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Testing Covariance Stationarity

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Abstract

In this paper, we show that the widely used stationarity tests such as the KPSS test have power close to size in the presence of time-varying unconditional variance. We propose a new test as a complement of the existing tests. Monte Carlo experiments show that the proposed test possesses the following characteristics: (i) In the presence of unit root or a structural change in the mean, the proposed test is as powerful as the KPSS and other tests; (ii) In the presence a changing variance, the traditional tests perform badly whereas the proposed test has high power comparing to the existing tests; (iii) The proposed test has the same size as traditional stationarity tests under the null hypothesis of stationarity. An application to daily observations of return on US Dollar/Euro exchange rate reveals the existence of instability in the unconditional variance when the entire sample is considered, but stability is found in subsamples.

1 Introduction

Since Nelson and Plosser (1982), a great deal of research attention has been focused on the debate over whether economic time series are best characterized as trend stationarity processes or unit root processes. For this reason, a number of testing procedures for the hypothesis of (trend) stationarity have been proposed in the last 15 years.

In econometrics, a widely used procedure in testing stationarity is the KPSS test, proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992) in the context of testing stationarity against the unit root alternative. Leybourne and

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McCabe (1994) suggested a similar test which differs from the KPSS test in its treatment of autocorrelation and applies when the null hypothesis is an AR(k) process. Xiao (2001) proposed testing stationarity by examining the fluctuations in the detrended time series and developed a Kolmogoroff-Smirnov type test for trend stationarity. More recently, Giraitis et al (2003) proposed a test based on the rescaled variance statistic. Also see Hobijn et al. (2004) for a recent generalization of the KPSS test.

Among various plausible alternatives in economic applications, arguably the two most popular alternatives are unit root models and models with structural changes. The aforementioned tests were originally designed to test stationarity against alternatives of unit root processes or long memory processes (Lo (1991)). But they are also widely used in testing structural breaks (see, inter alia, Ploberger and Kramer (1992)) and have power against alternatives with changes in the mean. In nowadays, these tests are widely used in testing the hypothesis of (trend) stationarity in many empirical applications.

Another important alternative model is the one of time series with changes in unconditional volatility. Time varying volatility has been an important subject of research in the last 20 years. The statistical literature on changes of variance can be dated back to Hsu, Miller and Wichern (1974) in modelling stock returns. Recently, there has been an increasing interest in the study of processes with time varying unconditional volatility. A partial list along this direction includes Engle and Rangel (2004), Starica and Mikosch (1999), Loretan and Phillips (1996), Pagan and Schwert (1990a), and Pagan and Schwert (1990b). In general, a statistical analysis of time series data requires some stationarity assumptions. Assumptions such as a functional central limiting theorem (FCLT) are frequently used in finding asymptotic results. In the presence of a change (or changes) in variance, a FCLT no longer holds and thus we lost the foundation of subsequent asymptotic analysis. In addition, many nonparametric and semiparametric estimators are constructed based on the implicit assumption of stationarity. If this assumption is violated, then one cannot justify the usage of such estimators based on asymptotic theory (Pagan and Schwert, 1990b). Other parametric models, such as stationary ARCH and GARCH can be immediately rejected as inappropriate if the time series is not stationary (Loretan and Phillips, 1996).

However, when the aforementioned traditional stationarity tests are applied to test stationarity, it is difficult to detect alternatives with unconditional volatility changes. In this paper, we propose a new test for the null hypothesis of (trend) covariance stationarity as a useful complement to the previous procedures. Comparing to the KPSS type tests, the proposed test is more "robust" in the sense that it not only has power against unit root alternative and alternatives with structural changes in the mean, but also has good power property in detecting changes in variance. The proposed test is simple and easy to calculate. Monte Carlo evidence indicates that the proposed test has good power against alternatives of unit root processes and processes with a changing variance, whereas traditional stationarity tests have very low power against changing variance. Moreover, the new test has empirical size similar to traditional stationarity tests when the null hypothesis of covariance stationarity is true. We provide an empirical application to illustrate the applicability of the proposed test. In particular, our results show that there is instability in the unconditional volatility of the returns on US Dollar/Euro exchange rate, but this instability is not captured by the traditional stationarity tests. Following the strategy used by Pagan and Schwert (1990b), we employ our new test to identify sub-samples in which unconditional variance is constant and, therefore, nonparametric estimators and volatility models that depend on the assumption of covariance stationarity can be correctly employed using observations from that sub-period. Our results show that instability in the unconditional variance is not present in the second half of our sample.

The outline of the paper is as follows. Section 2 defines the null model and brings an overview of the main stationarity tests used in applied work. Section 3 introduces our test for covariance stationarity. Monte Carlo experiments are conducted in section 4. Section 5 presents an empirical applications and section 6 concludes.

Notation is standard with weak convergence denoted by $\Rightarrow$ and convergence in probability by $\xrightarrow{p}$. Integrals with respect to Lebesgue measure such as $\int_0^1 W(s)ds$ are usually written as $\int_0^1 W$, or simply $\int W$ when there is no ambiguity over limits. All limits in the paper are taken as the sample size $n \to \infty$, except otherwise noted.

2 The Model and a Review on Existing Tests

2.1 The Model

Consider a time series y_t that can be written as the sum of a deterministic trend d_t and a stochastic component u_t:

$$y_t = d_t + u_t,\ t = 1, ..., n. \tag{1}$$

The deterministic trend d_t depends on unknown parameters and is specified as $d_t = \gamma' x_t$, where $\gamma = (\gamma_0, ..., \gamma_p)'$ is a vector of trend coefficient and x_t is a deterministic trend of known form, e.g., $x_t = (1, t, ..., t^p)'$. The leading cases of the deterministic component are (i) a constant term $x_t = 1$, and (ii) a linear time trend $x_t = (1, t)'$. u_t is the stochastic component of y_t. Under the null hypothesis H_0, u_t is covariance stationary and satisfies appropriate regularity assumptions that we will specify later in the paper.

We want to test the null hypothesis that y_t is stationary around a deterministic component d_t. In econometric applications, two types of alternative models have been widely studied. The first class of alternatives is

$H_1 : u_t$ is a unit root process.

Another type of alternative is

H_2: models with structural changes in unconditional mean (or deterministic trend).

Leading examples of models with structural changes in unconditional mean (H_2) include (1) H_{2A} : models with a discontinuous change in the mean, $d_t = \gamma_1' x_t$, for $t < \tau$ and $d_t = \gamma_2' x_t$ for $t \geq \tau$, $\gamma_1 \neq \gamma_2$ where τ is a point break; (2) H_{2B} : models with continuous change in the mean such as $d_t = \gamma(t/T)' x_t$, where $\gamma(t/T)$ is a continuous nonconstant function on [0,1]. We could also consider models with multiple (discontinuous) structural breaks in mean.

There is a third class of alternatives,

H_3: models with time-varying unconditional variance,

that has not received much attention in econometric applications. For example, we may consider leading alternatives similar to H_{2A}, H_{2B}, and H_{2C}, (1) H_{3A} : $\sigma_t^2 = \underline{\sigma}^2$, for $t < \tau$ and $\sigma_t^2 = \overline{\sigma}^2$, for $t \geq \tau$, $\gamma_1 \neq \gamma_2$ where τ is a point break and $\sigma_t^2 =$Var(u_t); (2) $H_{3B} : \sigma_t^2 = \sigma(t/T)$, where $\sigma(t/T)$ is a continuous nonconstant function on [0,1].

