



A Note On An Application of Arrow's Theorem: Sufficient Conditions for Lucas' Inflation and Welfare,^{*†}

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Abstract

Bellman's methods for dynamic optimization constitute the present mainstream in economics. However, some results associated with optimal control can be particularly useful in certain problems. The purpose of this note is presenting such an example. The value function derived in Lucas' (2000) shopping-time economy in Inflation and Welfare need not be concave, leading this author to develop numerical analyses to determine if consumer utility is in fact maximized along the balanced path constructed from the first order conditions. We use Arrow's generalization of Mangasarian's results in optimal control theory and develop sufficient conditions for the problem. The analytical conclusions and the previous numerical results are compatible .

1 Introduction

Bellman's methods for dynamic optimization constitute the present mainstream. However, some results associated with optimal control can be particularly useful in certain problems. The purpose of this note is presenting such an example.

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The value function derived in Lucas' (2000, Section 5) "Inflation and Welfare" need not be concave, leading this author to develop numerical analyses to determine the conditions under which the consumer utility is maximal. Lucas concludes that a possible failure of the first order conditions in characterizing a maximum could arise only for very low values of the coefficient of risk aversion (values lower than 0.01).

We use Arrow's generalization of Mangasarian's results in optimal control problems to derive sufficient conditions for the problem. Our analytical results are compatible with Lucas' previous numerical calculations.

2 The Model

To simplify the exposition, we make the product growth γ , which is exogenous in Lucas (2000), equal to zero. The consumer is supposed to maximize utility from the consumption (c) :

$$\int_0^\infty e^{-gt} U(c)dt \quad (1)$$

subject to the households budget constraint:

$$\dot{m} = 1 - (c + s) + h - \pi m \quad (2)$$

and to the transactions technology constraint:

$$-c + m\phi(s) \geq 0 \quad (3)$$

In these equations, s stands for the fraction of the initial endowment spent as transacting time, m for the real quantity of money, π for the rate of inflation, $U(c)$ for a strictly concave utility function, h for the (exogenous) real value of the flow of money transferred to the household by the government, and $g > 0$ for a discount factor. Regarding the technology function 3, we use here the same expression used by Lucas in his calculations, with $\phi(s) = ks, k > 0$.

Since constraint (3) will always hold with equality, we use it to substitute for c in the above equations and define the Hamiltonian:

$$H(m, s, \lambda) = U(kms) + \lambda(1 - (kms + s) + h - \pi m) \quad (4)$$

It is well known (Mangasarian , 1966) that the Pontryagin (1962) necessary conditions are sufficient for optimality if the Hamiltonian is jointly concave in the state and control variables. However, this condition is not fulfilled in the present case, due to the term sm in H . However, sufficient conditions can be generated with the use of Arrow's (1968) theorem¹. In Seierstad and Sydsater's (1987) version, the theorem reads as follows:

Lemma 1 (*Arrow sufficient theorem*): Let $(\bar{x}(t), \bar{u}(t))$ be a pair that satisfies the conditions (7) and (6) below, in the problem of finding a continuous control vector $u(t)$ and an associated continuous differentiable state vector variable $x(t)$, defined on the time interval $[t_0, t_1]$, that maximizes:

$$\int_{t_0}^{t_1} f_0(x(t), u(t), t) dt \quad (5)$$

subject to the differential equations:

$$\dot{x}_i(t) = f_i(x(t), u(t), t), \quad i = 1, 2, \dots, n \quad (6)$$

and to the conditions

$$\begin{aligned} x_i^0(t_0) &= x_i^0, \quad i = 1, 2, \dots, n \\ x_i(t_1) &= x_i^1, \quad i = 1, \dots, l \\ x_i(t_1) &\geq x_i^1, \quad i = l + 1, \dots, m \\ x_i(t_1) &\text{ free, } i = m + 1, \dots, n \\ u(t) &\in U \subset R^r \end{aligned} \quad (7)$$

Given the Hamiltonian function

$$H(x(t), u(t), p(t), t) = p_0 f_0(x(t), u(t), t) + \sum_{i=1}^n p_i f_i(x, u, t)$$

assume that there exist a piecewise continuous function $p(t) = (p_1(t), \dots, p_n(t))$ defined on $[t_0, t_1]$, which has a piecewise continuous derivative, $\dot{p}(t)$, except at a finite number of points, such that the following conditions are satisfied with $p_0 = 1$:

$$H(\bar{x}(t), \bar{u}(t), p(t), t) \geq H(\bar{x}(t), \bar{u}(t), p(t), t), \text{ for all } u \in U \quad (8)$$

$$\dot{p}_i(t) = -H_{x_i}^*(\bar{x}(t), \bar{u}(t), p(t), t), \quad i = 1, \dots, n \quad (9)$$

$$p_i(t_1) \text{ no conditions} \quad i = 1, \dots, l \quad (10)$$

$$p_i(t_1) \geq 0 \quad (= 0 \text{ if } \bar{x}^i(t_1) > x_i^1) \quad i = l + 1, \dots, m \quad (11)$$

$$p_i(t_1) = 0 \quad i = m + 1, \dots, n \quad (12)$$

¹Seidseter and Sydsater (1977) argue (p. 370) that the first published demonstration of this theorem, which was presented in Arrow and Kurz (1970), is not satisfactory, and that a correct proof did not seem to be available in the literature till the publication of their work. This theorem was first mentioned in Arrow (1968).

