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NASH EQUILIBRIUM UNDER KNIGHTIAN UNCERTAINTY: BREAKING DOWN BACKWARD INDUCTION

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ABSTRACT: We define Nash equilibrium for two-person normal form games in the presence of *uncertainty*, in the sense of Knight(1921). We use the formalization of uncertainty due to Schmeidler and Gilboa. We show that there exist Nash equilibria for any degree of uncertainty, as measured by the uncertainty aversion (Dow and Werlang(1992a)). We show by example that prudent behaviour (maxmin) can be obtained as an outcome even when it is not rationalizable in the usual sense. Next, we break down backward industion in the twice repeated prisoner's dilemma. We link these results with those on cooperation in the finitely repeated prisoner's dilemma obtained by Kreps-Milgrom-Roberts-Wilson(1982), and with the literature on epistemological conditions underlying Nash equilibrium. The knowledge notion implicit in this model of equilibrium does not display logical omniscience.

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1. INTRODUCTION

Among the well-documented phenomena that economic models do not fully explain is the fact that players of a finitely repeated game do not backward induct. The careful experiments performed by Neelin, Sonnenschein and Spiegel (1988) and McKelvey and Palfrey (1992), and the well known prisoner's dilemma tournament of Axelrod (1980) show that subjects do not act according to the logic of backward induction. Several attempts to model this behaviour exist in the literature. Some of them are based on bounded rationality (Radner (1980), Chou and Geanakoplos (1985), Aumann (1981), Neyman (1985), Meggido and Widgerson (1986), and the references therein). Another type of explanation requires evolutionary behaviour with a lower bound on the number of "mutants," as in Nachbar (1981). Undeniably, however, the most successful model to justify cooperation in the finitely repeated prisoners' dilemma (which is, in fact, stronger than violation of backward induction) is due to Kreps, Milgrom, Roberts and Wilson (1982). Essentially their argument is that there is always a small chance that one of the players will not act rationally. In the game they consider, there is a small chance that one of the players will cooperate. This will lead to significant levels of cooperation on the part of both players. Fudenberg and Maskin (1986) and Fudenberg and Levine (1989) explore the size of the set of equilibria obtained by means of the Kreps-Milgrom-Roberts-Wilson solution.

In this paper, we propose to extend the notion of Nash equilibrium to incorporate agents who act in a game as if they faced uncertainty in the sense of Knight (1921). We show that this equilibrium exists for any given degree of uncertainty aversion on the part of each player (as defined in Dow and Werlang (1992a)).

Our definition leads to the possibility of cooperation in the repeated prisoner's dilemma. We explain the relationship between this explanation and the Kreps-Milgrom-Roberts-Wilson explanation. At the same time, given that the phenomenon has been modelled by departing from the Bayesian view of Savage (1954), it is also true that the bounded *Bayesian* rationality justifications are compatible with our explanation (the evolutionary model is of a completely unrelated type, more suitable for analysis of economic institutions).

Our definition also demonstrates the possibility of prudent behaviour (maxmin) even when this is "inconsistent" with the knowledge of the fact that the other player is Bayesian rational, as in the example in Werlang (1986, chapter 3). Finally, the definition of equilibrium allows for the possibility that the same game may be played differently by the same player when facing different opponents (even if there is a unique rationalizable outcome).

The decision model on which our analysis is based is, like other non-expected-utility models, at a fairly early stage of development. The axiomatic foundations of the representation of the preference ordering (by a utility function and a non-additive probability) are well understood. Applications to economic settings are not so well developed, and a number of outstanding issues remain, for example the question of dynamic consistency (see Machina (1989), Hammond (1989), and Epstein and LeBreton (1992) or, for discussion in the context of games, Dekel, Safra and Siegel (1990)). Nevertheless, if these decision-theoretic models are worth taking seriously for economists, they must be applied to games and other economic situations. This paper is no more than a first step in the direction of a game-theoretic application of behaviour under uncertainty (Klibanoff (1992) also discusses similar issues). However, this step seems worth taking.

The fact that this model formalizes the concept of Knightian uncertainty gives it a particular intuitive appeal which differentiates it from other non-expected-utility models. On the other hand, it also makes the definition of Nash equilibrium more difficult since we cannot assume that mixed strategies are described by objective probability distributions, which then imply a best response function. That approach would be possible for other non-expected-utility models such as rank-dependent utility (see Weber and Camerer (1987) and Epstein (1990) for surveys and descriptions of these models). Here, we must use the subjective approach to mixed strategy Nash equilibrium, which imposes a consistency condition between a player's beliefs about his opponent's actions and the opponent's best response: all actions in the support of the opponent's belief must be best responses. Furthermore, with Knightian uncertainty the "support of an agent's belief" is not as simple to define as when we are dealing with the standard objective probabilities (Crawford (1990) defines and discusses equilibrium for games where agents have other types of non-expected-utility preferences, for which beliefs can be represented with standard probabilities).