In this paper, we focus on these three classes of alternatives.

2.2 Some Existing Tests

We review some existing stationarity tests for comparison to the proposed statistic.

2.2.1 KPSS Test

Kwiatkowski, Phillips, Schmidt, and Shin (1992) proposed a test of the null hypothesis of covariance stationarity against unit root. The KPSS statistic is defined as follows

$$KPSS = \frac{1}{(\widehat{\omega}n)^2} \sum_{k=1}^{n} \left(\sum_{j=1}^{k} (\widehat{u}_j) \right)^2, \tag{2}$$

where $\widehat{\omega}^2$ is an nonparametric estimator of the long run variance and $\widehat{u}_j$ is the detrended data.

2.2.2 V/S statistic

Giraitis et al (2003) proposed the following statistic to test the null hypothesis of covariance stationarity

$$V/S = \frac{1}{(\widehat{\omega}n)^2} \left[\sum_{k=1}^{n} \left(\sum_{j=1}^{k} \widehat{u}_j \right)^2 - \frac{1}{n} \left(\sum_{k=1}^{n} \sum_{j=1}^{k} \widehat{u}_j \right)^2 \right].$$

The V/S statistic can be re-written as

$$V/S = n^{-1} \frac{\widehat{Var}(S_1, S_2, ..., S_n)}{\widehat{\omega}^2}, \tag{3}$$

where $S_k = \sum_{1}^{k} (\widehat{u}_j)$ are the partial sums of the observations and $\widehat{Var}(S_1, S_2, ..., S_n)$ is the sample variance of the partial sums.

2.2.3 The KS Statistic

Notice that the KPSS statistic uses the Cramér-von Mises measure of the fluctuation in time series X_t. Xiao (2001) proposes testing for stationarity against the unit root alternative based on the Kolmogoroff-Smirnoff measure of fluctuation, that is:

$$KS = \underset{1\leq k\leq n}{\text{Max}} \frac{1}{\sqrt{n}} \frac{1}{\widehat{\omega}} \left| \sum_{t=1}^{k} \widehat{u}_t - \frac{k}{n} \sum_{t=1}^{n} \widehat{u}_t \right|.$$

3 A Test for Covariance Stationarity

In this section, we propose a test for covariance stationarity by looking at the fluctuation in the first two sample moments and reject the null hypothesis of covariance stationarity whenever there is excessive fluctuation in the data. The driving force behind the proposed test is as follows: If u_t is a covariance stationary time series, its first two moments are not changing over time. However, processes with a changing mean or variance do not satisfy this property, and unit root processes have unbounded variance and grows in a secular way over long period of time. This suggests that we can distinguish covariance stationary processes from plausible alternatives (such as processes with unit roots, or changing mean, or time-varying variances) by looking at the fluctuation in the first two sample moments in the demeaned (detrended) time series.[1]

If we denote the centered ${u_t}^2$ as v_t, i.e. $v_t = u_t^2 - \sigma_u^2$, and let

$$z_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

For convenience of asymptotic analysis, we assume that u_t satisfies regularity conditions so that appropriate invariance principles hold for the underlying time series. We discuss two types of regularity conditions that are commonly used in time series literature: the linear process assumption and mixing conditions.

[1] The proposed test is based on the assumption of finite fourth moment. Loretan and Phillips (1995) introduce fourth-moment failure through the restrictive assumption that the tails of the innovation distribution are of the asymptotic Pareto-Lévy type. In this case, if the assumption of finite fourth moment fails, then they show that conventional asymptotics based on the functional of Brownian bridge should be replaced by functionals of an asymmetric stable levy process.

The first type of regularity condition is based on linear process (Phillips and Solo, 1992). We assume that $u_t = C(L)\varepsilon_t$, where ε_t is a white noise process satisfying certain moment conditions and $C(L) = \sum_{j=0}^{\infty} c_j L^j$, $C(1) \neq 0$, whose coefficients satisfy summability conditions which ensure that u_t is stationary and has positive spectral density at the origin. In particular, we assume the following assumptions.

Assumption A_1: ε_t is iid with zero mean and finite fourth moment.

Assumption B: $\sum_{j=1}^{\infty} j^2 c_j^2 < \infty$.

The linear process condition is assumed for convenience of asymptotic analysis. It facilitates a straightforward asymptotic analysis by applications of the methods of Phillips and Solo (1992). Notice that the asymptotic analysis of linear processes holds under a variety of conditions, and the limiting results of our test can also be generalized to different classes of time series innovations. For example, under appropriate regularity assumptions, invariance principles still hold in the presence of conditional heteroskedasticity (see, e.g., Pantula (1986, 1989), Peters and Veloce (1988) Phillips (1987), Kim and Schmidt (1993) for related studies, also see our Monte Carlo results for cases with conditional heteroskedasticity). As an alternative to Assumption A_1, we may consider the following assumptions with martingale difference sequence innovations.

Assumption A_2: ε_t is a martingale difference sequence with respect to the natural filtration $\mathcal{F}_t$, in addition, there exists a dominating random variable $\boldsymbol{\varepsilon}$ such that $E(\boldsymbol{\varepsilon}^{4+\delta}) < \infty$, for some $\delta > 0$, and

$$P(|\varepsilon_t| \geq a) \leq cP(|\boldsymbol{\varepsilon}| \geq a),$$

for each t and $a \geq 0$ and some constant c, and

$$\frac{1}{n}\sum_{t=1}^{n} E\left(\varepsilon_t^4 | \mathcal{F}_{t-1}\right) \stackrel{a.s.}{\to} \kappa_4,$$

$0 < \kappa_4 < \infty$.

Linear processes satisfying the above assumptions include quite general classes

of time series models. The summability condition B is useful in validating expansions of the operator $C(L)$. For example, if we expand $C(L)$ as

$$C(L) = C(1) + \widetilde{C}(L)(L-1), \tag{4}$$

where $\widetilde{C}(L) = \sum_{j=0}^{\infty} \widetilde{c}_j L^j$ and $\widetilde{c}_j = \sum_{j+1}^{\infty} c_s$. This expansion gives rise to an explicit martingale difference decomposition of u_t

$$u_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t, \quad \text{with } \widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t, \tag{5}$$

The decomposition is sometimes called the martingale decomposition in the probability literature (see Hall and Heyde, 1980) because the first term in decomposition is a martingale difference and the partial sums $\sum_{s=1}^{t} u_s$ correspondingly have the leading martingale term $C(1)\sum_{s=1}^{t} \varepsilon_s$. Decompositions of this type (and second order decompositions) were justified by Phillips and Solo (1992), and can be used to prove that the partial sums of the time series u_t (or its second moment such as $u_t^2 - \sigma^2$) satisfy a functional central limit theorem.

Similar results could be obtained under, say, strong mixing conditions, which also ensure the necessary invariance principles.

Assumption M: $Z_{nt} = z_t/\sqrt{n}$ is L_2-near epoch depedent (NED)[2] of size -1/2 on a strong mixing random vector of size $-r/(r-2)$ and

$$\sup_{n \geq 1} \sup_{1 \leq t \leq n} \sqrt{n} \left(\|Z_{nt}\|_r + d_{nt} \right) < \infty$$

for some $r > 2$, (d_{nt} corresponds to the nonstochastic sequence in the definition of NED sequence).

