Money with Bank Networks*

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Abstract

We allow banks to choose between two networks in a simple version of the Cavalcanti and Wallace (1999) model of inside money. Members of a network have access to credit but must redeem banknotes issued by other members in random meetings. We find equilibria in which members of a particular network issue more valuable notes, but face the same ex-ante payoff as that of their competition . Banks are shown to be concerned with both credit externalities and with monetary liabilities. When the size of the bank sector is small, these two opposing forces may result in a stable equilibrium.

1 Introduction

Virtually all models of money abstract from issues relating bank competition to monetary stability. The main reason is that most existing models of banking are either aggregative or ignore the role of media of exchange. As a result, they offer few insights on how activities related to the provision of means of payments interact with measures of profitability and concentration in the banking industry. In this paper, we offer a simple alternative by amending an existing model of inside money, due to Cavalcanti and Wallace (1999), and allowing for the formation of two competing networks, instead of a single monopoly, in the provision of media of exchange.

Although the model is highly stylized, it is sufficiently rich to offer insights about two forces shaping the degree of concentration in the banking business. The first force is the externality associated with the size of a network. We assume that trades among members of the same network are conducted by credit, and that the network has the power to trigger a severe punishment of any member defecting from credit obligations. The larger the size of a network, the higher is the probability that a given member will trade by credit, which is a highly efficient mechanism for exchange. Although not emphasized by the authors, the *credit externality* is a distinguished feature of the Cavalcanti and Wallace (1999) model. This force alone leads banks to choose the largest available network and thus to the formation of a monopoly in the provision of inside money.

The second force is what we call a *money externality* and is related to the number of notes that a given network has in circulation. We assume that members of the same network must issue notes identified with this network. Moreover, in order to give notes a sustained value, we assume that members must redeem their network's notes, by producing for the nonbank public, or banks of a different network, that present their notes for redemption. With this kind of regulation in effect, it follows that a large network issues notes more often than a smaller one, so that the former faces a larger total liability in terms of potential redemptions than the latter. This force alone leads banks to avoid large networks and thus to compete in the provision of liquidity. We find that when the total measure of potential banks is sufficiently small, which is assumed to be a parameter of the model as it is in Cavalcanti and Wallace (1999), the money externality counteracts the credit externality and may yield an equilibrium which is stable in an ex-ante sense, to be defined formally below.

Although the study of the private provision of money in random-matching models is a fairly recent research topic, some of this recent work offers insights into related banking issues. Cavalcanti, Erosa and Temzelides (1999) ignore cooperation among banks, but emphasizes the redemption of liabilities as an important device for limiting bank profitability ("the law of reflux"). Cavalcanti and Wallace (1999) go to the other extreme and apply mechanism design, with a focus on optimal allocations, in which all banks are regulated as a single network. In

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this paper, we fix the regulation arbitrarily and assume that goods are indivisible in order to simplify the analysis. We discuss, however, how the results would change with different network rules, confronting a case in which one network accepts notes from its competition, with another in which it does not.

Williamson (1999), and Huang, He and Wright (2003), study a model of banking and media of exchange, but the bank sector in their models can be aggregated into a single unit. More recently, Monnet (2003) considers alternative providers of money, including private agents and suppliers of public goods, but there is no room for coalitions in his model due to the total absence of monitoring. The classic mechanism-design reference on the topic of bank intermediation is Diamond and Dybvig (1983), but theirs is also an aggregative model which abstracts from money. Cavalcanti (2003) shows that some aspects of the intermediation addressed by Diamond and Dybvig (1983) can be introduced into a version of the Cavalcanti and Wallace (1999) model, with deposits of money and capital, but he only studies optimal allocations and ignores bank competition.

In the rest of this paper we present our regulated banking environment, describe benchmark allocations, set participation constraints that this allocations must satisfy to be implementable and define as well as prove the existence of stable and unstable equilibria for this benchmark allocations.

2 A regulated banking environment

Like Cavalcanti and Wallace (1999), we amend a basic random-matching world to allow a fraction B of the population, where $B \in [0,1]$ is a parameter, to have their individual histories monitored. We shall allow, however, two coalitions to be formed: the *red network,* of measure x_0^r, and the *green network,* of measure $B - x_0^r$, where x_0^r is an endogenous variable to be set in equilibrium. We are thus departing from the strict mechanism-design approach of Cavalcanti and Wallace (1999), as it will be clear, since we are not necessarily studying optima, which essentially displays only one kind of network. Hence, we shall describe the physical environment together with a set of rules, best viewed as an exogenous banking regulation, that allows for some degree of bank competition through a choice of networks.

Time is discrete and the horizon is infinite. There are K types of perishable goods at each date and a $[0,1]$ continuum of each of K specialization types of people, where $K > 2$: a person whose type is k consumes only good k and produces only good $k+1$ (modulo K), for $k = 1, 2, ..., K$. Each person maximizes expected discounted utility with intertemporal discount factor $\beta \in (0,1)$. We assume, for simplicity, that each good is indivisible, and that each person can only produce either 0 or 1 unit in a given period. Utility for any type s is u, when consuming one unit of good s, and -1 when producing one unit of good $s+1$. Hence, period utilities are symmetric across types. We assume, moreover, that $u > 1$.

In addition to their distribution across the K specialization types, people are divided according to what becomes known about their trading histories. More specifically, in each period, people are randomly matched in pairs. Because $K > 2$, meetings are either *no-coincidence* meetings, when neither person produces what the other consumes, or *single-coincidence*, when type k person (the producer) meets a type $k+1$ person (the consumer). We call *bankers* the people whose actions in meetings are monitored, and can possibly be induced to produce in exchange for a future reward, and call *nonbankers* the people whose identities in meetings are unknown and, due to this anonymity, only produce in exchange for monetary assets. That is so because we also assume, as it is customary with this class of models, that people cannot commit to future actions. We shall also keep the tradition of restricting money holdings of individuals in the economy to either 0 or 1 units of money. However, since bankers are monitored, they can be induced to follow some network rules that result in limited forms of credit. Specialization types, money holdings and network membership of any individual are public information in meetings.

The main structure of the Cavalcanti and Wallace (1999) model is the fact that, using trigger strategies, society could induce a particular bank behavior by threatening banks to never consume after a defection to a planner's allocation is detected. Although individual's rationality allows a choice of no-trade in any meeting, a banker who refuses to produce can be identified and punished by others with autarky. One question in their model thus becomes what kind of allocations can be implemented when the participation constraints implied by the environment are imposed. In particular, they are interested either in a particular comparison between inside and outside money (when money creation is prohibited) or in the (optimal) allocation that maximizes the sum of all individual's utility.

Here, we borrow from the Cavalcanti-Wallace model some rules dictated by a planner, such as that all banks from a network must produce in exchange for money issued by that network. If there is just one network in equilibrium, then in some regards the equilibrium allocation has some optimality properties. However, because we study equilibria with two coexisting networks, our rules cannot be considered optimal, or derived from a mechanism-design analysis. Hence, the rules that we now turn to describe should be viewed as part of an exogenous, regulatory environment. Latter in the paper, we discuss how the results change when some key rules are modified.

All banks must make a lifetime appointment with a network at date 0, either the green or the red one. These colors also identify the notes that members are allowed to issue. The main regulation about networks is that a member much produce in exchange for any note issued by other members of the chosen network. Unlike the Cavalcanti and Wallace (1999) model, we do not ask, for simplicity, that banks produce gifts for nonbanks without a note. Moreover, banks are only asked to produce gifts for banks of the same network. Since we also check that production to other members satisfies participation constraints, then trade among banks of the same network can be viewed as a credit arrangement exactly as the production among banks in the Cavalcanti-Wallace model. Since banks are allowed to issue money to those that cannot engage in credit arrangements, the nonbanks or banks of the competing network, then the rules so far stated coincides with those implied by the Cavalcanti-Wallace approach when gifts to nonbanks are ignored.[1]

We also set some additional rules that have the potential to produce stable equilibria. We discuss later in the paper how the results change when we modify these rules. As a benchmark, we assume that green banks accept red notes from red banks or from nonbanks, but that red banks do not accept green notes. We proceed by looking for equilibria in which nonbanks consider the red notes more valuable than the green ones. More specifically, we look for equilibria in which a nonbank producer with a green note agrees to produce in exchange for a red note. We choose this behavior as a benchmark because it has been demonstrated that a swap of money holdings increases trade and welfare for some parameters of random-matching models with many currencies.[2]

Finally, we limit attention to steady states, and assume that banks choose networks according to a straight comparison of average, expected discounted utilities. Banks do so assuming that their individual decision will not affect the equilibrium size of each network, and make predictions using the fact that the average payoff associated to each network is a function of the aggregate measure x_0^r expected in equilibrium. Our notion of stability is a simple one. We ask whether a small increment in x_0^r/B, if $x_0^r/B < 1$, increase the difference between the red- and the green-bank payoffs. An equilibrium for which the answer is negative is considered *stable*. An equilibrium for which the answer is affirmative would be unstable because the marginal decision of small group of green banks to switch, in ex-ante terms, to the red network, would be profitable and lead to a new inflow of banks into this network. We present analytical results of stability in this local sense, but also perform numerical experiments displaying global stability in the obvious sense.

3 Stationary distributions

Throughout the paper, red banks do not accept green notes, and because they can always produce red notes, there is no need to give a state variable to a red bank. The benchmark allocations feature the green banks accepting red notes, and issuing a green note to a nonbank consumer in exchange for a red note. Red notes are also more valuable to nonbanks, in a way we make precise below.

We introduce now some notation that allow us to describe allocations formally. In general, we let production and consumption in a single-coincidence meeting be denoted $y_{ss'}^{ii'} \in \{0,1\}$. The superscripts, $i, i' \in \{r, g, n\}$, denote identities, where i, the first superscript, is the identity of the producer and i', the second, is that of the consumer. Identity r means red banker, identity g means green banker, and identity n means nonbanker. The subscripts $s, s' \in \{r, g, 0\}$ denote states, where s, the first subscript, is that of the producer and s', the second, is that of the consumer. The state of a nonbanker represents money holdings: r when holding a red note, g when holding a green note, and 0 when not holding a note. The state for a green banker is either r, when holding a red note, or 0, when not holding a red note. Green banks can always issue a green note, so that they never have to hold a note of their own network. The same applies to red bankers but, as a normalization, we only consider

[1] We document below, as a matter of fact, that these rules imply the optimum distribution of money of the Cavalcanti and Wallace (1999) model when participation constraints are not binding in their model and when only the red network is formed in our model.