Nash equilibrium implicitly depends on a notion of knowledge: each player knows the other will play a best response. The standard definition of Nash equilibrium, like other models of knowledge used in Economics, incorporates the property of *logical omniscience*: if an agent knows a fact, he immediately knows all the consequences of that fact. The literature on Philosophy and on Computer Science has recognized that knowledge models with logical ominscience fail to capture some essential espects of human knowledge, and has developed models without logical omniscience. We show that the knowledge notion implicit in our definition of Nash equilibrium under uncertainty does not display logical omniscience.

The paper is organized as follows. The next section discusses the basic uncertainty model, which is due to Schmeidler (1982, 1989) and Gilboa (1987). Section 3 gives the definition of Nash equilibrium under uncertainty and the theorem on the existence of Nash equilibrium for any given pair of values for the uncertainty aversion of the two players. Section 4 presents a sequence of examples. In Example 1, the definition is used in an example to obtain prudent behaviour which is incompatible with common knowledge of Bayesian rationality (common knowledge of Bayesian rationality is equivalent to rationalizability, in the sense of Bernheim (1984) and Pearce (1984), as shown in Tan and Werlang (1988)). Example 2 shows the distinction between our approach, and the standard (Bayes-Savage) approach with ϵ-trembles. Example 3 is more striking, showing the emergence of cooperation in a twice repeated prisoner's dilemma, under the our definition of Nash equilibrium under uncertainty. We use this example to relate the equilibrium notion with the Kreps-Milgrom-Roberts-Wilson model. Section 5 relates the knowledge notion implicit in the definition of Nash equilibrium under uncertainty to the one implicit in the standard (Bayes-Savage) definition. Section 6 concludes and points to directions for further research.

2. UNCERTAINTY

Schmeidler (1982, 1989) and Gilboa (1987) have developed an axiomatic model of rational decision making in which agents' behaviour distinguishes between situations where agents know the probability distributions of random variables and situations where they do not have this information. We refer to the former as *risk* and the latter as *uncertainty*, or Knightian

uncertainty. Synonyms that are used in the literature include *roulette lottery*, for risk, and *horse lottery* and *ambiguity*, for uncertainty. The standard model of uncertainty used in economics is that of Savage (1954), which reduces all problems of uncertainty to risk under a subjective probability. The Schmeidler-Gilboa axiomatization leads to different behaviour: behaviour under uncertainty is inherently different from behaviour under risk.

We now give a brief exposition of the main aspects of their model. The reader is referred to the papers by Schmeidler and Gilboa cited above for a complete description and for the underlying axioms, and to Dow and Werlang (1992a) which contains an example and an application to portfolio choice (it also includes a mathematical appendix with the basic material on non-additive probabilities). Dow and Werlang (1992b) has an application to stock price volatility, and Simonsen and Werlang (1991) also describe the implications for portfolio choice. Also, Wakker (1989) has a model which is very similar to Gilboa (1987).

Bewley (1986) presents a similar model which is also designed to capture Knightian uncertainty. His model predicts that uncertainty leads to inertia, a tendency to favour the *status quo*, while in Schmeidler-Gilboa there is a tendency to choose acts where the agent does not end up bearing uncertainty. In a decision problem, this will lead to different predictions unless the *status quo* is an act where the agent bears no uncertainty. In Game Theory, it is conventional not to distinguish any particular strategy as the *status quo*.

The Schmeidler-Gilboa model predicts that agents' behaviour will be represented by a utility function and a (subjective) non-additive probability distribution (also known as a *capacity*). A *non-additive probability P* reflecting aversion to uncertainty satisfies the condition

$$P(A) + P(B) \leq P(A \cap B) + P(A \cup B), \qquad (*)$$

rather than the stronger condition satisfied by (additive) probabilities:

$$P(A) + P(B) = P(A \cap B) + P(A \cup B).$$

In particular, $P(A) + P(A^c)$ may be less than 1; the difference can be thought of as a

measure of the uncertainty aversion attached by the agent to the event A. The *uncertainty aversion of P at event A* is $c(P,A) = 1 - P(A) - P(A^c)$ (Dow and Werlang (1991a)). We will say that P reflects *strict uncertainty aversion* if $c(P, A) > 0$ for all events A (except of course where A is the empty set or the set of all states).

