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First–Price Auction Symmetric Equilibria with a General Distribution

Paulo Klinger Monteiro

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Paulo Klinger Monteiro*†
FGV-EPGE
Praia de Botafogo 190 sala 1103
22250-900 Rio de Janeiro RJ
Brazil

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Abstract

In this paper I obtain the mixed strategy symmetric equilibria of the first-price auction for any distribution. The equilibrium is unique. The solution turns out to be a combination of absolutely continuous distributions case and the discrete distributions case.

1 Introduction

In the early literature[1] on auction theory there are two papers that deal with the sale of one object by sealed bid first price auctions: Vickrey (1961) and Griesmer, Levitan and Shubik (1967). Vickrey analyses a symmetric first-price auction with several bidders and a uniform distribution of types. He also analyses two asymmetric models. One with two bidders and two uniform distributions with distinct supports. His analysis is incomplete in this case and supposing one of the distributions degenerated he proceeds with a complete analysis. The paper by Griesmer, Levitan and Shubik gives a detailed treatment of the two firms with two distinct intervals of costs, uniformly distributed case. Perhaps these two papers justify the present predominance of the symmetric model. They show– by example– that even

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[1] Here I follow P. Klemperer's (2000).

the most natural generalization[2] originates a forest of complications and a need for a very careful analysis. In this paper I consider private values, symmetric, first-price auctions. My focus will be conceptual. To understand my motivation let me look at two usual although diametrically opposite cases. The most common assumption on the distribution of bidders valuations, $F : [a, b] \to [0, 1]$, is that it has a strictly positive density, $f = F'$. The equilibrium bidding function is easy to find[3] and has a nice interpretation: it is the expected value of the second highest valuation given that the bidder has the highest valuation. The opposite case of a discrete distribution is considered mainly for examples.[4] The equilibrium in the discrete case is in mixed strategies. There is also a kind of monotonicity in that a bidder with a higher valuation always bids higher than bidders with lower valuations. To see this briefly consider a two types distribution, say each type $v \in \{0, 1\}$ occurs with probability $\frac{1}{2}$. Then a bidder with $v = 0$ bids 0 and a bidder with $v = 1$ bids in the interval $[0, b]$ with probability $G(b) = \frac{b}{1-b}, 0 \leq b \leq \frac{1}{2}$. What is the equilibrium if the distribution, F, is not absolutely continuous and is not discrete? We will see that the symmetric equilibrium exists and has two parts. A pure strategy part at the points of continuity of F and a mixed strategy part at the points of discontinuities of F.

The second result of the paper is the unicity of the symmetric equilibrium. It is possible that the equilibrium be unique not only amongst the symmetric ones. This is probably very difficult to prove in the general case. For example the techniques of Maskin and Riley (2003) and more recently Lebrun (2006) uses differential equations.

The last result in the paper find the equilibrium if the set of types is multi-dimensional. This will be easy. Its main interest being to show that neither monotonicity nor continuity plays a role in the general case.

2 Preliminaries

In this section I collect some basic definitions and auxiliary results. I begin recalling the definition of a distribution.

Definition 1 *A function $F : \mathbb{R} \to [0, 1]$ is a distribution if*

1. *F is increasing: $x < y \Rightarrow F(x) \leq F(y)$;*

[2]Like distinct supports of uniform distributions.

[3]Namely $b(x) = \frac{\int_a^x yF^{n-1}(y)f(y)dy}{F^{n-1}(x)}$.

[4]Thus Riley (1989) use a discrete distribution to present in simple mathematical terms the revenue equivalence theorem.

2. F *is right-continuous:* $F(x) = \lim_{y \uparrow x} F(y)$ *and*

3. $\lim_{x \to -\infty} F(x) = 0$ *and* $\lim_{x \to \infty} F(x) = 1$.

