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COMMON KNOWLEDGE AND GAME THEORY

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1. INTRODUCTION

Until the mid seventies there was very little disagreement concerning the appropriateness of Nash equilibria as a solution concept for noncooperative games. To a large extent it was believed that for an arbitrary game all Nash equilibria were reasonable outcomes and that any non-Nash set of actions were unreasonable. Today the situation is very different. A literature has developed that deals with refinements of Nash and different game theorists believe in different refinements. Also, there are those that hold that non-Nash actions may be perfectly reasonable. In order to help to clarify the issues involved in arguing for refinement or relaxation of the Nash equilibrium concept one is lead to an investigation of the logical basis for solution concepts in general. We start with the idea that a solution concept should be based on assumptions regarding Bayesian rationality, what is known, what is common knowledge, and behavioural norms.

The aim of this essay is to provide foundations for different noncooperative solution concepts. To illustrate the point that the knowledge and common knowledge of certain characteristics of the players plays a central rôle in the choice of the solution concept, let us imagine that you are going to play a given bimatrix game with two alternative players. The payoffs of the game are in dollar terms. The first player is an intelligent acquaintance of yours, whom you know very well. The second player is a

stranger. She comes from the Himalaya, and the only relevant information you know about her is that she was taught the meaning of a bimatrix game (the rules of the game) and what a dollar can buy. For the sake of argument, let us say that : (i) there is a unique pure strategy Nash equilibrium, which gives you a thousand dollars; (ii) your security level is nine hundred dollars; and (iii) if the other player does not play his/her part of the Nash equilibrium you get at most five hundred dollars. How should you play against the two different opponents? It seems clear to me that everyone who is faced with this situation is much more likely to follow the Nash strategy when facing the acquaintance, than when facing the stranger. Thus a well defined game may be played in different ways by the same person. This fact indicates that, in the specification of a game, some additional information about the background of the players is essential for the solution of this game. By explicitly modelling knowledge and common knowledge of different attributes, for example, of Bayesian rationality, one obtains which solution concept is suitable for each situation considered.

In order to get started one has to understand formally the notion of common knowledge. Suppose that there are n players. One says that a statement is common knowledge if everyone knows it, everyone knows everyone knows it, ..., everyone knows (everyone knows)$^{m-1}$ it, and so on, for all m. Two ways of formalising the notion of common knowledge have been considered. One is given by Aumann(1976), and the other one is based on the idea of an infinite hierarchy of beliefs (Armbruster and Böge(1979), Böge and Eisele(1979), Mertens and Zamir(1985))[1]. Aumann defines the notion of an event being common knowledge; however, the definition requires that we

understand what it means for the structure of the uncertainty in a game to be common knowledge. In other words, a serious shortcoming of Aumann's formalisation is that it is self-referential. Section two begins by observing that the second definition of common knowledge overcomes this difficulty; that is, the second definition of common knowledge is not self-referential. When one assumes that the structure of uncertainty is common knowledge in the sense of that definition, an event is common knowledge in the sense of Aumann if and only if it is common knowledge in the sense of the second definition. Thus, Aumann's definition can be embedded in the more general framework.

Section three begins with a discussion of games and solution concepts. Bernheim(1984) and Pearce(1984) show that common knowledge of rationality is not enough to justify Nash behaviour. They introduce a noncooperative solution concept which is derived from the hypothesis that Bayesian rationality is common knowledge. They call their solution concept rationalisable strategic behaviour. The point I wish to emphasise is that Bernheim and Pearce derive their solution concept for games from assumptions about the behaviour of the players. One can quite generally approach the analysis of solution concepts in the same manner. Which are the implicit behavioural assumptions behind a given solution concept? From a Bayesian point of view, the decision of each player in a game is determined by this player's beliefs about the actions of other players. But, if, in their turn, other players' beliefs about other players' actions affect their own actions, then it must be that the beliefs one player has about the beliefs of other players also affect the decision of this player in the game. If

we carry this argument further, we see that the action taken by a player is determined by his infinite hierarchy of beliefs about actions of other players. The space of these infinite hierarchies of beliefs is the appropriate space for the study of behavioural assumptions about the players. Section three deals with this matter in detail, and poses formally the relationship between solution concepts and behavioural assumptions implicit in them.

As an illustration of this formalism, section four, taken from Tan and Werlang(1984), discusses the solution concept given by rationalisable strategic behaviour[2]. The main behavioural assumption to be considered is that of common knowledge of Bayesian rationality. Bernheim(1984) and Pearce(1984) argue that common knowledge of Bayesian rationality implies the choice of a rationalisable action. They also argue the converse: any rationalisable action can be chosen by players for whom Bayesian rationality is common knowledge. Although the proof of this result corresponds directly to the intuition when the action spaces of the players is finite, some subtle measurability issues arise for infinite action spaces.

The fifth section deals with Nash equilibrium behaviour. It starts by formally stating a justification for Nash behaviour which is closely related to the classical one. This allows one to see how strongly coordinated the players have to be. Not only rationality should be taken as common knowledge, but also the actions to be chosen. This allows one to see how strongly coordinated the players have to be. When one relaxes this hypothesis slightly, everything breaks down. Another behavioural assumption is studied: that each player "knows" the other players. When the game has

two players, Armbruster and Böge(1979) proved that this yields Nash equilibrium beliefs. We give an example that this is not enough for Nash equilibrium beliefs in the case of three (and consequently more than two) players. Then, we provide theorems that generalise the results which justify Nash behaviour. Finally, we point out how our analysis can be modified so that we can obtain Aumann's result on the foundations for correlated equilibrium (Aumann(1985)).

2. COMMON KNOWLEDGE

2.1 Aumann's Definition of Common Knowledge

To begin one must understand formally the notion of common knowledge. Suppose that there are n players. One says that a statement is common knowledge if everyone knows it, everyone knows everyone knows it, ..., everyone knows (everyone knows)$^{m-1}$ it, and so on, for all m. Two ways of formalising the notion of common knowledge have been considered. One is given by Aumann(1976), and the other one is based on the idea of an infinite hierarchy of beliefs (Armbruster and Böge(1979), Böge and Eisele(1979), Mertens and Zamir(1985)). Aumann defines the notion of an event being common knowledge; however, the definition requires that we understand what it means for the structure of the uncertainty in a game to be common knowledge. In other words, a serious shortcoming of Aumann's formalisation is that it is self-referential. We begin by presenting Aumann's model. Then we provide the reader with the mathematical background necessary for dealing with infinite hierarchies of beliefs. We give the second definition of common knowledge, which is based on the formalism of the infinite hierarchies of beliefs. We proceed to show that the second definition of common knowledge overcomes the self-referentiality of Aumann's definition : when one assumes that the structure of uncertainty is common knowledge in the sense of the second definition, an event is common knowledge in the sense of

Aumann if and only if it is common knowledge in the sense of the second definition.

Aumann(1976) has a model of knowledge very similar to Kripke(1963). First, we describe the basic ingredients of the model[3]:

(i) Ω : a set of states of nature. Kripke called it the set of possible worlds;

(ii) Σ : a σ-algebra of the set Ω. The elements of Σ are called events. When there is a topological structure one considers it to be the Borel σ-algebra. When Ω is finite one takes it to mean $\wp(\Omega)$, the power set of Ω;

(iii) $(\Pi_i)_{i\in N}$: an n-tuple of sub-σ-algebras of Σ, where $n=\#N$, and N is the set of agents. Each Π_i represents the information structure of agent i : whenever $\omega \in \Omega$ is the real state of nature, or the real world, agent i is told all the sets of Π_i which contain ω.

<u>2.1.1 Example</u> : Let $\Omega=\{1,2,3,4,5\}$, $\Sigma=\wp(\Omega)$, $\Pi_1=\sigma(\{\{1\},\{2,3\},\{4\},\{5\}\})$, $\Pi_2=\sigma(\{\{1,2\},\{3,4\},\{5\}\})$. The symbol $\sigma(X)$ means the smallest σ-algebra which contains X. When the set Ω is <u>finite</u> every σ-algebra can be generated by partitions of Ω, by taking unions of sets in the partition and adding the null set. Sometimes one takes Π_i by the underlying partitions. Thus suppose the real state of nature is $\omega=2$.

The smallest event agent 1 is told is {2,3}. Agent 2, similarly, is told that the event {1,2} occurred.

Let $P_i[X]$ be the "smallest" set of the σ-algebra Π_i which contains $X \in \Sigma$. If Ω is finite, $P_i[X]$ is <u>always</u> well defined. If Ω is infinite, and endowed with a topology which makes it compact and metric (for most applications one has to require it to be complete too), and Σ is the Borel σ-algebra generated by this topology, one defines $P_i[X]$ as the intersection of all sets which are elements of Π_i and contain X. In general this set may lie out of Π_i . To avoid this,we further require the Π_i's to be countably generated (see Williams(1979, II-68)). Then $P_i[X]$ is also well defined .

The fact about this structure which will interest us the most is : once $\omega \in \Omega$ has occurred, each agent knows $P_i[\{\omega\}]$ has occurred. Aumann's definition of common knowledge hinges on this interpretation.

<u>2.1.2 Definition</u> : Let $\Pi = \Pi_1 \cap \Pi_2 \cap ... \cap \Pi_n$ (Π is the intersection of the sub-σ-algebras Π_i , also called the "meet"). Suppose the true state of nature is $\omega \in \Omega$. An event $A \in \Sigma$ is said to be <u>common knowledge at ω</u> if $\exists B \in \Pi$, $\omega \in B$ and $A \supset B$.

2.1.3 Example : In the previous example (2.1.1) let us see whether the sets $A_1 = \{1,2,3\}$ and $A_2 = \{1,2,3,4\}$ are common knowledge at ω=2 (the true state of nature). By computing $\Pi = \Pi_1 \cap \Pi_2 = \sigma(\{\{1,2,3,4\},\{5\}\})$, and applying the definition above, it is easy to check that A_2 is common knowledge at ω=2, whereas A_1 is not. However, if ω=2 both agents know that the event A_1 occurred . In fact, given the real state of the world is ω=2, agent 1 observes {2,3}, and $A_1 \supset \{2,3\}$. Similarly, agent 2 observes {1,2}, and $A_1 \supset \{1,2\}$. Furthermore, agent 2 knows that agent 1 knows A_1. In fact, agent 2 observes {1,2}. Hence he knows the state of nature is either ω=1 or ω=2. As agent 2 knows that agent 1 conditions his priors on Π_1, agent 2 knows that agent 1 observed either {1} or {2,3}. But $A_1 \supset \{1\} \cup \{2,3\}$, so that it follows that agent 1 knows A_1. The same is not true of agent 1. Agent 1 cannot know for sure whether agent 2 knows A_1. This because agent 1 observes {2,3}. Thus he deduces the true state of nature is either ω=2 or ω=3. If it were ω=2, then knowing Π_2, agent 1 would know that {1,2} was seen by agent 2, and then he would be sure agent 2 knew A_1, because $A_1 \supset \{1,2\}$. However, if

the true state were 3, and knowing Π_2, agent 1 would know that {3,4} was observed by agent 2. Therefore agent 1, in this case, would not be sure whether or not agent 2 knew A_1={1,2,3}, since it is not true that {3,4} is contained in A_1.

In the example above it is possible to grasp the intuition behind 2.1.2 : in order to reason that agent 2 knows agent 1 knows A_1 it is necessary to assume that agent 2 knows agent 1 conditions his beliefs on Π_1. That is to say : it is necessary to assume that agent 2 knows Π_1. This is a general fact : Aumann's definition of common knowledge <u>implicitly assumes that the sub-σ-algebras $(\Pi_i)_{i\in N}$ are common knowledge themselves</u>. Aumann himself recognises the problem in the text of his 1976 article:

"Worthy of note is the implicit assumption that the information partitions P_1 and P_2 are themselves common knowledge." (Aumann(1976), p. 1237)

The self-referentiality of his definition makes it difficult to apply. In a situation where common knowledge has to be defined, one must exactly know the information sub-σ-algebras Π_i's which are common knowledge a priori. For the problems we are going to be dealing with, namely applications to game theory, it becomes difficult to specify the space Ω as well as the Π_i's. For example, suppose one wants to formalise the fact that Bayesian rationality is common knowledge. In this case the space Ω has

to be the space of "types" of players, and the event has to be the set of "types" of players which are Bayesian rational. Add to that the fact that the information sub-σ-algebras have to be chosen too. As one can see, this is not an easy task. In subsection 2.3 the second definition of common knowledge, which is based on the infinite hierarchies of beliefs, is provided. This definition simultaneously turns the concept of common knowledge into a more useful tool and solves the self-referentiality mentioned above. As an application of the framework which will be developed, section four shows how one can use this second definition of common knowledge to define common knowledge of Bayesian rationality. The next subsection contains the mathematical facts needed later in the text.

2.2 Mathematics of Infinite Recursion of Beliefs

This section is aimed at giving the basic results on infinite recursions of beliefs (also called hierarchies of beliefs) , the essential mathematical tool needed in the text. Let S be a compact metric space. From now on we will concentrate only on this case : all our spaces are compact and metric. This topology shall, furthermore, reflect the economic situation to be analysed. For example, Milgrom and Weber(1979) derive topologies which are relevant for games with incomplete information. Define the set of probability measures over S endowed with the Borel σ-algebra, as $\Delta(S)$. A natural topology over this set is the weak convergence of measures (see

Billingsley(1968) and Hildenbrand(1974)) . The main result is :

<u>2.2.1 Theorem</u> S compact and metric with the Borel σ-algebra. If $\Delta(S)$ denotes the set of probability measures on S , and is endowed with the topology of the weak convergence of measures, then $\Delta(S)$ is compact and metric.

This theorem follows from Billingsley (1968, pp. 238-240, Theorems 5 and 6).

The formal framework to be developed appeared before in Armbruster and Böge(1979), Böge and Eisele(1979), Mertens and Zamir(1985) and Myerson(1983).

Let the the set of possible states of nature, as perceived by agent i, be represented by a compact and metric set S_{0i} . In section 2, since we are interested in analysing common knowledge of events, we have $S_{0i} = \Omega$ for all i . In sections 3,4 and 5, since we are interested in games of complete information, $S_{0i} = A_{-i}$. In other problems the correct specification of these spaces is fundamental.

Given these sets of states of nature, agent i has subjective beliefs about the occurrence of a state in S_{0i} . This subjective belief is the first order belief, $s_{1i} \in \Delta(S_{0i})$. Set $S_{1i} = \Delta(S_{0i})$. The second order beliefs will be beliefs about beliefs of other agents. However, it is also possible to consider the possibility of these being correlated with agent i's beliefs about the states of nature he perceives.

Therefore $s_{2i} \in \Delta(S_{0i} \times \Pi_{k \neq i} S_{1k}) = S_{2i}$. Inductively, we define the m-th order beliefs of agent i as $s_{mi} \in \Delta(S_{0i} \times \Pi_{k \neq i} S_{m-1,k}) = S_{mi}$.

Notice that we could have modelled S_{mi} to include correlation among all the previous layers of beliefs. This is the approach followed by Mertens and Zamir(1985, p. 7, Th. 2.9), but given the consistency requirements they have (as we do below), their framework is equivalent to the one we use here.

Observe that an arbitrary m-th order belief contains information about all beliefs of order less than m. An obvious requirement that should hold is that the first order belief of agent i should be the marginal of his second order belief on his basic uncertainty space. Given a probability distribution $Q \in \Delta(C \times D)$, one defines the marginal of Q on C, and writes $\mathrm{marg}_C[Q]$, as the following probability defined over C : given any measurable X contained in C, $\mathrm{marg}_C[Q](X) = Q(X \times D)$. We will construct a way of determining the lower order beliefs, given a belief of a certain order. This is the approach of Myerson(1983,1984). We include Myerson's proof for completeness. Let us impose on an agent's beliefs the <u>minimal consistency requirement</u> : that if it is possible to evaluate the probability of an event through his m-th order beliefs and through his p-th order beliefs, with $m \neq p$, then both probability assessments agree. Define inductively the functions which will recover the (m-1)-th order beliefs, given the m-th order belief, by :

(i) for $m \geq 2$, $\Psi_{m-1,i} : S_{mi} \rightarrow S_{m-1,i}$;

(ii) if $m = 2$, $\Psi_{1i}(s_{2i})(E) = s_{2i}(E \times \Pi_{k \neq i} S_{1k})$ $\forall E$ contained in S_{0i}. As was said before, the first order beliefs are simply the marginal of the second;

(iii) if $m \geq 3$, by induction on m we assume $(\Psi_{m-2,k})_{k \in N}$ defined, and :

for all E contained in $S_{0i} \times \Pi_{k \neq i} S_{m-2,k}$, $\Psi_{m-1,i}(s_{mi})(E) =$

$= s_{mi}(\{(s_{0i}, (s_{m-1,k})_{k \neq i}) \in S_{0i} \times \Pi_{k \neq i} S_{m-1,k} \mid (s_{0i}, (\Psi_{m-2,k}(s_{m-1,k}))_{k \neq i}) \in E\})$.

