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## LIMITED MEMORY AND THE ESSENTIALITY OF MONEY

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# Limited Memory and the Essentiality of Money\*

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## Abstract

This paper investigates the relationship between memory and the essentiality of money. We consider a random matching economy with a large finite population in which commitment is not possible and memory is limited in the sense that only a fraction  $m \in (0, 1)$  of the population has publicly observable histories. We show that no matter how limited memory is, there exists a social norm that achieves the first-best regardless of the population size. In other words, money can fail to be essential irrespective of the amount of memory in the economy. This suggests that the emphasis on limited memory as a fundamental friction for money to be essential deserves a deeper examination.

Key Words: Fiat Money, Memory, Social Norms.

JEL Codes: E40, C73, D82, D83

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# 1 Introduction

Fiat money is essential if socially desirable allocations can only be achieved with its use. Since, by definition, fiat money does not provide any direct utility, its essentiality must come from the fact that it overcomes frictions in the trade process that alternative arrangements are unable to circumvent. Thus, if one wants to identify the conditions under which (fiat) money is essential, one must have a clear assessment of the role played by the different trading frictions in preventing non-monetary trade. Monetary theorists usually focus on two frictions: limited commitment and limited memory or record-keeping (Kiyotaki and Moore (2002), Kocherlakota (2002), Wallace (2001)). In this article we investigate the role that limited memory plays on the essentiality of money.

We consider a random matching economy with a large finite population in which commitment is not possible. Following Kocherlakota and Wallace (1998), we define memory as a technology that records the past actions of agents and makes this information public.<sup>1</sup> We say that memory is limited or imperfect if this technology only keeps track of the records of a fraction  $m < 1$  of the population. It is well-known that money is inessential if memory is perfect. Our main result is that no matter how limited is memory and how large is the population, there exists a social norm that achieves the first-best as long as the arrival rate of trading opportunities is high enough, where the lower bound on the arrival rate does not depend on the population size. Thus, even though limited memory is necessary, it is *never* sufficient to make money essential. This suggests that the emphasis on limited memory as a fundamental friction for money to be essential needs to be re-evaluated.

This work is not the first to look at the extent to which social norms can substitute the use of money.<sup>2</sup> Araujo (2004) analyzes a random matching economy with a finite population

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<sup>1</sup>The notion of memory we consider has been extensively used in the literature. See, for instance, Ales et al. (2008), Berentsen (2006), Cavalcanti and Wallace (1999a,b), Cavalcanti, Erosa, and Temzelides (1999), Martin and Schreft (2006), and Mills (2007, 2008). It is different from the notion of memory introduced by Kocherlakota (1998), though. We discuss the relationship between the two at the end of the paper.

<sup>2</sup>The basic references in the literature on social norms are Kandori (1992), Ellison (1994), and Okuno-Fujiwara and Postlewaite (1995). Recent contributions are Deb (2008) and Takahashi (2008).

and no memory and shows that there exists a social norm that achieves the first-best as long as the arrival rate of trading opportunities is sufficiently high. However, he also shows that for a fixed arrival rate of trading opportunities, there exists no social norm that achieves the first-best (or any non-autarkic allocation, for the matter) when the number of agents is large enough. So, money is essential in large economies if there is no memory. This makes it natural to consider the role of memory on the essentiality of money in large populations.<sup>3</sup>

Another related paper is Kocherlakota and Wallace (1998), who also look at the relationship between imperfect record-keeping and the essentiality of money in a decentralized economy with no commitment. They define imperfect memory as a technology that records the past actions of *all* agents in the economy and makes this information public with a lag, where the greater the lag, the more imperfect is the memory.<sup>4</sup> Their main result is that for a given arrival rate of trading opportunities, the optimal incentive-feasible allocation has some role for money if memory is imperfect enough. The nature of our exercise is different. We ask whether money is essential given a fixed (and limited) amount of memory. As described above, our main result is that for *any* amount of memory there is no role for money if the arrival rate of trading opportunities is high enough.

The paper is organized as follows. We introduce the environment in Section 2. In Section 3 we discuss the major difficulties in establishing our main result and outline the approach we take to deal with them. We prove the main result in Section 4 and address its robustness in Section 5. We conclude in Section 6. The Appendix contains omitted proofs and details.

## 2 Environment

The environment we consider is based on Shi (1995) and Trejos and Wright (1995). Time is discrete and indexed by  $t \geq 0$ . There is a finite set  $I = \{1, \dots, N'\}$  of infinitely-lived and

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<sup>3</sup>More recently, Aliprantis, Camera, and Puzzello (2007) study the role of social norms in environments, first introduced in Lagos and Wright (2005), where there is occasional trade in ‘centralized’ markets. See Lagos and Wright (2008) for a comment on their contribution. For another application of social norms to monetary theory, see Corbae, Temzelides, and Wright (2003).