If F is a distribution so is F^m for every $m > 0$. If F is a distribution define

$$\begin{aligned} \underline{v} &:= \inf\{x : F(x) > 0\} \text{ and} \\ \bar{v} &:= \sup\{x; F(x) < 1\}. \end{aligned}$$

Throughout this paper I suppose that $\infty > \bar{v} > \underline{v} \geq 0$. Abusing notation, I denote by F the restriction $F|_{[\underline{v},\bar{v}]}$. This entails no confusion. Define $\mathcal{C}$ as the set of continuity of $F|_{[\underline{v},\bar{v}]}$ and $\mathcal{D}$ the set of discontinuities of $F|_{[\underline{v},\bar{v}]}$. Then $\mathcal{D}$ is countable.[5] Moreover

$$\mathcal{D} = \{x \in (\underline{v}, \bar{v}]; F(x) > F(x-)\}.$$

As usual $F(x-) := \sup\{F(y); y < x\} = \lim_{y \uparrow x} F(y)$. The following function is the main ingredient of the equilibrium strategy:

Definition 2 *Define* $b_F : [\underline{v}, \bar{v}] \to \mathbb{R}$ *by* $b_F(\underline{v}) = \underline{v}$ *and*

$$b_F(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(y)\, dy}{F^{n-1}(v)}, v \in (\underline{v}, \bar{v}]. \tag{1}$$

Remark 1 *It is easy to check, using integration by parts that*

$$b_F(v) = \frac{\int_{\underline{v}-}^{v} y dF^{n-1}(y)}{F^{n-1}(v)}. \tag{2}$$

This expression shows that b_F *can be interpreted as the expected value of the second highest valuation given that the highest valuation is* v. *If* F *has a density* f *we may rewrite* $b_F(v) = \frac{\int_{\underline{v}}^{v} y(n-1)F^{n-2}(y)f(y)dy}{F^{n-1}(v)}$.

In the next lemma I prove the basic properties of b_F.

Lemma 1 *The following properties are true:*

(i) $b_F(v) < v$ *if* $v > \underline{v}$*;*

(ii) $F(v') < F(v'')$ *if and only if* $b_F(v') < b_F(v'')$*;*

[5] Since F is monotonic.

(iii) *b_F is right-continuous, increasing and*

$$\{x \in [\underline{v}, \bar{v}] ; b_F \textit{ is continuous at } x\} = \mathcal{C}.$$

(iv)

$$(v - b(v-)) F^{n-1}(v-) = (v - b(v)) F^{n-1}(v). \tag{3}$$

Proof:

(i) Suppose $v > \underline{v}$. The definition of $\underline{v}$ implies that $F(y) > 0$ in a neighborhood of $\underline{v}$. Thus $\int_{\underline{v}}^{v} F^{n-1}(y)\, dy > 0$ and from (1), $b_F(v) < v$.

(ii) Suppose $F(v') < F(v'')$. Then $v' < v''$ and writing $b = b_F$:

$$b(v'') = \frac{\int_{\underline{v}-}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')} = \frac{\int_{\underline{v}-}^{v'} y dF^{n-1}(y) + \int_{v'}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')} =$$
$$\frac{b(v') F^{n-1}(v') + \int_{v'}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')}.$$

Thus using **(i)** above, $\int_{v'}^{v''} y dF^{n-1}(y) > b(v') (F^{n-1}(v'') - F^{n-1}(v'))$ and $b(v'') > b(v')$. Now if $F(v') = F(v'')$ then $\int_{v'}^{v''} y dF^{n-1}(y) = 0$ and $b(v'') = b(v')$.

(iii) From (1) the right-continuity of b_F follows directly from the right-continuity of F. The left limit of b is $b(v-) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(y) dy}{F^{n-1}(v-)}$. Thus b is continuous at v if and only if F is continuous at v. Moreover b is discontinuous at v if and only if $b(v-) < b(v)$.

(iv) This item follows from

$$(v - b(v)) F^{n-1}(v) = \int_{\underline{v}}^{v} F^{n-1}(y)\, dy = (v - b(v-)) F^{n-1}(v-).$$

QED

The next lemmas finishes our preliminary work.