We then have :

<u>2.2.2 Proposition</u> Suppose all agents are aware that each of them satisfy the minimum consistency requirement. Then $\forall i$, $\forall m \geq 2$: $\Psi_{m-1,i}(s_{mi}) = s_{m-1,i}$.

<u>Proof</u> The proof goes by induction. For $m = 2$, let E be contained in S_{0i}. Then the event E (event = measurable set) is evaluated by s_{1i} as $s_{1i}(E)$.

However E is the same as $E \times \Pi_{k \neq i} S_{1k}$ evaluated by s_{2i}. Therefore, by the consistency requirement : $s_{1i}(E) = s_{2i}(E \times \Pi_{k \neq i} S_{1k}) = \Psi_{1i}(s_{2i})(E)$. Thus $s_{1i} = \Psi_{1i}(s_{2i})$. Agents also know that $s_{1k} = \Psi_{1k}(s_{2k})$, because other agents are also consistent. The first step of the induction process is proved. Let us assume it is true for $m \geq 2$. We will prove it is true for $m+1$. If it is true for m, we know that

$\forall k \in N$ $(N = \{1, ..., n\})$: $\Psi_{m-1,k}(s_{mk}) = s_{m-1,k}$. Given the event E contained in $S_{0i} \times \Pi_{k \neq i} S_{m-1,k}$, define E^* contained in $S_{0i} \times \Pi_{k \neq i} S_{mk}$ by :

$$E^* = \{ (s_{0i}, (s_{mk})_{k \neq i}) \in S_{0i} \times \Pi_{k \neq i} S_{mk} \mid (s_{0i}, (\Psi_{m-1,k}(s_{mk}))_{k \neq i}) \in E \}.$$

By the induction hypothesis we have that

$$E^* = \{ (s_{0i}, (s_{mk})_{k \neq i}) \in S_{0i} \times \Pi_{k \neq i} S_{mk} \mid (s_{0i}, (s_{m-1,k})_{k \neq i}) \in E \}.$$

Therefore E^* and E are the same events (same in the sense used before : one is true if and only if the other is). Hence by the minimum consistency requirement $s_{mi}(E) = s_{m+1,i}(E^*)$. But $\Psi_{mi}(s_{m+1,i})(E) = s_{m+1,i}(E^*)$, so that the result follows.

<u>QED.</u>

Given the proposition above, we will restrict ourselves to beliefs which satisfy the minimum consistency requirement. The set of all possible beliefs becomes, then :

$$S_i = \{ (s_{1i}, s_{2i}, ...) \in \Pi_{m \geq 1} S_{mi} \mid \forall m : \Psi_{mk}(s_{m+1,k}) = s_{mk} \}.$$

The proposition below is proved in Armbruster and Böge(1979, p. 19, Th. 4.2), Böge and Eisele(1979, p. 196, Th. 1), and Mertens and Zamir(1985, p. 7, Th. 2.9).

<u>2.2.3 Proposition</u> S_i is compact and metric, in the topology induced by the product

topology.

The proposition above just says that the space of characteristics of the agents is tractable. The proof of the proposition is simple. One has only to show that the functions Ψ are continuous.

Notice that one can look at the space of beliefs which are consistent and are of level up to m. By 2.2.2 the lower order beliefs are entirely determined by the highest order one. Then this is the same as the space S_{mi}. Moreover, there is an immediate way of recovering the lowest order beliefs : just apply successively the functions Ψ. The most important result of this section states this for the case where we consider the whole stream of beliefs. The result is proved in Armbruster and Böge(1979, p. 19, Th. 4.2), Böge and Eisele(1979, p. 196, Th. 1), Mertens and Zamir(1985, p. 7, Th. 2.9) and Brandenburger and Dekel(1985b, p. 10, Th. 3.2). The first proof of this result seems to have appeared in Böge(1974).

<u>2.2.4 Theorem</u> $\forall i$ there exists $\Phi_i : S_i \rightarrow \Delta(S_{0i} \times \Pi_{j \neq i} S_j)$, which is a homeomorphism. Throughout the essay we are going to be referring to the homeomorphism shown in the proof below.

<u>Sketch of the proof</u> (essentially taken from Brandenburger and Dekel(1985b)) Let $Y_i = \Pi_{m \geq 1} S_{mi}$. Then $Y_i \supset S_i = \{ (s_{mi})_{m \geq 1} \mid \forall m \geq 1 : s_{mi} = \Psi_{mi} (s_{m+1,i}) \}$. Suppose

$s_i = (s_{1i}, s_{2i}, \dots) \in S_i$. We will construct $\Phi_i(s_i)$. $S_{0i} \times \Pi_{j \neq i} Y_j = S_{0i} \times \Pi_{j \neq i} (\Pi_{m \geq 1} S_{mj})$ $= S_{0i} \times \Pi_{m \geq 1} (\Pi_{j \neq i} S_{mj})$, by exchanging the order of the cartesian product. First we will exhibit $\Phi_i(s_i)$ as a probability measure in $S_{0i} \times \Pi_{j \neq i} Y_j$. Then we will prove that the support of $\Phi_i(s_i)$ is contained in $S_{0i} \times \Pi_{j \neq i} S_j$. To construct this measure one invokes Kolmogorov's extension theorem (see Dellacherie and Meyer(1978, p. 68, III.51-52)). To construct a probability in a countably infinite product of spaces, it is necessary and sufficient to give all the finite dimensional marginals, provided these marginals are not contradictory. Furthermore, this probability is uniquely determined by these marginals. Given k, let q_k denote a probability defined on $S_{0i} \times \Pi_{1 \leq m \leq k} (\Pi_{j \neq i} S_{mj})$. For k=0, let $q_0 = s_{1i}$. For $k+1 \geq 1$, we define $q_{k+1} \in \Delta(S_{0i} \times \Pi_{1 \leq m \leq k+1} (\Pi_{j \neq i} S_{mj}))$ in the sets which are of the form $E_{0i} \times E_{1,-i} \times \dots \times E_{k,-i} \times E_{k+1,-i}$, where E_{0i} is an event in S_{0i}, $E_{1,-i}$ is an event in $\Pi_{j \neq i} S_{1j}$, and in general, $E_{m,-i}$ is an event in $\Pi_{j \neq i} S_{mj}$. This is enough to define the probability q_{k+1}. The functions Ψ_{mi}, which are given above, will be used. $q_{k+1}(E_{0i} \times E_{1,-i} \times \dots \times E_{k,-i} \times E_{k+1,-i}) =$

$s_{k+2,i} (\{ (s_{0i}, (s_{k+1,j})_{j \neq i}) \in E_{0i} \times E_{k+1,-i} \mid (s_{0i}, (\Psi_{kj} (s_{k+1,j}))_{j \neq i}) \in E_{0i} \times E_{k,-i},$

$(s_{0i}, (\Psi_{k-1,j} \circ \Psi_{kj} (s_{k+1,j}))_{j \neq i}) \in E_{0i} \times E_{k-1,-i}$, and, successively,

$(s_{0i}, (\Psi_{1j} \circ \ldots \circ \Psi_{k-1,j} \circ \Psi_{kj}(s_{k+1,j}))_{j\neq i}) \in E_{0i} \times E_{1,-i}\})$. By the minimum consistency requirement q_k is the marginal of q_{k+1} on the first k coordinates. By Kolmogorov's theorem there exists a unique probability, which we call $\Phi_i(s_i)$, on the space $S_{0i} \times \Pi_{m\geq 1}(\Pi_{j\neq i} S_{mj})$ whose marginals are given by the q_k. We now have to show that the support of $\Phi_i(s_i)$ is in $S_{0i} \times \Pi_{j\neq i} S_j$. Let $(s_{kj}, s_{k+1,j})$ be in the support of $\Phi_i(s_i)$ in $S_{kj} \times S_{k+1,j}$. We will show that $s_{kj} = \Psi_{kj}(s_{k+1,j})$. By the construction of $\Phi_i(s_i)$, we have $(s_{kj}, s_{k+1,j})$ in the support of q_{k+1}. Define B_{km} to be the closed ball of radius $1/m$ around s_{kj}. Analogously, let $B_{k+1,m}$ be the closed ball of radius $1/m$ around $s_{k+1,j}$. As $(s_{kj}, s_{k+1,j})$ is in the support of q_{k+1}, this means that for every $m\geq 1$ the joint probability of $(s_{kj}, s_{k+1,j}) \in B_{km} \times B_{k+1,m}$ is greater than zero. Thus, by the definition of q_k, $\Psi_{kj}(B_{k+1,m}) \cap B_{km}$ is nonempty for all $m\geq 1$. These sets are also compact (because Ψ_{kj} is continuous), so that the intersection over all m is nonempty. But by the definition of B_{km} only one point could be in this infinite intersection : s_{kj}. In the same manner the infinite intersection of $\Psi_{kj}(B_{k+1,m})$ for all $m\geq 1$ can consist only of $\Psi_{kj}(s_{k+1,j})$. Thus, $s_{kj} = \Psi_{kj}(s_{k+1,j})$, and the support of

$\Phi_i(s_i)$ is in the space $S_{0i} \times \Pi_{j\neq i}S_j$. Therefore the function Φ_i can be viewed as $\Phi_i : S_i \rightarrow \Delta(S_{0i} \times \Pi_{j\neq i}S_j)$. To check that Φ_i is a bijection between these two spaces, suppose we are given a probability q on the space $S_{0i} \times \Pi_{j\neq i}S_j$. The point $s_i = (s_{1i} , s_{2i}, ...) \in S_i$ such that $q = \Phi_i(s_i)$, can be obtained simply by taking marginals on S_{0i}, $S_{0i} \times \Pi_{j\neq i}S_{1j}$, ..., $S_{0i} \times \Pi_{j\neq i}S_{kj}$, ..., for all $k\geq 1$. The sequence $(s_{1i} , s_{2i}, ...)$ thus obtained satisfies the minimum consistency requirement, because the probability q has its support on $S_{0i} \times \Pi_{j\neq i}S_j$. Finally, we have to check that Φ_i is a homeomorphism. However, by 2.2.3 S_i is compact and metric. Thus, by 2.2.1 so is $\Delta(S_{0i} \times \Pi_{j\neq i}S_j)$. Therefore it is enough to prove that either Φ_i or $[\Phi_i]^{-1}$ is continuous. The continuity of $[\Phi_i]^{-1}$ is easily checked, because it is composed of marginals, and these are continuous (in the weak topologies, as we have in the spaces here).

QED.

Another way to view this result is by noting that any infinite stream of beliefs can be seen as a belief about the realisation of agent i's uncertainty and the characteristics of the other agents. One may interpret s_i as the agent himself/herself :

it is a "psychology", or "type", of agent i. The S_i's are also known as "psychology spaces", or as "type spaces".

2.3 Infinite Recursion of Beliefs and Common Knowledge

In this subsection the formalism of infinite recursion of beliefs, derived above, is used to provide a definition of common knowledge which directly corresponds to the statement : "agent i knows other agents know...other agents know... etc. that the event A has occurred".

The set of states of nature as perceived by each agent will be the same for everyone. As in subsection 2.1, it is Ω. One supposes it has the same mathematical structure as before. Using the terminology of last section, one says the first order beliefs of agent i are $s_{1i} \in \Delta(\Omega)$. This refers to his beliefs about the state of nature: a probability distribution over the possible states. Call this set of beliefs S_{1i}. The second order beliefs are points $s_{2i} \in \Delta(\Omega \times \Pi_{k \neq i} S_{1k})$,where $\Pi_{k \neq i}$ indicates the Cartesian product over all indices $k \neq i$. Inductively an m-th order belief, s_{mi} , is a belief on $\Omega \times \Pi_{k \neq i} S_{m-1,k}$. Formally one has : $s_{mi} \in \Delta(\Omega \times \Pi_{k \neq i} S_{m-1,k})$. Agent i's "psychology" (or "type") is summarised by his infinite recursion of beliefs,

$(s_{mi})_{m\geq 1} \in \Pi_{m\geq 1} S_{mi}$. As above one imposes the minimum consistency requirement: whenever there is an event whose probability can be evaluated by his m-th order belief as well as his p-th order belief, then the probability assessment given by both orders of beliefs must coincide. For example, this implies that his first order belief must be the marginal of his second order belief on Ω : $s_{1i} = \mathrm{marg}_{\Omega}(s_{2i})$. Hence agent i's psychology is characterised by a point s_i in the set :

S_i = { $(s_{mi})_{m\geq 1}$: the beliefs satisfy the minimum consistency requirement } .

By Theorem 2.2.4, this set of psychologies has a very important property : any s_i in it can be viewed as a joint distribution on the occurrence of nature and on the other agents' psychologies. In formal terms, there is a homeomorphism $\Phi_i : S_i \rightarrow \Delta(\,\Omega \times \Pi_{k\neq i} S_k\,)$. Moreover, this homeomorphism has very attractive properties. In fact there is a canonical way of deriving it, and in particular one has $\mathrm{marg}_{\Omega}[\Phi_i(s_i)] = s_{1i}$. The interpretation given to supp $\mathrm{marg}_{S_k}[\Phi_i(s_i)]$ (where the word supp stands for the support of the probability) is in the heart of the definition discussed below. It is the set of psychologies of the k-th agent which agent i considers possible. In other words, it is the set of k-th agents which are possible in the eyes of agent i .

Let R_k be a subset of S_k for all $k \in N$. Suppose one wants to formalise the

statement : "agent i knows that agent k is in R_k". Making use of the interpretation given to $\Phi_i(s_i)$, one has to say : every s_k which, in the eyes of agent i, could possibly be true, satisfies $s_k \in R_k$. Formally this means : for any $k \neq i$, for any $s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$, one has $s_k \in R_k$. By extending this idea further, so that one can formalise longer sentences where the verb "know" appears many times, one has :

<u>2.3.1 Definition</u> Given $s_i \in S_i$, and $(R_k)_{k \in N}$, one says that <u>in the eyes of agent i the sets $(R_k)_{k \in N}$ are common knowledge</u> if $s_i \in \cap_{m \geq 1} R_{mi}$, where R_{mi} is inductively defined by :

$\forall i : R_{1i}$ is defined to be R_i ;

$\forall m \geq 2 : R_{mi} = \{ s_i \in R_{m-1,i} \mid \forall\, k \neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in R_{m-1,k} \}$.

Sometimes it is convenient to consider common knowledge from the point of view of all agents :

<u>2.3.2 Definition</u> $(R_k)_{k \in N}$ <u>are common knowledge</u> if $\forall i : s_i \in \cap_{m \geq 1} R_{mi}$.

We will use this framework to define common knowledge of an event. Let

$A \in \Sigma$ be an event. The set A_i of psychologies of the i-th agent for which the event A has occurred (i knows[4] that the event A occurred) is

$A_i = \{ s_i \in S_i \mid A \supset \operatorname{supp} \operatorname{marg}_\Omega[\Phi_i(s_i)] \}$. Observe that for Ω finite this is the same as requiring $\operatorname{marg}_\Omega[\Phi_i(s_i)](A) = 1$. For Ω infinite this is a very mild strenghtening. By using the concept of common knowledge defined in 2.3.1 , one has :

2.3.3 Definition [5] An event $A \in \Sigma$ is said to be common knowledge in the eyes of agent i if $s_i \in \cap_{m \geq 1} A_{mi}$, where A_{mi} is inductively defined by :

$\forall i : A_{1i}$ is defined as above to be A_i ;

$\forall m \geq 2 : A_{mi} = \{ s_i \in A_{m-1,i} \mid \forall k \neq i : s_k \in \operatorname{supp} \operatorname{marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in A_{m-1,k} \}$.