<sup>4</sup>In our setting, the public histories are revealed without a lag. We make this assumption for simplicity. As we discuss later in the paper, our main result holds for any lag.

anonymous agents who discount future utility at rate  $\beta \in (0, 1)$ . We assume that  $N' = 2^\eta N$ , where  $\eta$  and  $N$  are positive integers. There is also one indivisible and perishable good that comes in many varieties. We discuss the indivisibility assumption in Section 5.

Agents are heterogeneous with respect to what varieties they can consume and produce in a period. Each agent can only consume a subset of the varieties and cannot consume any of the varieties that he produces. An agent who consumes  $y$  units of the good obtains utility  $u(y)$ ; the cost of producing  $x$  units of the good is  $x$ . We assume that  $u(x) - x$  has a unique maximizer, that we denote by  $x^*$ , and that  $x^* > 0$ . Moreover, we assume that there exists an upper bound  $\bar{x} \geq x^*$  on how much an agent can produce in each period.

Trade is decentralized and agents face frictions in the exchange process. More precisely, in each period agents are randomly and anonymously matched in pairs and at most one agent can produce in a meeting. The probability that an agent is a consumer in a match is  $\alpha \leq 1/2$  and it is the same as the probability that he is a producer.

Note that an increase in  $\beta$  corresponds to a reduction in the time interval between two consecutive periods. This amounts to an increase in the arrival rate of trading opportunities. We adopt this interpretation of an increase in  $\beta$  in the remainder of the paper.<sup>5</sup>

There are two types of agents in the economy: public and private. An agent is private, or type-1, if the only other agent who can observe his action in a period is his current partner. An agent is public, or type-0, if everyone else in the population can observe his actions in every period. We say a meeting is public if it involves at least one public agent, otherwise we say the meeting is private. An agent observes his partner's type in any meeting he participates. As we discuss in Section 4, this makes it more difficult to sustain cooperation between the agents. The number of public agents is  $N'_0 = 2^\eta N_0$ , with  $N_0 \in \{1, \dots, N\}$ . We denote the fraction of public agents by  $m = N_0/N$ . By definition,  $m$  is the amount of memory in the economy. Notice that an increase in  $\eta$  increases the population size while keeping the amount of memory fixed.

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<sup>5</sup>An alternative would be to consider a setting in which  $\beta$  is fixed, but the agents are randomly and anonymously matched in pairs  $q \geq 1$  times in each period. An increase in  $q$  would then amount to an increase in the arrival rate of trading opportunities. The results we obtain are the same.

We simplify the description of actions by assuming that in every match the participating agents simultaneously announce the amount  $x$  they are willing to produce. This announcement occurs before the agents know whether the meeting is a single-coincidence or not, and is binding within the meeting. In other words, an agent who announces  $x$  in a meeting commits to produce  $x$  units of the good to his partner if he is a producer. An agent can announce zero, so that this assumption does not violate his participation constraint.

The private history of a private agent is the list of his past actions together with the types of all the partners he had so far and the action choices of all the private partners he had so far. The private history of a public agent is the list of the types of all the partners he had so far together with the action choices of all the private partners he had so far. Let  $A = \{0, \dots, \bar{x}\}$  be the set of possible announcements,  $Y_0 = A \times \{0, 1\}$ , and  $Y_1 = A \times (\{0\} \cup (\{1\} \times A))$ . The set of period- $t$  private histories for a type- $\ell$  agent, with  $\ell \in \{0, 1\}$ , is then  $H_{\ell,t} = Y_{\ell}^t$ . We denote a typical element of  $H_{\ell,t}$  by  $h^{\ell,t}$ .

All agents in the economy share a common history, which is the list of all past action choices made by the public agents. A public observation is a map  $\psi$  from  $A$  into  $\{0, \dots, N'_0\}$ , where  $\psi(x)$  is the number of public agents who announce  $x$ . We denote the set of all such profiles by  $\Psi$ . A period- $t$  common history is then a list  $(\psi_0, \dots, \psi_t)$ , where  $\psi_s$ , with  $s \leq t$ , is the period- $s$  public observation and  $\psi_1 = \emptyset$  by convention. We denote the set of all period- $t$  common histories by  $\Omega_t = \Psi^t$  and a typical element of this set by  $\omega^t$ .

Let  $Z_{\ell,t} = H_{\ell,t} \times \Omega_t$  and denote a typical element of this set by  $z^{\ell,t} = (h^{\ell,t}, \omega^t)$ . By construction,  $Z_{\ell,t}$  is the set of period- $t$  histories for a type- $\ell$  agent. Now let  $Z_{\ell} = \bigcup_{t=0}^{\infty} Z_{\ell,t}$ . A behavior strategy for a type- $\ell$  agent is a map from  $Z_{\ell} \times \{0, 1\}$  into  $\Delta(A)$ , the set of mixed actions—an agent can condition his announcement in a meeting both on his history and on his partner's type. A profile of behavior strategies is symmetric if all agents of the same type use the same strategy.