Lemma 2 *For any $v \in [\underline{v}, \overline{v}]$,*

$$\max_{y} (v - b(y)) F^{n-1}(y) = (v - b(v)) F^{n-1}(v).$$

Proof: Rewrite

$$(v - b(y)) F^{n-1}(y) = vF^{n-1}(y) - b(y) F^{n-1}(y) =$$
$$\int_{\underline{v}-}^{y} v dF^{n-1}(x) - \int_{\underline{v}-}^{y} x dF^{n-1}(x) = \int_{\underline{v}-}^{y} (v - x) dF^{n-1}(x).$$

Since $v - x \geq 0$ if and only if $x \leq v$ the last integral is maximized at $y = v$.

QED

Lemma 3 *Suppose $v \in \mathcal{D}$ and $v > \underline{v}$. Then the following is true:*

1. *The function $G_v : [b(v-), b(v)] \to [0,1]$*

$$G_v(x) = \frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{v - b(v-)}{v - x} \right)^{\frac{1}{n-1}} \right), \tag{4}$$

is a continuous, strictly increasing distribution.

2. *For any $x \in [b(v-), b(v)]$,*

$$(v - x) \big(F(v-) + (F(v) - F(v-)) G_v(x) \big)^{n-1} = (v - b(v)) F^{n-1}(v). \tag{5}$$

Proof: (1) It is immediate that $G_v(b(v-)) = 0$ and that G_v is continuous. Moreover since $v - b(v-) > 0$ the function G_v is strictly increasing. Finally using (3) we have that

$$G_v(b(v)) = \frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{v - b(v-)}{v - b(v)} \right)^{\frac{1}{n-1}} \right) =$$
$$\frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{F^{n-1}(v)}{F^{n-1}(v-)} \right)^{\frac{1}{n-1}} \right) = 1.$$

(2) Suppose $x \in [b(v-), b(v)]$. Then

$$(F(v) - F(v-)) G_v(x) + F(v-) = F(v-) \left(\frac{v - b(v-)}{v - x} \right)^{\frac{1}{n-1}}.$$

Therefore

$$(v - x) ((F(v) - F(v-)) G_v(x) + F(v-))^{n-1} =$$
$$F^{n-1}(v-) (v - b(v-)) = F^{n-1}(v) (v - b(v)).$$

QED

3 The equilibrium

There are n bidders participating in a first-price auction. Values are private and bidders types are independent identically distributed according to the distribution $F : [\underline{v}, \overline{v}] \to [0, 1]$. The equilibrium is in mixed strategies. However it is not very wild. The mixed part occurs only at the discontinuities of F (which are countable). Moreover the support of the mixed strategies are non-intersecting and monotonic.

The equilibrium strategy is composed of two parts. First if $v \in [\underline{v}, \overline{v}] \cap \mathcal{C}$ the bidder bids $b(v)$ where $b = b_F$ is defined by (1). If $v \in \mathcal{D}$ the bidder bids the mixed strategy μ_{G_v}. Thus for every $x \in [b(v-), b(v)]$ he bids in the interval $[b(v-), x]$ with probability $G_v(x)$. Define $\mathbf{M} = (\mu_v)_{v \in [\underline{v}, \bar{v}]}$ where

$$\mu_v = \begin{cases} \text{pure strategy } b(v) & \text{if } v \in \mathcal{C}, \\ \text{mixed strategy } G_v & \text{if } v \in \mathcal{D}. \end{cases}$$

Thus the pure strategy $b(v)$ is played at the continuity points of the distribution and the mixed strategy G_v is played if the distribution is discontinuous at v. I need the distribution of bids generated by $\mathbf{M}$. The next two lemmas complete this step.