[2] See Cavalcanti (2000).

allocations in which red bankers do not accept green notes, so that their state is trivially set to 0. The relevant combinations of identities and states are thus listed in the set $IS \equiv \{(n,r),(n,g),(n,0),(g,r),(g,0),(r,0)\}$.

We let x_s^i with $(i,s) \in IS$, denote the fraction of each production-consumption specialization type who have identity i and state s. So, these fractions are nonnegative and satisfy, in a steady state,

$$x_0^g + x_r^g + x_0^r = B \text{ and } x_0^n + x_g^n + x_r^n = 1 - B. \tag{1}$$

The set of stationary and symmetric allocations in the benchmark case has the following features: in all single-coincidence meetings between nonbankers, there is production in exchange for a note if and only if the consumer has a more valuable note than the producer (the producer may have no notes whatsoever) and there is a swap of holdings (including the case in which the producer leaves with a note and the consumer leaves with none) that is consistent with preservation of note holdings in the meeting and the unit upper bound on holdings. A nonbanker that does not have any note produces for a bank and receives a note issued by the bank (with the color of the bank's network). A nonbanker that has a green note produces for a bank, if and only if, she receives a red note. A red bank produces for a nonbanker, if the nonbanker has a red note, which the bank receives and destroys. A green bank produces for a nonbanker holding a green note, which the bank receives and destroys, and produces for a nonbanker holding a red note, which the bank receives and holds, unless the bank has a red note already (when no trade takes place). A green bank always produces to another green bank. In this case, if the consumer has a red note and the producer does not, the note switch hands. Moreover, a green bank produces for a red bank and receives a red note if and only if she does not hold a red note. Finally, a red bank always produces to another red bank. In all other meetings nothing happens.

The stationarity across states implies that the inflow into states 0 and g must equal the outflow from states 0 and g, respectively, for nonbankers, and the inflow into state r must equal the outflow from state r for green bankers, that is,

$$x_g^n(B - x_0^r) + x_r^n x_0^r = x_0^n B, \tag{2}$$

$$x_0^n(B - x_0^r) + x_r^n x_0^g = x_g^n(B + x_r^g), \tag{3}$$

$$x_0^g(x_0^r + x_r^n) = x_r^g(x_g^n + x_0^r). \tag{4}$$

Notice that in the right-hand side of the second stationarity condition, since $B = x_0^r + x_0^g + x_r^g$, the term x_r^g appears twice, since the nonbankers in this case can consume by spending their green notes with the green banks with red notes, and can also produce to acquire red notes from them. As we shall see, a list (x_s^i) for each (i,s) in IS, together with the benchmark rules of trade and asset transfers, defines not only a steady-state distributions, but also defines value functions uniquely. Thus, since all goods in this economy are indivisible, the allocations, to be defined more formally below, have no intensive margins of interest. Before associating (x_s^i) to an allocation, we find it useful to think of it as a distribution, which informs whether competing networks coexist.

Definition. A (benchmark) distribution (x_s^i) is stationary if it satisfies (1-4), and is interior if all its coordinates are positive.

The following proposition states that all benchmark, interior (x_s^i) feature the same distribution of red notes outside the red network.

Proposition 1 *Let $a \in (0, B)$ and consider the benchmark rules. Then there is an unique stationary distribution such that $x_0^r = a$, which can be computed analytically. In that distribution, moreover, among those in the population who are not red bankers, one half hold red notes.*

Proof. See appendix A. ■

Of course, the total measure of red notes depends on the size of the red network. That size also pins down the distribution of green notes.

4 Stationary values

Equations (1-4) supply a system of nonlinear equations whose the solution is given in appendix A. The stationarity also implies that we can write

$$\begin{aligned} Kv_0^n &= -(x_g^n + x_r^n + x_0^g + x_r^g + x_0^r) + \\ &+\beta\left\{\left(x_g^n + x_0^g + x_r^g\right)v_g^n + \left(x_r^n + x_0^r\right)v_r^n + [K - (x_g^n + x_0^g + x_r^g + x_r^n + x_0^r)]v_0^n\right\}, \end{aligned} \tag{5}$$

$$\begin{aligned} Kv_g^n &= u(x_0^n + x_0^g + x_r^g) - (x_r^n + x_r^g + x_0^r) + \\ &\beta\left\{\left(x_0^n + x_0^g + x_r^g\right)v_0^n + \left(x_r^n + x_r^g + x_0^r\right)v_r^n + [K - (x_0^n + x_0^g + 2x_r^g + x_r^n + x_0^r)]v_g^n\right\}, \end{aligned} \tag{6}$$

$$\begin{aligned} Kv_r^n &= u(x_0^n + x_g^n + x_0^g + x_0^r) + \\ &\beta\left\{\left(x_0^n + x_0^r\right)v_0^n + \left(x_g^n + x_0^g\right)v_g^n + [K - (x_0^n + x_0^r + x_g^n + x_0^g)]v_r^n\right\}, \end{aligned} \tag{7}$$

$$\begin{aligned} Kv_0^g &= u(x_0^n + x_0^g + x_r^g) - (x_g^n + x_r^n + x_0^g + x_r^g + x_0^r) + \\ &\beta\left\{\left(x_r^n + x_r^g + x_0^r\right)v_r^g + [K - (x_r^n + x_r^g + x_0^r)]v_0^g\right\}, \end{aligned} \tag{8}$$

$$\begin{aligned} Kv_r^g &= u(x_0^n + x_g^n + x_0^g + x_r^g + x_0^r) - (x_g^n + x_0^g + x_r^g) + \\ &\beta\left\{\left(x_g^n + x_0^g + x_0^r\right)v_0^g + [K - (x_g^n + x_0^g + x_0^r)]v_r^g\right\}, \end{aligned} \tag{9}$$

and

$$Kv_0^r = u(x_0^n + x_g^n + x_0^g + x_0^r) - (x_r^n + x_0^r + x_r^g) + \beta v_0^r. \tag{10}$$

Notice that for a given allocation, the system (5-10) consist of six linear equations in the six expected discounted utilities. This system has a unique solution. Among the ways to establish that is by a trivial contraction mapping argument.

5 Network choice

We now define the criteria for network choice. Since banks must choose ex-ante between networks, if a bank observes at date 0 that, on average, the red sector has a better payoff, he chooses to join the red network. Otherwise, he agrees to join the green network. Therefore, a bank agrees to join the green network if $\pi(x_0^r) \leq 0$, where, for (x_s^i) stationary,

$$\pi(\bar{x}_0^r) = \lim \Pi(x_s^i, v_s^i) \equiv v_0^r - \frac{x_0^g v_0^g + x_r^g v_r^g}{x_0^g + x_r^g}, \tag{11}$$

where the limit is taken over the set of (x_s^i, v_s^i) such that (x_s^i) is interior and stationary, and (v_s^i) is the list of values associated to (x_s^i). Since the system of Bellman equations has a unique solution and, as a result of Proposition 1, there is a unique interior (x_s^i) for a given x_0^r, the function π is well defined and the limit can be shown to exist even for $\bar{x}_0^r = 0$ (see appendix). The value $\pi(x_0^r)$ is the expected difference between the red- and green-network payoffs, conditional on the measure of red banks being x_0^r.

In the same fashion, a bank agrees to join the red network if the relative payoff for the red sector is weakly higher, that is, $\pi(x_0^r) \geq 0$. The function π is continuous, and we are interested in the behavior of the function π, as B moves either towards 0 or towards 1, because stationary distributions are easier to describe in these limits, and because they give us insights about how equilibria change when the bank sector is small or large. Our equilibrium concept for network choice is as follows.

Definition A (benchmark) allocation is a stationary (x_s^i), which is the limit of a sequence of interior and stationary distributions, and such that there exists (v_s^i) satisfying Bellman equations (5-10) with (x_s^i). Let π denote the ex-ante relative payoff associated to the set of allocations, according to (11). An allocation $(\bar{x}_s^i)$ displays a network

equilibrium whenever $\pi(\bar{x}_0^r) = 0$ if $(\bar{x}_s^i) \in (0, B)$, $\pi(\bar{x}_0^r) \leq 0$ if $\bar{x}_0^r = 0$, and $\pi(\bar{x}_0^r) \geq 0$ if $\bar{x}_0^r = B$. The equilibrium is (ex-ante) stable if, for all allocations (x_s^i) in a neighborhood of $(\bar{x}_s^i)$, $\pi(x_0^r) \geq 0$ if and only if $x_0^r \leq \bar{x}_0^r$.

Before discussing implementability, we present our main results, relating critical values of B to stability. The proposition below uses the fact that the sign of π, as $x_0^r/B \to 0$, depends uniquely on the magnitude of B relative to a critical value $\bar{b}$ (approximately $.12\dot{4}$). The proposition makes no additional assumptions about β or K.

Proposition 2 *Consider the benchmark rules and let $\bar{b}$ denote the unique solution to $6b^2 + b + 6 = (2+b)\left[4b^2 - 4b + 9\right]^{\frac{1}{2}}$ for b in $(0,1)$. (i) If $B < \min\{\bar{b}, (2\sqrt{u}-1)^{-1}\}$, then all network equilibria are interior and there exists one which is stable. (ii) Moreover, if $B > \max\{\bar{b}, (2\sqrt{u}-1)^{-1}\}$, then there exist two corner equilibria which are stable, one with $x_0^r = B$ and the other with $x_0^r = 0$, and an interior equilibrium which is not stable.*

Proof. See appendix B. ■

The proof of the first part of the proposition proceeds by computing first the limit of $\pi(z)$ as z approaches 0. That limit is the product of a polynomial in B (for which $\bar{b}$ is the solution) and a positive term (proportional to the social surplus $u-1$). Thus, $\pi(0^+) \geq 0$ if and only if $B \leq \bar{b}$. In a second step, the limit of $\pi(z)$ as z approaches B (from the left) is computed. It is shown that $\pi(B^-) \leq 0$ if and only if $B \leq 1/(2\sqrt{u}-1)$. Since π is continuous, if the condition in (i) holds then π must cross the line $\pi = 0$ at least once from above. The proof of second part of the proposition follows similar lines, but with inequalities working in the opposite directions. We call attention now to the following side result.

