when positive, its influence on the participation depends on the levels of δ_S and δ_I. 6 shows $\alpha(\rho)$ practically constant for lower Post Real Plan levels of δ_I, and increasing for higher Pre Real Plan δ_I. Even a high correlation does not appear to affect α provided δ_I is low enough.

Summing up, we could see the intimate relationship among these variables. When the volatility of its underlying asset increases, the value of the option also increases.

In our contract, not affecting the guarantee term, this increase in asset volatility implies a reduction of the participation rate, since the option value increased. This makes sense. In fact, higher volatility means higher probability of not getting minimum guaranteed returns, and this risk is then taken by the company only, since the policyholder is guaranteed that minimum. For incurring in this higher risk, the company is compensated simply offering a lower share of asset performance, i.e., a lower participation rate to the policyholder. But the underlying asset we are referring to,

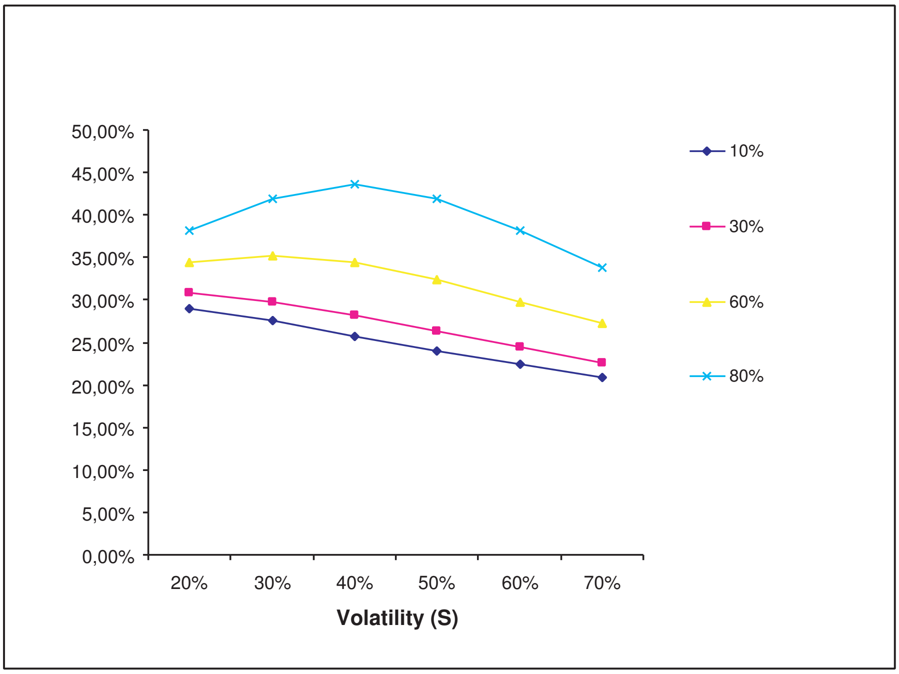

Figure 5: Participation α (alpha) as function of δ_S, varying ρ. $T = 5$ years, $g = 3\%$, $R = 6\%$, $\delta_I = 50\%$.

is a normalized asset entity, which volatility depends on the volatilities of S, I as well as their correlation. For a given investment fund volatility S, the effect of the volatility of I, depends first on its level. Then this first effect can be expanded or amortized depending on the correlation. Higher positive correlation to I, amortizes the effect of δ_I on the aggregate volatility, diminishing then the risk and possibly increasing the participation rate to the policyholder. On the contrary, negative correlation to δ_I expands the effect of δ_I on the aggregate volatility, and then causes lower participation to the policyholder. All these aspects are therefore intimately related.

6 Conclusions

The model proposed above shows good coherence with reference models' results. It shows also the importance of the relationship among asset volatility, price index volatility and their correlation, in determining fair levels of participation rates. We found that the effects of asset volatility on the fair level of participation rate are conditional to the volatility level of the index price and to the their correlation.

We now must give some words of caution. First, the model is applied to a

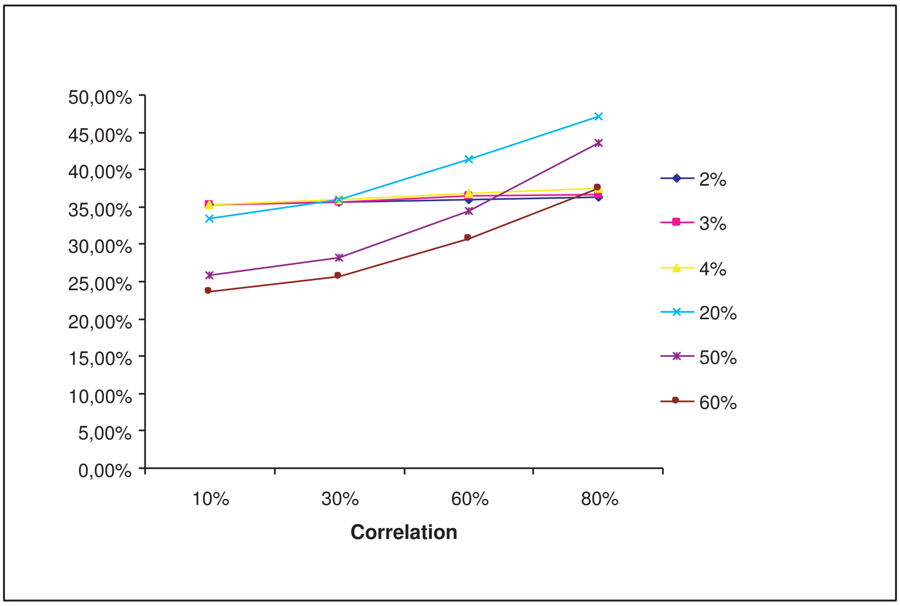

Figure 6: Participation α (alpha) as function of ρ, varying δ_I. $g = 3\%$, $T = 5$ years, $\delta_S = 40\%\ \rho$.

generic ideal product full of simplifications. Of course, early redemptions, reversions, periodic premiums and annuities will add a lot of complexity to the model. Also, the model assumptions put some simplification to the problem: constant real interest rates, geometric brownian motions both to the asset and price index dynamics. These are, of course limitations that clearly indicate future field of research. Notwithstanding, the use of this simplified structure will certainly be helpful in binding the price process of life insurance and pension products, precisely in terms of identifying financial sources of risk formerly treated in a deterministic and sometimes, overly conservative way.

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