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Full surplus extraction with dominant strategies[1]

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Abstract

In this note I consider the full surplus extraction in an auction with private but possibly correlated values. I show that full extraction in the continuum of types case is not possible in general. Neither is approximate full surplus extraction if the seller is budget constrained.

1 Introduction

Consider an indivisible good to be sold at auction to n potential bidders. Each bidder i is characterized by his type $t_i \in T_i$. Define $T = \Pi_{i=1}^{n} T_i$ and $F(t_1, \ldots, t_n)$ a distribution function on T. Similarly to Crémer and McLean (1988), I refer to (T, F) as an information structure. The value of the good for bidder i of type $x \in T_i$ is given by $u^i(x), u^i : T_i \to \mathbb{R}_+$. Crémer and McLean show that for almost all discrete information structure (T, F), full surplus extraction is possible for every evaluation function $u = (u^1, \ldots, u^n)$. Consider now the continuum of types case. Define also

$$H_i = \left\{ g : T_{-i} \to \mathbb{R}; g \in L^1(F_i(\cdot|t_i)) \text{ for almost every } t_i \in T_i \right\}.$$

In this case, Theorem 1A in Crémer and McLean guarantees full surplus extraction by dominant strategy auction for every integrable evaluation function u if and only if

Condition 1 *The range of the functional* $G : H_i \to L^1(F_i)$

$$G(g)(t_i) := \int_{T_{-i}} g(t_{-i})\, dF_i(t_{-i}|t_i), t_i \in T_i$$

contains[1] $L^1_+(F_i)$.

It turns out that Condition 1 cannot be satisfied in general and thus full surplus extraction with a continuum of types is not guaranteed. In this context, a natural question to ask is how typical is the absence of full surplus extraction if the set of possible evaluation functions is considered. Perhaps full surplus extraction could be typical although not true only for some pathological evaluation function. In this note, however, I show that full surplus extraction is atypical for nice evaluation functions.

2 The model.

The set of types is $T = [0, 1]^n$ and n is the number of bidders. The distribution of types is denoted by $F : T \to [0, 1]$ and has a density function f. If bidder i is of type $t_i \in [0, 1]$ we form the vector of types $t = (t_1, \ldots, t_n) \in T$.

[1] And thus the functional is onto $L^1(F_i)$.

The set $\Pi_{j\neq i}T_j$ is denoted T_{-i}. And t_{-i} denotes $(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$. The distribution density of t_i is $f_i(t_i) = \int f(t)\,dt_{-i}$ and the conditional distribution density of t_{-i} given t_i is denoted $f(t_{-i}|t_i) = \frac{f(t)}{f_i(t_i)}$. Bidder i's valuation of the object is $u^i(t_i)$, where $u^i : T_i \to \mathbb{R}$ is strictly increasing and $u^i(0) = 0$. The probability of bidder i receiving the object is denoted by $p^i(t)$. The auctioneer may choose not to sell the good. Thus $\sum_{i=1}^n p^i(t) \leq 1$. If the vector of types is t the seller receives[2] $x^i(t) \in \mathbb{R}$ from bidder i. The vector $(p^i(\cdot), x^i(\cdot))_{i=1}^n$ is called a direct mechanism. Using Myerson's revelation principle I consider only mechanisms that are individually rational and incentive compatible. Define $V^i(t) = p^i(t)\,u^i(t_i) - x^i(t)$ and assume that $E[|x^i|] < \infty$. The steps that follow are similar to those in Myerson(1981). The incentive compatibility constraint (in dominant strategies) is given by:

$$V^i(t) \geq p^i(s,t_{-i})\,u^i(t_i) - x^i(s,t_{-i}),\ \forall s \in [0,1], t \in T. \tag{IC'}$$

It can be rewritten in the more convenient form

$$V^i(t) - V^i(s,t_{-i}) \geq p^i(s,t_{-i})\left(u^i(t_i) - u^i(s)\right), \forall s, t_i \in T_i. \tag{IC}$$

Replacing s with t_i we obtain $V^i(s,t_{-i}) - V^i(t) \geq p^i(t)\,(u^i(s) - u^i(t_i))$ adding both inequalities we obtain

$$\left(u^i(t_i) - u^i(s)\right)\left(p^i(t) - p^i(s,t_{-i})\right) \geq 0.$$

Thus $s \to p^i(s,t_{-i})$ is non-decreasing if (IC) is satisfied. In a manner similar to Myerson(1981) we can show that (p,x) satisfy the incentive compatibility constraints if and only if:

1. $p^i(t)$ is non-decreasing in t_i;
2. $x^i(t) = p^i(t)\,u^i(t) - x^i(0,t_{-i}) - \int_0^{t_i} p^i(x,t_{-i})\,(u^i)'(x)\,dx$ for every $t \in T$.

The voluntary participation constraints are

$$E\left[V^i(t)\,|t_i\right] \geq 0, \forall t_i \in T_i. \tag{VP}$$

Thus we have

$$E\left[x^i(0,t_{-i}) + \int_0^{t_i} p^i(x,t_{-i})\left(u^i\right)'(x)\,dx \,\middle|\, t_i\right] \geq 0. \tag{1}$$

[2]The seller pays bidder i if $x^i(t) < 0$.