Corollary 3 *If B is sufficiently high then the conditions of the second part of Proposition 1 are satisfied. The stable equilibria can be described either as a monopoly in both currencies and networks (only red notes circulate), or as a monopoly in networks (only green network is formed) with competition between currencies (both green and red notes circulate).*

The fact that both red and green currencies circulate when $x_0^r = 0$ merits some explanations. We only look for corner equilibria that are limits of interior allocations, and for which our notion of stability is well defined. With interior allocations, the fact that red banks ignore green notes produces the result that the half of the people outside the red network hold red notes. As a result, on one hand, when $x_0^r = B$, the distribution of money is the same as the unconstrained optimum studied by Cavalcanti and Wallace (1999). In this case, there is a single type of bank and a single type of currency in circulation, exactly as in their model. On the other hand, all interior allocations in any sequence converging to a distribution with $x_0^r = 0$ displays the same property, and the mass of the green network just pins down how green notes are distributed within the half of the population without red notes. In particular, when B is set equal to zero, everyone is a nonbank, and half of the population hold red notes, and one fourth of the population hold green notes. This last case is an example of coexisting currencies, and for some parameters, as Cavalcanti (2000) has show in the case of divisible goods, the coexistence might produce an welfare improvement, when participation constraints are satisfied, because more trade takes place when notes of different colors are swapped.[3]

We provide below numerical illustrations describing the forces that determine network equilibria. When B is small, bank values are mainly determined by the distribution of money among nonbanks and, in particular, by the relative size of bank liabilities. When checking the satisfaction of participation constraints, we need to know the value of x_0^r in equilibrium, and for that we set $B = 0$ and study the distribution of money which would make (measure zero) banks indifferent between alternative networks. We find an explicit formula relating x_0^r to u in equilibrium, which does not depend on β. We label these restrictions, placed on equilibrium x_0^r, *money externalities*, in contrast to the *credit externalities* that become prominent as B increases. When B is low, banks choose the network with the most favorable liability situation, which deteriorates with network size. The money externality thus tends to produce stable equilibria. As B increases, banks place increasing attention to the credit arrangements internal to networks, and the gains from credit exchanges increase with network size. The credit externality thus tends to produce network monopolies or unstable equilibria.

All of the above discussion was pursued so far without consideration to participation constraints, which is a simplification allowed by the assumption that goods are indivisible. We conjecture that a similar approach applies if we had assumed divisible goods, but restricted attention to the case of output being set at an ex-ante optimal level (an approach partly pursued by Cavalcanti and Wallace, 1999).

[3] See also Aiyagari, Wallace and Wright (199?).

6 Participation constraints

An allocation is implementable if it satisfies some participation constraints. In order to express participation constraints in terms of discounted expected utilities (v_s^i).

The first constraint is the restriction on production by green bankers,

$$-1 + \beta \min\{v_0^g, v_r^g\} \geq 0, \tag{12}$$

which applies when the measure of green bankers is positive, and when bank g in state 0 is the producer in a meetings with (i, s) in the set $\{(n, g), (g, 0), (n, r), (g, r), (r, 0)\}$, or when bank g in state r is the producer in a meetings with (i, s) in $\{(n, g), (g, 0), (g, r), (n, r), (g, r), (r, 0)\}$. The next constraint is the restriction on production by red bankers,

$$-1 + \beta v_0^r \geq 0, \tag{13}$$

which applies when the measure of red bankers is positive, and $(r, 0)$ is the producer in meetings with (i, s) in $\{(n, r), (g, r), (r, 0\}$. The right-hand side of (12-13) are zero because we assume that banks are punished with autarky after violating their network's rules.

We also require that nonbankers have nonnegative gains from trade when they consume. The constraint for this case,

$$u \geq \beta \max\{v_r^n - v_0^n, v_r^n - v_g^n, v_g^n - v_0^n\}, \tag{14}$$

applies when (n, r) is the consumer in meetings with (i, s) in $\{(n, g), (n, 0), (g, 0), (r, 0)\}$, or when (n, g) is the consumer in meetings with (i, s) in $\{(n, 0), (g, 0), (g, r)\}$. We assume that nonbankers do not freely dispose off money. As a consequence, nonbankers who possess a red note will never produce until she consumes and gives up the note. The next set of constraints require that nonbankers have nonnegative gains from trade when they produce,

$$-1 + \beta \min\{v_r^n - v_0^n, v_r^n - v_g^n, v_g^n - v_0^n\} \geq 0, \tag{15}$$

which applies when (n, g) is the producer and $(i, s) \in \{(n, r), (g, r), (r, 0)\}$ is the consumer, or when $(n, 0)$ is the producer and $(i, s) \in \{(n, r), (n, g), (g, r), (g, 0), (r, 0)\}$ is the consumer. If green notes do not circulate, there is no need to enforce constraints involving subscript g in nonbank participation constraints.

Definition An allocation (x_s^i) is implementable with values (v_s^i) if it satisfies participation constraints (12-15).

Before investigating whether Proposition-1 equilibria are implementable, we find it useful to display some numerical examples whose qualitative properties do not depend on β.

7 Some examples

Figures 1-4 display the graphs of relative payoff π, values (v_s^i) and distributions (x_s^i), as functions of x_0^r in $[0, B]$, for two values of B; a low value ($B = .1$) where stability is expected, and a high value ($B = .6$) where the opposite is more likely. The other parameters in these examples are set as $\beta = .98$, $u = 6.36$ and $K = 3$. For this value for u, the critical values for B in Proposition 2 are roughly 1/8 and 1/4. Notice that the values of K and β are not important at this point.

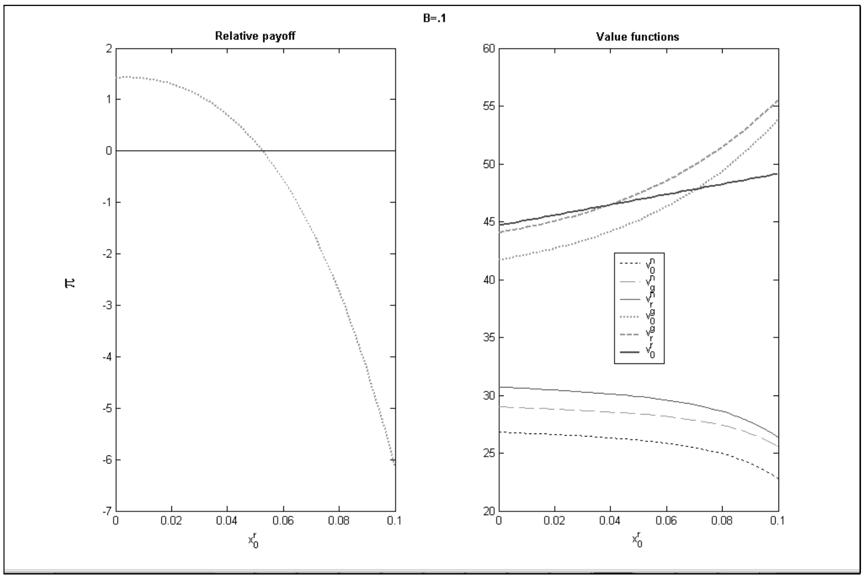

Figure 1 - Graphs of π and v_s^i for $B = .1$.
The interior equilibria in these examples feature a share of the red network (x_0^r/B) of about 53% and 51%, respectively for the low and for the high B.

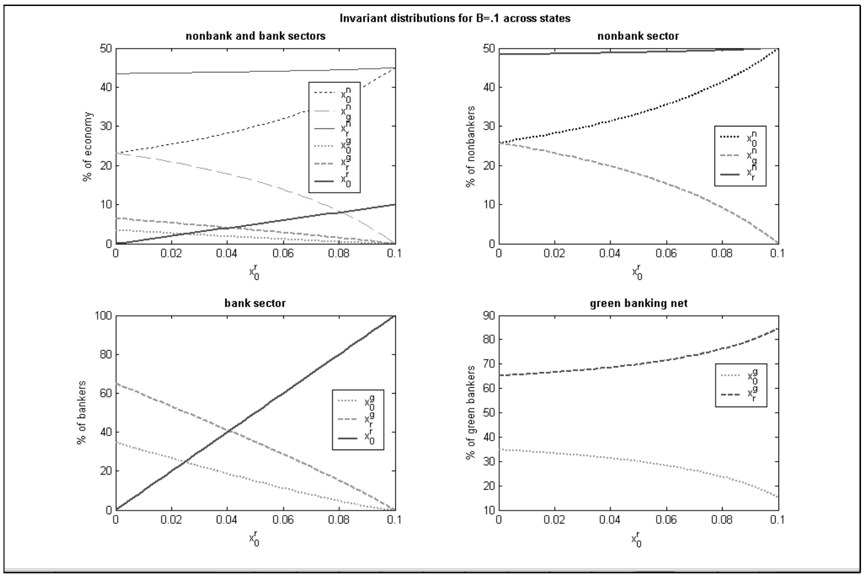

Figure 2 - Graphs of x_s^i for $B = .1$.

In Figures 1 and 2 we find an stable equilibrium for $B = .1$, described by $x_0^n = .3049$, $x_g^n = .1547$, $x_r^n = .4403$, $x_0^g = .0140$, $x_r^g = .0334$ and $x_0^r = .0526$.

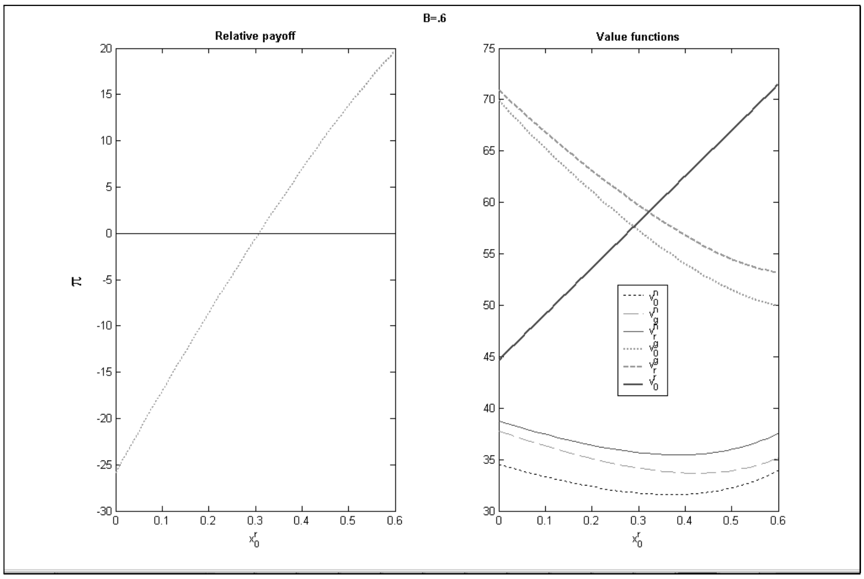

Figure 3 - Graphs of π and v_s^i for $B = .6$.