A (finite-state) automaton for an agent is a list  $(\Theta, \theta^0, f, \tau)$ , where: (a)  $\Theta$  is a finite set of states; (b)  $\theta^0$  is the initial state; (c)  $f : \mathbb{Z}_+ \times \Theta \times \{0, 1\} \rightarrow \Delta(A)$  is a decision rule where  $f(t, \theta, \ell)$  is the agent's action in period  $t$  if his state is  $\theta$  and his partner's type is  $\ell$ ; (d)

$\tau : \mathbb{Z}_+ \times \Theta \times A \times \{0, 1\} \times A \times \Psi \rightarrow \Theta$  is a transition rule where  $\tau(t, \theta, a, \ell, a', \psi)$  is the agent's state in  $t + 1$  if his state in  $t$  is  $\theta$ , he announces  $a$ , his partner is of type  $\ell$  and announces  $a'$ , and the public observation is  $\psi$ . In the remainder of this section we restrict attention to symmetric strategy profiles  $\sigma$  that are induced by a pair of automata, one for each type of agent. We can assume, without loss, that both automata have the same state space  $\Theta$ .

A profile of states is a map  $\pi = (\pi_0, \pi_1) : \Theta \rightarrow \{0, \dots, N'_0\} \times \{0, \dots, N'_1\}$ , where  $N'_1 = N' - N'_0$  and  $\pi_\ell(\theta)$  is the number of type- $\ell$  agents in state  $\theta$ . Denote the set of state profiles by  $\Pi$ . A strategy profile  $\sigma$  induces an evolution  $\{\lambda_t^\sigma\}_{t \geq 0}$  of probability distributions over state profiles.<sup>6</sup> A belief for a private agent is a map  $p_i : \Theta \times \{0, \dots, N'_0\} \times \{0, \dots, N'_1\} \rightarrow [0, 1]$  such that  $p(\theta, n_0, n_1)$  is the probability that the agent assigns to the event that there are  $n_\ell$  type- $\ell$  agents in state  $\theta$ . A belief for a public agent is defined in a similar way.

Let  $\Delta_\ell$  be the set of beliefs for a type- $\ell$  agent. A belief system for a type- $\ell$  agent is a map from  $Z_\ell$  into  $\Delta_\ell$ . If  $z^{\ell, t}$  has positive probability under  $\sigma$ , then a type- $\ell$  agent with history  $z^{\ell, t}$  can compute his belief after  $z^{\ell, t}$  from  $\lambda_t^\sigma$  by using Bayes' rule. Suppose  $\sigma$  is such that every history for both types of agents happens with positive probability. In this case, we can compute the belief system  $\mu_i(\sigma)$  of each agent  $i$  from the sequence  $\{\lambda_t^\sigma\}$  by applying Bayes' rule. Denote the profile of belief systems obtained in this way by  $\mu(\sigma) = (\mu_i(\sigma))_{i \in I}$ .

A decision rule  $f$  is fully mixed if it always assigns positive probability to every element of  $A$ . Consider an assessment  $(\sigma, \mu)$ , where  $\sigma_\ell = (\Theta, \theta_\ell^0, f_\ell, \tau_\ell)$  is the automaton used by the type- $\ell$  agents. The assessment  $(\sigma, \mu)$  is consistent if there exist sequences  $\{f_0^n\}$  and  $\{f_1^n\}$  of fully mixed decision rules such that  $f_\ell^n \rightarrow f_\ell$  for each  $\ell \in \{0, 1\}$  and  $\mu^n = \mu(\sigma^n) \rightarrow \mu$  pointwise, where  $\sigma^n = (\sigma_0^n, \sigma_1^n)$  is the strategy profile where the type- $\ell$  agents use the automaton  $\sigma_\ell^n = (\Theta, \theta_\ell^0, f_\ell^n, \tau_\ell)$ . The assessment  $(\sigma, \mu)$  is a sequential equilibrium if it is both consistent and sequentially rational.

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<sup>6</sup>Let  $\theta_\ell^0$  be the initial state of the type- $\ell$  agents. In  $t = 0$ , the state profile is  $\pi^0$  such that  $\pi_\ell^0(\theta_\ell^0) = N'_\ell$  for all  $\ell \in \{0, 1\}$ . So,  $\lambda_0^\sigma$  is the element of  $\Delta(\Pi)$  that assigns probability one to  $\pi^0$ . Now observe that there exists a map  $Q_t^\sigma : \Pi \rightarrow \Delta(\Pi)$  such that if the state profile in period  $t$  is  $\pi$ , then the probability that the state profile in period  $t + 1$  lies in  $B \subset \Pi$  is  $Q_t^\sigma(\pi)(B)$ . This implies that  $\lambda_{t+1}^\sigma = \sum_{\pi \in \Pi} Q_t^\sigma(\pi) \lambda_t^\sigma(\pi)$  for all  $t \geq 0$ .