All the non-additive probabilities considered in this paper will reflect uncertainty aversion, ie they will satisfy inequality (*). Also, we will restrict attention to the case of a finite set of states of the world.

The agent maximizes *expected utility under a non-additive distribution* where the expectation of a non-negative random variable X is defined as:

$$E(X) = \int_{\mathbf{R}^+} P(X \geq x)\, dx.$$

Associated with a non-additive probability P is a set of additive probabilities called the *core* of P, which is defined (analogously to the core in cooperative game theory) as the set of additive probability measures π such that $\pi(A) \geq P(A)$ for all events A. If the non-additive probability reflects aversion to uncertainty (inequality (*)), the core is non-empty.

A closely related model of behaviour under uncertainty (equivalent in many simple cases) is a *maxmin formulation*: the agent acts to maximize the minimum value, over the elements of the core, of expected utility (Gilboa and Schmeidler, 1989). Note that this is not the same as maxmin over outcomes (prudent behaviour), except in the special case where the core contains all the distributions giving probability one to a state.

EXAMPLE: Suppose the agent has to choose between lottery tickets whose payoffs depend on whether a blue or a red ball is drawn from an urn containing balls of the two colours, in unknown proportions. He may choose between having (i) a "red" lottery ticket paying £10 if a red ball is drawn, (ii) a "blue" ticket paying £10 if a blue ball is drawn, and (iii) a certain payoff of £4. His utility function is linear (risk-neutral): $U(w) = w$. His beliefs are represented by the following non-additive distribution:

$$p_{RED} = 0.45$$
$$p_{BLUE} = 0.45.$$

His expected utility from holding (i) the red ticket is given by the formula

$$E(U(w)) = \int_{\mathbf{R}^+} P(w \geq x)\, dx = 0 + (0.45)10 = 4.5,$$

since $P(w \geq x) = 1$ if $x \leq 0$, $P(w \geq x) = 0.45$ if $0 < x \leq 10$ and $P(w \geq x) = 0$ if $x > 10$.

Similarly, choice (ii), the blue ticket, is worth 4.5 and choice (iii), the safe payoff, is worth 4. So the agent is indifferent between holding either ticket, and prefers holding one of them to a safe payoff of £4.

In this example, the maxmin formulation using the core of the distribution is equivalent. The core of this distribution is the set of all (additive) probability distributions with chances of red of between 45% and 55%. So, using the max-min model we evaluate the payoff from choice (i), the red ticket, as:

$$\min \{10p + 0\,(1-p) \mid 0.45 \leq p \leq 0.55\} = 4.5,$$

as before (and similarly for (ii) and (iii)).

The support of a non-additive probability P may be defined analogously to the additive case. One might start by supposing the appropriate analogy to be smallest event A such that P(A) = 1 (the initial version of this paper, Dow and Werlang (1991), explored this notion in greater detail). However, notice that if P reflects strict uncertainty aversion then the entire set of all states is the only event with probability one. Intuitively, the interpretation of an event of non-additive probability zero is the same as in the additive case: it is an event which will almost never happen. With a non-additive probability, however, if a set has probability zero, that does not mean its complement is of probability one. But if the complement of a zero-probability set has positive probability (in principle it could be zero too) it has relatively infinitely more chance of happening than the original set. Hence, we

are led to a definition of support based on the idea that the complement of the support has zero probability. The motivation for this definition is further discussed in section 3 below when we define Nash equilibrium under uncertainty.

DEFINITION: a *support* of a non-additive probability P is an event A such that $P(A^c) = 0$ and $p(B^c) > 0$ for all events $B \subset A$, $B \neq A$.

It should be clear that there may be several supports, and that a support is always contained in the smallest set with probability one.

EXAMPLE: There are three states with:

$$p_1 = p_2 = p_3 = p \in (0, 1/3)$$
$$p_{12} = p_{13} = p_{23} = q \in (2p, 1-p).$$

where p_i is the probability of state i and p_{ij} is the probability of state i or state j. This example has constant uncertainty aversion. The (unique) support is the set of all states. {1, 2, 3}.

EXAMPLE: Again there are three states.

$$p_1 = p_2 = p \in (0, \tfrac{1}{3})$$
$$p_3 = 0$$
$$p_{12} = q \in (2p, 1-p)$$
$$p_{13} = p_{23} = p.$$

The (unique) support is the event {1, 2}.

EXAMPLE: this example does not have a unique support. Again there are three states.

$$p_1 = 0.5$$
$$p_2 = p_3 = 0$$
$$p_{12} = p_{13} = 0.6$$
$$p_{23} = 0.1.$$

The supports are $\{1, 2\}$ and $\{1, 3\}$.