In Figures 3-4 we find an unstable equilibrium in bank for $B = .6$ described by $x_0^n = .1342$, $x_g^n = .0826$, $x_r^n = .1833$, $x_0^g = .1296$, $x_r^g = .1637$ and $x_0^r = .3074$.

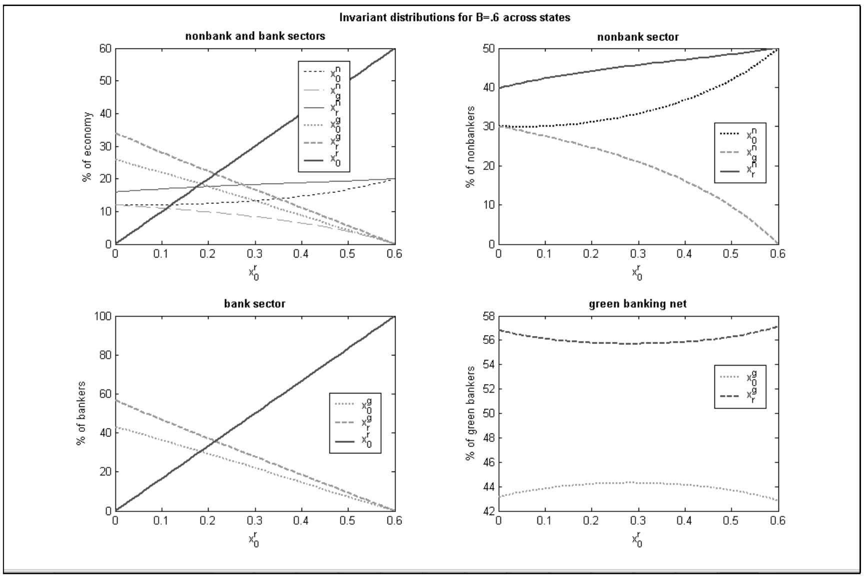

Figure 4 - Graphs of x_s^i for $B = .6$.

We find that the following algebra can also be helpful for documenting the presence of the money and credit externalities displayed in the examples. Let us define the relative distribution of red notes in the green network with the notation $\rho_r^g = x_r^g/(x_0^g + x_r^g)$. Using also $\rho_0^g = 1 - \rho_r^g$, and differentiating (11) with respect to x_0^r, yields

$$\frac{\partial \pi(x_0^r)}{\partial x_0^r} = \frac{\partial v_0^r}{\partial x_0^r} - \left[\rho_0^g \frac{\partial v_0^g}{\partial x_0^r} + \rho_r^g \frac{\partial v_r^g}{\partial x_0^r} + \frac{\partial \rho_0^g}{\partial x_0^r} v_0^g + \frac{\partial \rho_r^g}{\partial x_0^r} v_r^g\right] \tag{16}$$

or, since $\frac{\partial \rho_0^g}{\partial x_0^r} = -\frac{\partial \rho_r^g}{\partial x_0^r}$,

$$\frac{\partial \pi(x_0^r)}{\partial x_0^r} = \frac{\partial v_0^r}{\partial x_0^r} - \left[\rho_0^g \frac{\partial v_0^g}{\partial x_0^r} + \rho_r^g \frac{\partial v_r^g}{\partial x_0^r} + \frac{\partial \rho_r^g}{\partial x_0^r}(v_r^g - v_0^g)\right]. \quad (17)$$

Recall now that red-bank producers destroy all notes, while green-bank producers keep red notes, when they find them. When B is small enough, red-bank entry pushes the total measure of notes down. Therefore, total liabilities for the green network is then reduced and the number of potential nonbank producers, i.e., nonbankers without a note, increases. Provided that credit inside the green network, or more precisely, the welfare derived from production among green bankers, is not so large, all green bankers can improve with a red-network expansion. As a result, the term in square brackets on the right-hand side of equation (17) is positive and $\partial\pi(x_0^r)/\partial x_0^r < 0$ near the stable network equilibrium. This is the relationship between stability and the money externality.[4]

When B is large enough and the benefit from credit inside the green network is large, green bankers are worse off when the red network increases. As a result, the term in square brackets on the right-hand side of equation (17) is negative and $\partial\pi(x_0^r)/\partial x_0^r > 0$ near the unstable equilibrium. This why the credit externality leads to the lack of stability of interior equilibria.

Figure 3 also displays two stable corner-equilibria. On one hand, as stated before, when the total measure of green bankers collapse to zero, one finds the pure inside-money allocation of Cavalcanti and Wallace (1999). On the other hand, when the total mass of red bankers collapse to zero, one finds an allocation that is a mix of red outside-money and a green inside-money, and such that the banking sector accepts outside money.

The payoff function π is not necessarily monotone, and may not have interior solutions. In other examples (not displayed), we find a case with B low where there are two stable and one unstable equilibrium, and cases with B high where there is only one stable equilibrium, either the mixed one or the pure inside-money one with red notes.

8 Implementable equilibria

We discuss in this section the satisfaction of ex-post participation constraints.

Proposition 4 *If $u > 3$, B is sufficiently low, and β is sufficiently high, then benchmark equlibria are implementable.*

Proof. Appendix C. ■

It is shown in the proof that two nonbank constraints are particularly important. The first one, $\beta(v_g^n - v_0^n) > 1$, applies to the acceptance of green notes, and the second one, $\beta(v_r^n - v_g^n) > 1$, to an upgrade to red notes. As B approaches zero, the only distribution that matters is that of nonbanks, and in the limit, that distribution is $x_r^n = \frac{1}{2}$, $x_0^n = x$, $x_g^n = \frac{1}{2} - x$, where $x \in [0, \frac{1}{4}]$ depends on the limit of the ratio x_0^r/B, pinned down by equilibrium in network choice. If that limit equals zero then $x = \frac{1}{4}$, and if that limit equals the unity then $x = \frac{1}{2}$. Sufficient conditions for these inequalities can be obtained by expressing the participation constraints in terms of u when $\beta = 1$ and $B = 0$. When $\beta = 1$, the value of K does not matter. Acceptance of green notes then requires $u - 1 > \frac{1}{2x}$, a condition warranting that notes are sufficiently scarce. Since the lowest value of x is $\frac{1}{4}$ for limits of interior equilibria, then $u > 3$ suffices for satisfaction of $u - 1 > \frac{1}{2x}$. Regarding the upgrade to red notes, the requirement for $\beta = 1$ and $B = 0$ is $u - 1 > 2xu$, an inequality that is not satisfied for $x = \frac{1}{2}$. The requirement that $\pi = 0$ for interior equilibria imposes, however, a restriction on the value x, which can be expressed as the equality $4(1+u)x = 3 + 2u - \sqrt{5+4u}$. As a result, for $u > 1 + \sqrt{2}$, that equality implies $u - 1 > 2xu$. Since, according to Proposition 2, all equilibria are interior when B is sufficiently low, then $\pi = 0$ can be assumed indeed, and the proof follows for the stated values of u, B, and β, due to the continuity of network-equilibrium allocations with respect to these parameters.

Regarding values of B in the other extreme, that is, B close to the unity, we find sufficient conditions for interior equilibria and for the corner equilibrium with a single currency. The corner equilibrium with two currencies is not implementable if B is sufficiently high. The proof follow trivial steps and is thus omitted.

[4] The money externality arises because red bankers ignore green notes, and doing so, destroy green notes for "free" in some trades with nonbankers holding green notes. That discrimination against green notes by red bankers coul be an equilibrium outcome in a more general model in which green banks refuse to redeem green notes from red banks, and profit from the consequent reduction in the speed of clearing of their laibilities.

Proposition 5 *For B sufficiently high in the benchmark case, (i) the corner equilibrium with $x_0^r = B$ is implementable if β is sufficiently high, and (ii) the corner equilibrium with $x_0^r = 0$ is not implementable.*

When $x_0^r = B$, there are no green bankers, and implementability requires just β sufficiently high, as in a typical random-matching model with a single currency (the condition $\beta(v_g^n - v_0^n) > 1$ does not hold, but there are no green notes in circulation). When $x_0^r = 0$, there are no red bankers and, as B approaches the unity, nonbank values are essentially determined by meetings with green bankers, which accept green and red notes. These bankers sell goods for green notes, regardless of their own states, and those without red notes sell goods in exchange for a swap between red and green notes. Hence, there is no scarcity of buyers for green notes, but only half the population, the green banks without a red note, produce for a red note. Although the red note may allow consumption twice in row to a nonbank due to the swap in the first exchange, the green note has twice the demand, and has twice the supply (it is easier to be acquired again). Hence, nonbanks do not agree to give up their green notes, in order to produce in upgrades to red notes, thus violating $\beta(v_g^n - v_0^n) > 1$. The difference of this result to the finds that two currencies may enhance welfare is that, in this equilibrium, all banks discriminate against red notes, and yet the allocation recommends that red notes be more valuable. A natural conjecture is that implementability would be restored if green notes were considered more valuable by nonbanks, but that kind of allocation does not result from our assumptions about network choice.[5] Regarding interior equilibria, we find the following.

Proposition 6 *If $u > 3$, and both B and β is sufficiently high, then benchmark, interior equilibria are implementable.*

Proof. Appendix D. ■

The proof is based on expressions of participation constraints for $B = \beta = 1$. It is demonstrated that, if $B = 1$ then $\pi = 0$ if and only if $x_0^r = \frac{1}{2}$. The most important participation constraint is shown to be $\beta(v_r^n - v_g^n) > 1$, and once again $u > 3$ produces the desired result when $\beta = 1$.

We have learned that the use of two monies in general requires more than sufficient patience for implementability. In Cavalcanti (2000), a high social benefit from money use was necessary with divisible production. Here, the social surplus is positive when $u > 1$, but stronger restrictions on preferences are often necessary ($u > 3$). When the measure of the bank sector is high, however, benchmark equilibria without red banks are not implementable, and the only one both stable and implementable has a single currency.