### 3 Preliminary Discussion

The first-best is achieved when in every single-coincidence meeting the agent who is the producer transfers  $x^*$  units of the good to his partner. We want to determine whether the first-best allocation can be sustained (by a sequential equilibrium) when there is limited memory. We know from Araujo (2004) that this is possible even when there is no memory in the economy as long as the arrival rate of trading opportunities is sufficiently high.<sup>7</sup> However, as Proposition 1 below shows, without memory, for any  $\beta \in (0, 1)$ , the only allocation that can be sustained when the population is large enough is the autarkic allocation.

**Proposition 1.** *Suppose that  $m = 0$ . For each  $\beta$ , there exists  $N'(\beta)$  such that if  $N' \geq N'(\beta)$ , then autarky is the only Nash equilibrium outcome.*

We just provide a sketch of the proof of Proposition 1, since it is very similar to the proof of Proposition 2 in Araujo (2004). Any non-autarkic allocation has at least one agent,  $i$  let us say, producing  $x' \geq 1$  units of the good to his partner in some period  $t$ . This is only possible if  $i$  is punished for announcing zero in  $t$ . Define the sequence of (random) sets  $\{I_s(i)\}_{s=1}^\infty$  recursively as follows. Let  $I_1(i)$  be the singleton set with  $i$ 's partner in  $t$  and, for each  $s \geq 1$ , let  $I_{s+1}(i)$  be the set of agents  $j$  such that either  $j \in I_s(i)$  or  $j$  is matched with an agent from  $I_s(i)$  in  $t + s - 1$ . Notice that  $|I_s(i)| \leq \min\{2^{s-1}, N'\}$  for all  $s \geq 1$ . Since  $m = 0$ , agent  $i$  can only be punished for a defection in  $t$  when he meets with an agent from  $I_s(i)$  in period  $t + s$ , with  $s \geq 1$ . Moreover, because of discounting, these punishments have a deterrent effect only if they happen within a finite number  $T$  of periods from  $t$ , where  $T$  depends on  $\beta$ . However, the probability that  $i$  is not matched with someone from  $I_s(i)$  in  $t + s$  for all  $s \in \{1, \dots, T\}$  converges to one as the population size grows. Thus, autarky is the only Nash equilibrium outcome when  $N'$  is large enough.

The main result of the paper is in sharp contrast to Proposition 1. It shows that as long as there is some memory, it is possible to sustain the first-best if  $\beta$  is high enough no matter

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<sup>7</sup>Araujo (2004) considers the case where there is a unit upper bound on the amount of goods that can be produced in a single-coincidence meeting. It is straightforward to adapt his argument to our setting.

the population size (as long as it is large enough). In the remainder of this section we discuss the main challenges involved in establishing our main result and outline (in broad terms) the approach we use to deal with them.

We say that an agent *cooperates* if he announces  $x^*$  and *defects* if he announces any  $x$  different from  $x^*$ . Moreover, we say that a defection is public if it is done by a public agent, otherwise we say that it is private. It is straightforward to provide public agents with the incentive to cooperate: just punish a public defection with sufficiently many periods of global autarky. The problem lies with providing private agents with the incentive to do the same.

We know from the reasoning above that without public agents, the amount of time it takes for the information about a private defection to reach a substantial fraction of the population increases with the population size. This suggests that the role of the public agents in sustaining cooperation in arbitrarily large populations is to speed up the process by which the population learns about a private defection by doing a public defection.

In light of this last observation, a natural candidate for an equilibrium sustaining the first-best is the strategy profile where an agent who suffers a private defection always defects afterwards and the economy moves to global autarky for a long enough time once a public defection occurs. In this strategy profile, a private agent who defects starts a “contagion” process that eventually reaches a public agent, triggering global autarky. Moreover, how fast the punishment phase begins does not depend on the population size, but on the speed with which trading opportunities arrive (and on the fraction of public agents). So, if  $\beta$  is high enough, private agents have an incentive to always cooperate no matter the population size.

There are two (related) difficulties with the above strategy profile, though. The first is that a public agent who suffers a private defection may not have an incentive to defect. Indeed, since a public defection immediately triggers global punishment, such an agent is only willing to defect if he believes that other public agents have also suffered a private defection, which is unlikely early on in the game or when the fraction of public agents is small. Likewise, a private agent who suffered a private defection will only have an incentive to defect against a public agent if he believes that many other private agents have suffered

a private defection as well (in which case a public defection is likely to occur even if he does not defect). This is also not probable early on in the game or when the fraction of public agents is small.<sup>8</sup>

We solve the two problems by changing the strategy profile in the following way: a public agent who suffers a private defection only defects in periods that are positive multiples of some  $T > 1$ , where  $T$  is large enough, but independent of the population size. To see why this works, consider a public agent who suffers a private defection for the first time in some period  $s \in \{kT, \dots, k(T+1) - 1\}$ , with  $k \geq 0$ . When  $s$  is close to  $kT$ , the chance that before  $k(T+1)$  another public agent suffers a private defection is high, and so the agent finds it in his best interest to defect in  $k(T+1)$ .<sup>9</sup> On the other hand, when  $s$  is close to  $k(T+1)$ , the agent believes that the number of private agents who are defecting is large enough that the likelihood that another public agent has also suffered a defection sometime between  $kT$  and  $s$  is high. Indeed, public agents communicate a private defection that they observe in  $\{kT, \dots, k(T+1) - 1\}$  only in  $k(T+1)$ , which allows the number of private agents who are defecting to become large. Hence, the public agent also finds it optimal to defect in  $k(T+1)$ . A similar reasoning shows that a private agent who suffers a private defection now has an incentive to defect against any public agent that he meets.