Notice that in this framework one does not have information partitions $(\Pi_k)_{k \in N}$, or the true state of nature ω . It is possible to define common knowledge independently of these concepts. The primitive objects for us are the psychologies of the agents. Of course one could always interpret the s_i's as states of nature in a much larger uncertainty space : that which determines the agents.

2.4 Common Knowledge of Information Partitions

In this subsection it is shown how the two definitions of common knowledge relate to each other. In particular the self-referentiality problem of Aumann's definition is dealt with formally. We show that whenever the information sub-σ-algebras are common knowledge, in a sense very much in the spirit of the last section, and ω is the real state of the world, then A is common knowledge if and only if it is common knowledge at ω in Aumann's sense. This result can be interpreted as saying that the problem with the self-reference is solved in structures of infinite recursion of beliefs.

We feel the most direct definition 2.3.3 is more useful because : (i) it solves the problem of self-referentiality; (ii) it implicitly tells us which information sub-σ-algebras are the ones which are relevant to the question being analysed; and (iii) it allows a view of the world which may be different for each agent. Let us recall that given a set $X \in \Sigma$, $P_i[X]$ denotes the smallest element of Π_i which contains X.

Again, following 2.3.1, we will define common knowledge of the information sub-σ-algebras $(\Pi_k)_{k \in N}$. The set P_i of psychologies of the i-th agent for which an agent k (different from i) has information structure given by Π_k , is:

$P_i = \{s_i \in S_i \mid \forall\, k \neq i : (\omega, s_k) \in \text{supp marg}_{\Omega \times S_k}[\Phi_i(s_i)] \Rightarrow \text{supp marg}_{\Omega}[\Phi_k(s_k)] = P_k[\{\omega\}]\}$.

The interpretation is straightforward. If a state of the world ω is thought possible to occur jointly with a psychology s_k, it must be the case that s_k has a belief over Ω whose support coincides with the set of states of nature that agent k is told. This set is, by the interpretation given earlier on the text, $P_k[\{\omega\}]$. Again, notice that in the case of Ω infinite, this imposes more restrictions on the possible sub-σ-algebras Π_i's : not only do they have to be countably generated, but also it must be the case that $P_i[\{\omega\}]$ is closed for all i and all $\omega \in \Omega$.

<u>2.4.1 Definition</u> We say that the information sub-σ-algebras $(\Pi_k)_{k \in N}$ <u>are common knowledge in the eyes of agent i</u> [6] if $s_i \in \cap_{m \geq 1} P_{mi}$, where P_{mi} is inductively defined by : (i) $\forall i : P_{1i} = P_i$;

(ii) $\forall m \geq 2 : P_{mi} = \{ s_i \in P_{m-1,i} \mid \forall\, k \neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in P_{m-1,k}\}$.

To embed Aumann's framework in the infinite recursions of beliefs model, we still need agent i to behave as if the real state of the world ω has occurred :

<u>2.4.2 Definition</u> The state <u>ω occurred in the eyes of agent i</u> if

$$\text{supp marg}_{\Omega}[\Phi_i(s_i)] = P_i[\{\omega\}] \, .$$

Notice that agent i does not know whether ω actually occurred. He observes only $P_i[\{\omega\}]$. We are now ready for the main result : the equivalence of both common knowledge definitions, under the hypothesis of common knowledge of the sub-σ-algebras and the occurrence of state ω.

2.4.3 Theorem Suppose that in the eyes of agent i the information sub-σ-algebras $(\Pi_k)_{k\in N}$ are common knowledge and that ω occurred. Then $A\in\Sigma$ is common knowledge at ω in the sense of Aumann if and only if, in the eyes of agent i, A is common knowledge.

Before proving the result an example will help us understand the intution behind the proof.

2.4.4 Example Let us argue in the case of example 2.1.3 why common knowledge of information partitions and the fact that A_2 is common knowledge at ω=2 in the sense of Aumann, imply that A_2 is common knowledge in the sense of the infinite hierarchy of beliefs definition.

(i) The fact that the agents use their information partitions implies that both agents know that A_2 occurred.

In fact, $P_1[\{2\}] = \{2,3\}$, and $A_2 \supset \{2,3\}$, so that agent 1 knows A_2. Analogously, $P_2[\{2\}] = \{1,2\}$, and $A_2 \supset \{1,2\}$, so that agent 2 knows A_2.

(ii) The fact that one agent knows that the other agent's information structure is given by his information partition implies that agent 1 knows that agent 2 knows that A_2 occurred, and that agent 2 knows that agent 1 knows that A_2 occurred.

For agent 1 : since $\omega = 2$ is the real state of the world, agent 1 is told $\{2,3\}$. Agent 1 knows that agent 2's information structure is given by Π_2. Hence, agent 1 knows that agent 2 was told either $\{1,2\}$ or $\{3,4\}$. In any case, as $A_2 \supset \{1,2\} \cup \{3,4\}$, agent 1 knows that agent 2 knows A_2. In formal terms $A_2 \supset P_2 [P_1[\{2\}]]$.

Analogously, agent 2 is told $\{1,2\}$. By knowing that agent 1 has his information structure given by Π_1, agent 2 knows that agent 1 observed either $\{1\}$ or $\{2,3\}$. Again, $A_2 \supset \{1\} \cup \{2,3\}$, so that the assertion follows. In formal terms $A_2 \supset P_1 [P_2[\{2\}]]$.

(iii) The fact that agent 1 knows that the information structure of agent 2 is given by Π_2, and that agent 1 knows that agent 2 knows that the information structure of agent 1 is given by Π_1, implies that agent 1 knows that agent 2 knows that agent 1 knows that A_2 occurred.

By (ii) we know that agent 1 knows that agent 2 was told either $\{1,2\}$ or $\{3,4\}$. If agent 2

were told {1,2}, as agent 1 knows that agent 2 knows that the information structure of agent 1 is given by Π_1, then agent 2 would know that agent 1 was told either {1} or {2,3}. On the other hand, if agent 2 were told {3,4} then, by the same argument agent 2 knows that agent 1 is told either {2,3} or {4}. But $A_2 \supset (\{1\}\cup\{2,3\}) \cup (\{2,3\}\cup\{4\}) = = P_1 [P_2 [P_1[\{2\}]]]$.

(iv) By the same argument, higher orders of knowledge of A_2 can be seen to follow from higher orders of knowledge of the information structure, simply by checking that sequences of any length of the forms :

$\ldots P_1 [P_2 [P_1[\{2\}]]]\ldots$ and $\ldots P_2 [P_1 [P_2[\{2\}]]]\ldots$ are contained in A_2. In the example here, $A_2 = P_1 [P_2 [P_1[\{2\}]]]$ and $A_2 = P_2 [P_1 [P_2[\{2\}]]]$, and these sequences become constant after three steps, so that the common knowledge of A_2 in the sense of the infinite hierarchies of beliefs is established.

<u>Proof of the theorem ($\Rightarrow$)</u> We need to show that if there exists $B \in \Pi = \Pi_1 \cap \Pi_2 \cap \ldots \cap \Pi_n$ with $\omega \in B$ and $A \supset B$, then $s_i \in \cap_{m\geq 1} A_{mi}$ under the hypothesis that in the eyes of agent i ω has occurred and $(\Pi_k)_{k\in N}$ are common knowledge. We will prove an auxiliary result first. Given B as above, define

$B_{1i} = \{ s_i \in S_i \mid B \supset \text{supp marg}_\Omega[\Phi_i(s_i)] \}$, and for $m \geq 2$,

$B_{mi} = \{ s_i \in B_{m-1,i} \mid \forall k \neq i : s_k \in \text{supp marg}_{S_k} [\Phi_i(s_i)] \Rightarrow s_k \in B_{m-1,k} \}$.

<u>2.4.5 Lemma</u> $\forall i, \forall m$: $B_{mi} \cap P_{mi}$ is contained in $B_{m+1,i}$.

<u>Proof of 2.4.5</u> By induction on m. For m=1, $s_i \in B_{1i} \cap P_{1i} \Rightarrow s_i \in P_{1i}$. This implies that $\forall k \neq i : s_k \in \text{supp marg}_{S_k} [\Phi_i(s_i)] \Rightarrow \exists \omega' \in \text{supp marg}_\Omega[\Phi_i(s_i)]$ such that $\text{supp marg}_\Omega[\Phi_k(s_k)] = P_k[\{\omega'\}]$. Because $s_i \in B_{1i}$ we have that $B \supset \text{supp marg}_\Omega[\Phi_i(s_i)]$. Hence $\omega' \in B$. But B is Π_k-measurable, which implies that $B \supset P_k[\{\omega'\}]$, so that $B \supset \text{supp marg}_\Omega[\Phi_k(s_k)]$. Thus $s_i \in B_{2i}$, and the lemma is shown to be valid for m=1.

Suppose now it is true for $m \geq 1$. We have to show it is true for m+1.

If $s_i \in B_{mi} \cap P_{mi}$ then $\forall k \neq i : s_k \in \text{supp marg}_{S_k} [\Phi_i(s_i)] \Rightarrow$

$\Rightarrow s_k \in B_{m-1,k} \cap P_{m-1,k}$. By the induction hypothesis $s_k \in B_{mk}$. As $s_i \in B_{mi}$ as well, it immediately follows that $s_i \in B_{m+1,i}$.

<u>QED</u>(of 2.4.5)

Returning to the proof of ($\Rightarrow$) of 2.4.3, by the fact that $B \in \Pi$.

we have $B \in \Pi_i$. However, part of the hypothesis of Theorem 2.4.3 requires that ω has occurred in the eyes of agent i. Therefore, supp $\mathrm{marg}_{\Omega}[\Phi_i(s_i)] =$

$= P_i[\{\omega\}]$. But we have as hypothesis that $\omega \in B$, so that, as B is Π_i-measurable, $B \supset P_i[\{\omega\}]$. Hence $B \supset$ supp $\mathrm{marg}_{\Omega}[\Phi_i(s_i)] \Rightarrow s_i \in B_{1i}$.

By the hypothesis of the theorem, $s_i \in P_{mi}$ for every $m \geq 1$. Applying lemma 2.4.5 successively, we have $s_i \in \cap_{m\geq 1} B_{mi}$. However, since $A \supset B$, $\forall i, \forall m: \; A_{mi} \supset B_{mi} \Rightarrow s_i \in \cap_{m\geq 1} A_{mi}$. This means that in the eyes of agent i, the event A is common knowledge.

QED(of $(\Rightarrow)$).

Proof of $(\Leftarrow)$ To prove the converse we need to show that if ω occurred in the eyes of agent i, and moreover, if the sub-σ-algebras $(\Pi_k)_{k\in N}$ and the set A are, in the eyes of agent i, common knowledge, that is to say, $s_i \in \cap_{m\geq 1} A_{mi} \cap P_{mi}$, then there exists $B \in \Pi = \Pi_1 \cap \Pi_2 \cap \ldots \cap \Pi_n$ with $\omega \in B$ and $A \supset B$.

We shall construct such a set B. Let $B^1 = \text{supp marg}_\Omega[\Phi_i(s_i)]$.

Define $\forall m \geq 1$, $\forall k_1, k_2, \ldots, k_m$, each $k_j \in \{1, \ldots, n\}$, the set:

$B_{k_1 k_2 \ldots k_m} = P_{k_m}[P_{k_{m-1}}[\ldots P_{k_2}[P_{k_1}[B^1]]\ldots]]$. As before, $P_{k_j}[X]$ is the smallest set Π_{k_j} - measurable which contains the set X.

We first show that, $s_i \in A_{m+1,i} \cap P_{mi} \Rightarrow A \supset B_{k_1 k_2 \ldots k_m}$, by induction on $m \geq 1$. For $m=1$: $s_i \in A_{2i} \cap P_{1i} \Rightarrow \forall k \neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$ then $s_k \in A_{1k}$. Hence $A \supset \text{supp marg}_\Omega[\Phi_k(s_k)]$. Since $s_i \in P_{1i}$ as well, we have that $\forall s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$, $\exists \omega' \in \text{supp marg}_\Omega[\Phi_i(s_i)] = B^1$ with $\text{supp marg}_\Omega[\Phi_k(s_k)] = P_k[\{\omega'\}]$. In particular, for this ω' it follows that $A \supset P_k[\{\omega'\}]$. However, again from the definition of P_{1i}, $\forall \omega' \in B^1 = \text{supp marg}_\Omega[\Phi_i(s_i)]$ there exists a $s_k \in S_k$ such that $\text{supp marg}_\Omega[\Phi_k(s_k)] = P_k[\{\omega'\}]$. Thus $A \supset P_k[B^1]$. This shows the statement is valid for $m = 1$ in the case $k_1 \neq i$. If $k_1 = i$, as $\text{marg}_\Omega[\Phi_i(s_i)](A) = 1$, we have $A \supset B^1$. But $P_i[B^1] = B^1$, since the hypothesis that ω occurred implies that $B^1 = \text{supp marg}_\Omega[\Phi_i(s_i)] = P_i[\{\omega\}] = P_i[P_i[\{\omega\}]] = P_i[B^1]$. Therefore $A \supset P_i[B^1]$, and $A \supset P_{k_1}[B^1]$ $\forall k_1 \in \{1, \ldots, n\}$.

Suppose it is true for $m \geq 1$. We shall show that it is true for $m+1$.

In fact, $s_i \in A_{m+2,i} \cap P_{m+1,i} \Rightarrow \forall k \neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$ implies $s_k \in A_{m+1,k} \cap P_{mk}$. By the induction hypothesis :

$\forall k \neq i$: $A \supset {}_kB_{k1k2...km}$, where these ${}_kB_{k1k2...km}$ are defined identically to $B_{k1k2...km}$, with the sole difference being that B^1 is substituted with ${}_kB^1 = \text{supp marg}_\Omega[\Phi_k(s_k)]$. Since $s_i \in P_{m+1,i}$ it follows that $s_i \in P_{1i}$, as well. Thus, there is $\omega' \in \text{supp marg}_\Omega[\Phi_i(s_i)]$ such that ${}_kB^1 = P_k[\{\omega'\}]$. By the induction hypothesis $A \supset {}_kB_{k1k2...km} = P_{km}[P_{km-1}[\ldots P_{k2}[P_{k1}[{}_kB^1]]\ldots]]$. Hence, it follows that for this ω', $A \supset P_{km}[P_{km-1}[\ldots P_{k2}[P_{k1}[P_k[\{\omega'\}]]]\ldots]]$. But $s_i \in P_{mi}$ implies that for every $\omega' \in B^1 = \text{supp marg}_\Omega[\Phi_i(s_i)]$ there exists such $s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$. Therefore $\forall \omega' \in B^1$ we have $A \supset P_{km}[P_{km-1}[\ldots P_{k2}[P_{k1}[P_k[\{\omega'\}]]]\ldots]]$ which implies that $\forall k_1 \neq i$ $A \supset B_{k1k2...kmkm+1} = P_{km+1}[P_{km}[\ldots P_{k2}[P_{k1}[B^1]]\ldots]]$. The case $k_1 = i$ remains to be shown. This case is solved by the same argument used to deal with $k_1 = i$ in the first step of the induction.

Having proved the auxiliary result claimed at the beginning of

the proof, we are now in a position to construct the set $B \in \Pi =$ $= \Pi_1 \cap \Pi_2 \cap \ldots \cap \Pi_n$ with $\omega \in B$ and $A \supset B$, and this will complete the proof of 2.4.3.

Notice that $\forall m$: $A \supset B_{k_1 k_2 \ldots k_m k_{m+1}} \supset B_{k_1 k_2 \ldots k_m}$. Define, for $N=\{1,\ldots,n\}$, $B^2 = \cap_{k_1 \in N} B_{k_1}$, $B^3 = \cap_{k_2 \in N} \cup_{k_1 \in N} B_{k_1 k_2}$, and $\forall m$:

$B^{m+1} = \cap_{k_m \in N} \cup_{k_{m-1} \in N} \cdots \cup_{k_1 \in N} B_{k_1 \ldots k_{m-1} k_m}$. Then one has

$\ldots \supset B^m \supset \ldots \supset B^3 \supset B^2 \supset B^1$. Moreover, $\forall m$: $A \supset B^m$. Let $B = \cup_{m \geq 1} B^m$.