9 Alternative allocations

In this section we investigate the possibility that alternative rules, within the context of coexisting currencies, produce qualitatively different outcomes. We consider two departures of the model considered so far. In the first we assume that green bankers do not accept red notes. In the second, we return to the rule that green banks accept red notes, but assume instead that they attempt to reduce monetary liabilities, and do not swap green for red notes when red notes are presented to them by nonbank producers.

9.1 Green banks ignore red notes

This is the symmetric case, when green and red banks do not accept each other notes. Appendix E presents a detailed discussion of this case. A characterization, equivalent to Propositions 1, 2, and 3, is derived in which the critical value $\bar{b}$ for B takes now the value 1/5. If B is sufficiently low then there all equilibria are interior, and at least one is stable. Implementability is however lost. The value of x_0^n such that $\pi = 0$, as $B \to 0$, also implies that $1 < \beta(v_r^n - v_g^n)$ is violated, even for β and u sufficiently high. The result follows because, as u increases, the equality $\pi = 0$ requires an increase in the measure of buyers willing to produce to green banks, that is, a higher x_0^n, otherwise the relative profit for green banks drop. But as x_0^n increases, the mass of green notes goes down, eliminating the advantage of red notes over green notes, since holders of red notes find a reduced measure of swapping opportunities. As a result, the stated participation constraint is violated. In summary, equilibria

[5] The corner equilibrium with $x_0^r = B$, and B close to the unity, is one in which the currency issued by banks is more valuable. However, in that equilibrium, the other currency (green) disappears because the red bankers end up destroying them.

with low B and two currencies are no longer implementable because u and x_0^n cannot vary in opposite directions, and mantain $\pi = 0$ at the same time.

Equilibria with B approaching one and green banks ignoring red notes are not appealing as well. The new green-bank behavior is such that $\pi = 0$ also at $x_0^r = 1/2$ in the limit, but the interior equilibrium is not only unstable, but features no trade between networks. That equilibrium also violates the condition $1 < \beta(v_r^n - v_g^n)$. At the corners, if $x_0^r = 0$ or $x_0^r = B$, the two currencies cannot coexist as well, since the whole bank sector ignores one of them.

9.2 Green banks reduce liabilities

We find that a more appealing alternative to the benchmark allocation takes place when green banks accept red notes, but economize on note issue by not giving a green note to a nonbank consumer, if the consumer has a red note. This alternative sounds appealing because that behavior tends to reduce the liabilities of the green network, and for a fixed x_0^r, tends to increase its relative payoff. A detailed examination of this alternative set of allocations is presented in Appendix F. We find, however, that the necessary reduction in x_0^r that brings π back to zero, after the reduction in the green liabilities, end up reducing the green network welfare below that of the benchmark case, when B is close to zero.

As a result, the initial goal of increasing network profits through a reduction in liabilities becomes unfeasible as a result of competition. Table 1 illustrates with examples this reduction in welfare, as well as other results documented in the Appendix. Results similar to Propositions 1, 2 and 3, hold. However, there is now a critical value of u, called $\bar{u}$ in Appendix F, such that equilibria are interior when $B \to 0$ and $u > \bar{u}$, but only a corner equilibirum (with $x_0^r/B = 0$) exists when $B \to 0$ and $u \in (1, \bar{u}]$. In general, u must increase beyond $\bar{u}$ for higher values of B in order to allow the existence of interior equilibrium, and for B above a critical value, that possibility disappears as in the benchmark case. In Table 1, when u assumes the same value as that of previous examples, it follows that $B = .025$ is consistent with interior stabilitity, but $B = .05$ is not. In the first case, the interior equilibrium found has a drop in x_0^r/B from a ratio of about 55% in the bechmark case, to one of 16% in the alternative case. The consequent reduction in x_0^r/B reduces x_0^n because red bankers in smaller numbers destroy less green notes. The increase in size of the green network produces an increase in their holdings of red notes, and a reduction in the holdings of red notes by nonbankers. That reduction ends up reducing the average welfare of the economy, since red notes command the highest value among nonbank states.

Table 1 also conveys information about the case $B = 0$ as a reference point. It is shown in Appendix F, that as long as $u > \bar{u}$, the interior equilibrium as $B \to 0$ features the same distribution of nonbankers for the benchmark and the alternative case; the equilibrium x_0^n is the same, but the equilibrium x_0^r/B is not, as explained above. The corner solution with $B = .05$ features an increase in average welfare, but a reduction in bank values. Regarding implementability, the same difficulties are found when $B \to 1$ (regarding banks, green liabilities are not an issue in that case, and regarding nonbanks, the reduced swapp opportunities forces now $u > 5$ for implementability in the interior case). When $B \to 0$, $u > \bar{u}$ suffices for implementability (since, it can be shown, $\bar{u} > 3$).

In summary, the case for this alternative allocation is not supported by an increase in bank profits.

	$B=0$		$B=.025$		$B=.05$	
	Benchmark	Alternative	Benchmark	Alternative	Benchmark	Alternative
mean v_s^i	27.0928	27.0928	27.8406	27.5580	28.6011	28.6072
$\frac{x_0^r}{B}$	.5573	.3305	.5503	.1676	.5433	.0000
x_0^n	.3466	.3466	.3361	.3250	.3258	.3075
x_g^n	.1534	.1534	.1542	.1675	.1545	.1786
x_r^n	.5000	.5000	.4847	.4825	.4697	.4639
x_0^g	.0000	.0000	.0028	.0054	.0061	.0139
x_r^g	.0000	.0000	.0084	.0154	.0167	.0361
x_0^r	.0000	.0000	.0138	.0042	.0272	.0000
ρ_0^g	.2348	.2348	.2520	.2607	.2678	.2780
ρ_r^g	.7652	.7652	.7480	.7393	.7322	.7220
v_0^n	24.9014	24.9014	25.1946	25.2315	25.4845	25.4490
v_g^n	27.1690	27.1690	27.4576	27.4897	27.7462	27.7609
v_r^n	28.5883	28.5883	28.8971	28.9466	29.2024	29.1923
v_0^g	43.0693	43.0693	43.6780	43.2609	44.2766	44.0576
v_r^g	45.1769	45.1769	45.8419	45.4368	46.4880	46.2235
v_0^r	44.6820	44.6820	45.2969	44.8691	45.8961	44.6823
mean v_s^g	44.6820	44.6820	45.2969	44.8691	45.8961	45.6216

Table1 - Equilibrium values for $u = 6.3618$, $\beta = .98$ and $K = 3$.

10 Concluding remarks

In this paper we ask the question of whether two coalitions of banks, issuing money with different rates of return, can compete and coexist in a stable stationary equilibrium. We show that the model in Cavalcanti and Wallace (1999) can be amended to address bank competition and the issue of private instruments with different velocities. We find that the answer to this question depends crucially on the aggregate measure of banks B, a parameter with the same interpretation as that given by Cavalcanti and Wallace. Although their strict subset result, a comparison between outside and inside money allocations, is invariant to the value of B, the coexistence of networks depends on B in our model because high values of B increases credit externalities leading to the monopoly in currency issue. By contrast, low values of B brings to the scene the bank liabilities in the form of competing currencies in circulation: large networks have large quantities of money in circulation redeemable in services that cause disutilities. The relationship between B and the equilibrium profile of bank liabilities introduces new effects that we call money externalities, and which can lead to the formation of competing bank networks of small sizes. It is important to highlight that not only is the discussion of competing networks new in the literature, but also the concept of equilibrium stability that we bring to the discussion, which, in a loose sense, has a tradition to show up in monetary models as questions about robustness, essentiality or tenuousness of equilibria.

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Appendix A

Substituting $B = B_r + B - B_r$ in the right-hand side of (2) and rearranging terms, yields an expression of condition (2) as

$$(x_0^n + x_g^n - x_r^n)x_0^r = x_g^n B - x_0^n(B - x_0^r).$$

Likewise, conditions (3) and (4) can be rewritten as

$$x_g^n B - x_0^n(B - x_0^r) = x_r^n x_0^g - x_g^n x_r^g$$

and

$$x_r^n x_0^g - x_g^n x_r^g = (x_r^g - x_0^g)x_0^r,$$

respectively. Adding now, side by side, these three equalities, yields

$$(x_0^n + x_g^n - x_r^n)x_0^r = (x_r^g - x_0^g)x_0^r$$

and the conclusion that, for stationary distributions in the closure of those with $x_0^r > 0$, the distribution of red notes is as stated: $x_r^n + x_r^g = (1 - x_0^r)/2$.

Regarding the solution of the system for a given value of x_0^r, we start by noticing that, condition (2) can be rewritten once again as

$$(x_r^n - x_g^n)x_0^r = (x_0^n - x_g^n)B,$$

so that $x_0^r/B = 0$ implies $x_0^n = x_g^n$, while $x_0^r/B = 1$ implies $x_0^n = x_r^n$. Results for interior distributions require some additional algebra. Substituting now $x_0^n = 1 - B - x_r^n - x_g^n$ for x_0^n in the right-hand side of (2) yields, after some simple algebra,

$$x_g^n(2B - x_0^r) + x_r^n(x_0^r + B) = B(1 - B). \tag{18}$$

Substituting $x_0^g = B - x_0^r - x_r^g$ for x_0^g in the left-hand side of (4) yields

$$(B - x_0^r - x_r^g)(x_0^r + x_r^n) = x_r^g x_0^r + x_r^g x_g^n. \tag{19}$$

Eliminating x_g^n between (18) and (19) yields now a polynomial in x_r^n and x_r^g in which the term of highest order has the product of x_r^n and x_r^g. The coefficient of this term is zero if $x_0^r = B/2$, in which case the polynomial becomes a linear equation. Otherwise, the equality $x_r^n + x_r^g = (1 - x_0^r)/2$, together with the polynomial, can be used to produce a quadratic equation in x_r^g for $x_0^r \in (0, B)$, which has a unique solution in the interval $(0, B - x_0^r)$. Some tedious but straightforward algebra demonstrates that that solution is

$$\begin{aligned} x_r^g \;=\; & \frac{B(3 + 2B + x_0^r) - 2x_0^r}{4(B - 2x_0^r)} - \\ & \frac{-\sqrt{4B^4 + 4x_0^{r2}(1 + 2x_0^r)^2 - 4B^3(1 + 3x_0^r) + B^2(9 + 54x_0^r + 57x_0^{r2}) - 4Bx_0^r(3 + x_0^r(15 + 14x_0^r))}}{4(B - 2x_0^r)}. \end{aligned}$$

Appendix B

When $x_0^r \to 0$, from appendix A, we have that the limit distribution is described by $x_0^n = x_g^n = \frac{5-2B-\left[4B^2-4B+9\right]^{\frac{1}{2}}}{8}$, $x_r^n = \frac{-1-2B+\left[4B^2-4B+9\right]^{\frac{1}{2}}}{4}$, $x_0^g = \frac{-3+2B+\left[4B^2-4B+9\right]^{\frac{1}{2}}}{4}$, and $x_r^g = \frac{3+2B-\left[4B^2-4B+9\right]^{\frac{1}{2}}}{4}$. This distribution, together with equations (11), and (8) up to (10), yields

$$\pi(0^+) = \Gamma\left(B\right) \frac{u - 1}{8K(1 - \beta)B},$$

where $\Gamma\left(B\right) = -6 - B + (2 + B)\left[4B^2 - 4B + 9\right]^{\frac{1}{2}} - 6B^2$. Hence, $\pi(0^+) > 0 \Leftrightarrow \Gamma\left(B\right) > 0 \Leftrightarrow 0 < B < \bar{b} \simeq 0.12405$.