## 4 Main Result

We now establish the main result of the paper. Notice that the normalized lifetime payoff to the agents in the first-best is  $V_{FB} = \alpha[u(x^*) - x^*]$ .

**Proposition 2.** *For all  $m \in (0, 1)$ , there exists  $\beta' \in (0, 1)$  and  $\eta' \geq 1$  such that the first-best can be sustained when  $\beta \geq \beta'$  for all  $\eta \geq \eta'$ .*

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<sup>8</sup>A third difficulty, which is central in Ellison (1994) and Araujo (2004), is that a private agent who suffered a private defection may find it optimal to delay the contagion process by not defecting against another private agent. We are going to see that this is not important when the population size is large.

<sup>9</sup>This shows that the amount of memory in the environment dictates the choice of  $T$ ; the smaller memory is, the larger  $T$  needs to be.

For the proof of Proposition 2, let  $\sigma^*(T)$  be the strategy profile described by the following automaton. The state space is  $\Theta = \{C, A, D, D_0, \dots, D_K\}$ , and all agents start in state C.

- **State C.** An agent in state C always cooperates. The agent stays in state C if he observes no defection (public or private), and moves to state A if there is a public defection. If there is no public defection, a public agent in state C moves to state D if his partner defects. If there is no public defection, a private agent in state C moves to: (i) state D if his partner defects; (ii) state  $D_0$  if he defects and his partner is public; (iii) state  $D_1$  if he defects and his partner is private and does not defect.
- **State A.** An agent in state A always announces  $x = 0$ . This state is absorbing.
- **States D and  $D_0$ .** A private agent in states D and  $D_0$  always announces  $x = 0$ . A public agent in state D always cooperates if  $t$  is not a positive multiple of  $T$  and always defects if  $t$  is a positive multiple of  $T$ . An agent in states D and  $D_0$  moves to state A if there is a public defection, otherwise he stays in the same state.

By construction, a public agent can never be in states  $D_0$  to  $D_K$ . Moreover, a private agent can be in one of these states only if he deviates in state C and his partner does not. Thus, a private agent in states  $D_1$  to  $D_K$  need not behave in the same way as a private agent in state D, since the latter does not know the starting point of the contagion (defection) process unless he observes a private defection in  $t = 0$ . It turns out that how a private agent behaves in states  $D_1$  to  $D_K$  is not relevant for behavior in states C, A, D, and  $D_0$ . For this reason, we describe the state transitions and action choices in the states  $D_1$  to  $D_K$  in the Appendix. We return to this point at the end of the section.

Before we start with the proof of Proposition 2, let us show that  $\sigma^*(1)$  can never be part of a sequential equilibrium when the population is large enough. It is easy to show that under  $\sigma^*(1)$ , a private agent in state C has no profitable one-shot deviation only if  $\beta > \beta^* = x^*/u(x^*)$ .<sup>10</sup> Consider now a public agent in state D in some  $t \geq 1$ . He has no

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<sup>10</sup>Since a public defection immediately triggers global autarky, a one-shot deviation by a public agent in

profitable one-shot deviation if, and only if,

$$\beta q'(1 - q'')u(x^*) \leq x^*, \quad (1)$$

where  $q'$  is the probability that no other public agent is in state D in  $t$  and  $q''$  is the probability that the public agent's partner in  $t + 1$  is in state D when no public defection occurs in  $t$ . Suppose that  $t = 1$ . For any  $\mu$  such that  $(\sigma^*(1), \mu)$  is a consistent assessment, the public agent believes that there is only one private agent in state D and no other public agent in this state. Thus,  $q' = 1$  and  $q'' \leq 2/(N' - 1)$ , which implies that (1) cannot be satisfied when  $\beta > \beta^*$  for  $N'$  large enough.

We proceed as follows. We start with some preliminary results. Then, we construct a belief system  $\mu^*$  such that the assessment  $(\sigma^*(T), \mu^*)$  is consistent and establish some properties of  $\mu^*$ . Following that, we analyze behavior, first on the path of play, and then off the path of play. The task is to show that for each  $m \in (0, 1)$ , we can find a value of  $T$  such that  $(\sigma^*(T), \mu^*)$  is a sequential equilibrium for arbitrarily large populations when the arrival rate of trading opportunities is large enough.

#### *Preliminary Results.*

Let  $N_{\eta,t}(n, T; m)$  be the number of private agents in state D in  $t \leq T - 1$  when in  $t = 0$  there are  $n \geq 1$  such agents in state D and all the other agents are in state C. The first result we establish is useful when we describe beliefs.