Lemma 4 *For every $x \in [\underline{v}, b(\bar{v})]$ there exists the smallest $\omega \in [\underline{v}, \bar{v}]$ such that $b(\omega-) \leq x \leq b(\omega)$.*

Proof: Define for $x \in [\underline{v}, b(\bar{v})]$, $\omega^r = \omega^r(x) := \inf\{\omega; b(\omega) \geq x\}$. From the right-continuity of b we conclude that $x \leq b(\omega^r)$. For any $\omega < \omega^r$ it is true that $b(\omega) < x$ and therefore $b(\omega^r-) \leq x$. That ω^r is the smallest is clear from its definition. QED

The following corollary is follows immediately:

Corollary 1 *Let B_i be the random variable of bidder i bids. The distribution of B_i is*

$$\Pr(B_i \leq x) = F(\omega^r-) + (F(\omega^r) - F(\omega^r-))\, G_{\omega^r}(x).$$

In particular $\Pr(B_i = x) = 0$ *for every $x > \underline{v}$.*

Theorem 1 *The mixed strategy* $\mathbf{M}$ *is a symmetric equilibrium of the first-price auction.*

Proof: Suppose bidders $i = 2, \ldots, n$ bids the mixed strategy $\mathbf{M}$. Suppose bidder 1 has valuation v. The corollary above shows that if bidder 1 bids

x the probability of a tie is null. Thus if he bids $x = b(y-) + \delta$, $0 \leq \delta \leq b(y) - b(y-)$, his expected utility is

$$\phi = (v-x)\left(F(y-) + (F(y) - F(y-))\,G_y(x)\right)^{n-1}.$$

From lemma (5),

$$(y-x)\left(F(y-) + (F(y) - F(y-))\,G_y(x)\right)^{n-1} = (y - b(y))\,F^{n-1}(y).$$

Therefore

$$\phi = \frac{v-x}{y-x}\,(y - b(y))\,F^{n-1}(y).$$

Since[6] ϕ increases in x if and only if $v > y$ it follows that:

$$\begin{aligned}
&\text{if } v > y,\ \phi \leq (v - b(y))\,F^{n-1}(y);\\
&\text{if } v < y,\ \phi \leq \frac{v - b(y-)}{y - b(y-)}\,(y - b(y))\,F^{n-1}(y) =\\
&(v - b(y-))\,F^{n-1}(y-) = (v - b(y))\,F^{n-1}(y).
\end{aligned}$$

In any case

$$\phi \leq (v - b(y))\,F^{n-1}(y) \leq (v - b(v))\,F^{n-1}(v).$$

Thus $y = v$ is the best reply. And if v is a point of discontinuity, $x \in [b(v-), b(v)]$ is bid accordingly to $G_v(\cdot)$ is a best response. QED

4 Unicity of the mixed strategy equilibrium

In this section I show that the mixed strategy equilibrium $\mathbf{M}$ is unique.

Theorem 2 *Suppose* $\Upsilon = (\tau_v)_{v \in [\underline{v}, \bar{v}]}$ *is a mixed strategy symmetric equilibrium. Then* $\Upsilon = \mathbf{M}$.

To simplify the notation a bit I suppose $\underline{v} = 0$. Define H^i as the distribution of bids of i when Υ is played. Thus $H = H^1 = \ldots = H^n$ and

$$H(x) = \int_{0-}^{\bar{v}} \tau_y[0, x]\,dF(y).$$

Define also G as the distribution of the maximum bid of bidders $j \neq i$. Thus $G(x) = H^{n-1}(x)$. Denote by $\mathcal{P}$ the set of Borelean probabilities measures on $\mathbb{R}$. The following lemma is basic.

[6]Note that $\frac{d}{dx}\left(\frac{v-x}{y-x}\right) = \frac{v-y}{(y-x)^2}$.

Lemma 5 *Suppose* $\phi : \mathbb{R} \to \mathbb{R}$ *is measurable and bounded,* $\bar{\mu} \in \mathcal{P}$ *and that*

$$\int \phi(z)\, d\bar{\mu}(z) = \sup_{\mu \in \mathcal{P}} \int \phi(z)\, d\mu(z).$$

Then $\phi^{\max} := \max_{z \in \mathbb{R}} \phi(z)$ *exists and* $\bar{\mu}(\{z; \phi(z) \neq \phi^{\max}\}) = 0$.