For the deterministic component, we assume that there is a standardizing matrix D such that $Dx_{[nr]} \to X(r)$ uniformly over r, as $n \to \infty$. For example, if x_t is a p-th order polynomial trend, $D = diag[1, n^{-1}, \cdots, n^{-p}]$ and $X(r) = (1, r, \cdots, r^p)$. Again, for convenience of asymptotic analysis, we make the following assumption on $X(r)$.

Assumption C: $X(r)$ is a continuously differentiable function on $[0, 1]$.

[2] z_t is L_2-near epoch depedent of size -1/2 on a strong mixing random vector ξ_t of size $-r/(r-2)$ if $\left\| z_t - E\left(z_t | \mathcal{F}_{t-m}^{t+m} \right) \right\|_2 \leq d_{nt}\nu(m)$, where d_{nt} is a nonstochstic triangular array, and $\nu(m) = O(m^{-r/(r-2)-\epsilon})$ for some $\epsilon > 0$.

Assumption C implies that the limiting function is of bounded variation. Consequently it ensures convergence to stochastic integrals such as $\int X(s)dW(s)$, where $W(s)$ is a Brownian motion.

Under Assumption M, or A_1 (or A_2) and B , z_t satisfies a bivariate invariance principle $n^{-1/2}\sum_{t=1}^{[nr]} z_t \Rightarrow B(r) = (B_1(r), B_2(r))' = BM(\Omega)$, where $B(r)$ is a vector Browning motion with the following variance matrix

$$\Omega = \begin{bmatrix} \omega_u^2 & \omega_{uv} \\ \omega_{uv} & \omega_v^2 \end{bmatrix}, \tag{6}$$

where ω_u^2 and ω_v^2 are the long-run variance of the process $\{u_t\}$ and $\{v_t\}$, respectively. The parameter ω_{uv} is the long-run covariance of $\{u_t\}$ and $\{v_t\}$.

If u_t were observable, we might consider the following generalized CUSUM test

$$C_n = \max_{1\le k\le n}\left\|\widehat{\Omega}^{-1/2}\frac{1}{\sqrt{n}}\sum_{t=1}^{k} z_t\right\|,$$

where $\widehat{\Omega}$ is a consistent estimator of Ω, and $\|\cdot\|$ is an appropriate norm of vectors.

It will be convenient in what follows to make the following high level assumption about the nonparametric estimate $\widehat{\Omega}$ that we use in our development.

Assumption D: $\widehat{\Omega} \to_p \Omega$ as $n \to \infty$.

There is a large literature on the study of HAC (heteroskedastic and Autocorrelation Consistent) estimators. For example, we may consider the following kernel estimates (see, e.g., Phillips, 1995):

$$\widehat{\omega}_u^2 = \sum_{h=-q}^{q} k\left(\frac{h}{q}\right)\widehat{\gamma}_{uu}(h), \quad \widehat{\omega}_v^2 = \sum_{h=-q}^{q} k\left(\frac{h}{q}\right)\widehat{\gamma}_{vv}(h), \quad \widehat{\omega}_{vu} = \sum_{h=-q}^{q} k\left(\frac{h}{q}\right)\widehat{\gamma}_{uv}(h), \tag{7}$$

which are nothing else than the conventional spectral density estimators. In (7), $k(\cdot)$ are kernel functions, q is the lag truncation parameter, and the quantities $\widehat{\gamma}_{\mathrm{uu}}(h)$, $\widehat{\gamma}_{vv}(h)$, and $\widehat{\gamma}_{uv}(h)$ are sample covariances defined by $n^{-1}\sum' u_t u_{t-h}$, $n^{-1}\sum' v_t v_{t-h}$, $n^{-1}\sum' u_t v_{t-h}$where $\sum'$ signifies summation over $1 \le t-h, t \le n$. When u_t is unobservable, we calculate $\widehat{\gamma}_{\mathrm{uu}}(h)$, $\widehat{\gamma}_{vv}(h)$, and $\widehat{\gamma}_{uv}(h)$ based on estimated u_t and v_t defined later in this paper. The following condition of kernel

functions and the bandwidth are convenient for consistency of the nonparametric estimates:

ASSUMPTION K: *The kernel k has support $[-1,1]$, $k(0)=1$, is symmetric about zero and continuous at 0 and all but a finite number of points. In addition, $\int k(u)du = 1$, and $\int |\psi(s)|\, ds < \infty$,* where $\psi(s) = (2\pi)^{-1}\int_{-\infty}^{\infty} k(x)e^{isx}dx$.

ASSUMPTION W: $\lim_{n\to\infty}\{q^{-1}, n^{-1/2}q\} = 0$.

Many kernel functions satisfy the assumption K. When we use the Bartlett kernel $k(x) = 1 - |x|$, the estimators of ω_u^2, ω_v^2, and ω_{vu}^2, will have, respectively, the following form

$$\begin{aligned}
\widehat{\omega}_u^2 &= \widehat{\gamma}_{\mathrm{uu}}(0) + 2\cdot\sum_{h=1}^{q}(1-\frac{h}{q+1})\cdot\widehat{\gamma}_{\mathrm{uu}}(h), \qquad (8)\\
\widehat{\omega}_v^2 &= \widehat{\gamma}_{vv}(0) + 2\cdot\sum_{h=1}^{q}(1-\frac{h}{q+1})\cdot\widehat{\gamma}_{vv}(h),\\
\widehat{\omega}_{uv}^2 &= \widehat{\gamma}_{uv}(0) + \sum_{h=1}^{q}(1-\frac{h}{q+1})\widehat{\gamma}_{vu}(h) + \sum_{h=1}^{q}(1-\frac{h}{q+1})\widehat{\gamma}_{uv}(h).
\end{aligned}$$

Under Assumptions M, K and W, $\widehat{\Omega}$ is a consistent estimator of Ω and thus Assumption D holds. (under Assumptions A_1 and B, the process z_t satisfies the L_2-near epoch dependence assumption of de Jong and Davidson (2000), in addition with Assumptions K and W, a consistent estimator of Ω can be obtained). For related study on covariance matrix estimation, also see Hannan, 1970, Andrews, 1991, Phillips 1995, and Jansson 2002).[3] Thus we have

$$\widehat{\Omega}^{-1/2}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} z_t \Rightarrow W(r) = \begin{bmatrix} W_1(r) \\ W_2(r) \end{bmatrix},$$

where $W(r)$ is a 2-dimensional standardized Brownian motion, and

$$C_n \Rightarrow \sup_{0\le r\le 1}\|W(r)\|.$$

However, u_t is unobservable since the deterministic component $d_t = \gamma' x_t$ is unknown. The leading cases being (i) a constant term where $d_t = \gamma_0$, and

[3] Phillips (1990) considers estimation of the long-run variance under the absence of finite second moment. We do not consider this case in this paper, since the proposed test is based on the assumption of finite fourth moment.