**Lemma 1.**  $\lim_{\eta \rightarrow \infty} E[N_{\eta,t}(1, T; m)] = (2 - m)^t$  for all  $1 \leq t \leq T - 1$  and  $m \in (0, 1)$ .

The intuition behind Lemma 1 is straightforward. Suppose there are  $N_s$  private agents in state D in  $s \leq T - 2$ . When the population is large, the probability that any two such agents meet is negligible. Thus, each private agent in state D either meets with a public agent, which happens with probability roughly equal to  $m$ , or meets with a private agent

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state C is not profitable if, and only if,  $(1 - \beta)\alpha u(x^*) \leq V_{FB}$ , that is, if, and only if,  $\beta \geq \beta^* = x^*/u(x^*)$ . Now observe that the payoff from a one-shot deviation in C is larger for a private agent in a private meeting than for a public agent, since a defection in a private meeting does not immediately lead to global autarky.

in state C, which happens with probability roughly equal to  $1 - m$ . Hence,  $E[N_{s+1}|N_s] \approx mN_s + (1 - m)2N_s = (2 - m)N_s$ , which implies the desired result.

Now let  $\varepsilon_\eta(n, T; m)$  be the probability that there are no public agents in state D in period  $T$  when in  $t = 0$  there are  $n \geq 1$  private agents in state D and all the remaining agents are in state C. It is immediate to see that these probabilities are decreasing in  $T$ . It is also easy to see that the greater the number of private agents in state D in  $t = 0$ , the smaller the probability that a public defection does not occur in  $T$ . In other words,  $\varepsilon_\eta(n, T; m)$  is also decreasing in  $n$ .

**Lemma 2.**  $\varepsilon_\eta(n, T; m)$  is decreasing in  $n$  and  $T$  for all  $\eta \geq 1$  and  $m \in (0, 1)$ .

Lemma 2 is a step towards proving that for each  $m \in (0, 1)$ ,  $T \cdot \sup_\eta \varepsilon_\eta(1, T; m)$  converges to zero as  $T$  increases to infinity. This implies that when a private defection occurs, the expected number of periods it takes for a public defection to take place can be taken to be *independent* of the population size, a fact that is crucial for our main result.

**Lemma 3.**  $\lim_{T \rightarrow \infty} T \cdot \sup_\eta \varepsilon_\eta(1, T; m) = 0$  for all  $m \in (0, 1)$ .

### Beliefs

Denote the decision rule under  $\sigma^*(T)$  by  $f^*$  and the mixed action that assigns the same probability to every announcement by  $\mathbf{e}$ . Let  $\sigma_n^*(T)$  be the profile of behavior strategies that is obtained from  $\sigma^*(T)$  when  $f^*$  is replaced with  $f_n^*$ , where  $f_n^*(\cdot) = (1 - \frac{1}{n})f^*(\cdot) + \frac{1}{n}\mathbf{e}$ . By construction, if  $\mu^* = \lim_n \mu(\sigma_n^*(T))$ , then  $(\sigma^*(T), \mu^*)$  is a consistent assessment. In what follows, we denote the probability of an event  $B$  conditional on an event  $C$  by  $P_\eta^n(B|C)$  if the agents play according to  $\sigma_n^*(T)$  and  $P_\eta(B|C)$  if the agents play according to  $\sigma^*(T)$ . We omit the dependence of these conditional probabilities on  $m$  for simplicity.

Fix an agent. For each  $t \in \mathbb{Z}_+$ ,  $k \leq t - 1$ , and  $(s_0, \dots, s_k) \in \mathbb{Z}_+^{k+1}$ , with  $s_0 < \dots < s_k \leq t$ , let  $\mathcal{O}^t(s_0, \dots, s_k)$  be the period- $t$  event that no public defection occurs in or before  $t$  and the agent observes a private defection in some  $s \in \{s_0, \dots, s_k\}$ ; the agent observes a private defection in  $s_i$  if his partner in this period is private and defects. Now, for each  $t \in \mathbb{Z}_+$ ,  $k \leq t - 1$ ,  $(s_0, \dots, s_k) \in \mathbb{Z}_+^{k+1}$ , and  $(d_0, \dots, d_k) \in \mathbb{N}^{k+1}$ , with  $s_0 < \dots < s_k \leq t$  and

$d_0 + \dots + d_k \leq N'_1 - 1$ , let  $\mathcal{D}^t(d_0, s_0; \dots; d_k, s_k)$  be the period- $t$  event that no public defection occurs in or before  $t$  and  $d_i$  private agents in the rest of the population start a defection process in  $s_i$ . For ease of notation, let  $\mathcal{O}_s^t = \mathcal{O}^t(s)$  and  $\mathcal{D}_s^t = \mathcal{D}^t(1, s)$ . Notice that as  $n$  increases, the events  $\mathcal{D}^t(d_0, s_0; \dots; d_k, s_k)$  where either  $d_0 + \dots + d_k \geq 2$  or  $k \geq 1$  become infinitely less likely than the events  $\mathcal{D}_s^t$ . Hence, for each  $r \leq s \leq t$ ,

$$\lim_{n \rightarrow \infty} \left\{ P_\eta^n(\mathcal{D}_r^t | \mathcal{O}_s^t) - \frac{P_\eta^n(\mathcal{O}_s^t | \mathcal{D}_r^t) P_\eta^n(\mathcal{D}_r^t)}{\sum_{q=0}^s P_\eta^n(\mathcal{O}_s^t | \mathcal{D}_q^t) P_\eta^n(\mathcal{D}_q^t)} \right\} = 0.$$