Proof: Define $M = \sup \phi(\mathbb{R})$. Let δ_x denote the Dirac measure at $x \in \mathbb{R}$. Then

$$\left\{\int \phi(z)\, d\mu(z); \mu \in \mathcal{P}\right\} \supset \left\{\int \phi(z)\, d\delta_x(z); x \in \mathbb{R}\right\} = \{\phi(x); x \in \mathbb{R}\}.$$

Thus

$$M \geq \sup_{\mu \in \mathcal{P}} \int \phi(z)\, d\mu(z) \geq M.$$

Therefore

$$\int (M - \phi(z))\, d\bar{\mu}(z) = M - \int \phi(z)\, d\bar{v}(z) = 0.$$

Hence $\phi(z) = M$ for almost every z with respect to $\bar{\mu}$ and therefore the supremum is achieved and $\bar{\mu}(\{z; \phi(z) \neq \phi^{\max}\}) = 0$. QED

Let us now consider bidder i with valuation v. Since Υ is an equilibrium the best reply is τ_v. If there is a tie we suppose that the tie is solved with equal probability amongst the winners. Thus if a bidder i bids x he wins with probability

$$\widetilde{G}(x) = \sum_{j=0}^{n-1} \frac{(n-1)!}{j!\,(n-1-j)!} H^{n-1-j}(x-) \cdot \frac{(H(x) - H(x-))^j}{j+1}. \tag{6}$$

Thus

$$\int (v - b)\, \widetilde{G}(b)\, d\tau_v(b) = \sup_{\tau \in \mathcal{P}} \int (v - b)\, \widetilde{G}(b)\, d\tau(b).$$

The lemma above implies that

$$A_v = \left\{\bar{x} \geq 0; (v - \bar{x})\, \widetilde{G}(\bar{x}) = \max_{x \geq 0} (v - x)\, \widetilde{G}(x)\right\} \neq \emptyset, \tag{7}$$

and $\tau_v(A_v^c) = 0$.

Lemma 6 *For every* $v > 0$, $H(v) > 0$.

Proof: Let $\tilde{v} = \inf\{v; H(v) > 0\}$. Suppose $\tilde{v} > v \geq 0$. If $\tau_v(\tilde{v}, \bar{v}] > 0$ then there is a $b' > \tilde{v}$ such that $b \in A_v$. However $(v - b')\,\widetilde{G}(b') < 0$ since $\widetilde{G}(b') \geq H^{n-1}(b'-) > 0$. Thus $\tau_v(\tilde{v}, \bar{v}] = 0$. From

$$H(\tilde{v}) = \int_{0-}^{\bar{v}} \tau_y[0, \tilde{v}]dF(y) \geq F(\tilde{v}-) > 0$$

we conclude that $H(\tilde{v}) > 0$. Thus since $\widetilde{G}(\tilde{v}) \geq H^{n-1}(\tilde{v}) > 0$ the reasoning above implies $\tilde{v} \notin A_v$. Hence $\tau_v[0, \tilde{v}) = 1$ for every $v < \tilde{v}$. Now

$$0 = H(\tilde{v}-) = \int_{0-}^{\bar{v}} \tau_y[0, \widetilde{v})dF(y) \geq F(\tilde{v}-) > 0,$$

a contradiction. Therefore $\tilde{v} = 0$. QED

Lemma 7 *For any v the distribution H is continuous on A_v.*

Proof: First note from expression (6) that $H(x) = H(x-)$ if and only if $\widetilde{G}(x) = H^{n-1}(x)$. Now if $\bar{b} \in A_v$ and $b_m \downarrow \bar{b}$ through points of continuity of H then $\widetilde{G}(b_m) = H^{n-1}(b_m)$ and

$$\widetilde{G}\left(\bar{b}\right) = \lim_{m\to\infty} \widetilde{G}(b_m) = \lim_{m\to\infty} H^{n-1}(b_m) = H^{n-1}\left(\bar{b}\right).$$