(ii) a linear time trend where $d_t = \gamma_0 + \gamma_1 t = \gamma' x_t$, where $x_t = (1\ t)'$. In order to test H_0, we estimate u_t (by detrending or demeaning y_t) first and then test stationarity in the demeaned (detrended) data[4]. Assume that there is a standardizing matrix D such that $Dx_{[nr]} \to X(r)$ as $n \to \infty$. For example, if x_t is a p-th order polynomial trend, $D = diag[1, n^{-1}, \cdots, n^{-p}]$ and $X(r) = (1, r, \cdots, r^p)$. We detrend the time series y_t by, say, least-squares regression and denote

$$\widehat{z}_t = \begin{bmatrix} \widehat{u}_t \\ \widehat{v}_t \end{bmatrix}, \tag{9}$$

$$\widehat{u}_t = y_t - \widehat{\gamma}' x_t,\ \widehat{v}_t = \widehat{u}_t^2 - \widehat{\sigma}_u^2, \text{ with } \widehat{\sigma}_u^2 = \frac{1}{n}\sum_{j=1}^{n} \widehat{u}_j^2.$$

We consider the following statistic based on the estimated vector $\widehat{z}_t$,

$$\widehat{C}_n = \max_{1\le k\le n} \left\| \widehat{\Omega}^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{k} \widehat{z}_t \right\|.$$

The asymptotic property of the proposed test is summarized in Theorem 1.

THEOREM 1: *Under Assumptions A_1, B, C, D, or A_2, B, C, D, or M and C, D, as $n \to \infty$,*

$$\widehat{C}_n = \max_{1\le k\le n} \left\| \widehat{\Omega}^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{k} \widehat{z}_t \right\| \Rightarrow \sup_{0\le r\le 1} \left\| \widetilde{W}(r) \right\|,$$

where $\widehat{z}_t$ is defined by Eq. (9), and

$$\widetilde{W}(r) = \begin{bmatrix} W_1(r) - \left[\int_0^1 dW_1(s)X(s)'\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1}\int_0^r X(s)ds \\ W_2(r) - rW_2(1) \end{bmatrix}.$$

Similar to the previous testing procedures and other tests in the unit root literature, the asymptotic distribution of $\widehat{C}_n$ is free of nuisance parameter. Given a choice of the deterministic component, the limiting distribution of $\sup_r \left\| \widetilde{W}(r) \right\|$ can be easily calculated using simulation. In the leading special case when the deterministic component is a constant term (i.e.: $x_t = 1$) the limiting variate reduces to the 2-dimensional standardized Brownian bridge:

$$\widetilde{W}(r) = W(r) - rW(1) = \begin{bmatrix} W_1(r) - rW_1(1) \\ W_2(r) - rW_2(1) \end{bmatrix}.$$

[4] Hence, when $x_t = 1$, H_0 corresponds to a level stationary process, whereas when $x_t = (1, t)'$, H_0 corresponds to a trend stationary process.

In the case when the deterministic component is a linear time trend $x_t = (1, t)'$,

$$\widetilde{W}(r) = \begin{bmatrix} W_1(r) - rW_1(1) + 6r(1-r)\left[\frac{1}{2}W_1(1) - \int_0^1 W_1(s)ds\right] \\ W_2(r) - rW_2(1) \end{bmatrix}.$$

For the choice of norm, we may simply choose, say, for $x = (x_1, \cdots, x_k)$:

$$\|x\| = |x_1| + \cdots + |x_k|,$$

where $|x_i|$ is the absolute value of x_i.[5]

Critical values for the test were simulated using 10,000 Gaussian time series of length 1,000. Table I displays 1%, 5% and 10% critical values for both demeaned and detrended cases, that is, for the cases where $\widehat{u}_t = y_t - \widehat{\gamma}' x_t$, with $x_t = 1$ and $x_t = (1\ t)'$, respectively.

Table 1: Critical Values

critical value	demeaned	detrended
1%	2.40	2.00
5%	2.07	1.74
10%	1.89	1.59

Remark 1 *In the proposed test, we measure the fluctuation in $\widehat{z}_t$ based on the CUSUM process and thus the test is of Kolmogorov type. In principle, other types of functionals (say, the Cramer von Mises type) can be applied to the partial sum process $n^{-1/2}\sum_{t=1}^{k}\widehat{z}_t$ and consistent tests can be constructed.*

Remark 2 *Like the KPSS test, the proposed testing procedure in this paper is a consistent test under regularity conditions that ensure invariance principles to hold and long-run variance matrix be estimated consistently. The regularity assumptions are typically used in the literature and are sufficient, but not necessary, for, say, the invariance principles. Without sufficient restrictions on the model, Pötscher (2002) show that the minimax risk for estimating the value of the long-run variance is infinite and thus it is impossible to consistently discriminate between I(0) and I(1) processes. Also see Faust (1996), Dufour (1997), and Müller (2005) for related discussion on the "impossibility" issue.*

[5] In principle there are a lot of choices for the norm. We choose this norm due to the important fact that the modulus function $|x_i|$ is less sensitive to outlying observations.

In practice, a bandwidth value q has to be selected in the construction of the tests. Consequently, the finite sample performance of the aforementioned tests depends on the choice of the bandwidth. The most popular bandwidth choice is probably the data-dependent automatic bandwidth

$$q = \mu_k \widehat{\delta}(f, k) n^{1/(2p+1)}, \tag{10}$$

where μ_k is a constant associated with the kernel function k, $\delta(f, k)$ is a function of the unknown spectral density and is estimated using a plug-in method, and p is the characteristic exponent of k. This bandwidth choice has been studied by Andrews (1991) in the estimation of a covariance matrix for stationary time series and is now widely used in econometrics applications. It has the advantage that it partially adapts to the serial correlation in the underlying time series through the data-dependent component $\widehat{\delta}(f, k)$. An example is the AR(1) plug-in estimator, which is frequently used in applications:

$$q = \left[\left(\frac{3n}{2} \right)^{1/3} \cdot \left(\frac{2\widehat{\rho}}{1 - \widehat{\rho}^2} \right)^{2/3} \right], \tag{11}$$

where $[\cdot]$ represents an integer number, n is the sample size, and $\widehat{\rho}$ is an estimate of the first-order autoregression coefficient of the demeaned (detrended) data, $\widehat{u}_t$.

In a recent paper, Lima and Xiao (2006) show that using the data-dependent bandwidth choice (10) is inappropriate for the inference problem of distinguishing between I(0) and I(d), $0 < d \leq 1$. They propose a partially data-dependent bandwidth choice, which is the data-dependent plug-in bandwidth (10) coupled with an upper bound, that is:

$$M = \min\{\mu_k \widehat{\delta(f, k)} n^{1/(2p+1)}, B(n)\},$$

where $B(n)$ is an upper bound function. In the next section, we show that the test proposed in this article has very good finite-sample performance when the above partially data-dependent bandwidth choice is used to compute the long-run covariance matrix.[6]

[6] We used the same bandwidth value to compute all the elements of the long-run covariance matrix.

4 Monte Carlo

In this section we conduct monte-carlo experiments to assess the performance of the new test. We assume that the data were generated from the following DGP

$$
\begin{aligned}
y_t &= \alpha y_{t-1} + \varepsilon_t, \\
\varepsilon_t &= \sqrt{\lambda_t} v_t, \\
\lambda_t &= 1 + \beta \varepsilon_{t-1}^2 + \gamma \lambda_{t-1},
\end{aligned}
$$

where $\{v_t\}_{t=1}^n$ is a sequence of independent Gaussian random variables with zero mean and variance equals to $(1 + c \cdot t)$, where $t = 1, 2, ..., n$, and n is the sample size[7]. Moreover, ε_{t-1} and v_t are independent of each other. The larger is β, the larger is the response of λ_t to new information ε_t. The parameter γ controls the autoregressive persistence displayed by λ_t. Hence, the innovation ε_t is an GARCH(1,1) process with zero mean and unconditional variance equal to $(1 + c \cdot t) \cdot \frac{1}{1-(\beta+\gamma)}$. Notice that although the GARCH process ε_t has a conditional variance that varies over time, λ_t, its unconditional variance will always be constant if $c = 0$.