Note that in each of these three examples, the smallest set of probability 1 is the set of all states.

3. NASH EQUILIBRIUM UNDER UNCERTAINTY

We restrict attention to two-person finite normal form games $\Gamma = (A_1, A_2, u_1, u_2)$, where the A_i are pure strategy sets and the u_i are utilities (payoffs).

In the standard theory, a mixed strategy Nash equilibrium is defined as follows. Let (μ_1, μ_2) be a pair of (additive) probability measures and let $\text{supp}[\mu_i]$ denote the support of μ_i. In Nash equilibrium, every $a_1 \in \text{supp}[\mu_1]$ is a best response to μ_2, (ie a_1 maximizes the expected utility of player 1 given that player 2 is playing the mixed strategy μ_2); conversely, every $a_2 \in \text{supp}[\mu_2]$ is a best response to μ_1.

A subjective interpretation can be given to the Nash equilibrium: the mixed strategy of player 1, μ_1, may be viewed as the beliefs that player 2 has about the pure strategy play of player 1. Conversely, the mixed strategy of player 2, μ_2, may be viewed as the belief player 1 has about the pure strategy play of player 2. This subjective interpretation is suitable for the generalization we will introduce in this paper. (The alternative interpretation, which we do not consider here, is that players act as if they actually use random-number generators to implement mixed strategies. Models of Knightian uncertainty are not suitable for describing this objective interpretation.)

DEFINITION: NASH EQUILIBRIUM UNDER UNCERTAINTY. A pair (P_1, P_2) of non-additive probabilities P_1 over A_1 and P_2 over A_2 is a *Nash Equilibrium under Uncertainty* if there exist a support of P_1 and a support of P_2 such that:

(i) for all a_1 in the support of P_1, a_1 maximizes the expected utility of player 1, given that P_2 represents player 1's beliefs about the strategies of player 2;

and conversely;

(ii) for all a_2 in the support of P_2, a_2 maximizes the expected utility of player 2 given that P_1 represents player 2's beliefs about the strategies of player 1.

This definition reduces to the standard definition of Nash equilibrium whenever there is no uncertainty (ie when the P's are additive).

Clearly the definition rests upon our definition of support of a non-additive distribution. One could speculate what would be the implications of replacing the support as we have defined it by other sets. For example, the smallest set of probability 1 is "too large," and the equilibrium notion that would result would be too strong. The reason is that such an approach would ignore the intuitive strength of the non-additive model: that an event may be infinitely more likely than its complement, but still have probability less than 1. Take for example the case of a strategy set with two elements, a and b. Let the other player's beliefs about the strategies be $P(a) = 0.8$ and $P(b) = 0$. Intuitively, b has no chance of happening, but we cannot guarantee that a will happen. If we want to be "sure" that an event will happen, this event can only be the whole strategy set $\{a, b\}$.

The reader may wish to compare the discussion above with the intuitive argument used by Schmeidler as the starting point for his derivation of the decision theory of non-additive probabilities: if an agent has symmetric information about a number of mutually exclusive and exhaustive possible events, they should be assigned equal "probabilities," but the "probabilities" need not sum to one (Schmeidler (1989), pp 571 and 572).

It might be enquired why the above definition is presented intuitively rather than derived axiomatically. The reason is that the standard Nash equilibrium itself has only the weakest axiomatic foundations (Tan and Werlang (1988), Bernheim (1987)). Axiomatic

considerations lead instead to the concept of rationalizability (Bernheim (1984), Pearce (1984), Tan and Werlang (1988)). Nevertheless, Nash equilibrium rather than rationalizability is the standard solution concept in game-theoretic applications, presumably because of its intuitive appeal. Thus we cannot hope for a strong axiomatic foundation for the Nash equilibrium under uncertainty.

Clearly, a standard mixed strategy Nash equilibrium is also a Nash equilibrium under uncertainty, so existence of at least one Nash equilibrium under uncertainty is no problem. But, as we discuss in section 5 below, it is desirable to be able to view an agent's uncertainty aversion as a parameter in the description of the game. The theorem that follows allows us to do this.

Note that in the statement of the theorem, the reason for the interchange of the subscripts in the notation is that P_1 represents player 2's beliefs about what player 1 will do, so that the uncertainty aversion of P_1 is a characteristic of player 2, and vice versa. Incidentally, note also that while we compute equilibrium where the beliefs are of the form $P(A) = (1 - c)\ Q(A)$ for additive Q, there are beliefs with constant uncertainty aversion which are not of this form.