Thus $H\left(\bar{b}\right) = H\left(\bar{b}-\right)$. QED

Lemma 8 *If $v' < v''$ then $\sup A_{v'} \leq \inf A_{v''}$.*

Proof: Suppose $v' < v''$ and that there exist $b' \in A_{v'}, b'' \in A_{v''}$ such that $b'' \leq b'$. It is always true that

$$\begin{aligned} (v' - b')\,\widetilde{G}(b') &\geq (v' - b'')\,\widetilde{G}(b''), \text{ and} \\ (v'' - b'')\,\widetilde{G}(b'') &\geq (v'' - b')\,\widetilde{G}(b'). \end{aligned}$$

Adding and collecting terms $(v'' - v')\left(\widetilde{G}(b'') - \widetilde{G}(b')\right) \geq 0$. Hence $H^{n-1}(b'') = \widetilde{G}(b'') \geq \widetilde{G}(b') = H^{n-1}(b')$. Thus $H(b') = H(b'') > 0$. This implies

$$v' - b' \geq v' - b''$$

and therefore $b' \leq b''$. Thus $b' = b''$. QED

Lemma 9 *Suppose now that $\#A_v > 1$. Then $v \in \mathcal{D}$.*

Proof: Take $b', b'' \in A_v,\ b' < b''$. Thus $(v - b')\, \widetilde{G}(b') = (v - b'')\, \widetilde{G}(b'')$. Therefore

$$H^{n-1}(b'') = \widetilde{G}(b'') > \widetilde{G}(b') = H^{n-1}(b').$$

Thus $H(b'') > H(b')$. Now if $\epsilon < b'' - b'$l,

$$0 < H(b'' - \epsilon) - H(b') = \int (\tau_y([0, b'' - \epsilon]) - \tau_y([0, b'])) \, dF(y) =$$

$$\int \tau_y(b', b'' - \epsilon] dF(y) = \tau_v(b', b'' - \epsilon] \, (F(v) - F(v-)). \tag{8}$$

In (8) I used Lemma 8. Thus $F(v) - F(v-) > 0$. QED

Lemma 10 *For every $b \geq 0, v \in \mathcal{D}$ we have that $\tau_v(b) = 0$. In particular H is continuous in* $(\inf A_v, \sup A_v)$.

Proof: If $b \notin A_v$ then $0 \leq \tau_v(b) \leq \tau_v(A_v^c) = 0$. If $b \in A_v$ then

$$0 = H(b) - H(b-) = \int \tau_y\{b\} \, dF(y).$$

Therefore $\tau_y\{b\} = 0$ for almost every y with respect to F. Hence $\tau_v\{b\} = 0$. QED

Lemma 11 *Suppose $b' < b''$ are elements of $A_v, v \in \mathcal{D}$. Then $(b', b'') \cap A_v \neq \emptyset$.*

Proof: Since $(v - b')\, H^{n-1}(b') = (v - b'')\, H^{n-1}(b'')$ and $b' < b''$ it follows that $H(b'') > H(b')$. Now

$$0 < H(b'') - H(b') = \int \tau_y(b', b''] dF(y) = \tau_v(b', b''] \, (F(v) - F(v-))$$

and therefore $\tau_v(b', b'') = \tau_v(b', b''] > 0$ ending the proof.

Lemma 12 *For every $v \in \mathcal{D}$, $A_v \supset$* $(\inf A_v, \sup A_v)$.

Proof: Suppose $v \in \mathcal{D}$. For any $x \in (\inf A_v, \sup A_v)$ define

$$\bar{x} = \inf\{b \in A_v; b > x\}.$$

If $\bar{x} \notin A_v$ there exist $b_l \in A_v$ $b_l \downarrow \bar{x}$. Then if we define

$$\phi^{\max} = \max\left\{(v - b)\, \widetilde{G}(b); b \geq 0\right\}$$

it is true that

$$\phi^{\max} = (v - b_l)\, H^{n-1}(b_l) \to (v - \bar{x})\, H^{n-1}(\bar{x}-) = (v - \bar{x})\, \widetilde{G}(\bar{x})\,.$$