Moreover,  $P_\eta^n(\mathcal{D}_r^t)/P_\eta^n(\mathcal{D}_q^t)$  converges to 1 for all  $r, q \in \{0, \dots, t\}$ . Thus,

$$P_\eta(\mathcal{D}_r^t | \mathcal{O}_s^t) = \lim_{n \rightarrow \infty} P_\eta^n(\mathcal{D}_r^t | \mathcal{O}_s^t) = \lim_{n \rightarrow \infty} \frac{P_\eta^n(\mathcal{O}_s^t | \mathcal{D}_r^t)}{\sum_{q=0}^s P_\eta^n(\mathcal{O}_s^t | \mathcal{D}_q^t)} = \frac{P_\eta(\mathcal{O}_s^t | \mathcal{D}_r^t)}{\sum_{q=0}^s P_\eta(\mathcal{O}_s^t | \mathcal{D}_q^t)}. \quad (2)$$

By construction,  $P_\eta(\mathcal{D}_r^t | \mathcal{O}_s^t)$  is the conditional probability of  $\mathcal{D}_r^t$  given  $\mathcal{O}_s^t$  when the belief system is given by  $\mu^*$ .

The first result we establish about beliefs allows us to reduce the problem of checking incentives off the path of play for every period  $t$  to the problem of checking these incentives only when  $t \in \{0, \dots, T\}$ . For each  $t \in \mathbb{Z}_+$ ,  $1 \leq k \leq t$ , and  $(s_1, \dots, s_k) \in \mathbb{Z}_+^k$ , with  $s_1 < \dots < s_k \leq t$  if  $k > 1$ , let  $F_{\eta, s_1, \dots, s_k}^t(r; m) = \sum_{q=0}^r P_\eta(\mathcal{D}_q^t | \mathcal{O}^t(s_1, \dots, s_k))$  for all  $r \leq s_1$ .

**Lemma 4.** *Suppose that  $s_1 < \dots < s_k \leq t$ , with  $s_1, t \in \{jT, \dots, (j+1)T - 1\}$  for some  $j \geq 1$ . Then  $F_{\eta, s_1, \dots, s_k}^t(r; m) \geq F_{\eta, s_1 - jT, \dots, s_k - jT}^{t-jT}(r - jT; m)$  for all  $jT \leq r \leq s_1$ .*

The next result is intuitive. In any  $t \leq T$ , an agent who observes a private defection for the first time in  $s < t$  and then observes other private defections assigns a greater probability to the event that the defection process started earlier than an agent who only observes a private defection in  $s$ .

**Lemma 5.**  $F_{\eta, s, s_1, \dots, s_k}^t(r; m) \geq F_{\eta, s}^t(r; m)$  for all  $r \leq s \leq t - 1$ .

The last result about beliefs is a straightforward consequence of equation (2) together with Lemma 1 and the fact that for all  $r, q \leq s$ , with  $s \leq \min\{t, T - 1\}$  and  $t \leq T$ ,

$$\lim_{\eta \rightarrow \infty} \frac{P_\eta(\mathcal{O}_s^t | \mathcal{D}_r^t)}{P_\eta(\mathcal{O}_s^t | \mathcal{D}_q^t)} = \lim_{\eta \rightarrow \infty} \frac{\mathbb{E}[N_{\eta, s-r}(1, T; m)]}{\mathbb{E}[N_{\eta, s-q}(1, T; m)]}.$$

**Lemma 6.** For each  $r \leq s \leq \min\{t, T-1\}$  and  $t \leq T$ ,

$$\lim_{\eta \rightarrow \infty} F_{\eta,s}^t(r; m) = \frac{(2-m)^{s+1}}{(2-m)^{s+1} - 1} \left\{ 1 - \frac{1}{(2-m)^{r+1}} \right\} = F_s(r; m).$$

*On-the-equilibrium-path behavior*

We now show that a one-shot deviation by a public agent in state C is not profitable if, and only if,  $\beta \geq \beta^*$  (see Footnote 10). Consider now a private agent in the same state. Regardless of the population size and the value of  $T$ , a private defection by him eventually leads to a public defection. So, a one-shot deviation is not profitable as long as  $\beta$  is sufficiently large. This is not enough for our results, though, since the lower bound on  $\beta$  may depend on the population size. We show that this is not the case.