We claim that $B \in \Pi_k$, $\forall k \in N$. We show this by demonstrating that $\forall m$, $\forall k$, there exists $D_{mk} \in \Pi_k$ with $B^{m+2} \supset D_{mk} \supset B^m$. Now

$B^m = \cap_{k_{m-1} \in N} \cup_{k_{m-2} \in N} \cdots \cup_{k_1 \in N} B_{k_1 k_2 \ldots k_{m-1}}$. But $P_k[B_{k_1 k_2 \ldots k_{m-1}}] \supset$ $\supset B_{k_1 k_2 \ldots k_{m-1}}$, and $P_k[B_{k_1 k_2 \ldots k_{m-1}}] \in \Pi_k$. Also $P_k[B_{k_1 k_2 \ldots k_{m-1}}] =$ $= B_{k_1 k_2 \ldots k_{m-1} k}$. So, by making $D_{mk} =$

$= \cap_{k_{m-1} \in N} \cup_{k_{m-2} \in N} \cdots \cup_{k_1 \in N} B_{k_1 \ldots k_{m-2} k_{m-1} k}$, one has $D_{mk} \supset B^m$, and $D_{mk} \in \Pi_k$. Furthermore,

$$B^{m+2} = \cap_{k_{m+1} \in N} \cup_{k_m \in N} \cup_{k_{m-1} \in N} \cdots \cup_{k_1 \in N} B_{k_1 \ldots k_{m-1} k_m k_{m+1}} \supset$$

$\supset \cap_{k_{m-1} \in N} \cup_{k_{m-2} \in N} \cdots \cup_{k_1 \in N} B_{k_1 \ldots k_{m-2} k_{m-1} k} = D_{mk}$. Hence

$A \supset B^{m+2} \supset D_{mk} \supset B^m$ and $D_{mk} \in \Pi_k$, for all m and k. Thus

$B = \cup_{m \geq 1} B^m = \cup_{m \geq 1} D_{mk}$. As Π_k is a sub-σ-algebra the set $\cup_{m \geq 1} D_{mk}$

is an element of Π_k. Therefore $B \in \Pi_k$ for all k. Also, by construction,

$A \supset B$.

Finally, we need to show that $\omega \in B$. Since ω occurred in agent i's eyes, it follows that $\omega \in B^1$, since $B^1 = \text{supp marg}_\Omega[\Phi_i(s_i)] = P_i[\{\omega\}]$. By construction $B \supset B^1$. Thus $\omega \in B$, and the theorem is proved.

QED.

The reader familiar with Aumann(1976) will recognise that the proof of the converse of the theorem is essentially a formalisation of the notion of "reachability" in Aumann's article.

2.5 Relation with the Literature

One can interpret the main theorem proved in here as the following : given a structure of knowledge in terms of a space of events and sub-σ-algebras, that is to say, a structure à la Aumann, we generate a structure of knowledge in the sense of an infinite recursion of beliefs which handles the same common knowledge problem.

In a recent article, Brandenburger and Dekel(1985b) proved a converse of this result. Given a description of a common knowledge problem in the framework of infinite recursion of beliefs, one can generate a space of events, n sub-σ-algebras and a state of the world (in other words, an Aumann's structure of knowledge) which handles the same common knowledge problem. However, there is a subtlety. The set of states of the world which they obtain is infinite. Since they use a model where priors exist (see footnotes 2 and 5), there are some additional requirements for the definition of common knowledge à la Aumann. There are two ways of handling the problem which are similar : that of Nielsen(1984) and that of Brandenburger and Dekel(1985a).

It is important to notice that the analysis we present is more complete. Not only we generate an infinite hierarchy of beliefs which handles the same common knowledge problem, but also this infinite hierarchy of beliefs has the property that the information sub-σ-algebras are common knowledge. On the other hand, the information sub-σ-algebras that Brandenburger and Dekel(1985b) obtain are not common knowledge in the infinite hierarchy of beliefs that they started with. In any case, they solve a converse of our theorem. Thus, one is lead to the conclusion that both ways of viewing common knowledge are equivalent, the difficulty with Aumann's

definition being that of self-referentiality : the information structure he begins with must be common knowledge in order for his definition to make sense.

2.6 Conclusion

This section presented the study of common knowledge by means of probabilistic models. We began by giving Aumann's definition of common knowledge of events. The shortcomings of this definition were discussed, the main one being a self-referentiality problem. Then another definition, based on the framework of infinite recursion of beliefs, was provided. This definition has been previously given in Armbruster and Böge(1979), Böge and Eisele(1979) and Mertens and Zamir(1985). We then proved that this definition not only solves Aumann's self-referentiality problem but also that it is completely equivalent to Aumann's definition when the latter is embedded in the former.

The problem above has also been discussed in more abstract, logic-theoretic models. Aumann's model is very similar to Kripke(1963). Our framework is somewhat similar to Fagin,Halpern and Vardi(1984). Fagin, Halpern and Vardi also note the equivalence between Kripke's definition and theirs, but they argue that their definition, which takes into account the several layers of knowledge, is more adequate for the purposes of artificial intelligence.

The next sections show how to apply the formalism developed here to foundations of noncooperative solution concepts.

3. FOUNDATIONS OF NONCOOPERATIVE SOLUTION CONCEPTS

We begin by repeating the example given in the introduction. Imagine that you are going to play a given bimatrix game with two alternative players. The payoffs of the game are in dollar terms. The first player is an intelligent acquaintance of yours, whom you know very well. The second player is a stranger. She comes from the Himalaya, and the only relevant information you know about her is that she was taught the meaning of a bimatrix game (the rules of the game) and the what a dollar can buy. For the sake of argument, let us say that : (i) there is a unique pure strategy Nash equilibrium, which gives you a thousand dollars; (ii) your security level is nine hundred dollars; and (iii) if the other player does not play his/her part of the Nash equilibrium you lose at least a hundred dollars. How should you play the same game against the two different opponents? It seems clear to me that everyone who is faced with this situation is much more likely to follow the Nash strategy when facing the acquaintance, than when facing the stranger.

The fact that a well defined game may be played in different ways by the same person, indicates that in the specification of a game some additional information about the background of the players is essential for the solution of this game. Therefore, a solution concept which depends only on the payoff matrix, as Nash equilibrium does, needs an interpretation. This points out the need for foundational analysis of solution concepts. In this section we establish a general methodology for the analysis of solution concepts. In particular, we point out which extra information the

players need to play a game. In sections four and five we apply this methodology.

A fairly widespread notion among theorists these days is the lack of a noncooperative solution concept [7] (not to speak of cooperative ones) which will describe the behaviour of players in a game in a satisfactory way. The main noncooperative solution concept is that of Nash equilibrium. The literature on noncooperative solution concepts follows two distinct trends. One states that there are too many Nash equilibria, and suggests alternative ways of refining Nash's notion. Some of the most important refinements of the Nash equilibrium concept are : subgame perfection, Selten(1965); perfection, Selten(1975); properness, Myerson(1978); sequentiality, Kreps and Wilson(1982); tracing procedure, Harsanyi(1975) and Harsanyi and Selten(1980-84); persistent equilibrium, Kalai and Samet(1982); strategic stability, Kohlberg and Mertens(1985); justifiable beliefs, McLennan(1985); forward induction equilibrium, Cho(1985); perfect sequential equilibrium, Grossman and Perry(1985). In the context of signalling games other refinements were proposed : intuitive criterion, Kreps(1985); divinity, Banks and Sobel(1985); neologism-proof, Farrell(1985). For a more complete account of the literature on refinements of Nash equilibrium, see van Damme(1983). A discussion of the recent progress in the area can be found in Kohlberg and Mertens(1985) and in Cho(1985).

The other trend followed by the literature is that of expanding the concept of Nash equilibrium. The important contributions in the area are : correlated equilibrium, Aumann(1974, 1985); refinements of correlated equilibrium, Myerson(1985); and

rationalisable strategic behaviour, Bernheim(1984) and Pearce(1984).

In view of the many existing suggestions, as the list above exemplifies, which solution concept should be chosen to solve a specific game? We will not answer this question directly. Alternatively, we propose a methodology which enables us to analyse solution concepts. Bernheim(1984) and Pearce(1984) show that common knowledge of rationality is not enough to justify Nash behaviour. They introduce a noncooperative solution concept which is derived from the hypothesis that Bayesian rationality is common knowledge. They call their solution concept rationalisable strategic behaviour. The point I wish to emphasise is that Bernheim and Pearce derive their solution concept for games from assumptions about the behaviour of the players. One can generally approach the analysis of solution concepts in the same manner. Which are the implicit behavioural assumptions behind a given solution concept?

In order to answer this question, let us be more precise. We shall be concerned with the following complete information simultaneous game :

3.1Definition A game with n players, u, is a 2n-tuple

$(A_1, ..., A_n, u_1, ..., u_n)$, where :

(i) each A_i, the set of strategies or actions available to player i, is a compact metric space;

(ii) let $A = A_1 \times ... \times A_n$. Then $u_i : A \to R$, is a function which gives the payoff to player i, for each possible combination of strategies

of all players. For each i, u_i is assumed to be continuous.

Let U mean the set of all n-tuples of payoff functions. By an abuse of notation, we say $u \in U$ is a given game, where u represents the n-tuple $(u_1, \ldots, u_n)$.

3.2 Definition A solution concept (also called equilibrium notion) is a correspondence $\Gamma : U \to A$.

A solution concept is a correspondence that assigns to each game a set of prescribed action profiles. The interpretation given to a solution of a game u, is that among the n-tuples of actions in $\Gamma(u)$, there is one which will be chosen when players play the game u. Hence, we are allowing that different players playing the same game could choose different n-tuples of actions, as long as these n-tuples are in $\Gamma(u)$.

From a Bayesian point of view, the decision of each player in a game is determined by this player's beliefs about the actions of other players. But, if, in their turn, other players' beliefs about other players' actions affect their actions, then it must be that the beliefs one player has about the beliefs of other players also affect the decision of this player in the game. If we carry this argument further, we see that the action taken by a player is determined by his infinite hierarchy of beliefs about actions of other players, beliefs about beliefs about other players' actions, and so on. The space of these infinite hierarchies of beliefs is the appropriate space for the study of

behavioural assumptions about the players. For every i, let

$A_{-i} = A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n$. We follow the mathematical formalism of

subsection 2.2. The set of states of the world, as perceived by player i, is A_{-i}. A first

order belief of player i is a point $s_{1i} \in \Delta(A_{-i}) = S_{1i}$. In general, the m-th order belief of

player i is a point $s_{mi} \in \Delta(A_{-i} \times \Pi_{k \neq i} S_{m-1,k}) = S_{mi}$. As in subsection 2.2, we impose

the minimum consistency requirement. Thus, by Theorem 2.2.4, the infinite hierarchy

of beliefs s_i can be viewed as a joint belief about what other players play, and which

hierarchies of beliefs other players have. This is done through the homemorphisms Φ_i.

As before the infinite hierarchy of beliefs s_i is interpreted as the "psychology" (or

"type") of player i. It embodies all relevant decision-theoretic variables which are

necessary for understanding how player i will play the game u. The space of these

psychologies, S_i, is the appropriate space for the study of behavioural assumptions

about player i.

We have to determine how each different psychology s_i will play the game u.

First we define a general Bayesian decision problem.

<u>3.3 Definition</u> A <u>Bayesian decision problem</u> for player i is given by :

(i) T_i a compact metric probability space endowed with the Borel σ-algebra.

It represents all the elements of uncertainty for player i;

(ii) A_i a compact set of actions available to player i;

(iii) $U_i : A_i \times T_i \to R$, his subjective utility function;

(iv) $Q_i \in \Delta(T_i)$, his subjective prior on S_i.

Given a decision problem, one can derive the structure above from more basic facts as in Savage(1954). It is important to note that U_i and Q_i characterise player i.

Let $V_i : A_i \times \Delta(T_i) \to R$ be the expected subjective utility for player i, when he takes an action a_i, and has prior Q_i : $V_i(a_i, Q_i) = \int_{S_i} U_i(a_i, t_i)\, dQ_i(t_i)$. To avoid unnecessary notation, we will simply write $V(a_i, Q_i)$ instead of $V_i(a_i, Q_i)$.

<u>3.4 Definition</u> Player i is <u>Bayesian rational</u> when, faced with a Bayesian decision problem, he chooses an action $\tilde{a}_i \in A_i$ such that the expected subjective utility is maximised : $V(\tilde{a}_i, Q_i) \geq V(a_i, Q_i)$, $\forall a_i \in A_i$.

To determine how a given psychology s_i plays the game u, we have :

<u>3.5 Definition</u> Given a game u and psychology s_i, we define

The Bayesian decision problem associated with u and s_i as :

(i) $T_i = A_{-i} \times S_{-i}$, where $S_{-i} = \Pi_{k \neq i} S_k$;

(ii) A_i is the same as A_i for the game u ;

(iii) $U_i (a_i , t_i) = u_i (a_i , \mathrm{Proj}_{A_{-i}}(t_i))$;

(iv) $Q_i \in \Delta(T_i)$ is given by $\Phi_i(s_i)$.

The viewpoint of this approach to game theoretical situations can be summarised by :

3.6 Axiom The decision problem player i faces in the game u when its psychology is s_i, is the same as the Bayesian decision problem associated with u and s_i . In other words : all that we need to know about player i to determine her/his behaviour in the game u , is given by the psychology s_i .

Let us comment a little about the above. One sees that the only relevant probability distribution for player i is the first order belief $s_{1i} = \mathrm{marg}_{A_{-i}}[\Phi_i(s_i)]$. since in the expected value function $V(a_i,t_i)$ only the projection in the actions of the other players are considered relevant. This seems to tell us that the only important part of s_i is the first order belief. Ultimately that is so. However, we cannot forget that

higher order beliefs influence the lower order beliefs. By an abuse in notation, one defines $V(a_i,\Phi_i(s_i)) = V(a_i,s_i)$.

Now we are ready to state our methodology. Let us consider a given game u fixed. That is to say, we will concentrate in the correspondence Γ restricted to a singleton $\{ u \}$ contained in U . We are not interested in global properties of $\Gamma(.)$, as, for example, Kohlberg and Mertens(1985). In the case of a fixed game $u \in U$, a solution concept Γ is simply a subset of $A = A_1 \times \ldots \times A_n$. Associated with Γ, we want to find a subset $B(\Gamma)$ contained in $S_1 \times \ldots \times S_n$, a subset of the set of psychologies of all players, such that :

(i) $\forall (a_1, \ldots, a_n) \in \Gamma, \exists (s_1, \ldots, s_n) \in B(\Gamma)$ such that for every i : a_i is an action which maximises the subjective utility for player s_i (according to 3.5);

(ii) $\forall (s_1, \ldots, s_n) \in B(\Gamma), \exists (a_1, \ldots, a_n) \in \Gamma$ such that for every i : a_i is an action which maximises the subjective utility for player s_i (according to 3.5).

In words: the first statement says that any n-tuple of actions in the solution set Γ can be played by some psychology in $B(\Gamma)$. Conversely, the second statement says that any psychology in $B(\Gamma)$ can play an action in Γ. This means that for every solution

Γ we associate a set of psychologies which corresponds to Γ. The set $B(\Gamma)$ can be interpreted as a set of behavioural assumptions behind the solution concept Γ. It is important to notice that the set $B(\Gamma)$ is not uniquely determined. Each different $B(\Gamma)$ represents a different set of behavioural assumptions under which the solution concept Γ is justified. However, if $B_1(\Gamma)$ and $B_2(\Gamma)$ both satisfy (i) and (ii), so does $B_1(\Gamma) \cup B_2(\Gamma)$. Hence, there is a maximal set $BM(\Gamma)$ satisfying (i) and (ii). This set is to be interpreted as the set of all behavioural assumptions which justify the solution concept Γ.

The ultimate aim of the methodology described here is to obtain sets $B(\Gamma)$ for every solution concept Γ. This would facilitate the selection of the solution concept : one should see which behavioural assumptions apply to the economic situation being modelled, and choose the solution concept accordingly.

In section four we consider Γ = rationalisable strategic behaviour. In section five we consider Γ = Nash equilibrium.

4. RATIONALISABLE STRATEGIC BEHAVIOUR

This section intends to give a complete answer to the second question posed in section three : which psychologies correspond to Γ being the map that associates to each game the subset of actions given by the rationalisable actions. In other words : we compute a set $B(\Gamma)$ of behavioural assumptions behind the solution concept given by Γ = rationalisable strategic behaviour. The answer, as Bernheim(1984) and Pearce(1984) argued, is simple. The psychologies which will play rationalisable actions are those for which rationality is common knowledge. Thus, this section proves, in a formal setting, that their arguments hold. Furthermore, we are able to prove an extension of their results for compact and metric spaces. In this case there are subtle measurability problems, which can be dealt with in our framework. We will come back to this point in example 4.6.