Thus $\bar{x} \in A_v$. Analogously we define $\underline{x}$:

$$\underline{x} = \sup\{b \in A_v; b < x\}\,.$$

Thus $\underline{x} \in A_v$. Now if $\underline{x} < \bar{x}$ then $(\underline{x}, \bar{x}) \cap A_v = \emptyset$ a contradiction. Hence $\underline{x} = \bar{x} = x$. QED

Define $b(v)$ as the pure strategy played when $v \in \mathcal{C}$. And if $v \in \mathcal{D}$ define $b(v) = \inf_{\omega > v} b(\omega)$. Thus b is increasing and right-continuous. We have that

$$H(b(v)) = \int \tau_y([0, b(v)])\, dF(y) = F(v)\,.$$

Theorem 3 *For every v, $b(v) = b_F(v)$.*

Proof: Suppose $v \in \mathcal{C}$. Then for every $\omega \in \mathcal{C}$,

$$(v - b(v))\, F^{n-1}(v) \geq (v - b(\omega))\, F^{n-1}(\omega)\,.$$

By the right-continuity of b and F this is also true for every ω and for every v. The inequality above is equivalent to

$$v\left(F^{n-1}(v) - F^{n-1}(\omega)\right) \geq b(v)\, F^{n-1}(v) - b(\omega)\, F^{n-1}(\omega)\,.$$

Interchanging v with ω we get:

$$\omega\left(F^{n-1}(\omega) - F^{n-1}(v)\right) \geq b(\omega)\, F^{n-1}(\omega) - b(v)\, F^{n-1}(v)\,.$$

Thus for every v and ω :

$$\begin{aligned} & v\left(F^{n-1}(v) - F^{n-1}(\omega)\right) \geq b(v)\, F^{n-1}(v) - b(\omega)\, F^{n-1}(\omega)\,, \\ & b(v)\, F^{n-1}(v) - b(\omega)\, F^{n-1}(\omega) \geq \omega\left(F^{n-1}(v) - F^{n-1}(\omega)\right). \end{aligned} \qquad (\#)$$

Take $\omega_0 = \underline{v} < \omega_1 < \ldots < \omega_N = \bar{v}$ a partition of $[\underline{v}, \bar{v}]$ such that $\max_j |\omega_{j+1} - \omega_j| < \frac{1}{N}$. We have that

$$\omega_{j+1}\left(F^{n-1}(\omega_{j+1}) - F^{n-1}(\omega_j)\right) \geq b(\omega_{j+1})\, F^{n-1}(\omega_{j+1}) - b(\omega_j)\, F^{n-1}(\omega_j)$$

$$\int \sum_{j=0}^{N-1} \omega_{j+1} \chi_{(\omega_j, \omega_{j+1}]}(y)\, dF^{n-1}(y) = \sum_{j=0}^{N-1} \omega_{j+1}\left(F^{n-1}(\omega_{j+1}) - F^{n-1}(\omega_j)\right) \geq$$

$$\sum_{j=0}^{N-1}\left(b(\omega_{j+1})\, F^{n-1}(\omega_{j+1}) - b(\omega_j)\, F^{n-1}(\omega_j)\right) = b(v)\, F^{n-1}(v)\,.$$

Since

$$\sup_{\underline{v}\leq y\leq \bar{v}}\left|\sum_{j=1}^{N}\omega_{j+1}\chi_{(\omega_j,\omega_{j+1}]}(y)-y\right|\leq \max_j|\omega_{j+1}-\omega_j|<\frac{1}{N}$$

by making $N\to\infty$ we get:

$$\int_{\underline{v}-}^{v} ydF^{n-1}(y)\geq b(v)F^{n-1}(v). \tag{9}$$

The other inequality is obtained from the inequality in (#). QED

Thus the pure strategy part is unique. The unicity of the mixed strategy is proved in an analogous manner.