THEOREM: *let* $\Gamma = (A_1, A_2, u_1, u_2)$ *be a two-person finite game. For all* $(c_1, c_2) \in [0, 1] \times [0, 1]$, *there exists a Nash equilibrium under uncertainty* (P_1, P_2), *such that* c_1 *is the uncertainty aversion of* P_2, *and* c_2 *is the uncertainty aversion of* P_1.

PROOF: By Dow and Werlang (1992a) if $P(A) = (1 - c)\ Q(A)$ for some additive probability Q and for all events A (other than the entire set of states of the world), then P exhibits constant uncertainty aversion c. In other words, P is a "uniform squeeze" of Q. In this case, one has:

$$E_P(X) = c \min X + (1 - c)\ E_Q(X). \qquad (**)$$

(Note that in case $c = 1$ this reduces to maxmin behavior.) We modify the original game Γ to $\Gamma_{(c1,\, c2)} = (A_1, A_2, u_1, u_2)$, where

$$v_i(a_i, a_j) = c_i \min_{a \in A_j} u_i(a_i, a) + (1 - c_i)\, u_i(a_i, a_j) \text{ for } i = 1, 2 \text{ and } j \neq i. \quad (***)$$

Let (Q_1, Q_2) be a standard mixed strategy Nash equilibrium of the modified game $\Gamma_{(c1, c2)}$. We will show that the pair (P_1, P_2) where

$$P_1(A) = (1 - c_2)Q_1(A) \text{ for any event } A \neq A_1, \text{ and}$$

$$P_2(A) = (1 - c_1)Q_2(A) \text{ for any event } A \neq A_2,$$

(and naturally $P_1(A_1) = P_2(A_2) = 1$), is a Nash equilibrium under uncertainty for the original game Γ, with the specified levels of uncertainty aversion. It is immediate to verify that the uncertainty aversion of P_2 is c_1 and that of P_1 is c_2.

To verify that this is a Nash equilibrium under uncertainty, note that (except in case $c_i = 1$) the support of P_i is unique, and coincides with the support of Q_i. Since (Q_1, Q_2) is a standard mixed strategy Nash equilibrium for $\Gamma_{(c1, c2)}$, it follows that any $a_i \in \text{supp}[Q_i]$ is a best response to Q_j (for the modified utility v_i). In other words, a_i maximizes the following expression over $a \in A_i$:

$$E_{Qj}[v_i(a, \cdot)] = \int_{Aj} v_i(a, a_j)\, dQ_j(a_j)$$

$$= c_i \int_{Aj} [\min_{a^* \in Aj} u_i(a, a^*)]\, dQ_j(a_j) + (1-c_i) \int_{Aj} u_i(a, a_j)\, dQ_j(a_j) \qquad \text{by } (***)$$

$$= c_i \min_{a^* \in Aj} u_i(a, b_j)] + (1-c_i)\, E_{Qj}[u_i(a, \cdot)]$$

$$= E_{Pj}[u_i(a, \cdot)]. \qquad \text{by } (**).$$

Thus a_i is also a best response in the original game Γ. There remains the possibility that $c_i = 1$. In this case, any singleton $\{a_j\}$ is a support of P_j. Therefore from the viewpoint of the maxmin player i, any choice for player j is a best response.

Thus (P_1, P_2) is a Nash equilibrium under uncertainty for Γ. QED

4. Examples

Example 1: Non Rationalizable Maxmin Behaviour May Occur

The game shown in Figure 1 below has a unique rationalizable equilibrium (which, therefore, coincides with the (standard) Nash equilibrium), given by (u, a). Further, if player 1 knows that player 2 is rational (observe that this requires just one level of elimination of strictly dominated strategies) then she knows that player 2 can never play b, because it is a strictly dominated strategy. Thus she should play u. Let us imagine, however, that 10 stands for 10 million dollars, or, equivalently, that the parameter ϵ is very small. This game is very similar to one in Werlang (1986, chapter 3), who asked the following question: would you play u in this game? Note that strategy d gives a payoff very similar to the payoff obtained in the Nash equilibrium, without any "risk" that the other player does not play his part. Our definition of Nash equilibrium under uncertainty leads to the "prudent" decision d even for low uncertainty aversion of player 1.