In order to assess the performance of the new test in small sample sizes, we consider the following setups: (i) $\beta = 0$ and 0.05; (ii) $\gamma = 0, 0.45, 0.85$ and 0.90,[8] so that $(\beta + \gamma) = 0, 0.5, 0.90,$ and 0.95; (iii) $\alpha = 0.5$ and 1, and; (iv) $c = 0$, 0.005, and 0.5. Provided that $\beta + \gamma < 1$, the time series y_t is stationary when $|\alpha| < 1$ and $c = 0$. In the case of $|\alpha| < 1$ and $c \neq 0$, the time series y_t does not have a unit root but does have a time varying unconditional variance and, therefore, it is not a stationary process. In the case of $a = 1$, and $c = 0$, y_t is the conventional unit root process. Hence, considering the above DGP and the stationarity tests described in the previous sections, we say that empirical size is obtained when $|\alpha| < 1$ and $c = 0$, and empirical power is obtained when $\alpha = 1$ or $c \neq 0$.[9]

[7] Under the null hypothesis $c = 0$ and, therefore, the fundamental innovation v_t is i.i.d under H_0. In this case, ε_t will also be i.i.d. when $\beta = \gamma = 0$.

[8] Bollerslev (1986) in Theorem 2 of his paper shows that the necessary and sufficient condition for the existence of the fourth moment of a GARCH(1,1) process is $3\beta^2 + 2\beta\gamma + \gamma^2 < 1$. Thus, the values of β and γ we considered in this Monte-Carlo simulation satisfy such a restriction.

[9] In this Monte Carlo experiment we allowed volatility to have continuous changes, a situ-

Recall that Assumption A_1 assumes that ε_t is an i.i.d innovation sequence with finite fourth moment. In the GARCH(1,1) model, however, ε_t is not independent because $E_{t-1}\varepsilon_t^2 = \lambda_t \neq 1$ when $\beta \neq 0$ and/or $\gamma \neq 0$.[10] Notice, however, that assumption A_2 remedies this problem. Nonetheless, if $(\beta + \gamma) \approx 1$, the innovation ε_t will have a very large finite second moment, i.e., $E\varepsilon_t^2 = (1 + c \cdot t) \cdot \frac{1}{1-(\beta+\gamma)} \approx \infty$, when $(\beta + \gamma) \approx 1$. Thus, it is interesting to investigate the impact of such an event on the performance of the new test.

We analyzed power and size of 5% tests. We considered the KPSS, V/S, KS and the proposed C test. All test statistics are computed using demeaned observations $\widehat{u}_t = y_t - \frac{1}{n}\sum_{t=1}^{n} y_t$, which means that we are using 5% critical value for demeaned data. We generated 5000 time series with length n =200 , 400, and 800. The long-run covariance matrix Ω is consistently estimated using Eq. (8). To evaluate the effect of the bandwidth choice on the performance of the new test, we considered

$$q_1 = \min\{q, B(n)\},$$

where $B(n) = [8 \cdot (n/100)^{1/3}]$ and

$$q_2 = \min\{q, D(n)\},$$

where $D(n) = [12 \cdot (n/100)^{1/3}]$. In both choices we set

$$q = \left[\left(\frac{3n}{2} \right)^{1/3} \cdot \left(\frac{2\widehat{\rho}}{1 - \widehat{\rho}^2} \right)^{2/3} \right],$$

where $[\cdot]$ represents an integer number, and $\widehat{\rho}$ is an estimate of the first-order autoregression coefficient of $\widehat{u}_t$

ation that may be justified by persintent microstructural impacts as pointed out by Loretan and Phillips (1996). The unconditional variance can, however, present structural breaks such as, say, $e_t \sim iidN(0,1)$ if $t < (\tau * n)$ and $e_t \sim N(0, 1 + c)$ if $t \geq (\tau * n)$ with $0 < \tau < 1$ and $c \neq 0$. We did consider such structural breaks in our Monte Carlo experiments, but the conclusions coming from this alternative DGP are similar to the one obtained considering continuous changes in the unconditional volatility. Hence, we decided not to report these results to save space.

[10] $E_{t-1}\varepsilon_t^2 = E(\varepsilon_t^2 | \psi_{t-1}) = \lambda_t$, where ψ_t is the information set $(\sigma - field)$ of all information through time t.

4.1 Size of Tests

Empirical size is shown in Panel 1 of the Tables 2 and 3. The results displayed in Table 2 (Table 3) were obtained using the bandwidth q_1 (q_2) to estimate the long-run covariance matrix. Recall that the null model is characterized by $|\alpha| < 1$ and $c = 0$. In the case that $(\beta+\gamma) = 0$, the new test does have empirical size not only close to the nominal size of 5%, but also close to the empirical size of KPSS, V/S and KS tests. We notice that the empirical size of the C test always seems to converge to the nominal size of 5% as the sample size increases. Indeed, for sample sizes of moderate size, say $n = 400$ or $n = 800$, the empirical size of the new test is always close to 5% no matter the bandwidth choice. If sample is too small, say $n = 200$, and $(\beta + \gamma) = 0$, the C test seems to be undersized, specially when q_2 is employed to compute the long-run covariance matrix.

When $(\beta + \gamma) = 0.5$, $|\alpha| < 1$, and $c = 0$, we say that the time series y_t is stationary but possesses GARCH innovations. In this case, our monte-carlo simulations indicate that the size of the new test is still close to 5%, meaning that the presence of GARCH innovations with moderate persistence does not cause size distortions in the C test. Problems arise when $(\beta+\gamma) = 0.9$ or $(\beta+\gamma) = 0.95$. In this case $(\beta + \gamma)$ is too close to unity and the innovation process will have a very large second moment. Since the C test is based on the fluctuation of the first two sample moments, its size becomes larger than 5% when $(\beta + \gamma) \approx 1$. The same does not happen to the existing stationarity tests because they are only based on the fluctuation of the first sample moment. Notice, however, that this size distortion can be reduced if an appropriate bandwidth parameter is used. For example, if q_1 is considered and $(\beta + \gamma) = 0.95$, then the size of the C test is about 11% (the size distortion is pretty stable across sample sizes), but it reduces to about 8% when q_1 is replaced by q_2. We will see next that this reduction of the size distortion obtained using q_2 does not cause too much loss of power.

It is important to mention that the problem of size distortion is also found in the existing stationarity tests, such as the KPSS test. In that case, the tests will be oversized when the autoregressive coefficient α gets close to unity. Again, this happens because the existing tests are based on the fluctuation of the first sample moment, which becomes very unstable when α is too close to

one. Since our new test is based on the fluctuation of the first two sample moments, oversizing will always appear whenever α and/or $(\beta + \gamma)$ approaches unity.

In general, the empirical size of the new test is satisfactory, even under the presence of moderate GARCH effects in the innovation process. This means that the C test is able to identify a stationary process even when it exhibits conditional heteroskedasticity. The empirical size of the C, KPSS, KS and V/S tests gets close to one another as the sample size increases. This result suggests that if the null hypothesis of covariance stationarity is true, then the new test performs as good as the existing stationarity tests. This happens because under H_0 and large samples, the KPSS, V/S, KS and C test statistics are all correctly specified.