We look at a slightly different formulation of rationalisable strategic behaviour from that defined by Bernheim(1984) and Pearce(1984). The essential difference is that we permit player i to believe that the other players may be correlating their strategies. Therefore, it follows from Pearce(1984) that rationalisable actions are obtained after successive elimination of strictly dominated strategies[8]. It should be clear, however, that an additional assumption that players suppose it is common knowledge that the others act independently, would result in a framework identical to theirs.

<u>4.1 Definition</u> <u>Rationalisable actions</u> : Let $A_{0i} = A_i$, and for $m \geq 1$ define

$A_{mi} = \{ a_i \in A_i \mid \exists \mu \in \Delta(\Pi_{k \neq i} A_{m-1,k})$, such that $a_i \in \arg\max E_\mu[u_i(\tilde{a}_i, a_{-i})] \}$,

where E_μ is the expectations operator with respect to the probability μ which

ranges over the variables $a_{-i} \in \Pi_{k \neq i} A_k$, and the maximisation problem has $\tilde{a}_i$

as variable. Then the rationalisable actions of player i are the actions a_i

in the set $\cap_{m \geq 0} A_{mi}$.

Next we restrict beliefs of each player to reflect the fact that Bayesian rationality is common knowledge and derive a theorem stating that under this restriction, a player would take only rationalisable actions.

<u>4.2 Definition</u> <u>Bayesian Rationality is Common Knowledge</u>

K^1 : player i knows everyone is rational $\Leftrightarrow$ $s_i \in K_{1i} =$

$= \{ s_i \in S_i \mid \forall k \neq i : (a_k, s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)] \Rightarrow$

$\Rightarrow a_k \in \text{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, s_k) \}$, where V was defined at the end of 3.2 ;

K^m : player i knows everyone (knows everyone)$^{m-1}$ is rational $\Leftrightarrow$ $s_i \in K_{mi} =$

$= \{ s_i \in K_{m-1,i} \mid \forall k \neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in K_{m-1,k} \}$;

Bayesian rationality is common knowledge for player i if and only if all the statements K^m are true, that is to say, $s_i \in \cap_{m \geq 1} K_{mi}$. One defines Bayesian rationality to be common knowledge, if it is common knowledge for all players $i \in N$.

Informally, $(a_k, s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$ means that player i believes that k can have psychology s_k , and that he/she will choose the action a_k . Hence K^1 implies that a_k must be an action which maximises player k's expected subjective utility when she/he has psychology s_k . That is to say, player i believes player k is Bayesian rational. K^2 is then a restriction on the second layer of beliefs of player i requiring that he/she believes that other players believe that everyone is rational. Obviously this intuitive interpretation carries on to the higher layers of beliefs.

Next, we show how the restrictions on each layer of beliefs affect the first layer of beliefs - and therefore the choice of actions.

<u>4.3 Theorem</u> $\forall i$, $\forall m$: $s_i \in K_{mi} \Rightarrow \Pi_{k \neq i} A_{mk} \supset \text{supp marg}_{A_{-i}}[\Phi_i(s_i)]$.

<u>4.4 Theorem</u> If a player i is Bayesian rational and $s_i \in K_{mi}$, he takes an

action $a_i \in A_{m+1,i}$.

Let us interpret these results before proving them. Theorem 4.3 states that if player i knows everyone (knows everyone)$^{m-1}$ is rational, she/he is able to conduct m rounds of elimination of non-best responses in the manner of Bernheim-Pearce. In addition, if player i is Bayesian rational, Theorem 4.4 states that he/she will choose an action that resists m+1 rounds of elimination.

Proof of 4.3 By induction. Let $s_i \in K_{1i}$. Then $(a_k,s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$

$\Rightarrow V(a_k,s_k) = \max_{A_k} V(\,.\,,s_k) \Rightarrow a_k \in A_{1k}$ since the required μ in the definition

of A_{1k} is $\text{marg}_{A_{-k}}[\Phi_k(s_k)]$. Since this is true for all $k \neq i$, then

$a_{-i} \in \text{supp marg}_{A_{-i}}[\Phi_i(s_i)] \Rightarrow a_{-i} \in \Pi_{k \neq i} A_{1k}$. Hence

$\Pi_{k \neq i} A_{1k} \supset \text{supp marg}_{A_{-i}}[\Phi_i(s_i)]$. Assume the inductive step for m :

$\forall i, \forall m : s_i \in K_{mi} \Rightarrow \Pi_{k \neq i} A_{mk} \supset \text{supp marg}_{A_{-i}}[\Phi_i(s_i)]$. We shall show it

is true for m+1. Let $s_i \in K_{m+1,i}$. Then for $(a_k,s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$

it follows from the definition of $K_{m+1,i}$, that $s_k \in K_{mk}$. Thus, by the induction

hypothesis, $\Pi_{p \neq k} A_{mp} \supset \text{supp marg}_{A_{-k}}[\Phi_k(s_k)]$. Now $s_i \in K_{m+1,i}$ implies

$s_i \in K_{1i}$, by definition, so that $V(a_k,s_k) = \max_{A_k} V(\,.\,,s_k)$. Hence,

$a_k \in A_{m+1,k}$ since $\mathrm{marg}_{A_{-k}}[\Phi_k(s_k)] \in \Delta(\Pi_{p\neq k}A_{mp})$ is the required belief μ in the definition of $A_{m+1,k}$. Since this holds for all $k\neq i$, we have that

$a_{-i} \in$ supp $\mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] \Rightarrow a_{-i} \in \Pi_{k\neq i}A_{m+1,k}$, and this concludes the proof.

QED.

Proof of 4.4 By the result above, $s_i \in K_{mi} \Rightarrow \Pi_{k\neq i}A_{mk} \supset$ supp $\mathrm{marg}_{A_{-i}}[\Phi_i(s_i)]$.

Hence $\mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] \in \Delta(\Pi_{k\neq i}A_{mk})$. Since i is rational, he takes an action a_i such that $V(a_i,s_i) = \max_{A_i}V(\,.\,,s_i)$. But $V(\,.\,,s_i)$ is simply $E_\mu[u_i(\,.\,,a_{-i})]$, where $\mu = \mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] \in \Delta(\Pi_{k\neq i}A_{mk})$ is the required in the definition of $A_{m+1,i}$. Therefore $a_i \in A_{m+1,i}$.

QED.

The following proposition is an immediate consequence of 4.4.

4.5 Theorem $s_i \in \cap_{m\geq 1}K_{mi}$ and player i is rational $\Rightarrow$

$\Rightarrow$ i chooses $a_i \in \cap_{m\geq 1}A_{mi}$.

Theorem 4.5 provides the formal derivation of rationalisable strategies

from the assumption that Bayesian rationality is common knowledge.

We now proceed to derive the converses toTheorems 4.4 and 4.5. Theorem 4.12 below says that for <u>any</u> action in $A_{m+1,i}$, there is a belief s_i for player i which rationalises the action and furthermore satisfies knowledge of rationality up to level m, i.e. $s_i \in K_{mi}$. Theorem 4.13 similarly establishes that <u>any</u> rationalisable action can indeed be supported by beliefs which satisfy "rationality is common knowledge". As a result, if "rationality is common knowledge" is the only restriction which a game theorist imposes on a solution concept, then that solution concept must correspond to the notion of rationalisable strategic behaviour.

The definition of a rationalisable action requires that this action remains after iterated elimination of nonbest responses. However, it may be that a belief which supports an action (we say that a belief supports an action when this action is a best response for this belief) in one iteration of the elimination procedure is "inconsistent" with the belief which will support this action in the next iteration. Theorems 4.12 and 4.13 prove that there is always a consistent way of supporting a rationalisable action, that is to say, this rationalisable action is supported by an infinite hierarchy of beliefs which satisfies the minimum consistency requirement, and for which rationality is common knowledge. The example below shows how to construct a consistent infinite hierarchy of beliefs which supports a rationalisable action, and also explains what is the potential source of inconsistencies.

<u>4.6 Example</u> Consider the two-person game below. $A = \{a_1, a_2, a_3, a_4\}$ is the action

space of player 1, and $B = \{b_1, b_2\}$ is the action space of player 2. The payoff matrix is:

	b_1	b_2
a_1	(3,1)	(2,3)
a_2	(-3,3)	(4,-2)
a_3	(6,2)	(-3,0)
a_4	(1,0)	(3,4)

This game has the property that all actions of both players are rationalisable:

(i) For player 1 : a_1 is rationalised by belief $(0.5\, b_1, 0.5\, b_2)$; a_2 is rationalised by action b_2; a_3 is rationalised by action b_1; and a_4 is rationalised by belief $(0.25\, b_1, 0.75\, b_2)$.

(ii) For player 2 : b_2 is rationalised by a_1; and for the purposes of the example we will consider two different ways of rationalising b_1: it can be rationalised by action a_3 and by belief $(0.5\, a_2, 0.5\, a_4)$.

Every action has many ways of being rationalised, and this multiplicity is the potential source of inconsistencies. For the time being, let us consider fixed, for each action

different from b_1, the beliefs which rationalise them, as given by (i) and (ii). We will construct two infinite hierarchies of beliefs, which will satisfy the minimum consistency requirement, that will support the action a_1. These hierarchies of beliefs will correspond to the two different ways of rationalising the action b_1 given in (ii). Let us call the two infinite hierarchies of beliefs $s_1 = (s_{11}, s_{21}, s_{31}, \ldots)$ and $t_1 = (t_{11}, t_{21}, t_{31}, \ldots)$. The infinite hierarchy s_1 will support the action a_1 and rationalise b_1 by the first belief given in (ii): the action a_3. Analogously, t_1 will support action a_1 and rationalise b_1 by the second belief given in (ii): the belief $(0.5\, a_2, 0.5\, a_4)$. The first order beliefs s_{11} and $t_{11} \in \Delta(A_2) = S_{11}$ are the same: $s_{11} = t_{11} = (0.5\, b_1, 0.5\, b_2)$. The second order beliefs will differ : in s_{21} we rationalise b_1 by a_3, and in t_{21} we rationalise b_1 by $(0.5\, a_2, 0.5\, a_4)$. We have that s_{21} and $t_{21} \in \Delta(A_2 \times S_{12}) = S_{21}$, where $S_{12} = \Delta(A_1)$, are given by:

(a) $s_{21} = (\,0.5(\{b_1\}\times\{\delta_{a3}\}),\, 0.5(\{b_2\}\times\{\delta_{a1}\})\,)$, where δ_{a1} and δ_{a3} are the probability measures concentrated in a_1 and a_3, respectively; and

(b) $t_{21} = (\,0.5(\{b_1\}\times\{(0.5\, a_2, 0.5\, a_4)\}),\, 0.5(\{b_2\}\times\{\delta_{a1}\})\,)$.

These beliefs are consistent with s_{11} and t_{11} : $\mathrm{marg}_{A2}[s_{21}] = (0.5\, b_1, 0.5\, b_2) = s_{11}$

and $\text{marg}_{A_2}[t_{21}] = (0.5\, b_1, 0.5\, b_2) = t_{11}$. To construct the third order beliefs we must have the second order beliefs of player 2 which will rationalise the beliefs δ_{a3}, δ_{a1} and $(0.5\, a_2, 0.5\, a_4)$. Those are $s_{22}(1)$, $s_{22}(2)$, $t_{22}(1)$ and $t_{22}(2) \in \Delta(A_1 \times S_{11}) = S_{22}$:

$s_{22}(1) = \delta(\{a_3\}\times\{\delta_{b1}\})$;

$s_{22}(2) = \delta(\{a_1\}\times\{(0.5\, b_1, 0.5\, b_2)\})$;

$t_{22}(1) = (0.5(\{a_2\}\times\{\delta_{b2}\}), 0.5(\{a_4\}\times\{(0.25\, b_1, 0.75\, b_2)\}))$; and

$t_{22}(2) = \delta(\{a_1\}\times\{(0.5\, b_1, 0.5\, b_2)\})$.

The third order belief of the hierarchies s_1 and t_1 are s_{31} and $t_{31} \in \Delta(A_2 \times S_{22}) = S_{31}$, given by:

$s_{31} = (0.5(\{b_1\}\times\{s_{22}(1)\}), 0.5(\{b_2\}\times\{s_{22}(2)\}))$; and

$t_{31} = (0.5(\{b_1\}\times\{t_{22}(1)\}), 0.5(\{b_2\}\times\{t_{22}(2)\}))$.

It is immediate to verify that $\Psi_{21}(s_{31}) = s_{21}$ and $\Psi_{21}(t_{31}) = t_{21}$, so that the third order beliefs are consistent with the second order beliefs. The process above carries on to higher order of beliefs in a straightforward way: whenever it is necessary to rationalise the choice of an action by one of the players, we use the beliefs given by (i)

and (ii), which are fixed throughout. Thus, assume we constructed in this inductive manner the infinite hierarchies of beliefs s_1 and t_1. Define the infinite hierarchy of beliefs $r_1 = (s_{11}, s_{21}, t_{31}, t_{41}, \dots)$. This is an infinite hierarchy of beliefs which has rationality as common knowledge, and also supports action a_1. But it does not satisfy the minimum consistency requirement : $\Psi_{21}(r_{31}) = \Psi_{21}(t_{31}) = t_{21} \neq s_{21} = r_{21}$. The reason is clear: we are rationalising b_1 in two different ways in the different layers of beliefs.

The example above also shows a general procedure for proving Theorem 4.13 in the case of finite action spaces. First find the set of rationalisable actions. Every rationalisable action can be rationalised by a belief whose support consists only of rationalisable actions. For any rationalisable action fix a unique belief which rationalises this action, and whose support consists only of rationalisable actions. Then build up the orders of beliefs by rationalising each action always by the same belief. The result of this process will be an infinite hierarchy of beliefs which is consistent and for which rationality is common knowledge. Such a procedure will not work in the case of infinite action spaces. This is because if for each action we fix a belief that rationalises this action, then the set of beliefs necessary to rationalise all actions may be a nonmeasurable set. This is a nontrivial subtlety, which the lemmata below will solve: instead of fixing one belief for each action, we always consider the set

of all possible beliefs which rationalise that action. In general, the set of all possible beliefs which rationalise a compact set of actions is compact, and, therefore, Borel measurable. Several technical results are required before we can establish Theorems 4.12 and 4.13. These may be of interest in themselves for future applications of these kinds of structures.

<u>4.7 Lemma</u> Let A and B be compact metric spaces. Let $f : A \times B \to R$ be a continuous function. Consider the set:

$BR(A) = \{ a \in A \mid \text{there exists } \mu \in \Delta(B) \text{ with } a \in \mathrm{argmax}_{\tilde{a} \in A} \int f(\tilde{a}, b)\, d\mu(b) \}$

where the integral is over the set B whose generic element is b. [This set is the set of actions in A which are best responses against some mixed behaviour on B.] Then $BR(A)$ is nonempty, compact and metric.

<u>Proof</u> $BR(A) \neq \emptyset$ because given any $\mu \in \Delta(B)$, $\int f(.\,, b)\, d\mu(b)$ is a continuous function on the compact set A. It is sufficient then to check that $BR(A)$ is closed since it is a subset of the compact and metric space A. Moreover, it is enough to check sequentially. Let $a_n \in BR(A)$ converge to $a \in A$.

Then there exists $\mu_n \in \Delta(B)$ such that a_n is a best response to μ_n. $\Delta(B)$ is sequentially compact so there is a subsequence $\{ \mu_{nk} \}_{k \geq 0}$ which tends to

a measure $\mu \in \Delta(B)$. Let us rename this subsequence μ_n. By definition,

$\forall n : \int f(a_n, b)\, d\mu_n(b) \geq \int f(\bar{a}, b)\, d\mu(b) \quad \forall \bar{a} \in A$. Define $f_n : B \to R$ by

$f_n(b) = f(a_n, b)$ and $f : B \to R$ by $f^*(b) = f(a,b)$. Then $f_n \to f^* \quad \forall b \in B$. Also,

f_n is dominated by the function $g(b) \equiv \max_{A \times B} f(\cdot, \cdot)$. Therefore, by

Hildenbrand(1974) Chapter 1, D.I.42, we have $\int f_n d\mu_n \to \int f^* d\mu$. Thus,

taking the limits on both sides of the inquality above gives :

$\int f(a, b)\, d\mu(b) \geq \int f(\bar{a}, b)\, d\mu(b) \quad \forall \bar{a} \in A$. This implies $a \in BR(A)$.