Let P_1 be described by probability p_u, player 2's belief player 1 will play u, and probability p_d, his belief 1 will play d, with $p_u + p_d \leq 1$. Similarly, P_2 is described by q_a and q_b, with $q_a + q_b \leq 1$. Notice that from the point of view of player 2, the beliefs P_1 are irrelevant: player 2 always chooses a. Will that mean that player 1 will necessarily play u? The answer is no. We will look for Nash equilibria under uncertainty of the form $q_b = 0$ and $1 > q_a > 0$. Since {a} is a support (in this simple case the only one) we only have to check that a is a best response, a fact that we already know. The strategy b, since it lies outside the support, need not be a best response, again a fact that we already know. What we would like to obtain is the set of parameters which will yield d as the equilibrium action of player 1. Computing the expected values shows that she will play d if $q_a \leq 1 - \epsilon/20$. Notice that this is exactly consistent with the intuition that just a little bit ($\epsilon/20$) of uncertainty aversion is enough to make player 1 turn from the only rationalizable action u to the prudent, and intuitively sensible action d. Notice also that the smaller ϵ is, the less uncertainty averse player 1 has to be in order to play d. Finally, note that we could also justify this action by the argument of Kreps-Milgrom-Roberts-Wilson: if there were a small chance δ of player 2 being crazy, player 1 would behave cautiously. The following example shows, however,

that the notion of Nash equilibrium under uncertainty does not coincide with the δ-craziness solution. The relationship to Kreps-Milgrom-Roberts-Wilson is discussed further after example 3 below.

EXAMPLE 2: NASH UNDER UNCERTAINTY IS NOT EQUIVALENT TO δ-CRAZINESS

Consider the modification of example 1 shown in figure 2 below. Now b is a strictly dominant strategy for player 2, and we would expect player to play d. Suppose we impose $\delta > 0$ probability that player 2 will play a. It is easy to see that for $0 < \delta < 1 - \epsilon/20$, the standard Nash equilibrium yields b as the unique best response for player 1. On the other hand if $\delta > 1 - \epsilon/20$, her unique best response is u in the standard Nash equilibrium. The "δ-craziness" approach of changing the game description by postulating that with δ probability the other player will actually play the other strategy (perhaps because of having a "crazy" different utility function) admits u as a best response in some cases. We can compare this behaviour with the prediction of our theory. In Nash equilibrium under uncertainty, it is immediate to check that playing u is never a best response in any equilibrium, regardless of the level of uncertainty aversion. The reason for this difference in behaviour is that the δ-craziness solution here imposes a strategy choice for player 2 (with exogenous probability δ) that turns out in this example to be "optimistic" from the point of view of player 1. On the other hand, in Nash equilibrium under uncertainty preferences are always pessimistic: they give more weight to undesirable outcomes. Thus the criteria are quite distinct even in the simplest games. The difference is discussed further following example 3 below.

EXAMPLE 3: BREAKING DOWN BACKWARD INDUCTION

We now show that cooperation may arise in the twice repeated prisoner's dilemma, thereby demonstrating that (Knightian) rational agents may not backward induct. Consider the version of the prisoner's dilemma of Kreps, Milgrom, Roberts and Wilson (1982), as shown in Figure 3 below, where $a = 1.25$ and $b = -0.5$, so that $a + b < 2$ as they require. The strategies are F (for "fink") and C (for "cooperate").

In the twice repeated version of this game, there are eight strategies for each of the players. Four are the unconditional, or history independent strategies: F^2, FC, CF, C^2, which stand for, respectively, F in both rounds, F in the first round and then C in the second round, C and then F, and C in the both rounds. There are four history dependent strategies, which we name W, X, Y and Z, for convenience. W is: start with F. If the other player played C in the first round, then play C in the second round. Otherwise play F in the second round. The letter X stands for: start with F, and if the other player played C, then play F, otherwise play C. The letter Y stands for the tit-for-tat: start with C, and play C in the second round if the other player played C in the first round. Otherwise, play F. Finally, Z stands for: start with C, and play F in the second round if C was also played by the other player. Otherwise, play C. To summarize:

Strategy	First Round	Second Round if other player's First Round move was: C	F
F^2	F	F	F
FC	F	C	C
CF	C	F	F
C^2	C	C	C
W	F	C	F
X	F	F	C
Y	C	C	F
Z	C	F	C

Consider the game without discounting (the payoffs are sums of payoffs of each one-shot game). It may be verified that the following is a non-additive probability P which reflects uncertainty aversion (inequality (*) in section 2 above):

(1) $P(\{F^2, CF, C^2, W\}) = 1$

(2) $P(\{F^2, CF, W\}) = 0.8$

(3) $P(\{C^2, CF, W\}) = P(\{F^2, CF, C^2\}) = 0.4$

(4) $P(\{F^2, CF\}) = P(\{CF, W\}) = P(\{CF, C^2\}) = 0.4$

(5) $P(W) = P(C^2) = P(F^2) = P(\{F^2, C^2\}) = P(\{C^2, W\}) = P(\{F^2, C^2, W\}) = 0$

(6) $P(CF) = 0.4$.