4.2 Power of Tests

The null hypothesis of covariance stationarity can be violated by unit root (or long memory) as well as time-varying unconditional variance alternatives.[11] Tables 2 and 3 display the power of 5% tests for bandwidth choices q_1 and q_2, respectively. Panel 4 in both tables shows that all the test statistics deliver good power against the unit root alternative. As expected, the power increases with n because these tests are consistent under the alternative hypothesis of unit root. However, the null hypothesis of covariance stationarity is also violated when $c \neq 0$, and this may happen even if the root is not unity. Panels 2 and 3 show that the KPSS, V/S and KS statistics have power close to nominal size when $|\alpha| < 1$ and $c \neq 0$. The power is small even for large n and c. For example, when $n = 800$ and $c = 0.5$, the power of the existing tests is no larger than 0.07 in Table 2 and no larger than 0.06 in Table 3. These tests seem to be even biased (power less than size) in some cases. These results suggest that the KPSS, V/S and KS statistics are not adequate to test the null hypothesis of stationarity against the alternative of time-varying unconditional variance.

Unlike traditional tests for stationarity, the C test is based on the fluctuation in the first two sample moments. Therefore, if the second moment of the time series exhibits some instability, then we might expect that the new test would

[11] The null hypothesis can also be violated by structural breaks in the deterministic component d_t, but, in order to simplify the presentation of the results, we decided not to consider this alternative hypothesis in our experiments.

reject the null hypothesis of stationarity even if the process does not have a unit root. This is confirmed by the Monte Carlo results: if $c \neq 0$, then results in Table 2 and 3 tell us that the C test has good power even for small pertubations in the unconditional variance. The power increases with n because the new test is also consistent under the alternative model of changing variance. We stress the fact that the power of the C test does not decrease too much when we replace the bandwidth parameter q_1 by q_2. This result comes as a good news since we have showed before that we can reduce size distortion by using q_2 in place of q_1. Another important result is that the presence of GARCH innovations does not seem to affect the power of the new test. Indeed, no matter whether $(\beta+\gamma)=0$ or $0<(\beta+\gamma)<1$, the C test has power always above 0.90 for $n=800$ and $c \neq 0$. Even when sample size is small, say $n=200$, the C test delivers high power if there are strong changes in the unconditional variance, i.e., $c=0.5$.

In a recent paper, Engle and Rangel (2004) develop an GARCH model with time-varying unconditional variance, but did not offer a statistical method that can be used to distinguish an GARCH process with constant unconditional variance ($c=0$) from another one with time-varying unconditional variance ($c \neq 0$). The results presented in this section indicate, however, that as long as $|\alpha|<1$, the C test can be used to distinguish the two aforementioned processes. We believe that this possibility may be helpful for applied researches that are built on conditional heteroskedasticity models.

In sum, our Monte Carlo results seem to suggest that: (i) Under the null hypothesis, the proposed test has similar empirical size to other tests such as KPSS; (ii) In the presence of unit root or a structural change in the mean, the proposed test is as powerful as the existing stationarity tests; (iii) In the presence of a changing unconditional variance, the traditional tests perform badly whereas the proposed test deliver high power comparing to the other stationarity tests and; (iv) the conclusions (i), (ii) and (iii) are relatively robust against the presence of GARCH innovations.

In the next section, we illustrate the applicability of the proposed C test using real-life data.

Table 2: Power and Size of Tests

n = 200					n = 400					n = 800				
Painel 1														
α = 0.5 and c = 0														
β+γ	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**
0.0	0.028	0.029	0.047	0.057	**0.0**	0.036	0.036	0.053	0.054	**0.0**	0.039	0.039	0.040	0.044
0.5	0.035	0.028	0.046	0.054	**0.5**	0.044	0.036	0.053	0.054	**0.5**	0.047	0.042	0.055	0.049
0.90	0.065	0.031	0.046	0.050	**0.90**	0.072	0.036	0.053	0.052	**0.90**	0.070	0.040	0.055	0.049
0.95	0.104	0.031	0.046	0.048	**0.95**	0.110	0.037	0.053	0.048	**0.95**	0.109	0.041	0.054	0.049
Painel 2														
α = 0.5 and c = 0.005														
β+γ	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**
0.0	0.114	0.032	0.047	0.054	**0.0**	0.500	0.037	0.051	0.058	**0.0**	0.986	0.052	0.054	0.052
0.5	0.128	0.032	0.047	0.054	**0.5**	0.464	0.039	0.051	0.058	**0.5**	0.975	0.051	0.054	0.053
0.90	0.172	0.033	0.046	0.050	**0.90**	0.460	0.037	0.051	0.057	**0.90**	0.932	0.052	0.055	0.054
0.95	0.231	0.035	0.046	0.048	**0.95**	0.489	0.038	0.051	0.055	**0.95**	0.918	0.054	0.055	0.055
Painel 3														
α = 0.5 and c = 0.5														
β+γ	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**
0.0	0.595	0.048	0.047	0.058	**0.0**	0.951	0.060	0.050	0.058	**0.0**	1.000	0.064	0.057	0.058
0.5	0.565	0.048	0.048	0.057	**0.5**	0.923	0.059	0.049	0.059	**0.5**	1.000	0.063	0.057	0.059
0.90	0.550	0.046	0.046	0.056	**0.90**	0.901	0.058	0.052	0.057	**0.90**	0.999	0.066	0.058	0.059
0.95	0.563	0.048	0.047	0.056	**0.95**	0.895	0.058	0.051	0.056	**0.95**	0.999	0.066	0.057	0.059
Painel 4														
α = 1 and c = 0														
β+γ	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**	**β+γ**	**C**	**KS**	**V/S**	**KPSS**
0.0	0.780	0.745	0.849	0.804	**0.0**	0.895	0.868	0.936	0.904	**0.0**	0.966	0.956	0.986	0.964
0.5	0.784	0.748	0.848	0.802	**0.5**	0.895	0.861	0.938	0.897	**0.5**	0.964	0.952	0.983	0.959
0.90	0.794	0.751	0.851	0.807	**0.90**	0.896	0.861	0.940	0.898	**0.90**	0.966	0.952	0.984	0.959
0.95	0.792	0.762	0.855	0.809	**0.95**	0.898	0.863	0.942	0.902	**0.95**	0.965	0.953	0.985	0.960