When $x_0^r \to B$, from appendix A, we have that the limit distribution is given by $x_0^n = x_r^n = \frac{1-B}{2}$, $x_g^n = x_0^g = x_r^g = 0$, and $x_0^r = B$. This distribution, together with equations (11), and (8) up to (10), now yields,[6]

[6]It can be usefull to note that $\lim\limits_{B_r \longrightarrow B_-} \frac{x_\cdot^G}{x_\cdot^G + x_R^G} = \frac{2B}{3B+1}$ and $\lim\limits_{B_r \longrightarrow B_-} \frac{x_R^G}{x_\cdot^G + x_R^G} = \frac{B+1}{3B+1}$.

$$\pi(B^-) = \frac{4B^2u - (B+1)^2}{2K(1-\beta)(3B+1)}.$$

Hence, $\pi(B^-) < 0 \Leftrightarrow u < \left(\frac{B+1}{2B}\right)^2$.

In conclusion, (i) if $0 < B < \bar{b}$ and $u < \left(\frac{B+1}{2B}\right)^2$, then $\pi(0^+) > 0$ and $\pi(B^-) < 0$; (ii) if $\bar{b} < B < 1$ and $u > \left(\frac{B+1}{2B}\right)^2$, then $\pi(0^+) < 0$ and $\pi(B^-) > 0$.

Appendix C

When $B \to 0$, from appendix A, it is straightforward to see that the limit of the distribution of nonbankers holding red notes is $x_r^n = \frac{1}{2}$. Suppose that $x = x_0^n$, where $x \in (0, \frac{1}{2})$ is to be determined in that limit. We know that $0 \leq x_g^n = \frac{1}{2} - x \leq \frac{1}{2}$. From equation (3), $\rho_0^g = \frac{x_0^g}{x_0^g + x_r^g} \to \frac{1-2\,x}{2(1-\,x)}$ and $\rho_r^g = \frac{x_r^g}{x_0^g + x_r^g} \to \frac{1}{2(1-\,x)}$.

This distribution values, in addition to equations (5-10) yields

$$\begin{aligned}
v_0^r &= \frac{1}{K(1-\beta)}\left\{\frac{1}{2}(u-1)\right\}; && (20)\\
v_0^g &= \frac{1}{K(1-\beta)}\left\{xu - (1-x) + \beta\frac{1}{2}\Delta\right\}; && (21)\\
v_r^g &= \frac{1}{K(1-\beta)}\left\{\frac{1}{2}u - \left(\frac{1}{2}-x\right) - \beta\left(\frac{1}{2}-x\right)\Delta\right\}; && (22)\\
v_0^n &= \frac{1}{K(1-\beta)}\left\{-(1-x) + \beta\frac{1}{2}\Delta_{r0} + \beta\left(\frac{1}{2}-x\right)\Delta_{g0}\right\}; && (23)\\
v_g^n &= \frac{1}{K(1-\beta)}\left\{xu - \frac{1}{2} - \beta x\Delta_{g0} + \beta\frac{1}{2}\Delta_{rg}\right\}; && (24)\\
v_r^n &= \frac{1}{K(1-\beta)}\left\{\frac{1}{2}u - \beta x\Delta_{r0} - \beta\left(\frac{1}{2}-x\right)\Delta_{rg}\right\}. && (25)
\end{aligned}$$

where,

$$\begin{aligned}
\Delta &= v_r^g - v_0^g = \frac{\left(\frac{1}{2}-x\right)u + \frac{1}{2}}{K(1-\beta) + \beta(1-x)}; && (26)\\
\Delta_{r0} &= v_r^n - v_0^n = \frac{\frac{1}{2}u + 1 - x}{K(1-\beta)+\beta}; && (27)\\
\Delta_{rg} &= v_r^n - v_g^n = \frac{\left(\frac{1}{2}-x\right)u + \frac{1}{2}}{K(1-\beta)+\beta}; && (28)\\
\Delta_{g0} &= v_g^n - v_0^n = \frac{xu + \left(\frac{1}{2}-x\right)}{K(1-\beta)+\beta}. && (29)
\end{aligned}$$

After some algebra, we find that

$$\pi = 0 \Leftrightarrow x = F(u) \tag{30}$$

and $x \in [1/4, 1/2]$, where

$$F(u) \equiv \frac{3 + 2u - \sqrt{5+4u}}{4(1+u)}.$$

It is easy to see, from equations (20-29), that in a neighborhood of an interior allocation, $v_r^n > v_g^n > v_0^n$, $v_0^r > 0$, $v_r^g > v_0^g$ and, when $\beta \to 1$,

$$u > \frac{5}{4} \Rightarrow v_0^g > 0 \ \text{ and } u > 1 \Rightarrow v_0^n > 0 \tag{31}$$

In order to assure that constraints (**??**) up to (**??**) do not bind, we must have $1 < \beta\left(v_r^n - v_g^n\right)$, $1 < \beta\left(v_g^n - v_0^n\right)$, $1 < \beta v_0^g$, $1 < \beta v_0^r$ and $u > \beta\left(v_r^n - v_0^n\right)$. Now, when $\beta \to 1$, for

$$G(u) \equiv \frac{1}{2}\left(1 - \frac{1}{u}\right)$$

then

$$\begin{aligned} x &< G(u) \Rightarrow 1 < \beta\left(v_r^n - v_g^n\right), & (32) \\ u &> 3 \Rightarrow 1 < \beta\left(v_g^n - v_0^n\right), & (33) \\ u &> 1 \Rightarrow 1 < \beta v_0^g \text{ and } 1 < \beta v_0^r, & (34) \\ u &> \frac{3}{2} \Rightarrow u > \beta\left(v_r^n - v_0^n\right). & (35) \end{aligned}$$

One can see from (30) and (32) that the graph of F cuts that of G only at $u = 1 + \sqrt{2}$. For lower values of u, F majorizes G, and for higher values, the converse takes place. Hence, provided that $u > 3$, the result follows by a continuity argument.

Appendix D:

When $B \to 1$, using appendix A, it is straightforward to see that the limit of the distribution of bankers is $x_0^g = x_r^g = \frac{1-x_0^r}{2}$. From equation (4), $\rho_0^g = \frac{x_0^g}{x_0^g + x_r^g} \to \frac{1}{2}$ and $\rho_r^g = \frac{x_r^g}{x_0^g + x_r^g} \to \frac{1}{2}$.
This limit distribution, in addition to equations (8-10) yields

$$\begin{aligned} v_0^r &= \frac{1 + x_0^r}{K(1-\beta)}\left\{\frac{1}{2}(u-1)\right\} & (36) \\ v_0^g &= \frac{1}{K(1-\beta)}\left\{(1 - x_0^r)u - 1 + \beta\frac{(1 + x_0^r)}{2}\Delta\right\} & (37) \\ v_r^g &= \frac{1}{K(1-\beta)}\left\{u - (1 - x_0^r) - \beta\frac{(1 + x_0^r)}{2}\Delta\right\}, & (38) \end{aligned}$$

where,

$$\Delta = v_r^g - v_0^g \ . \tag{39}$$

After some algebra, we find that

$$\pi = 0 \Leftrightarrow x_0^r = \frac{1}{2}. \tag{40}$$

From (5-7) and (36-39), then, as $B \to 1$ with $x_0^r = \frac{1}{2}$,

$$
\begin{aligned}
v_0^r &= \frac{3}{4K(1-\beta)}\{u-1\}; && (41)\\
v_0^g &= \frac{1}{K(1-\beta)}\left\{\left[\frac{1}{2}+\frac{3\beta}{4[2K(1-\beta)+3\beta]}\right]u-1+\frac{3\beta}{4[2K(1-\beta)+3\beta]}\right\}; && (42)\\
v_r^g &= \frac{1}{K(1-\beta)}\left\{\left[1-\frac{3\beta}{4[2K(1-\beta)+3\beta]}\right]u-\frac{1}{2}-\frac{3\beta}{4[2K(1-\beta)+3\beta]}\right\}; && (43)\\
\Delta &= v_r^g-v_0^g=\frac{u+1}{2K(1-\beta)+3\beta}; && (44)\\
v_0^n &= \frac{8K^2(1-\beta)^2 1+(1-\beta)\beta K[19-5u]+\beta^2[9-8u]}{2K(\beta-1)[2K(\beta-1)-3\beta]^2}; && (45)\\
v_g^n &= \frac{4K^2(1-\beta)^2[3-2u]+(1-\beta)\beta K[29-23u]+2\beta^2[9-8u]}{4K(\beta-1)[2K(\beta-1)-3\beta]^2}; && (46)\\
v_r^n &= \frac{-12K^2(1-\beta)^2 u+(1-\beta)\beta K[11-29u]+2\beta^2[9-8u]}{4K(\beta-1)[2K(\beta-1)-3\beta]^2}; && (47)\\
\Delta_{g0} &= v_g^n-v_0^n=\frac{4K(1-\beta)[1+2u]+\beta[9+13u]}{4[2K(\beta-1)-3\beta]^2}; && (48)\\
\Delta_{rg} &= v_r^n-v_g^n=\frac{4K(1-\beta)[3+u]+2\beta[9+3u]}{4[2K(\beta-1)-3\beta]^2}; && (49)\\
\Delta_{r0} &= v_r^n-v_0^n=\frac{4K(1-\beta)[4+3u]+\beta[27+19u]}{4[2K(\beta-1)-3\beta]^2}. && (50)
\end{aligned}
$$

It is easy to see from equations (41-50) that $v_r^n > v_g^n > v_0^n$, $v_0^r > 0$, $v_r^g > v_0^g$ and, when $\beta \to 1$, that

$$u > 1 \Rightarrow v_0^g > 0 \quad \text{and} \quad u > \frac{9}{8} \Rightarrow v_0^n > 0 \qquad (51)$$

In order to assure that constraints (12) up to (15) do not bind, we must have $1 < \beta\left(v_r^n - v_g^n\right)$, $1 < \beta\left(v_g^n - v_0^n\right)$, $1 < \beta v_0^g$, $1 < \beta v_0^r$ and $u > \beta\left(v_r^n - v_0^n\right)$. When $\beta \to 1$ now,

$$
\begin{aligned}
u &> 3 \Rightarrow 1 < \beta\left(v_r^n - v_g^n\right), && (52)\\
u &> \frac{27}{13} \Rightarrow 1 < \beta\left(v_g^n - v_0^n\right), && (53)\\
u &> 1 \Rightarrow 1 < \beta v_0^g \text{ and } 1 < \beta v_0^r, && (54)\\
u &> \frac{27}{17} \Rightarrow u > \beta\left(v_r^n - v_0^n\right). && (55)
\end{aligned}
$$

Hence, as long as $u > 3$, the result follows once again by a continuity argument.