(7) For all events B, $P(B) = P(B \cap \{F^2, CF, C^2, W\})$

It is easy to see that the only support is {CF}. There is a Nash equilibrium under uncertainty in which both players have these beliefs. The expected payoffs for each of the players, given the belief P about the other player's actions is, for each strategy: $U(F^2)=0.5$, $U(FC)=0.2$, $U(CF)=0.6$, $U(C^2)=0.2$, $U(W)=0.3$, $U(X)=0.2$, $U(Y)=0.3$ and $U(Z)=0.4$. Thus, the equilibrium above has CF as the prediction, with the joint probabilities of all other strategies together being zero. This means that cooperation in the first round may occur, again with (Knightian) rationality.

COMMENT: RELATION TO KREPS-MILGROM-ROBERTS-WILSON

Kreps, Milgrom, Roberts and Wilson (1982) obtain cooperation in the finitely repeated prisoner's dilemma using an intuitively appealing argument, of the type we have described above as the δ-craziness approach. They consider a small "risk" that one of the players will always play tit-for-tat. They show that this generates cooperation in several stages of the repeated game. The irrational agent (tit-for-tat) that they "added" to the game is added exogenously, but clearly the modellers' choice of that specific type of irrationality was not exogenous. It was motivated intuitively in a way which was endogenous to the particular game under analysis.

A similar analysis is given in Kreps and Wilson (1982) and Milgrom and Roberts (1982) in their model of the chain-store paradox. They suggest that potential entrants may be afraid of encountering a "maniac" who positively enjoys playing an apparently irrational strategy, in this case, starting a ruinous price war in the chain-store entry game.

In example 3, we showed that uncertainty can lead to cooperation. There we did not use the tit-for-tat strategy to achieve cooperation, but our point was just to show that cooperation could arise. In our model the potentially "irrational" behaviour of the other agent is generated by the fact that, in the presence of uncertainty, the players tend to give more weight to the potential losses from taking an action. Thus in our model, the players endogenously decide which sort of agent they are "afraid" of meeting. In other words, we have a theory that explains how the "irrationality" appears in the model.

However, there are two important differences between the Kreps-Milgrom-Roberts-Wilson approach and the approach described in this paper. The first difference was described in example 2 - where we showed that adding a "benevolent" maniac might induce agents to take less cautious decisions, whereas Nash equilibrium under uncertainty would not.

The second difference is that, in Nash equilibrium under uncertainty, the type of opponent players are "afraid" of meeting may be different at different strategies. In the Kreps-Milgrom-Roberts-Wilson approach, the type of opponent who has been added to the model is constant when evaluating the payoff of different strategies. In Nash under uncertainty, agents systematically evaluate payoffs relative to the worst strategy of their opponent - which will change depending on their own strategy.

5. Remarks on Logical Omniscience

The models of knowledge common in the Economics literature (see for example Aumann (1976), Bacharach (1985), Brandenburger and Dekel (1985, 1987) Geanakoplos and Polemarchakis (1982), Milgrom and Stokey (1982) Rubinstein and Wolinsky (1990), Samet (1990), Tan and Werlang (1986, 1988, 1992) and Werlang (1989)) all represent behavior of logically omniscient agents. Logical omniscience means that if an agent knows a fact, and knows that this fact implies another fact, then the agent knows that other fact. This seemingly innocuous property has powerful consequences. For example, a logically omniscient agent who knows the basic rules of propositional logic and the Peano axioms must know all mathematical results proven, and ever to be proven.

Lack of logical omniscience may seem unfamiliar or even odd to economists, although the fact that human knowledge is not logically omniscient is well accepted among philosophers. Most readers of this paper know the rules of logic together with Peano's axioms, without knowing all possible mathematical results: clearly, logical omniscience is a strong property that fails to capture some essential aspects of human knowledge. For references to the extensive discussion of logical omniscience in the philosophy and artificial intelligence literature, see Fagin, Halpern and Vardi (1990).

The definition of Nash equilibrium implicitly presupposes a notion of knowledge: it assumes implicitly that an event is known when it contains a support. In other words, each player knows the opponent will play a best response. When probabilities are additive, this knowledge notion is very similar to the existing models in the Economics literature (see Brandenburger and Dekel (1985)). However, some interesting and less usual properties arise from our definition of Nash equilibrium under uncertainty. In this case there are two sources of the lack of logical omniscience in the knowledge notion.