Table 3: Power and Size of Tests

n = 200					n = 400					n = 800				
Painel 1														
$\alpha = 0.5$ and c = 0														
$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS
0.0	0.016	0.017	0.026	0.048	**0.0**	0.032	0.029	0.038	0.048	**0.0**	0.036	0.034	0.045	0.044
0.5	0.031	0.017	0.026	0.047	**0.5**	0.040	0.029	0.038	0.049	**0.5**	0.042	0.034	0.044	0.045
0.90	0.059	0.016	0.024	0.041	**0.90**	0.060	0.028	0.037	0.046	**0.90**	0.060	0.033	0.045	0.045
0.95	0.087	0.020	0.024	0.037	**0.95**	0.086	0.029	0.038	0.043	**0.95**	0.088	0.033	0.046	0.044
Painel 2														
$\alpha = 0.5$ and c = 0.005														
$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS
0.0	0.095	0.020	0.026	0.047	**0.0**	0.421	0.031	0.039	0.049	**0.0**	0.967	0.043	0.046	0.049
0.5	0.095	0.020	0.026	0.047	**0.5**	0.400	0.031	0.039	0.049	**0.5**	0.940	0.042	0.046	0.049
0.90	0.148	0.020	0.024	0.041	**0.90**	0.378	0.030	0.037	0.048	**0.90**	0.870	0.044	0.045	0.048
0.95	0.192	0.020	0.024	0.040	**0.95**	0.397	0.031	0.036	0.046	**0.95**	0.830	0.045	0.045	0.048
Painel 3														
$\alpha = 0.5$ and c = 0.5														
$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS
0.0	0.467	0.032	0.025	0.047	**0.0**	0.851	0.046	0.039	0.051	**0.0**	1.000	0.055	0.047	0.052
0.5	0.453	0.031	0.024	0.048	**0.5**	0.812	0.046	0.037	0.050	**0.5**	0.998	0.055	0.048	0.052
0.90	0.431	0.031	0.023	0.047	**0.90**	0.760	0.046	0.038	0.050	**0.90**	0.990	0.055	0.047	0.053
0.95	0.440	0.034	0.023	0.047	**0.95**	0.745	0.044	0.038	0.049	**0.95**	0.975	0.055	0.048	0.054
Painel 4														
$\alpha = 1$ and $c = 0$														
$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS	$\beta+\gamma$	C	KS	V/S	KPSS
0.0	0.623	0.578	0.703	0.712	**0.0**	0.797	0.764	0.877	0.812	**0.0**	0.910	0.887	0.951	0.921
0.5	0.630	0.577	0.701	0.715	**0.5**	0.799	0.763	0.875	0.817	**0.5**	0.911	0.887	0.953	0.920
0.90	0.632	0.586	0.708	0.718	**0.90**	0.800	0.766	0.877	0.822	**0.90**	0.915	0.888	0.952	0.919
0.95	0.640	0.598	0.714	0.722	**0.95**	0.805	0.775	0.879	0.822	**0.95**	0.912	0.892	0.951	0.919

5 An Application to Financial Data

The assumption of stationarity is frequently employed in much applied work because its statistical convenience. Thus, the usage of such estimators and models cannot be justified if the hypothesis of constancy in the unconditional variance is violated. In this section, we investigate the validity of the hypothesis of covariance stationarity in financial time series. We consider the data $y_t = \log(E_t/E_{t-1})$ where E_t is the daily US Dollar/Euro exchange rate from 01/04/1999 to 12/31/2003, which gives 1004 observations. Note that y_t is the return series. Figure 1 shows realizations of y_t across time. One can easily note that the process $\{y_t\}$ seems to exhibit mean reversion, suggesting that it does not have a unit root. However, the absence of unit root is not a sufficient condition for stationarity. Because stationarity implies that unconditional variance of the data is constant over time, Pagan and Schwert (1990b) investigated the

likelihood of such constancy by using the recursive estimates of the variance of the series against time, as originally proposed by Mandelbrot (1963). In other words, if $\widehat{u}_t$ is the difference between y_t and its mean, then

$$\mu(t) = t^{-1} \sum_{k=1}^{t} \widehat{u}_k^2$$

is the recursive estimate of the unconditional variance at time t. Figure 2 displays the plot of $\mu(t)$ against time. There are three distinct phases. In the first, ending around the 200th observation, the unconditional variance estimate is quite erratic. After that, the estimate seems to increase continuously until the 530th observation. As pointed out by Loretan and Phillips (1996), this continuous change in the unconditional volatility may be explained by the temporal evolution of microstructural factors like the speed at which information reaches traders and their ability to interpret new information. Finally, the third phase, starting at 531st observation and ending at the last observation, seems to be very stable with the estimate of the unconditional variance being almost constant along this period. In sum, if we consider the time series y_t as a whole, then we may suspect that the unconditional variance is changing over time, but we also suspect that there are sub-periods within which the unconditional variance is constant and, therefore, nonparametric estimators and volatility models that depend on the assumption of covariance stationarity can be correctly employed using the observations from that sub-period. For this reason, the test proposed in this paper may be helpful: it can correctly identify sub-periods in which the unconditional variance is statistically constant.

Table 4 exhibits the results of our stationarity analysis. The test statistics were computed using the demeaned time series $\widehat{u}_t$. The notation $KPSS_{q_i}$, V/S_{q_i}, KS_{q_i} and C_{q_i}, $i = 1, 2$ is used to indicate that each test statistic is computed using the bandwidth parameters $q_1$and q_2. We considered observations from the entire sample and observations from a subsample (which starts at the 531st observation and ends at the last sample observation). This subsample corresponds to the third phase displayed in Figure 2 in which the unconditional variance apparently to be constant. When we look at the results based on the entire sample, Table 4 clearly shows the non-rejection of the null hypothesis by the KPSS, V/S and KS tests, meaning that the process y_t does not contain a unit root. However, as discussed previously, even if y_t does not have a unit root,

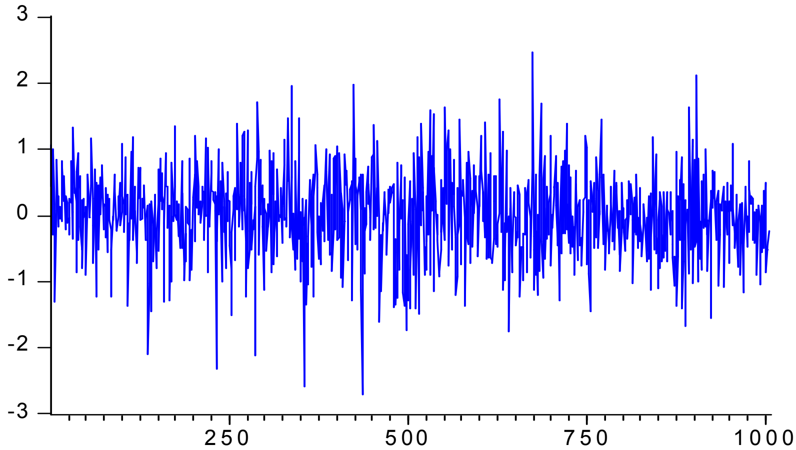

Figure 1: US Dollar/Euro Exchange Rate Return

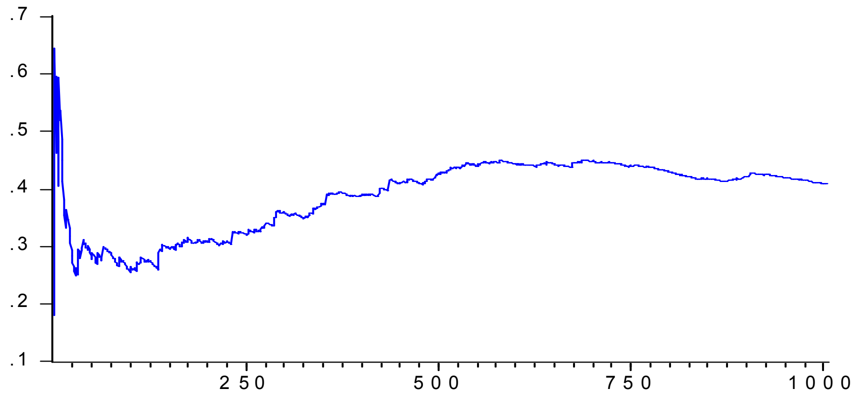

Figure 2: Recursive Estimate of the Unconditional Variance

it is not necessarily stationary since changes in the unconditional variance can be masked in the data. The monte-carlo results displayed in section 4 show that the existing stationarity tests are unable to reveal changes in the unconditional variance. When the C test is applied to the entire sample, the result indicates that we cannot accept the null hypothesis at 5% level of significance, suggesting that the process y_t is not stationary. In sum, when we look at the entire sample we can conclude that the US Dollar/Euro exchange rate return does not have a unit root but it does have a changing variance, so the description of y_t in the entire sample cannot be carried out by estimators and statistical models that assumes stationarity.