Appendix E: Green banks ignore red notes

Suppose that green banks do not accept red notes. In this case, their only state is trivially set as 0, and the distribution to be determined satisfies

$$x_0^g + x_0^r = B \text{ and } x_0^n + x_g^n + x_r^n = 1 - B, \qquad (56)$$

where $x_0^r \in [0, B]$. Stationarity now implies that, for nonbankers, inflows into states 0 and g must equal outflows from states 0 and g, so that

$$
\begin{aligned}
x_g^n(B - x_0^r) + x_r^n x_0^r &= x_0^n B && (57)\\
x_0^n(B - x_0^r) &= x_g^n B. && (58)
\end{aligned}
$$

Equations (56-58) determine a system of linear equations whose the solution is

$$
\begin{aligned}
x_0^n &= \frac{(1-B)\,B}{2\,(2B-x_0^r)}; && (59)\\
x_g^n &= \frac{(1-B)\,(B-x_0^r)}{2\,(2B-x_0^r)}; && (60)\\
x_r^n &= \frac{1-B}{2}; && (61)\\
x_0^g &= B-x_0^r; && (62)\\
x_0^r &= x_0^r. && (63)
\end{aligned}
$$

A competitive equilibrium can also be found when $\pi(x_0^r)=0$, where now

$$
\pi(x_0^r) \equiv v_0^r - v_0^g \tag{64}
$$

with the understanding that an interior x_0^r determines uniquely an allocation and the values associated to it, and that only allocations on the closure of the set of interior allocations are considered.

Proposition 7 *When green banks ignore red notes, the following holds. (i) If $B \in (0,.2)$ and $u < \frac{B+1}{2B}$, then $\pi(0^+) > 0$ and $\pi(B^-) < 0$; (ii) if $B \in (.2,1)$ and $u > \frac{B+1}{2B}$, then $\pi(0^+) < 0$ and $\pi(B^-) > 0$.*

Proof. When $x_0^r \to 0$, we know from (59-63) that the limit distribution is given by $x_0^n = x_g^n = \frac{1-B}{4}$, $x_r^n = \frac{1-B}{2}$ and $x_0^g = B$. This distribution, together with equation (64) and the value functions, implies

$$
\pi(0^+) = \frac{(1-5B)\,(u-1)}{4K(1-\beta)B}.
$$

Hence, $\pi(0^+) > 0 \Leftrightarrow 0 < B < .2$.

When $x_0^r \to B$, we know from (59-63) that the limit distribution is $x_0^n = x_r^n = \frac{1-B}{2}$ and $x_g^n = x_0^g = x_r^g = 0$. This distribution, in addition to equation (64) and the value functions, yields

$$
\pi(B^-) = \frac{2Bu-(B+1)}{2K(1-\beta)}.
$$

Hence, $\pi(B^-) < 0 \Leftrightarrow u < \frac{B+1}{2B}$. ∎

We find,however, that an interior equilibrium is not implementable if B is sufficiently low.

Proposition 8 *If $B \in (0,1)$ is sufficiently small then the interior equilibrium is not implementable.*

Proof. When $B \to 0$, it is straightforward to see, from equations (59-63), that the limit of the distribution of nonbankers holding red notes is $x_r^n = \frac{1}{2}$. Suppose then that $x = x_0^n \in (0,1/2)$ and $x_g^n = \frac{1}{2} - x$. This distribution, in addition to the value functions yields[7]

$$
\begin{aligned}
v_0^r &= \frac{1}{K\,(1-\beta)}\frac{1}{2}\,(u-1) && (65)\\
v_0^g &= \frac{1}{K\,(1-\beta)}\left\{xu-\left(\frac{1}{2}-x\right)\right\} && (66)\\
v_0^n &= \frac{1}{K\,(1-\beta)}\left\{-(1-x)+\beta\frac{1}{2}\Delta_{r0}+\beta\left(\frac{1}{2}-x\right)\Delta_{g0}\right\} && (67)\\
v_g^n &= \frac{1}{K\,(1-\beta)}\left\{xu-\frac{1}{2}-\beta x\Delta_{g0}+\beta\frac{1}{2}\Delta_{rg}\right\} && (68)\\
v_r^n &= \frac{1}{K\,(1-\beta)}\left\{\frac{1}{2}u-\beta x\Delta_{r0}-\beta\left(\frac{1}{2}-x\right)\Delta_{rg}\right\} && (69)
\end{aligned}
$$

[7] Here we omitted the value functions, but one can write them easily in the same fashion as (5)-(10).

where,

$$\begin{aligned}
\Delta_{r0} &= v_r^n - v_0^n = \frac{\frac{1}{2}u + 1 - x}{K(1-\beta)+\beta} &(70)\\
\Delta_{rg} &= v_r^n - v_g^n = \frac{\left(\frac{1}{2} - x\right)u + \frac{1}{2}}{K(1-\beta)+\beta} &(71)\\
\Delta_{g0} &= v_g^n - v_0^n = \frac{xu + \frac{1}{2} - x}{K(1-\beta)+\beta}. &(72)
\end{aligned}$$

After some algebra, we find that

$$\pi(x_0^r) = 0 \Leftrightarrow v_0^r = v_0^g \Leftrightarrow x = H(u). \tag{73}$$

where

$$H(u) = \frac{1}{2\left(1 + \frac{1}{u}\right)}.$$

It is easy to see from equations (65-72) that $v_r^n > v_g^n > v_0^n$, $v_0^r = v_0^g > 0$ and, when $\beta \to 1$, that

$$u > 1 \Rightarrow v_0^n > 0. \tag{74}$$

In order to assure that constraints[8] do not bind, we must have $1 < \beta\left(v_r^n - v_g^n\right)$, $1 < \beta\left(v_g^n - v_0^n\right)$, $1 < \beta v_0^g$, $1 < \beta v_0^r$ and $u > \beta\left(v_r^n - v_0^n\right)$. But, when $\beta \to 1$,

$$\begin{aligned}
x &< G(u) \Leftrightarrow 1 < \beta\left(v_r^n - v_g^n\right), &(75)\\
u &> 3 \Rightarrow 1 < \beta\left(v_g^n - v_0^n\right), &(76)\\
u &> 1 \Rightarrow 1 < \beta v_0^g = \beta v_0^r, &(77)\\
u &> \sqrt{2} \Rightarrow u > \beta\left(v_r^n - v_0^n\right), &(78)
\end{aligned}$$

where $G(u) \equiv \frac{1}{2}\left(1 - \frac{1}{u}\right)$. One can see from (73-75) that $H(u) > G(u)$ for all $u \geq 1$. Hence, even if $\beta \to 1$, the interior equilibrium is not implementable, since nonbankers holding green notes would never produce to receive a red note. ■

A negative results also holds for B sufficiently high.

Proposition 9 *If B is sufficiently high then the interior equilibrium is not implementable.*

Proof. When $B \to 1$, it is straightforward to see from equations (59-63) that the limit of the distribution of bankers is pinned down by $x_0^g = 1 - x_0^r$. This distribution, together with the value functions, yields

$$\begin{aligned}
v_0^r &= \frac{x_0^r}{K(1-\beta)}(u-1), &(79)\\
v_0^g &= \frac{1 - x_0^r}{K(1-\beta)}(u-1). &(80)
\end{aligned}$$

As a result,

$$v_0^r = v_0^g \Leftrightarrow x_0^r = \frac{1}{2}. \tag{81}$$

[8] Here we omitted the constraints, but one can write them easily in the same fashion as (??)-(??).