This first source was noted in a more general context by Lipman (1992b). The essential idea is that when an agent learns a fact, this has the of simultaneously learning the fact together with another a priori unknown state of the world. Lipman (1992b) refers to these states as "impossible possible worlds." Gilboa and Schmeidler (1992) give a related model: they show that in a decision problem under uncertainty, one can interpret non-additive probabilities as if they were additive, but defined over a suitably extended space of states of the world.

This may be illustrated by Example 1 of section 4. Since r is the rational action for player 2, and the (unique) support of P_2 is $\{r\}$, it follows by our definition that player 1 knows that "player 2 is rational." On the other hand, player 1 is rational and knows she is rational. Therefore, if the choice is between an action that would yield 10 and another action that would yield $10 - \epsilon$, she should choose the action that yields 10. Thus, player 1 knows that "if player 2 were to choose r, she should choose u." The lack of logical omniscience comes from the fact that player 1 chooses d in equilibrium. Hence player 1 cannot know she plays u, even though this is the local conclusion of the two facts that we argued player 1 does know. Here is an agent who knows a fact (that "player 2 chooses b"), and knows that this logically implies something else (that "if player 2 were to choose b, she - player 1 - should choose u), but does not know their implication (action u should be chosen).

The second source of lack of logical omniscience is the possible multiplicity of supports. Take the non-additive probability of the last example of section 2 above. Consider the events $A = \{1, 2\}$ and $D = \{1, 3\}$. An agent whose beliefs are represented by p knows A and knows D, in the sense that both A and D are supports of p. However, the event $\{1\} = A \cap D$ does not contain any support of p, so that it is not known by the agent. The

implications of the non-closure of the knowledge operator with respect to intersections of events are immediate. It is quite possible that both the event A and the event $A^c \cup B$ (which is equivalent to "A implies B") are known, but B is not known. For example, take $A = \{1, 2\}$ and $B = \{1\}$. Both A and $A^c \cup B$ are known, but B is not. This second source of lack of logical omniscience did not arise in Example 1 of section 4 above, since the support of player 1's belief is unique.

We have shown that the knowledge notion implicit in our definition of Nash equilibrium under uncertainty does not have local omniscience. In contrast to the majority of existing knowledge models without logical omniscience (see Fagin, Halpern and Vardi (1990) for references), our notion is operational: if an agent's behavior is described by a non-additive belief, a fact is known if it contains a support of the distribution. Thus, one reason why Knightian uncertainty can be regarded as an interesting behavioral model is that it implicitly, and without additional modelling effort, does away with logical omniscience.

6. Concluding remarks

The definition of Nash equilibrium under uncertainty provided here explains a number of economic phenomena which, to date, have not been modelled in a satisfactory way. For example, it is possible to provide a rationale, along the lines of the above examples, for the experimental results of Neelin, Sonnenschein and Spiegel (1988) and McKelvey and Palfrey (1992). Neelin, Sonnenschein and Spiegel (1988) ran bargaining experiments with iterated offers and complete information (similar to the models of Stahl (1972) and Rubinstein (1982)). They found that players behaved as if they applied only two rounds of backward induction, even when the game was repeated up to six rounds. McKelvey and Palfrey (1992) document similar violations of backward induction. The definition of Nash equilibrium under uncertainty provided here may readily be used to construct examples where backward induction breaks down.

An area of future research concerns rationalizability, rather then Nash equilibrium, under uncertainty. We would define the rationalizable outcomes as the set of strategies which are

compatible with common knowledge (using our notion of knowledge) of Knightian rationality. There then remains to explore the relation between this rationalizability concept and Nash equilibrium under uncertainty.

There is also the extension to n players.

Finally, we end by noting that researchers have often referred to a "missing parameter" in the description of the game. Such a parameter would allow a player to play the same game differently when facing different opponents (even when the opponents have the same payoffs). We suggest that uncertainty aversion may serve as such a missing parameter.

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Player 2

		a	b
Player 1	u	$10,\ 10$	$-10,\ 10-\alpha$
	d	$10-\varepsilon,\ 10$	$10-\varepsilon,\ 10-\alpha$

$\alpha > 0$

Figure 1

Player 2

		a	b
Player 1	u	$10, 10-\alpha$	$-10, 10$
	d	$10-\varepsilon, 10-\alpha$	$10-\varepsilon, 10$

$\alpha > 0$

Figure 2

		Col	
		F	C
Row	F	0, 0	a, b
	C	b, a	1, 1

Figure 3

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