Theorem 4 *The mixed strategy τ_v is unique for each $v\in\mathcal{D}$.*

Proof: Let us consider $v\in\mathcal{D}$. Suppose $b\in A_v, b<\sup A_v$. Then

$$H(b)-H(b(v-))=\int\tau_y(b(v-),b]dF(y)=\tau_v(b(v-),b](F(v)-F(v-)).$$

Therefore

$$H(b)=F(v-)+\tau_v(b(v-),b](F(v)-F(v-)).$$

Since τ_v cannot have mass points,

$$(v-b)\big(F(v-)+\tau_v(b(v-),b](F(v)-F(v-))\big)^{n-1}=(v-b(v-))F^{n-1}(v-).$$

Therefore

$$\tau_v(b(v-),b]=\left(-1+\left(\frac{v-b(v-)}{v-b}\right)^{\frac{1}{n-1}}\right)\frac{F(v-)}{F(v)-F(v-)}.$$

QED

5 Example and application.

I now show how the general multi-dimensional set of types case is reduced to a one dimensional case in complete generality. Suppose the set of types is the probability space $(T,\mathcal{T},P)$. A bidder with type $t\in T$ has a utility $U(t)$ when receiving the object. The function $U:T\to\mathbb{R}$ is bounded and measurable. Define

$$F(x)=\Pr(U(t)\leq x), x\in U(T)$$

the distribution of U. Define $\underline{v} = \inf U(T)$ and $\bar{v} = \sup U(T)$. Define also $b_U = b_F \circ U$ and $G_t^U = G_{U(t)}$. The equilibrium is then to bid $b_U(t)$ if F is continuous at $U(t)$ and the mixed strategy G_t^U if F is discontinuous at $U(t)$. I finish with an example showing how to calculate the mixed strategies support.

Example 1 *Suppose there are two bidders and three possible valuations* $v \in \{0, 1, 2\}$. *And*

$$\begin{cases} \Pr(v = 0) = a, \\ \Pr(v = 1) = b \\ \Pr(v = 2) = 1 - a - b \end{cases}, a > 0, b > 0, a + b < 1.$$

The bidders with a zero valuation bids 0. *A bidder with valuation* 1 *bids in the interval* $[b(1-), b(1)]$,

$$b(1-) = 0, b(1) = \frac{\int_{0-}^{1} ydF(y)}{F(1)} = \frac{b}{a+b}.$$

A bidder with valuation 2 *bids in the interval* $[b(2-), b(2)]$:

$$b(2-) = \frac{\int_{0-}^{2-} ydF(y)}{F(2-)} = \frac{b}{a+b}, b(2) = 2(1 - a - b) + b = 2 - 2a + b.$$

If $a = b = 1/2$ *we recover the example in the introduction.*

References

1. Griesmer,J. H., R. E. Levitan and M. Shubik, 1967. Towards a Study of Bidding Processes, Part IV: Games with Unknown Costs. Naval Research Logistics Quarterly 14: 415-433.

2. Klemperer, P., 2000. The economic theory of auctions vol. I, Elgar Reference Collection.

3. Lebrun, B., 2006, Uniqueness of the equilibrium in first-price auctions. Games and Economic Behavior 55, 131-151.

4. Maskin, E. and J. Riley, 2000, Equilibrium in sealed high-bid auctions. Review of Economic Studies 67, 439-454

5. Maskin, E. and J. Riley, 2003. Uniqueness of equilibrium in sealed high-bid auctions. Games and Economic Behavior 45, 395-409

6. Riley, J. G., 1989, Expected Revenue from Open and Sealed Bid Auctions. Journal of Economic Perspectives 3 (3), 41-50.

7. Vickrey , W., 1961. Counterspeculation, Auctions and Competitive Sealed Tenders. Journal of Finance 16, 8-37.

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[613] Pedro Cavalcanti Gomes Ferreira e Carlos Hamilton Vasconcelos Araújo. *On the Economic and Fiscal Effects of Infrastructure Investment in Brazil*. Ensaios Econômicos da EPGE 613, EPGE–FGV, Mar 2006.

[614] Aloisio Pessoa de Araújo, Mario R. Páscoa, e Juan Pablo Torres-Martínez. *Bubbles, Collateral and Monetary Equilibrium*. Ensaios Econômicos da EPGE 614, EPGE–FGV, Abr 2006.

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