Using now the value functions and equations (79-80), when $B \to 1$ and $x_0^r = \frac{1}{2}$, we find that

$$
\begin{aligned}
v_0^r &= v_0^g = \frac{1}{2K(1-\beta)}(u-1), && (82)\\
v_0^n &= \frac{-8K^2(1-\beta)^2 + K(1-\beta)\beta(4u-14) + \beta^2(4u-5)}{4K(1-\beta)[K(1-\beta)+\beta][2K(1-\beta)+3\beta]}, && (83)\\
v_g^n &= \frac{4K^2(1-\beta)^2(u-1) + K(1-\beta)\beta(8u-10) + \beta^2(4u-5)}{4K(1-\beta)[K(1-\beta)+\beta][2K(1-\beta)+3\beta]}, && (84)\\
v_r^n &= \frac{4K^2(1-\beta)^2 u + K(1-\beta)\beta(8u-4) + \beta^2(4u-5)}{4K(1-\beta)[K(1-\beta)+\beta][2K(1-\beta)+3\beta]}, && (85)\\
\Delta_{g0} &= v_g^n - v_0^n = \frac{u+1}{2K(1-\beta)+3\beta}, && (86)\\
\Delta_{rg} &= v_r^n - v_g^n = \frac{K(1-\beta)+\frac{3\beta}{2}}{[K(1-\beta)+\beta][2K(1-\beta)+3\beta]}, && (87)\\
\Delta_{r0} &= v_r^n - v_0^n = \frac{u[K(1-\beta)+\beta] + 2K(1-\beta) + \frac{5}{2}\beta}{[K(1-\beta)+\beta][2K(1-\beta)+3\beta]}. && (88)
\end{aligned}
$$

Equations (82-88) now imply $v_r^n > v_g^n > v_0^n$, $v_0^r = v_0^g > 0$ and, when $\beta \to 1$, that

$$u > \frac{5}{4} \Rightarrow v_0^n > 0. \quad (89)$$

In order to assure that all participation constraints do not bind, we must have $1 < \beta\left(v_r^n - v_g^n\right)$, $1 < \beta\left(v_g^n - v_0^n\right)$, $1 < \beta v_0^g$, $1 < \beta v_0^r$ and $u > \beta\left(v_r^n - v_0^n\right)$. When $\beta \to 1$,

$$
\begin{aligned}
u &> 2 \Rightarrow 1 < \beta\left(v_g^n - v_0^n\right), && (90)\\
u &> 1 \Rightarrow\ 1 < \beta v_0^g = \beta v_0^r, && (91)\\
u &> \frac{5}{4} \Rightarrow u > \beta\left(v_r^n - v_0^n\right). && (92)
\end{aligned}
$$

However, even when $\beta \to 1$, the inequality $1 < \beta\left(v_r^n - v_g^n\right)$ does not hold. Hence, even if $\beta \to 1$, the interior equilibrium is not implementable, since nonbankers holding green notes would never produce to receive a red note. ■

Appendix F: Green banks reduce liabilities

Suppose now that green banks return to the policy of trading with red notes, but do not offer a green note, in exchange for a red note, in meetings with nonbank consumers. The new strategy tends to reduce liabilities of the green network, but other effects have to be considered. Regarding stationary distributions, only the inflow-outflow conditions of nonbanks have to be revisited. Equations (2) and (3) change to

$$
\begin{aligned}
x_g^n(B - x_0^r) + x_r^n x_0^r + x_r^n x_0^g &= x_0^n B, && (93)\\
x_0^n(B - x_0^r) &= x_g^n(B + x_r^g), && (94)
\end{aligned}
$$

respectively. Since the first equation results from adding the (inflow into state 0) $x_r^n x_0^g$ to the left-hand side of (2), and the second results from subtracting it from the left-hand side of (3), these changes cancel out when the two equations are aggregated, so that the argument in Appendix A, deriving the distribution of red notes, remains in effect. Thus, as in the benchmark, $x_r^n + x_r^g = (1 - x_0^r)/2$. Similar steps can be used to derive the invariant distribution, as a function of x_0^r. As B approaches one, the benchmark and the new distributions coincide. As B approaches 0, the case $x_0^r/B = 1$ also remains as before, while the limit distribution when $x_0^r/B = 0$ is easily verified to be described now by $x_r^n = 1/2$, $x_g^n = (\sqrt{3}-1)/4$ and $x_0^n = (3-\sqrt{3})/4$. The mass of green notes in this case is indeed reduced from $1/4$ to $(\sqrt{3}-1)/4$, but without changes in the distribution of red notes since the behavior of red bankers has not been affected.

Interior allocations, as B approaches 0, are only stationary when x_0^r/B is consistent with $x_0^n \in ((3-\sqrt{3})/4, 1/2)$. By contrast, that interval for x_0^n in the benchmark case is $(1/4, 1/2)$. The interior equilibrium in network choices, as B approaches 0, can described as follows. When B approaches zero, bank payoffs depend only on the limit distribution of nonbankers. Moreover, the distribution of red notes is not affected by the change in behavior of green bankers; as a result, if x_r^n turns out to be the same now as that of the benchmark case, then $x_r^g/(x_r^g + x_0^g)$ must also be the same. The conclusion is that, for x_0^n in the intersection of $(1/4, 1/2)$ and $((3-\sqrt{3})/4, 1/2)$, that is, for $x_0^n \in ((3-\sqrt{3})/4, 1/2)$, the average payoff of a green banker must be the same as that of a red banker. If such a value of x_0^n implied $\pi = 0$ in the benchmark case, it must imply $\pi = 0$. Let us now, using the function $F(u)$ constructed in Appendix C, define $\bar{u}$ such that $u > \bar{u}$ implies $\pi = 0$ for some $x_0^n \in ((3-\sqrt{3})/4, 1/2)$ in the benchmark case. Hence, for $u > \bar{u}$, $\pi = 0$ also holds now for the same equilibrium value of x_0^n of the benchmark case. What remains to be discussed, is the new limit of x_0^r/B in the interior equilibrium when $B \to 0$. That is so because the values of π associated to $x_0^n \in ((3-\sqrt{3})/4, 1/2)$ remain the same, but the values $\pi(x_0^r)$ corresponding to each x_0^r/B are now different, as a result in the changes of stationarity conditions (2-3) noted above.

The new equilibrium values of x_0^r/B must lie below the old one, as we now show. The payoff profile of a red banker, as a function of x_0^r/B, is the same as before, since red bankers ignore the distribution of green notes and are thus not affected by the change in the green-bank behavior, and is an increasing function of x_0^r/B. Thus, the new value of x_0^r/B results from a shift in the function representing the average payoff of green bankers, and its intersection with the fixed one of red bankers. When $x_0^r/B = 1$, then distribution of nonbankers is the same as before, and so is the payoff of green bankers. When x_0^r/B assumes the value taken by the interior equilibrium in the benchmark case, the payoff of green bankers is now superior to that of red bankers because the new rules reduce the liabilities of the green network. By continuity, the new value of x_0^r/B, for which $\pi = 0$ in the interior case, as $B \to 0$, must be inferior to the benchmark one.

In addition to the above characterization of interior equilibria, as B approaches zero, we call attention to what happens when $u \leq \bar{u}$. While implementability could be violated (it can be shown that $\bar{u} = 2 + \sqrt{3}$), the only possibility now is a corner equilibrium with $x_0^r/B = 0$, while equilibria under the benchmark rules must be interior when $B \to 0$.

We now turn to verify the participation constraints. Since nonbankers without a note continue to produce in exchange for a red note, without swaps involving a green note, there are no new participation constraints to be considered, except that the values taking part in the old ones may have changed. Relative to a given distribution, however, the only value that changes is that of a nonbanker holding a red note, due to the meetings with green-bank producers, since these bankers do not issue a green note for a swap. The value of v_r^n now satisfies

$$\begin{aligned} Kv_r^n &= u(x_0^n + x_g^n + x_0^g + x_0^r) + \\ &\quad \beta\left\{(x_0^n + x_0^r + x_0^g)\, v_0^n + x_g^n v_g^n + [K - (x_0^n + x_0^r + x_g^n + x_0^g)]v_r^n\right\}, \end{aligned} \tag{95}$$

instead of (7), since x_0^g should be removed from the term multiplying v_g^n, and should be included in that multiplying v_0^n.

When $B \to 0$, the new distributions are as discussed above. Since nonbanks do not meet with green banks when $B = 0$, the values and participation constraints derived in Appendix C apply. Hence, the satisfaction of participation constraints, when $\beta \to 1$, require $u > 3$. The function F, however, implies that $\pi = 0$ if and only if $u \geq \bar{u} = 2 + \sqrt{3}$. Thus $u > 2 + \sqrt{3}$ and β sufficiently high are sufficient for satisfaction of participation constraints when B is sufficiently low.

When B approaches one there is nothing new to be said about nonbank values since the new rules do not affect the distribution of bank states or the network equilibrium as $B \to 1$. That equilibrium, as demonstrated in Appendix D, is again reached at $x_0^r = 1/2$. Using now the equilibrium values of the limit distribution, the nonbank values and participation constraints can be computed for interior equilibria when $B \to 1$. These values

are now

$$
\begin{aligned}
K(1-\beta)v_0^r &= \frac{3}{4}(u-1),\\
K(1-\beta)v_0^g &= \left[\frac{1}{2}+\frac{3\beta}{4\left[2K\left(1-\beta\right)+3\beta\right]}\right]u-1+\frac{3\beta}{4\left[2K\left(1-\beta\right)+3\beta\right]},\\
K(1-\beta)v_r^g &= \left[1-\frac{3\beta}{4\left[2K\left(1-\beta\right)+3\beta\right]}\right]u-\frac{1}{2}-\frac{3\beta}{4\left[2K\left(1-\beta\right)+3\beta\right]},\\
\Delta &= v_r^g-v_0^g=\frac{u+1}{2K\left(1-\beta\right)+3\beta},\\
K(1-\beta)v_0^n &= \frac{32K^2\left(1-\beta\right)^2+4\left(1-\beta\right)\beta K(19-5u)+3\beta^2(13-10u)}{2\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2},\\
K(1-\beta)v_g^n &= \frac{8K^2\left(1-\beta\right)^2\left(3-2u\right)+2\left(1-\beta\right)\beta K(29-23u)+3\beta^2(13-10u)}{2\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2},\\
K(1-\beta)v_r^n &= \frac{-24K^2\left(1-\beta\right)^2u+6\left(1-\beta\right)\beta K(4-9u)+3\beta^2(13-10u)}{2\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2},\\
\Delta_{g0} &= v_g^n-v_0^n=\frac{4K(1-\beta)\left[1+2u\right]+\beta(9+13u)}{\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2},\\
\Delta_{rg} &= v_r^n-v_g^n=\frac{4K(1-\beta)(3+u)+\beta(17+4u)}{\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2},\\
\Delta_{r0} &= v_r^n-v_0^n=\frac{4K(1-\beta)(4+3u)+\beta(26+17u)}{\left[4K\left(1-\beta\right)+6\beta\right]^2+\beta^2}.
\end{aligned}
$$

so that, as $\beta\to 1$, $u>1\Rightarrow v_0^g>0$ (as before) and $u>13/10\Rightarrow v_0^n>0$. Moreover, $u>5\Rightarrow 1<\beta(v_r^n-v_g^n)$, $u>28/13\Rightarrow 1<\beta(v_g^n-v_0^n)$, $u>1\Rightarrow 1<\beta v_0^g$ and $1<\beta v_0^r$, and $u>13/10\Rightarrow u>\beta(v_r^n-v_0^n)$, as long as β is sufficiently high. Therefore, the new critical value for u in this case is given by the inequality $u>5$. Put in words, the upgrade to red notes now require a higher u since it becomes more difficult to nonbank holding a red note to find a trade that includes a swap for a green note.