QED.

4.8 Lemma Let A, B and f be given as in lemma 4.7. Define the correspondence $\nu : BR(A) \to \Delta(B)$ to be the following:

$\forall a \in BR(A)$, $\nu(a) = \{ v \in \Delta(B) \mid a \in \operatorname{argmax}_{\bar{a} \in A} \int f(\bar{a}, b)\, dv(b) \}$. Then

ν is upper semi-continuous.

Proof By Hildenbrand(1974) Chapter 1, B.III.Th.1, it is enough to check that

$a_n \to a$, $v_n \in \nu(a_n)$, $v_n \to v \quad \Rightarrow \quad v \in \nu(a)$. But since $v_n \in \nu(a_n)$, we have

as in lemma 4.7 : $\int f(a_n, b)\, dv_n(b) \geq \int f(\bar{a}, b)\, dv_n(b) \quad \forall \bar{a} \in A$. Therefore

by arguments identical to those of lemma 4.7 we have $v \in v(a)$.

QED.

Let G be the graph of the correspondence of the lemma above:

$G = \{ (a, v) \in BR(A) \times \Delta(B) \mid v \in v(a) \}$.

4.9 Lemma For $\mu_C \in \Delta(A)$ such that $\text{supp}\,\mu_C = C$ (where C is an arbitrary closed subset of $BR(A)$), there is a $\mu \in \Delta(A \times \Delta(B))$ such that :

(i) $\text{marg}_A[\mu] = \mu_C$;

(ii) $\text{supp}\,\mu$ is contained in the set $G_C = \{ (a, v) \in G \mid a \in C \}$.

Proof Given D a closed set contained in $\Delta(A \times \Delta(B))$, define $\mu(D) = \mu_C(\text{Proj}_A(D \cap G))$. This is possible because given $BR(A)$ is closed in A, one can view G as a closed subset of $A \times \Delta(B)$. Hence μ is well defined. Then extend μ to all the Borel subsets of $\Delta(A \times \Delta(B))$. Properties (i) and (ii) are easily verifiable.

QED.

4.10 Proposition $\forall i$, $\forall m$: A_{mi} is compact and metric.

Proof Straightforward repeated application of lemma 4.7.

QED.

4.11 Proposition $\forall i$, $\forall m$: K_{mi} is compact and metric.

Proof It is sufficient to show that K_{mi} is closed since it is a subset of S_i, which is compact and metric. We demonstrate the proposition by induction.

Fix i and consider m=1. $K_{1i} = \{ s_i \in S_i \mid \forall k \neq i : (a_k, s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$ $\Rightarrow a_k \in \text{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, s_k) \}$. Again it is enough to check that $s_{in} \to s_i$ and $s_{in} \in K_{1i} \Rightarrow s_i \in K_{1i}$. But as Φ_i is an homemorphism, $\Phi_i(s_{in}) \to \Phi_i(s_i)$. This implies that $\text{marg}_{A_k \times S_k}[\Phi_i(s_{in})] \to \text{marg}_{A_k \times S_k}[\Phi_i(s_i)]$. Let $(a_k, s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$ and let $B_\varepsilon(a_k, s_k)$ be an open ball of radius $\varepsilon > 0$ around it in $A_k \times S_k$. By weak convergence :

$\liminf \mu_n(B_\varepsilon(a_k, s_k)) \geq \mu(B_\varepsilon(a_k, s_k)) > 0$ where $\mu_n = \text{marg}_{A_k \times S_k}[\Phi_i(s_{in})]$ and $\mu = \text{marg}_{A_k \times S_k}[\Phi_i(s_i)]$. Therefore, for n large enough, $\mu_n(B_\varepsilon(a_k, s_k)) > 0 \Rightarrow$

$\Rightarrow \exists (a_{kn}, s_{kn}) \in \text{supp}\, \mu_n$, with $(a_{kn}, s_{kn}) \to (a_k, s_k)$. But then, since $s_{in} \in K_{1i}$,

$a_{kn} \in \text{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, s_{kn})$. By the closedness of the maximality condition,

we have that, taking limits, $a_k \in \text{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, s_k)$. Thus $s_i \in K_{1i}$.

Let us assume the induction hypothesis that $\forall i$, $\forall m$: K_{mi} is closed. We shall show that $K_{m+1,i}$ is closed. This is an argument very similar to the one for $m = 1$. The closedness of this restriction will come from the induction hypothesis : K_{mk} is closed for all k.

QED

We are now in the position to state and prove :

4.12 Theorem $\forall i$, $\forall m$, $\forall a_i \in A_{m+1,i}$, there exists $s_i \in K_{mi}$ such that

$a_i \in \text{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, s_i)$.

Proof We shall proceed by induction. For $m = 1$, by the definition of A_{2i}, if

$a_i \in A_{2i}$, there exists $\mu_i(a_i) \in \Delta(\Pi_{k \neq i} A_{1k})$ such that

$a_i \in \text{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, \mu_i(a_i))$ [an abuse of notation : $V(. , \mu_i(a_i))$ here simply means the expected value with respect to the first order belief $\mu_i(a_i)$].

Define $Y_{1k} : A_{1k} \to \Delta(A_{-k} \times S_{-k})$ by $Y_{1k}(a_k) =$

$\{ \mu \in \Delta(A_{-k} \times S_{-k}) \mid a_k \in \mathrm{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, \mu) \}$. Y_{1k} is clearly non-empty since by the definition of A_{1k}, for $a_k \in A_{1k}$, there exists $\mu_k(a_k)$ such that $a_k \in \mathrm{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, \mu_k(a_k))$. Then any $\mu \in \Delta(A_{-k} \times S_{-k})$ whose marginal on A_k is $\mu_k(a_k)$, is in Y_{1k}. Moreover, Y_{1k} is upper semi-continuous by lemma 4.8. Define $R_{1k} : A_{1k} \to S_k$ by $R_{1k} = [\Phi_k]^{-1} \circ Y_{1k}$. Since $[\Phi_k]^{-1}$ is continuous, R_{1k} is upper semi-continuous. Let GR_{1k} be the graph of R_{1k}. Then by a construction identical to that used in lemma 4.9, there is $\mu \in \Delta(A_{-i} \times S_{-i})$ such that $\mathrm{marg}_{A_{-i}}[\mu] = \mu_i(a_i)$ and $\mathrm{supp}[\mu] = \prod_{k \neq i} GR_{1k}$. Now let $s_i = [\Phi_i]^{-1}(\mu)$. By construction, $a_i \in \mathrm{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, s_i)$. Furthermore,

$(a_k, s_k) \in \mathrm{supp}\ \mathrm{marg}_{A_k \times S_k}[\Phi_i(s_i)] \Rightarrow (a_k, s_k) \in GR_{1k} \Rightarrow$

$a_k \in \mathrm{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, s_k)$. Hence, $s_k \in K_{1k}$. Thus $s_i \in K_{2i}$. Now we shall assume that the induction hypothesis is true for every k and $m-1 \geq 1$. We need to show it is true for $m \geq 2$. For $a_i \in A_{m+1,i}$ there exists $\mu_i(a_i) \in \Delta(\prod_{k \neq i} A_{mk})$ such that $a_i \in \mathrm{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, \mu_i(a_i))$. Define

$Y_{mk} : A_{mk} \to \Delta(\prod_{p \neq k} A_{m-1,p} \times \prod_{p \neq k} K_{m-2,p})$ by $Y_{mk}(a_k) =$

$= \{ \mu \in \Delta(\Pi_{p \neq k} A_{m-1,p} \times \Pi_{p \neq k} K_{m-2,p}) \mid a_k \in \text{argmax}_{\tilde{a}_k \in A_k} V(\tilde{a}_k, \mu) \}$. In this expression $K_{0p} = S_p$ for all p. This is non-empty since by the induction hypothesis there exists $s_k \in K_{m-1,k}$ such that $a_i \in \text{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, s_k)$.

By Theorem 4.3 $\Pi_{p \neq k} A_{m-1,p} \supset \text{supp marg}_{A_{-k}}[\Phi_k(s_k)]$. Moreover, by the definition of $K_{m-1,k}$, $s_p \in \text{supp marg}_{S_p}[\Phi_k(s_k)] \Rightarrow s_p \in K_{m-2,p}$, so that $s_k \in Y_{mk}(a_k)$. Furthermore, by lemma 4.8, Y_{mk} is upper semi-continuous. Notice also that $\mu \in Y_{mk}(a_k) \Rightarrow [\Phi_k]^{-1}(\mu) \in K_{m-1,k}$ by construction. Define $R_{mk} : A_{mk} \to K_{m-1,k}$ by $R_{mk} = [\Phi_k]^{-1} \text{o}\, Y_{mk}$. R_{mk} is upper semi-continuous.

Again by a construction identical to lemma 4.9, there exists $\mu \in \Delta(A_{-i} \times S_{-i})$ such that $\text{marg}_{A_{-i}}[\mu] = \mu_i(a_i)$ and $\text{supp}[\mu] = \Pi_{k \neq i} GR_{mk}$, where GR_{mk} is the graph of R_{mk}. Let $s_i = [\Phi_i]^{-1}(\mu)$. By construction, $a_i \in \text{argmax}_{\tilde{a}_i \in A_i} V(\tilde{a}_i, s_i)$.

Moreover, $(a_k, s_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)] \Rightarrow (a_k, s_k) \in GR_{mk} \Rightarrow$

$\Rightarrow s_k \in K_{m-1,k}$. Hence, $s_i \in K_{mi}$.

<u>QED.</u>

The final theorem of this section demonstrates that any rationalisable strategic behaviour can be justified by beliefs which satisfy "rationality is common knowledge".

<u>4.13 Theorem</u> Let $a_i \in \cap_{m\geq 1} A_{mi}$. Then there exists $s_i \in S_i$ such that

$s_i \in \cap_{m\geq 1} K_{mi}$ and $V(a_i,s_i) = \max_{\hat{a}_i \in A_i} V(\hat{a}_i,s_i)$.

<u>Proof</u> Since $a_i \in \cap_{m\geq 1} A_{mi}$, for all m, $a_i \in A_{m+1,i}$. So, by Theorem 4.12, there exists $s_{im} \in K_{mi}$ which rationalises a_i. Observe that $s_{i,m+p} \in K_{mi}$ for all $p\geq 0$. Since K_{mi} is compact and metric, we can find a convergent subsequence of the sequence $\{ s_{i,m+p} \}_p$ which converges to $s_i \in K_{mi}$. Moreover, s_i rationalises a_i and it is the limit of beliefs which satisfy successively higher orders of rationality. Let us show that $s_i \in K_{m+q,i}$ for all $q \geq 0$. Fix q. Then for $p\geq q$, $s_{i,m+p} \in K_{m+q,i}$. Therefore the subsequence has another subsequence which converges to a point in $K_{m+q,i}$. However, since the original subsequence was convergent to s_i, it must mean that $s_i \in K_{m+q,i}$ for all $q \geq 0$. Thus

σ_i satisfies rationality is common knowledge and rationalises a_i .

QED.

5. NASH EQUILIBRIUM BEHAVIOUR

5.1 Coordination and Nash Behaviour

In this section we consider the foundations for Nash equilibrium. We begin by exhibiting the result most theorists have in mind when they try to justify the use of Nash's noncooperative solution concept. Then, using the framework of Section 3 one we will describe the assumptions that underlie the Nash solution correspondence.

We will be dealing with two alternative manners of interpreting the concept of Nash equilibrium. The first, the classical view, is that the players should choose a Nash action. The second, a subjective interpretation, is that every player can be Bayesian rational and believe that everyone else follow their Nash actions. Some Nash equilibria are such that the Nash actions are not unique best responses against the beliefs that the other players follow their Nash actions. Therefore, the subjective interpretation does not imply the classical interpretation. This point is exemplified in subsection 5.3. In subsections 5.1 and 5.2 we will focus on the classical interpretation of Nash equilibrium. In subsections 5.3 and 5.4 we will focus on the subjective interpretation.

For simplicity, in this subsection, we will look at a selection (call it Γ_N) from

the Nash solution correspondence. This function associates to every n-tuple of payoff functions, a pure strategy Nash equilibrium. Obviously for this purpose we are looking at games where pure strategy Nash equilibria exist. We call a function with these properties a Nash theory.

The usual justification for the Nash equilibrium concept is that no player has any incentive to deviate from the action prescribed by the theory, if this player believes the other players are going to fulfill their rôle. This is expressed in the classical quote below, taken from Luce and Raiffa(1957, page 173) :

"Nonetheless, we continue to have one very strong argument for equilibrium points : if our non-cooperative theory is to lead to an n-tuple of strategy choices, and if it is to have the property that knowledge of the theory does not lead one to make a choice different from that dictated by the theory, then the strategies isolated by the theory must be equilibrium points".

As one can see, this justification is a simple restatement of the definition of a Nash equilibrium. In this subsection we give an alternative interpretation to Nash equilibrium points. The Nash equilibria are the only n-tuples of actions which are consistent with common knowledge of the actions taken, as well as of rationality. If one takes a theory to be single-valued, then the Nash equilibria are the only n-tuples of actions which are consistent with common knowledge of the theory and of rationality.

Fix a game $u \in U$. The formalisation of the knowledge of a theory by the players, is simply the fact that the actions this theory predicts are the only actions which are considered possible by the players. The notation is the same as in sections three and four. In particular, if one wants to refer to "knowledge of a theory Γ", where Γ is contained in $A = A_1 \times \ldots \times A_n$, we have :

<u>5.1.1 Definition</u> Given Γ contained in A, a theory, we say that <u>player i knows a theory</u> Γ when $s_i \in \Gamma_{1i} = \{ s_i \in S_i \mid Proj_{A_{-i}}\Gamma \supset \text{supp marg}_{A_{-i}}[\Phi_i(s_i)] \}$. In other words: player i knows a theory when he thinks other players are going to fulfill their rôle in this theory.

<u>5.1.2 Definition</u> <u>A theory Γ is common knowledge in the eyes of player i</u> if :

$s_i \in \cap_{m\geq 1}\Gamma_{mi}$, where: Γ_{1i} is given above, and

$\forall m\geq 2 : \Gamma_{mi} = \{ s_i \in \Gamma_{m-1,i} \mid \forall\, k\neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in \Gamma_{m-1,k} \}$.

The following theorem express this point in formal terms:

<u>5.1.3 Theorem</u> Assume that $\Gamma = \{(\bar{a}_1, \ldots, \bar{a}_n)\}$, that is to say, Γ is a single-valued

theory. Suppose there exists $i \in N$ such that rationality and the theory Γ are common knowledge in the eyes of player i. Then Γ is a Nash theory (that is to say: $(\tilde{a}_1, ..., \tilde{a}_n)$ is a Nash equilibrium of the game). Moreover, any Nash theory, Γ_N, is compatible with common knowledge of the theory and common knowledge of rationality.

Proof Since player i knows that player k is rational and player k knows the theory, it follows that $\tilde{a}_k$ is a best response to $\tilde{a}_{-k}$, for all $k \neq i$. To check that $\tilde{a}_i$ is a best response to $\tilde{a}_{-i}$, it is enough to carry the same argument above one more layer.

Observe that it was necessary to use only $s_i \in K_{2i} \cap \Gamma_{3i}$. The second part of the theorem is immediate.

QED.

The result above gives one set of behavioural assumptions which justifies the Nash equilibrium concept. This set of assumptions is the main thrust of Nash equilibrium. However, we feel that the theorem above also shows the weakness of the concept. In fact, the Nash equilibrium is played when the actions which are going to be taken are common knowledge, before they have been taken. It shows the strong need

for coordination in obtaining Nash behaviour. This is the rôle played by several of the "stories" to justify Nash equilibrium behaviour : they are mere coordination mechanisms. Famous examples of these stories are the "book of Nash" and the "gentlemen's club". The former is well known. The latter is simply a revised version of the former : every player should belong to the same gentlemen's club, where the club's statute tells them how to behave in a game-theoretic situation. As they are gentlemen (and very possibly English), they all give their word of honour they will follow this statute (Binmore(1984)).

The main purpose of the rest of this section is to give alternative sets of behavioural assumptions under which Nash equilibrium behaviour is justified.