Given that stationarity fails over the entire sample, is there some interval in which one can apply models that assume unconditional homoskedasticity? We tried to answer this question using a strategy that combines the recursive estimate of the unconditional variance with the statistical test proposed in this paper. In other words, we first look at Figure 2 and identify an interval in which the unconditional variance is apparently constant and, second, we apply the new test to observations within that interval. Pagan and Schwert (1990b) employed similar strategy to identify intervals in which their nonparametric kernel estimator of conditional volatility could be applied without violating the assumption of covariance stationarity. The results are showed in Table 4. Differently from what was found using the entire sample, the null hypothesis of stationarity cannot be rejected by all tests in the subsample[12], even at 10% level of significance. Thus, econometric models that assume constancy in unconditional variance would have descriptive accuracy and validity if they were estimated using observations within this specific subsample.

[12] Again, the sub-sample starts at the 531st observation and ends at the last sample observation.

Table 4: Analysis of Covariance Stationarity

Test	Entire Sample	Subsample
$KPSS_{q_1}$	0.42	0.40
$KPSS_{q_2}$	0.41	0.39
V/S_{q_1}	0.11	0.07
V/S_{q_2}	0.10	0.09
KS_{q_1}	1.21	1.05
KS_{q_2}	1.20	1.11
C_{q_1}	2.26**	1.76
C_{q_2}	2.13**	1.74

6 Conclusion

This paper develops a test for the null of covariance stationarity against alternatives of unit roots, structural changes in the mean, as well as alternatives with time-varying unconditional variance. The proposed test complements conventional residual-based procedures in testing covariance stationarity. In an empirical application, we test whether the return on US Dollar/Euro exchange rate is covariance stationary or not. Our results suggest the absence of unit root in this time series but we were unable to accept the null hypothesis of covariance stationarity due to instability in the unconditional variance. This empirical finding confirms earlier work by Pagan and Schwert (1990a) and Loretan and Phillips (1996) and cast doubt on the validity of estimators and econometric models that assume constancy in the unconditional variance.

7 Appendix: A sketch of Proof for Theorem 1

If we detrend the time series y_t by OLS, then

$$\begin{aligned}\widehat{\gamma} &= \left[\sum_{t=1}^{n} x_t x_t'\right]^{-1}\left[\sum_{t=1}^{n} x_t y_t\right] \\ &= \gamma + \left[\sum_{t=1}^{n} x_t x_t'\right]^{-1}\left[\sum_{t=1}^{n} x_t u_t\right].\end{aligned}$$

Thus, under assumption A_1 (or A_2), B, and C, or assumptions M and C,

$$\begin{aligned}\sqrt{n}D^{-1}\left(\widehat{\gamma}-\gamma\right) &= \left[\frac{1}{n}\sum_{t=1}^{n} Dx_t x_t' D\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{t=1}^{n} Dx_t u_t\right] \\ &\Rightarrow \left[\int_0^1 X(r)X(r)'dr\right]^{-1}\int_0^1 X(s)dB_1(s).\end{aligned}$$

(see, Hamilton (1994 for a discussion on the leading case of a linear trend, and Phillips (2005) for more discussion on general cases.) By definition $\widehat{u}_t = y_t - \widehat{\gamma}' x_t$, $\widehat{v}_t = \widehat{u}_t^2 - \widehat{\sigma}_u^2$, with

$$\widehat{\sigma}_u^2 = \frac{1}{n}\sum_{j=1}^{n}\widehat{u}_j^2.$$

Thus we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{v}_t &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\left[v_t + (\widehat{u}_t^2 - u_t^2) - (\widehat{\sigma}_u^2 - \sigma_u^2)\right] \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} v_t + \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}(\widehat{u}_t^2 - u_t^2) - \frac{[nr]}{\sqrt{n}}(\widehat{\sigma}_u^2 - \sigma_u^2) \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} v_t + \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}([u_t - (\widehat{\gamma}-\gamma)'x_t]^2 - u_t^2) - \frac{[nr]}{\sqrt{n}}(\widehat{\sigma}_u^2 - \sigma_u^2) \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} v_t - \frac{[nr]}{\sqrt{n}}(\widehat{\sigma}_u^2 - \sigma_u^2) \\ &\quad + \frac{1}{\sqrt{n}}((\widehat{\gamma}-\gamma)'D^{-1}\sqrt{n}\left[\sum_{t=1}^{[nr]}\frac{Dx_t}{\sqrt{n}}\frac{x_t{}'D}{\sqrt{n}}\right]\sqrt{n}D^{-1}(\widehat{\gamma}-\gamma) \\ &\quad - \frac{2}{\sqrt{n}}\left[(\widehat{\gamma}-\gamma)'D^{-1}\sqrt{n}\sum_{t=1}^{[nr]}\frac{Dx_t u_t}{\sqrt{n}}\right] \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]} v_t - \frac{[nr]}{\sqrt{n}}(\widehat{\sigma}_u^2 - \sigma_u^2) + O_p(n^{-1/2}),\end{aligned}$$

thus

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{z}_t &= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\begin{bmatrix}\widehat{u}_t\\ \widehat{v}_t\end{bmatrix}\\
&= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\begin{bmatrix}u_t-(\widehat{\gamma}-\gamma)'x_t\\ \widehat{u}_t{}^2-\widehat{\sigma}_u^2\end{bmatrix}\\
&= \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\begin{bmatrix}u_t-(\widehat{\gamma}-\gamma)'x_t\\ v_t-(\widehat{\sigma}_u^2-\sigma_u^2)\end{bmatrix}+o_p(1)\\
&= \begin{bmatrix}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}u_t-(\widehat{\gamma}-\gamma)'\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}x_t\\ \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}v_t-r\sqrt{n}(\widehat{\sigma}_u^2-\sigma_u^2)\end{bmatrix}+o_p(1)\\
&= \begin{bmatrix}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}u_t-(\widehat{\gamma}-\gamma)'D^{-1}\sqrt{n}\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}Dx_t/\sqrt{n}\\ \frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}v_t-r\sqrt{n}(\widehat{\sigma}_u^2-\sigma_u^2)\end{bmatrix}+o_p(1).
\end{aligned}
$$

Therefore, under assumptions A_1(or, assumptions A_2), B, C and D, as $n\to\infty$,

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=1}^{[nr]}\widehat{z}_t &\Rightarrow \begin{bmatrix}B_1(r)-\left[\int_0^1 dB_1(s)X(s)'\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1}\int_0^r X(s)ds\\ B_2(r)-rB_2(1)\end{bmatrix}\\
&= \Omega^{1/2}\begin{bmatrix}W_1(r)-\left[\int_0^1 dW_1(s)X(s)'\right]\left[\int_0^1 X(s)X(s)'ds\right]^{-1}\int_0^r X(s)ds\\ W_2(r)-rW_2(1)\end{bmatrix}.
\end{aligned}
$$

Thus, in addition with Assumption D, we have

$$
\widehat{C}_n = \max_{1\le k\le n}\left\|\widehat{\Omega}^{-1/2}\frac{1}{\sqrt{n}}\sum_{t=1}^{k}\widehat{z}_t\right\| \Rightarrow \sup_r\left\|\widetilde{W}(r)\right\|.
$$

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