$H^*(x, p(t), t) = \max_{u \in U} H(x, u, p, t)$ exists and is a concave function of x for all t . Then, $(\bar{x}(t), \bar{u}(t))$ solves problem (5)-(7) above.

In this theorem, the concavity of the maximized Hamiltonian with respect to the state variables substitutes for the concavity of the Hamiltonian in both the state *and* control variables in the theorem due to Mangasarian. Also, notice that this theorem is written for a finite horizon. For infinite horizon, Arrow and Kurz (Proposition 8, 1970, p.49) show that the transversality conditions

$$\begin{aligned} \lim_{t \rightarrow \infty} p_i(t) &\geq 0 \\ \lim_{t \rightarrow \infty} p_i(t)x_i(t) &= 0 \end{aligned} \tag{13}$$

must be added to the sufficiency hypotheses.

3 Sufficient Conditions

In our original problem, s is the control variable (u) and m the state variable (x). Relatively to the above theorem, $l = m = 0$, $n = 1$, $U = [0, 1]$, $f_0(x, u, t) = U(c)$, $f_1(x, u, t) = 1 - (kms + s) + h - \pi m$, $[t_0, t_1] = [0, \infty]$. Given the way we wrote the Hamiltonian in (4), equation (9) must read $-\dot{\lambda}(t) + \rho\lambda = H_m(s, m)$ and, in the above transversality conditions, $e^{-\rho t}\lambda(t)$ substitutes for $p_1(t)$.

In order to derive the maximized Hamiltonian used in the theorem, we make the first derivative of (4), with respect to the control variable s , equal zero. For

$$\begin{aligned} U(c) &= c^{1-\sigma}/(1-\sigma), \sigma \neq 1, \sigma > 0 \\ U(c) &= \ln c \text{ (case } \sigma = 1 \text{)} \end{aligned}$$

which corresponds to the utility function used by Lucas. This leads, in both cases above, to:

$$s = \frac{1}{km} \left(\frac{km+1}{km} \lambda \right)^{-1/\sigma} \tag{14}$$

where $\lambda = U'(c)km/(1+km) > 0$. This value of s maximizes the Hamiltonian, since $H_{ss} = U''(c)m^2 < 0$. Using (14) in (4), the maximized Hamiltonian is equal to:

$$H^*(m, \lambda) = \frac{\sigma}{1-\sigma} \left(\frac{km}{\lambda(1+km)} \right)^{(1-\sigma)/\sigma} + \lambda(1+h-\pi m), \sigma \neq 1 \quad (15)$$

$$H^*(m, \lambda) = \log \frac{km}{\lambda(1+km)} + \lambda(1-1/\lambda+h-\pi m), \text{ (case } \sigma = 1 \text{)} \quad (16)$$

The next step in the application of the theorem is showing that the maximized Hamiltonian is concave with respect to the state variable m . This is trivially satisfied in the case when $\sigma = 1$ since, given k and λ , $\log \frac{km}{\lambda(1+km)}$ is a composite function of two monotone increasing concave functions. When $\sigma \neq 1$, first notice that the derivative of (15) with respect to m is given by:

$$H_m^*(m, \lambda) = \left(\frac{km}{\lambda(1+km)} \right)^{(1-\sigma)/\sigma} \frac{1}{m(1+km)} \quad (17)$$

Taking the derivative of the above expression, one easily concludes that $H_{mm}^*(m, \lambda) < 0$ iff:

$$\sigma > \frac{1}{2+2km} \quad (18)$$

In this case H is strictly concave in m and the interior balanced path is unique. Also, since $km > 0$, a sufficient condition that does not depend on m is given by

$$\sigma > 1/2$$

As one concludes from (18), lower values of σ may also imply $H_{mm}^*(m, \lambda) < 0$.

We are interested in finding the conditions under which the (unique) stationary point $(\bar{m}, \bar{\lambda})$ satisfying the conditions of the Arrow theorem, and characterized by the first order conditions, is optimal. We claim that, if $m \rightarrow \bar{m}$ and $\lambda \rightarrow \bar{\lambda}$, they constitute an optimal path. It suffices to notice that the transversality condition $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) m_i(t) = 0$. But by assumption $\lambda(t)$ and $m(t)$ converge to finite limits, and $\lim_{t \rightarrow \infty} e^{-\rho t} \rightarrow 0$.

In Lucas' paper, figures 9 and 10 are used to report numerical calculations designed to check if consumer utility is in fact maximized along the balanced path constructed from the first order conditions of the dynamic program. The numerical simulations are carried out for $k = 400$. From his numerical simulations, Lucas concludes that possible problems could only arise for values of $\sigma < 0.01$. Given the range of values assumed by m in the problem, this value is compatible with those given by (18).

4 Conclusion

We exemplify how a result developed in the optimal-control framework can be useful in dealing with problems approached, by the researcher, with Bellman's techniques. The conclusions of the analytical approach are compatible with those previously obtained with the numerical simulations.

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