The auctioneer's expected revenue is given by

$$E\left[\sum_{i=1}^{n} x^{i}(t)\right] = E\left[\sum_{i=1}^{n} p^{i}(t)\, u^{i}(t_i)\right] - \sum_{i=1}^{n} E\left[x^{i}(0, t_{-i}) + \int_0^{t_i} p^{i}(x, t_{-i}) \left(u^{i}\right)'(x)\, dx\right] \leq \tag{2}$$

$$E\left[\max_{1\leq i\leq n} u^{i}(t_i)\right].$$

The surplus is by definition $\max_i u^i(t_i)$. The inequality (1) and the equality (2) shows immediately[3] that to extract all surplus it is necessary and sufficient that:

Lemma 1

1. *For almost all* $t_i \in T_i$, $E\left[x^i(0, t_{-i})\,|t_i\right] = -E\left[\int_0^{t_i} p^i(x, t_{-i}) \left(u^i\right)'(x)\, dx|t_i\right]$
2. *The good is delivered amongst the Bidders* l *such that* $u^l(t_l) = \max_i u^i(t_i)$. *Formally* $p^i(t) = 0$ *if* $u^i(t) < \max_l u^l(t_l)$. *And* $\sum_{l\in L} p^l(t) = 1$, *where* $L = \left\{l; u^l(t) = \max_i u^i(t_i)\right\}$.

I now show that Condition 1 is never satisfied if f is continuous.

Proposition 1 *Suppose the density function* f *is continuous. Then Condition 1 is not true.*

Proof. For every $g \in H_i$ the function $G(g)(t_i) := \int_{T_{-i}} g(t_{-i})\, f(t_{-i}|t_i)\, dt_{-i}$ is continuous. Thus G cannot be onto $L_+^1(F_i)$. QED

This proposition suggests that we should examine full surplus extraction in the class of continuous valuations. The following theorem shows that even this will not be enough.

Theorem 1 *There is a dense set of density functions such that there is no full surplus extraction in dominant strategies.*

[3]This is the same as lemma 1A in Crémer and McLean.

Proof. Consider a continuous density function of the form $f(t) = \sum_{l=1}^{L} a_l(t_i) b_l(t_{-i})$.

Then

$$E\left[x^i(0, t_{-i})|t_i\right] = \int x^i(0, t_{-i}) \frac{f(t)}{f_i(t_i)} dt_{-i} = \sum_{l=1}^{L} \left(\int x^i(0, t_{-i}) b_l(t_{-i}) dt_{-i} \right) \frac{a_l(t_i)}{f_i(t_i)}.$$

Thus the set

$$\left\{E\left[x^i(0, t_{-i})|\cdot\right]; x^i \in L^1(F_{-i})\right\} = \left\{\sum_{l=1}^{L} \lambda_l \frac{a_l(\cdot)}{f_i(\cdot)}; \lambda \in \mathbb{R}^L\right\}.$$

The right hand side of (1) in Lemma 1 is omitting the minus signal:

$$E\left[\int_0^{t_i} p^i(x, t_{-i}) \left(u^i\right)'(x)\, dx \middle| t_i\right] = E\left[\left(u^i(t_i) - \max_{l\neq i} u^l(t_l)\right)^+ \middle| t_i\right] =$$
$$\int \left(u^i(t_i) - \max_{l\neq i} u^l(t_l)\right)^+ \frac{f(t)}{f_i(t_i)} dt_{-i} =$$
$$\sum_{l=1}^{L} \left(\int \left(u^i(t_i) - \max_{l\neq i} u^l(t_l)\right)^+ b_l(t_{-i})\, dt_{-i} \right) \frac{a_l(t_i)}{f_i(t_i)}.$$

Since in general the function,

$$x \rightarrow \left(\int \left(u^i(x) - \max_{l\neq i} u^l(t_l)\right)^+ b_l(t_{-i})\, dt_{-i} \right) \frac{a_l(x)}{f_i(x)},$$

isl not a linear combination of $\left\{\frac{a_l(\cdot)}{f_i(\cdot)}; 1 \leq l \leq L\right\}$, the proof is finished.

QED

Remark 1 *Condition $f(t) = \sum_{l=1}^{L} a_l(t_i) b_l(t_{-i})$ guarantees in the common-value setting the full extraction of the surplus. See McAfee, McMillan and Reny (1989).*

Remark 2 *Although I did not investigate whether approximate surplus extraction is possible, if the auctioneer is budget constrained (i.e., $x^i(t)$ is bounded) then the results of Page(1998) guarantee the existence of the optimal auction which would extract all surplus were approximate surplus extraction possible.*

The next theorem complements the last remark.

Theorem 2 *Suppose the auctioneer never disburse any money (i.e.,* $\sum_{i=1}^{n} x^i(t) \geq 0$*). Then the optimal auction yields expected revenue at most equal to*

$$E\left[\max_{1\leq i\leq n}\left\{u^i(t_i) - \frac{1-F_i(t_i|t_{-i})}{f(t_i|t_{-i})}\left(u^i\right)'(t_i)\right\}\right].$$

The proof is omitted. I finish this note with an example:

Example 1 *Suppose* $n = 2$, $f(t,s) = \frac{3(t^2+s^2)}{2}$ *and* $u^i(t_i) = t_i, i = 1, 2$. *Then*

$$\begin{aligned}\int x^1(0,s) f(t,s)\, ds &= \int x^1(0,s)\frac{3(s^2+t^2)}{2}ds = \\ &\int \frac{3s^2 x^1(0,s)}{2}ds + t^2\int\frac{x^1(0,s)}{2}ds.\end{aligned}$$

On the other hand

$$\int (t-s)^+ f(t,s)\, ds = \int_0^t (t-s)\frac{3(t^2+s^2)}{2}ds = \frac{7}{8}t^4 \notin \left\{\lambda\cdot\left(1,t^2\right); \lambda\in\mathbb{R}^2\right\}.$$

3 References

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