5.2 Common Knowledge that Players May Play Nash Equilibrium

If one restricts the class of games to be considered, the coordination mechanism required to achieve a Nash equilibrium may be very reasonable. Bernheim(1984, section 5) and Moulin(1984) give examples of classes of games for which the set of rationalisable strategies and Nash equilibrium strategies coincide. For these restricted classes of games, common knowledge of rationality is enough to ensure that a Nash equilibrium is played. An important game that belongs to this class

is the Cournot duopoly with linear demand and constant marginal costs. However, this class of games is very restricted. If one considers the oligopoly above with three firms, instead of two, the result is not true anymore: there is a continuum of rationalisable actions, while only one Cournot-Nash equilibrium.

<u>5.2.1 Example</u> (Cournot oligopoly with linear demand and constant marginal costs.) Let there be n identical firms, each of them with maximum capacity 10. Suppose marginal costs are constant and equal to 1. The market inverse demand function is given by $P(Q) = \max\{10\text{-}Q,0\}$. The firms play with quantities in the fashion of Cournot. The strategy set of firm i is : $A_i = [0,10]$, with generic element q_i. The payoffs are given by the profit functions $\Pi_i(q_1, \ldots, q_n) = P(\Sigma q_K).q_i - q_i$. This game has a unique Cournot-Nash solution : all firms produce the quantity $q_i = 9/(n+1)$. When n=2 the only rationalisable action for a firm is the Cournot-Nash equilibrium $q_i = 3 = 9/3$ (see Bernheim(1984) and Moulin(1984)). For n=3 the set of rationalisable actions for each firm is the interval $[0, 9/2]$, that is to say, any quantity between zero and the monopoly level is rationalisable.

In this subsection we consider a weakening of the assumption that a Nash theory is common knowledge. We will assume that it is common knowledge that the

players may play a Nash theory. At the same time, we maintain the assumption of common knowledge of rationality. Therefore, the class of games for which these two assumptions are a sufficient coordination mechanism to achieve Nash equilibrium, is potentially larger than the classes of games considered by Bernheim and Moulin. We will show that this new class of games is indeed larger than theirs.

5.2.2 Definition Given Γ contained in $A = A_1 \times \ldots \times A_n$, a theory, we say that player i knows that other players may play the theory Γ when: $s_i \in \Gamma P_{1i} =$

$= \{ s_i \in S_i \mid \mathrm{Proj}_{A_{-i}} \Gamma \cap \mathrm{supp}\ \mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] \neq \varnothing \}$.

5.2.3 Definition (Knowledge and Common Knowledge that other players may play a Nash theory.) Given a game u and $(\bar{a}_1, \ldots, \bar{a}_n)$ a Nash equilibrium (in pure strategies) of this game, we say player i knows that other players may play it, if :

$s_i \in N_{1i} = \{ s_i \in S_i \mid \bar{a}_{-i} \in \mathrm{supp}\ \mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] \}$. In the same way we say that it is common knowledge in the eyes of player i that the Nash theory $(\bar{a}_1, \ldots, \bar{a}_n)$ may be played by other players, if $s_i \in \cap_{m \geq 1} N_{mi}$, where :

$\forall m\geq 2 : N_{mi} = \{ s_i \in N_{m-1,i} \mid \forall k\neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in N_{m-1,k} \}$.

The next proposition shows that the class of games for which the common knowledge of rationality and the common knowledge that players may play the Nash equilibrium is a sufficient coordination mechanism to attain Nash behaviour, is strictly larger than those classes of games provided by Bernheim and Moulin. We do this by showing that, in the Cournot oligopoly example seen above, when the number of firms is three, the common knowledge of rationality, as well as the common knowledge of the fact that the players may play their Cournot-Nash actions, yields the Cournot-Nash outcome .

5.2.4 Proposition Let the game be as in example 5.2.1, with $n=3$. Assume that rationality is common knowledge and that the possibility of playing the Cournot equilibrium is also common knowledge. Then, the only possible action taken by a rational firm is the Cournot-Nash equilibrium (which is $q_i = 9/4$).

Proof. The requirement above is that $s_i \in (\cap_{m\geq 1} K_{mi}) \cap (\cap_{m\geq 1} N_{mi})$ (*) and that every player is rational (the sets N_{mi} are generated according to definition 5.2.3, taking as $(\bar{a}_1, \ldots, \bar{a}_n)$ the triple (9/4 , 9/4 , 9/4)). First point to notice is that due to the symmetry of

the game, it is enough to concentrate the analysis in one particular firm. We are going to show that the only action which is compatible with rational behaviour and condition (*) is 9/4 . Rearranging (*) : $s_i \in \cap_{m\geq 1}(K_{mi} \cap N_{m+1,i})$. This allows us to reinterpret the assumption of the theorem. For example, $K_{1i} \cap N_{2i}$ means that not only player i thinks k is rational, but also that any action k takes may be rationalised by beliefs which contain the Cournot actions of the other players in the support.

One can easily see that (*) is verifyed if and only if : (i) i thinks the others may play (9/4, 9/4); and (ii) all actions i thinks k may take have to be rationalised by beliefs which comprehend the Cournot actions in the support, and using the symmetry of the game, every action in this support has to be rationalised by beliefs which contain the Cournot actions in the support, and so on. Let us study what happens in each mental interaction described above. Let q_i be an action which is a best response to a belief $\mu \in \Delta([0,10] \times [0,10])$. The first thing to notice is that $q_i \notin]9/2, 1]$. In fact, suppose not. One can check that the action 9/2 will give a higher payoff. Let us go to the second round (notice : we still did not use the fact that (9/4, 9/4) is in the support of μ). Then $[0,9/2]^2 \supset \mathrm{supp}[\mu]$, from the analysis above. In this case the response function can be computed and it is :

$q_i(\mu) = (1/2).(9-E_\mu(\Sigma_{k\neq i}q_k))$

Let q_{inf} and q_{sup} be the infimum and the supremum of the of the support after the infinite recursion. By the above formula, for every $q \in \text{supp}[\mu]$:

$q \leq (1/2).(9 - 2q_{inf})$ (A) , and $q \geq (1/2).(9 - 2q_{sup})$ (B) .

But the beliefs which support q must contain (9/4, 9/4) in the support. Thus inequality (A) must be strict if $q_{inf} \neq 9/4$, and inequality (B) in case $q_{sup} \neq 9/4$. Suppose one of the strict inequalities above holds, let us say (A). One can take q to be q_{sup} in (A) and q to be q_{inf} in (B), since the support is a closed set. Hence :

$q_{sup} < 9/2 - q_{inf}$ and $q_{inf} \geq 9/2 - q_{sup}$. This is a contradiction. Thus $q_{inf} = q_{sup} =$ $= 9/4$, and the proposition follows.

<u>QED</u>

The result above does not generalise. For the case of four firms we do not obtain the Cournot-Nash equilibrium as the only possible outcome:

<u>5.2.4 Example</u> (Common knowledge of rationality and of the possibility of a Nash theory being played not enough to obtain Nash equilibrium.) Consider the same game as above, with $n=4$. In this case the Nash equilibrium is (9/5, 9/5, 9/5, 9/5). We

show, for example, that 0 can be an outcome in this game. This follows from the observations in the proof of the proposition above, plus the fact that :

(i) 0 is the best response to a belief which assigns probability 18/23 to (10/3, 10/3, 10/3), and 5/23 to actions (9/5, 9/5, 9/5);

(ii) 10/3 is the best response against a belief which assigns probability 46/81 to (0, 0, 0) and 35/81 to (9/5, 9/5, 9/5);

From this example one sees the need to investigate further on the foundations of Nash behaviour : the mere common knowledge of the possibility of a Nash theory being played does not imply Nash behaviour, even in an example with a unique Nash equilibrium (with or without mixtures) whose actions have the property of being unique best responses given the actions of the others. The next subsection will present another set of behavioural assumptions which will yield Nash behaviour for any two- person game. The assumptions and the main result are taken from Armbruster and Böge(1979).

5.3 The Knowledge of the Other Players and Nash Equilibrium

In this subsection we will focus on an alternative justification for the concept of

Nash equilibrium. We use a subjective interpretation of mixed strategy Nash equilibria. In this interpretation the belief of every player about other players coincides with the mixed strategy part of the other players in the Nash equilibrium. It is important to note that this does not imply that the players should play his/her part of the Nash equilibrium. The example below, due to Myerson, illustrates the point. There are two players, with action spaces given by $A_1 = \{u, d\}$ and $A_2 = \{l, r\}$. The payoff functions are given by:

		II	
		l	r
I	u	(1, 1)	(1, 1)
	d	(1, 1)	(0, 0)

In this example the Nash equilibrium (u, l) could be the only possible belief in both players' minds. However, the two Bayesian rational players could actually play (d, r) which is not a Nash equilibrium of this game. This problem arises because the Nash actions are not unique best responses. For this reason the subjective interpretation of Nash equilibrium is not as compelling as the classical interpretation. Nevertheless, this subjective view sheds light in some properties of Nash equilibria, as we can see in

this and the next subsections. To differentiate the subjective view from the classical view, we will define the former as being a belief that the Nash equilibrium is played.

<u>5.3.1 Definition</u> Let $(\mu_1, ..., \mu_n)$ be a mixed strategy Nash equilibrium for the game u, where $\mu_i \in \Delta(A_i)$. We say that <u>the n-tuple of psychologies $(s_1, ..., s_n)$ believes the Nash equilibrium</u> $(\mu_1, ..., \mu_n)$ if for all i : $marg_{A_{-i}}[\Phi_i(s_i)] = \otimes_{k \neq i} \mu_k =$

$= \mu_1 \otimes ... \otimes \mu_{i-1} \otimes \mu_{i+1} \otimes ... \otimes \mu_n$.

The main result of this subsection is due to Armbruster and Böge(1979). It says that for two players, if rationality is common knowledge, and if each player knows the other player, then they play a mixed strategy Nash equilibrium.

<u>5.3.2 Definition</u> Given an n-tuple $(s_1, ..., s_n)$ of psychologies, we say that <u>player i knows the other players</u> if : supp $marg_{S_{-i}}[\Phi_i(s_i)] = \{s_{-i}\}$.

This definition simply says that player i thinks that the only possible (n-1)-tuple of psychologies of other players is the actual one: $s_{-i} =$

$= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

The following theorem is a characterisation of Nash equilibria in two-person games. The first part of the theorem below is in Armbruster and Böge(1979).

5.3.3 Theorem Let u be a two-person game. Suppose rationality is common knowledge, and that player 1 knows player 2 and player 2 knows player 1. Then they believe a mixed strategy Nash equilibrium of the game u. Conversely, if (μ_1, μ_2) is a mixed strategy Nash equilibrium of u, there are psychologies (s_1, s_2) such that rationality is common knowledge, and each player knows each other, with the property that (s_1, s_2) believes (μ_1, μ_2) in the sense of definition 5.3.1.

Proof Consider the pair $\mu_1 = \text{marg}_{A1}[\Phi_2(s_2)]$ and $\mu_2 = \text{marg}_{A2}[\Phi_1(s_1)]$. We know that $\forall a_1 \in \text{supp}\, \mu_1$, a_1 is a best response to μ_2, since player 2 thinks player 1 is rational, and $\text{marg}_{S1}[\Phi_2(s_2)] = \{s_1\}$. Similarly, $\forall a_2 \in \text{supp}\, \mu_2$, a_2 is a best response to μ_1. Thus (μ_1, μ_2) is a mixed strategy equilibrium of the game u. Conversely, suppose (μ_1, μ_2) is a mixed strategy Nash equilibrium of the game u. One can

construct the infinite hierarchies of beliefs (s_1, s_2) which will believe (μ_1, μ_2) by rationalising in each round every point in the support of one of the mixed strtagies by the mixed strategies of the opponent. These infinite hierarchies of beliefs will obviously satisfy the requirements of the theorem.

QED.

Unfortunately the result above is not true for games with more than two players. Consider a situation with three players. Each player has beliefs over the actions of the other players. Suppose these beliefs satisfy the following condition: for each player i, the support of the beliefs on the actions of player k ($k \neq i$) is contained in the set of best responses of player k against player k's beliefs over actions of players who are not k. If there were only two players, the condition above would imply that the two players believed a mixed strategy Nash equilibrium, according to definition 5.3.1. With three players, the situation changes. It is not necessarily true that these players have a common prior. Thus, even when all the three players know each other, it is possible that they do not believe a Nash equilibrium: this because they may hold priors about the actions of others which are not consistent with a common prior. The next example will illustrate this point in formal terms.

5.3.4 Example (Common Knowledge of Rationality and Knowledge of Each Other

Does Not Imply Nash Beliefs in Three-person Games.) There are three players. The pure strategy sets are: $A_1 = \{u, d\}$, $A_2 = \{a, b\}$ and $A_3 = \{L, R\}$. The payoffs are given by the two matrices below. The matrix on the left corresponds to player three playing L; the matrix on the right, R.

		II	
		a	b
I	u	(3, 2, 0)	(2, 4, 2)
	d	(1, 3, 2)	(3, 2,-4)

III - L

		II	
		a	b
I	u	(4,-3, 1)	(0,-1, 0)
	d	(0, 1,-3)	(5, 0, 6)

III - R

Define $\mu_{ij} \in \Delta(A_j)$, for $i \neq j$, and $i,j = 1,2,3$, by:

$\mu_{12} = (1/2\ a,\ 1/2\ b)$, $\mu_{13} = (1/2\ L,\ 1/2\ R)$;

$\mu_{21} = (1/3\ u,\ 2/3\ d)$, $\mu_{23} = (1/3\ L,\ 2/3\ R)$;

$\mu_{31} = (2/3\ u,\ 1/3\ d)$, $\mu_{32} = (2/3\ a,\ 1/3\ b)$.

Then, we have:

A_1 = set of best responses to $\mu_{12} \otimes \mu_{13} = v_1$;

A_2 = set of best responses to $\mu_{21} \otimes \mu_{23} = v_2$;

A_3 = set of best responses to $\mu_{32} \otimes \mu_{31} = \nu_3$.

We now construct three infinite hierarchies of beliefs (s_1, s_2, s_3) such that for every i :

$\text{marg}_{A_j \times A_k}[\Phi_i(s_i)] = \mu_{ij} \otimes \mu_{ik}$ for $j,k \neq i$, with $j \neq k$. These hierarchies of beliefs will be such that rationality is common knowledge and for all i :

$\text{supp marg}_{S_j \times S_k}[\Phi_i(s_i)] = \{(s_j, s_k)\}$ for $j \neq k$, and $j,k \neq i$ (this means that each player knows the other two players). The construction is simultaneous. The first order beliefs, s_{11}, s_{12}, s_{13} are given by ν_1, ν_2, ν_3, respectively. The higher order beliefs will be all constructed in the same fashion as the second order beliefs. For example,

$s_{21} \in \Delta(A_2 \times A_3 \times S_{12} \times S_{13})$ is given by : $s_{21} = s_{11} \otimes \delta\{(s_{12}, s_{13})\}$, where $\delta\{\,.\,\}$ is the probability measure which puts mass 1 on the set $\{\,.\,\}$. The hierarchies of beliefs thus built are clearly consistent and satisfy the properties required above. However,

$\mu_{21} \neq \mu_{31}$, $\mu_{12} \neq \mu_{32}$, $\mu_{13} \neq \mu_{23}$. Therefore the triple (s_1, s_2, s_3) does not believe a mixed strategy Nash equilibrium.

This subsection presented a very intuitive set of behavioural assumptions

under which Nash equilibrium is played in a two-person game. This same set of assumptions is not sufficient to generate Nash belief in a three-person game (and, therefore, n-player, n>2). The next subsection will provide sufficient conditions for Nash equilibrium which are a generalisation of the conditions of Theorem 5.3.3 and of Theorem 5.1.3 for two-person games. Also, a set of sufficient conditions for Nash behaviour is provided for n-person strictly concave games which generaliseTheorem 5.1.3 when applied to strictly concave games.

5.4 The Exchangeability Hypothesis and Nash Equilibrium

In this subsection we generalise Theorem 5.3.3 and Theorem 5.1.3 (in the case of two-person games or strictly concave games). There is an assumption about psychologies which is crucial for these generalisations. This is the exchangeability hypothesis. Formally, we have:

5.4.1 Definition The Exchangeability Hypothesis. We say that the exchangeability condition holds for player i if $s_i \in E_{1i} = \{ s_i \in S_i \mid \forall k \neq i : \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)] = C_k \times D_k$ for some C_k contained in A_k, and D_k in $S_k \}$. We say it is common

knowledge in the eyes of agent i when $s_i \in \cap_{m\geq 1} E_{mi}$, where :

$\forall m\geq 2 : E_{mi} = \{ s_i \in E_{m-1,i} \mid \forall k\neq i : s_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)] \Rightarrow s_k \in E_{m-1,k} \}$.

In words : the exchangeability hypothesis means that if an action by player k, a_k, is considered possible by player i, then he also considers it possible when player k is of any of the types s_k he believes player k can be. This is certainly a very strong hypothesis, but it is weaker than requiring that the beliefs of player i about actions of other players and types (or psychologies) of other players are independently distributed.

The first result that we provide generalises Theorem 5.3.3 and Theorem 5.1.3 for the case of only two players. An additional assumption about beliefs is needed. This assumption says that each player considers possible that the infinite hierarchy of beliefs of the other player is what it really is. That is to say, the players are not totally wrong about each other:

5.4.2 Definition An n-tuple of players' psychologies $(s_1, ..., s_n)$ is said to satisfy direct consistency when for all i , it happens that $s_{-i} \in \text{supp marg}_{S_{-i}}[\Phi_i(s_i)]$.

With these two hypotheses we have, then:

<u>5.4.3 Theorem</u> Let u be a two-person game. Suppose (s_1, s_2) are such that :

(i) $s_i \in K_{1i} \cap E_{1i}$, for $i = 1,2$; and (ii) (s_1, s_2) are directly consistent. Then (s_1, s_2) believes a Nash equilibrium. Conversely, any Nash equilibrium can be believed by psychologies which obey (i) and (ii).

<u>Proof</u> Let $a_2 \in$ supp $\mathrm{marg}_{A2}[\Phi_1(s_1)]$. By $s_1 \in K_{11} \cap E_{11}$, a_2 is a best response against any belief in supp $\mathrm{marg}_{S2}[\Phi_1(s_1)]$. In particular, by direct consistency, a_2 is a best response to belief s_2. So that a_2 is best response to $\mathrm{marg}_{A2}[\Phi_1(s_1)]$. In the same way, $a_1 \in$ supp $\mathrm{marg}_{A1}[\Phi_2(s_2)]$ implies a_1 is a best response to $\mathrm{marg}_{A1}[\Phi_2(s_2)]$. Thus, (μ_1, μ_2) given by $(\mathrm{marg}_{A1}[\Phi_2(s_2)], \mathrm{marg}_{A2}[\Phi_1(s_1)])$ is a Nash equilibrium. Hence, (s_1, s_2) plays the mixed strategy Nash equilibrium (μ_1, μ_2). The converse follows from the converse of Theorem 5.3.3.

<u>QED.</u>

We can also generalise Theorem 5.1.3 when applied for strictly concave

games. This involves the exchangeability hypothesis, as well as a plausible assumption: the assumption that each player thinks that the other players may think that a Nash equilibrium is being played.

<u>5.4.4 Theorem</u> Let $s_i \in K_{1i} \cap E_{2i}$. Suppose u is a game where $(\tilde{a}_1, ..., \tilde{a}_n)$ is a Nash equilibrium such that every action $\tilde{a}_j$ is the unique best response against $\tilde{a}_{-j}$ (in particular any strictly concave game will do). Assume that s_i is an element of the set

$\{ s_i \in S_i \mid \forall k \neq i$: there exists $s_k \in$ supp $\mathrm{marg}_{S_k}[\Phi_i(s_i)]$, such that

$\mathrm{marg}_{A_{-k}}[\Phi_k(s_k)] = \delta_{\{\tilde{a}_{-k}\}}$. Then, if player i is rational, he will choose $\tilde{a}_i$, the Nash action. Notice that uniqueness of Nash equilibiria is not required. Conversely, any Nash equilibrium with the properties above may be played by hierarchies of beliefs with the properties above.

<u>Proof</u> Since there exists $s_k \in$ supp $\mathrm{marg}_{S_k}[\Phi_i(s_i)]$, such that $\mathrm{marg}_{A_{-k}}[\Phi_k(s_k)] = \delta_{\{\tilde{a}_{-k}\}}$, and since $\tilde{a}_k$ is unique best response to $\tilde{a}_{-k}$, then supp $\mathrm{marg}_{A_k}[\Phi_i(s_i)] = \{ \tilde{a}_k \}$ by K_{1i} and E_{2i}. But player i is rational,and again, $\tilde{a}_i$ is unique best response to $\tilde{a}_{-i}$, so that player i chooses $\tilde{a}_i$. The converse of the theorem is a direct consequence of the

converse of Theorem 5.1.3.

<u>QED.</u>

It is interesting to notice that there are several instances where Nash equilibria are believed (in the sense of definition 5.3.1) in which the exchangeability hypothesis is necessary. To see that, suppose $(s_1, \ldots, s_n)$ are psychologies of an n-person game u. Assume that $(\mu_1, \ldots, \mu_n)$ is a mixed strategy Nash equilibrium of the game u, and that $(s_1, \ldots, s_n)$ believes $(\mu_1, \ldots, \mu_n)$ in the sense of definition 5.3.1, that is to say: for all i $\mathrm{marg}_{A_{-i}}[\Phi_i(s_i)] = \otimes_{k \neq i} \mu_k$. Two hypotheses will imply the necessity of exchangeability. The first hypothesis assumes that every $t_k \in$ supp $\mathrm{marg}_{S_k}[\Phi_i(s_i)]$ is such that t_k thinks the Nash equilibrium $(\mu_1, \ldots, \mu_n)$ is believed. This hypothesis requires very little justification : it is very unlike Nash equilibrium to suppose it is being played without supposing other people think so also. To contradict it would be the same as saying that the players got to the Nash point by mere coincidence, which sounds extremely odd. The other assumption is less intuitive. It is a principle of a priori ignorance. Given that a belief t_k is considered possible by player i, any a_k

which is a best response to t_k must be considered possible of being played by t_k, in the eyes of player i. Notice that we do not require player i to consider all best responses equally likely. We only need player i to consider that all actions which are best responses for t_k are possible of being played by t_k. We conclude this section by stating the "necessity" of the exchangeability hypothesis:

5.4.5 Theorem Suppose $(s_1, ..., s_n)$ is such that they believe the mixed strategy Nash equilibrium $(\mu_1, ..., \mu_n)$. Assume that the players think that other players think this Nash equilibrium is believed. Finally, suppose that the ignorance principle holds, that is to say: $\forall i, \forall k \neq i : t_k \in \text{supp marg}_{S_k}[\Phi_i(s_i)]$, and if a_k is a best response to $\text{marg}_{A_{-k}}[\Phi_k(t_k)]$ then $(a_k, t_k) \in \text{supp marg}_{A_k \times S_k}[\Phi_i(s_i)]$. Then: $\forall i: s_i \in E_{1i}$.

Proof Immediate.

QED.

5.5 Correlated Equilibrium - A Result by Aumann

Aumann(1985) studies the concept of correlated equilibrium in the same way we analyse rationalisable strategic behaviour. It is possible to provide quite thorough foundations for correlated equilibrium behaviour. In order to do this, the framework of section three has to be modified to include player i's actions in his/her own beliefs. Thisis necessary because correlated equilibrium assumes cooperative-like behaviour : although player i is a selfish maximiser, she/he knows that everyone correlates decisions. One can prove theorems for correlated equilibrium which are exact analogues of Theorems 4.5 and 4.13. For this one has to add the hypothesis that there is a common prior on the actions of everyone, and this prior is common knowledge.

The result proved in Aumann(1985) is the following analogue to Theorem 4.5: "if rationality as well as a common prior is common knowledge, then the players play a correlated equilibrium". In the same way as we proved a converse of Theorem 4.5 (that is to say, Theorem 4.13), we can prove a theorem which is the converse of Aumann's: "any correlated equilibrium can be played by players for whom rationality is common knowledge and a common prior is common knowledge". These two theorems give a set of behavioural assumptions under which correlated equilibria are justified.

Every Nash equilibrium is a correlated equilibrium where the common prior is independent across players. Thus, using the results above, one can give an alternative foundation for Nash equilibria. It is enough to assume that rationality as

well as an independent common prior are common knowledge. In other words, in addition to assuming that a common prior is common knowledge, one must also assume that it is common knowledge that the agents act independently.

5.6 Conclusion

This essay dealt with the foundations of noncooperative solution concepts in simultaneous games. It is emphasised that the knowledge and common knowledge of certain characteristics of the players plays a central rôle in the choice of the solution concept. Suppose that there are n players. One says that a statement is common knowledge if everyone knows it, everyone knows everyone knows it, ..., everyone knows (everyone knows)$^{m-1}$ it, and so on, for all m. Two ways of formalising the notion of common knowledge have been considered. One is given by Aumann(1976), and the other one is based on the idea of an infinite hierarchy of beliefs (Armbruster and Böge(1979), Böge and Eisele(1979), Mertens and Zamir(1985)). Aumann defines the notion of an event being common knowledge; however, the definition requires that we understand what it means for the structure of the uncertainty in a game to be common knowledge. In other words, a serious shortcoming of Aumann's formalisation is that it is self-referential. In section two we began by observing that the second definition of

common knowledge overcomes this difficulty; that is, the second definition of common knowledge is not self-referential. When one assumes that the structure of uncertainty is common knowledge in the sense of that definition, an event is common knowledge in the sense of Aumann if and only if it is common knowledge in the sense of the second definition. Thus, Aumann's definition can be embedded in the more general framework.

Section three began with a discussion of games and solution concepts. Bernheim(1984) and Pearce(1984) show that common knowledge of rationality is not enough to justify Nash behaviour. They introduced a noncooperative solution concept which is derived from the hypothesis that Bayesian rationality is common knowledge. They call their solution concept rationalisable strategic behaviour. The point I wish to emphasise is that Bernheim and Pearce derived their solution concept for games from assumptions about the behaviour of the players. One can quite generally approach the analysis of solution concepts in the same manner. Which are the implicit behavioural assumptions behind a given solution concept? From a Bayesian point of view, the decision of each player in a game is determined by this player's beliefs about the actions of other players. But, if, in their turn, other players' beliefs about other players' actions affect their actions, then it must be that the beliefs one player has about the beliefs of other players also affect the decision of this player in the game. If

we carry this argument further, we see that the action taken by a player is determined by his infinite hierarchy of beliefs about actions of other players, beliefs about beliefs about other players' actions, and so on. The space of these infinite hierarchies of beliefs is the appropriate space for the study of behavioural assumptions about the players. Section three deals with this matter in detail, and poses formally the relationship between solution concepts and behavioural assumptions implicit in them.

As an illustration of this formalism, section four, taken from Tan and Werlang(1984), discussed the solution concept given by rationalisable strategic behaviour[2]. The main behavioural assumption to be considered is that of common knowledge of Bayesian rationality. Bernheim(1984) and Pearce(1984) argue that common knowledge of Bayesian rationality implies the choice of a rationalisable action. They also argue the converse: any rationalisable action can be chosen by players for whom Bayesian rationality is common knowledge. Although the proof of this new result corresponds directly to the intuition when the action spaces of the players is finite, some subtle measurability issues arise for infinite action spaces.

The fifth section dealt with Nash equilibrium behaviour. It started by formally stating a justification for Nash behaviour which is closely related to the classical one. Not only rationality should be taken as common knowledge, but also the actions to be chosen. This allows one to see how strongly coordinated the players have to be.

When one relaxes this hypothesis sligthly, everything breaks down. Another behavioural assumption is studied: that each player "knows" the other players. When the game has two players, Armbruster and Böge(1979) proved that this yields Nash equilibrium beliefs. We give an example that this is not enough for Nash equilibrium beliefs in the case of three (and consequently more than two) players. Then, we provide theorems that generalise the results which justify Nash behaviour. Finally, we point out how our analysis can be modified so that we can derive foundations for another equilibrium concept : correlated equilibrium. This is a result obtained by Aumann(1985).

I hope that the method proposed in the thesis proves useful in deciding among the several noncooperative equilibrium concepts suggested in the literature.

FOOTNOTES

1. The relation with the recent results of Brandenburger and Dekel(1985b) is discussed in subsection 2.5.

2. We use a slightly modified version of Bernheim and Pearce's definition of rationalisable action. See section four, second paragraph.

3. The approach I am going to take here does not require the existence of a prior P on the space Ω (or of different priors $P^1, \ldots, P^n$ for each of the agents). One could require the existence of such priors. In this case definitions 2.4.1 and 2.4.2 would be different. See footnote number 6.

4. There are differences between the words believe and know. For example, it is possible to believe true something which is false, but it is not possible to know something which is false. For this see Hintikka(1962) and Fagin, Halpern and Vardi(1984). These differences are, however irrelevant in our case : we assume agent i behaves according to things he believes true, being it or not false. Thus

knowledge and belief are the same concept in these probabilistic models.

5. This definition was formally stated in Tan and Werlang(1985a,1985b). It was given in Armbruster and Böge(1979), Böge and Eisele(1979) and Mertens and Zamir(1985). In a recent paper, Brandenburger and Dekel(1985b), a definition of common knowledge of an event is provided which is equivalent to definition 2.3.3.

6. If priors $P^1, \ldots, P^n$ are included in the specification of the model, the main theorem remains valid, as long as Ω is finite and $P^i(\{\omega\}) > 0$ for all i and ω. In this case, the definitions 2.4.1 and 2.4.2 have to be changed to include these priors. There are two ways of defining the common knowledge of the information partitions together with the priors. One takes an ex-post view of the occurrence of the state of the world. The other an ex-ante view. They are :

(i) The ex-post definitions :

a. In 2.4.1, P_{1i} should be read as

$$P_{1i} = \{ s_i \in S_i \mid \forall k \neq i : (\omega, s_k) \in \operatorname{suppmarg}_{\Omega \times S_k}[\Phi_i(s_i)] \Rightarrow$$

$$\Rightarrow \forall X \in \Sigma, \operatorname{marg}_{\Omega}[\Phi_k(s_k)](X) = P^k(X \mid \Pi_k)(\omega) \}.$$

b. In 2.4.2 , ω occured in the eyes of agent i when

$$\forall X \in \Sigma\,,\ \mathrm{marg}_{\Omega}[\Phi_i(s_i)](X) = P^i(X \mid \Pi_i)(\omega).$$

(ii) The ex-ante definitions

a. In this case, instead of the P_{1i} in definition 2.4.1, one should have :

$$P_{1i} = \{ s_i \in S_i \mid \forall k \neq i : s_k \in \mathrm{supp}\ \mathrm{marg}_{S_k}[\Phi_i(s_i)] \Rightarrow$$

$$\Rightarrow \forall X \in \Sigma\,,\ \mathrm{marg}_{\Omega}[\Phi_k(s_k)](X) = \int P^k(X \mid \Pi_k)(\omega')\, d\, \mathrm{marg}_{\Omega}[\Phi_i(s_i)](\omega') \}\,.$$

b. Definition 2.4.2 would be the same as (i)b.

The intuition behind the first definition is clear. The intuition behind the second definition is also simple. It says that agent i is aware that Π_k is the information sub-σ-algebra of agent k , but since the real state of the world is unknown to agent i , the best he can do is impute to agent k the weighted average of the conditional probabilities - the weights coming from i's prior on the state of nature. When Ω is infinite, several additional assumptions are needed on the topology of Ω, and on the sub-σ-algebras. A method following closely that of Brandenburger and Dekel(1985a), or Nielsen(1984) is necessary.

7. Binmore in two delightful recent papers, Binmore(1984,1985), discusses issues which are very related to the ones in here. See also Reny(1985) and Basu(1985).

8. If there are only two players, the requirement that one player may believe the others are correlating is irrelevant. However, in the case of $n \geq 3$ players, if correlation of the strategies of the other players is not allowed, it is not the case that a strategy which is never a best response is a strictly dominated strategy. The example which follows (taken from Pearce(1982)) proves that this is so: there are three players, the first of whom has pure strategies a_1, a_2 and a_3. Players two and three each play H or T. Only the payoffs to player one matter here. They are the payoffs shown below:

	HH	HT	TH	TT
a_1	6	6	6	6
a_2	10	10	10	0
a_3	0	10	10	10

In this game, if players two and three cannot correlate their choices, there is no belief over two's and three's strategies for which a_1 is a best response. However, one can

easily see that a_1 is not a dominated action.

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