Suppose the private agent defects in  $t = jT + s$ , with  $j \in \mathbb{Z}_+$  and  $s \in \{0, \dots, T-1\}$ , and let  $\varepsilon_s^k$ , with  $k \geq 1$ , be the probability that there is no public defection in  $(j+k+1)T$  when no public defection takes place in  $t \in \{(j+1)T, \dots, (j+k)T\}$ . An upper bound to the agent's lifetime payoff from a one-shot deviation in  $t$  is then given by

$$(1 - \beta) \sum_{j=0}^{2T-s} \beta^j \alpha u(x^*) + \beta^{2T-s+1} \sum_{k=1}^{\infty} (\prod_{j=1}^k \varepsilon_s^j) \beta^{(k-1)T} (1 - \beta^T) \alpha u(x^*). \quad (3)$$

This upper bound is obtained when no public defection occurs in  $(j+1)T$ , and the agent always meets with someone in state C as long as no public defection takes place.

Now observe, by Lemma 2, that  $\varepsilon_s^k \leq \varepsilon_\eta(2, T; m)$  for all  $k \geq 1$  and  $s \in \{0, \dots, T-1\}$ . By construction, the agent moves to state D<sub>0</sub> in  $(k+1)T$  if there is no public defection in this period. Thus, (3) is bounded above by

$$\bar{V} = (1 - \beta^{2T-s+1}) \alpha u(x^*) + \beta^{2T-s+1} \varepsilon_\eta(2, T; m) \frac{1 - \beta^T}{1 - \beta^T \varepsilon_\eta(2, T; m)} \alpha u(x^*).$$

Since, by Lemma 3,  $\lim_T \sup_\eta \varepsilon_\eta(1, T; m) = 0$ , we then have the following result:

**(I)** There exists  $T_1 = T_1(m)$  such that if  $T \geq T_1$ , then there exists  $\beta' = \beta'(T)$  such that a private agent in state C has no profitable one-shot deviation for all  $\beta \geq \beta'$  regardless of the population size.



*Off-the-equilibrium-path behavior*

Consider a private agent in state D in  $t \in \{1, \dots, T\}$  and let  $s_0 \leq t - 1 \leq T - 1$  be the first period in which he observed a private defection. There are two ways in which he gains by not defecting in  $t$ . First, this reduces the probability that a public defection occurs in the future. Second, this increases the chance that he meets with private agents in state C before a public defection takes place. In what follows we show that both effects can be made as small as one wants when the population is large enough.

Let  $\varepsilon^j$ , with  $j \geq 1$ , be the probability that there is no public defection in  $jT$  when there is no public defection in  $t \in \{T, \dots, (j-1)T\}$ . These probabilities depend on how the agent behaves in  $t$  (and after). Suppose the agent does a one-shot deviation in  $t$ . By Lemma 2, the probabilities  $\varepsilon^j$ , with  $j \geq 2$ , increase at most from zero to  $\varepsilon_\eta(1, T; m) \leq \varepsilon_\eta(1, T/2; m)$ . We need to determine (an upper bound on) the impact on  $\varepsilon^1$ . For this, assume that  $T$  is even. If  $s_0 \leq T/2$ , Lemma 2 implies that  $\varepsilon^1$  increases at most from zero to  $\varepsilon_\eta(1, T/2, m)$ . If  $s_0 \geq T/2 + 1$ , Lemmas 2 and 5 imply that  $\varepsilon^1$  increases at most from zero to

$$\begin{aligned} F_{\eta, s_0}^t(T/2; m)\varepsilon_\eta(1, T/2; m) + (1 - F_{\eta, s_0}^t(T/2; m))\varepsilon_\eta(1, T - s_0; m) \\ \leq \varepsilon_\eta(1, T/2; m) + 1 - F_{\eta, s_0}^t(T/2; m). \end{aligned}$$

A one-shot deviation in  $t$  also increases the chance that from  $t + 1$  to  $T$  the agent meets with agents who do not defect. At best, there is one more such agent in  $t + 1$ , two more such agents in  $t + 2$ , and so on. Hence, an upper bound to the payoff gain from a one-shot deviation in  $t$  is

$$\begin{aligned} (1 - \beta) \sum_{s=1}^{T-t+1} \beta^s \frac{2^{s-1}}{N' - 1} \alpha u(x^*) + \beta^{T-t+1} \frac{\varepsilon_\eta(1, T/2; m)}{1 - \beta^T \varepsilon_\eta(1, T/2; m)} (1 - \beta^T) \alpha u(x^*) \\ + \mathbf{I}_{\{s \geq T/2+1\}}(s_0) \beta^{T-t+1} (1 - \beta^T) (1 - F_{\eta, s_0}^t(T/2; m)) \alpha u(x^*), \end{aligned}$$

where  $\mathbf{I}_{\{s \geq T/2+1\}}$  is the indicator function of the set  $\{T/2 + 1, \dots, T - 1\}$ . Thus, this deviation is not profitable if

$$\beta \left\{ \frac{(2\beta)^T - 1}{2\beta - 1} + \frac{1 - \beta^T}{1 - \beta} \left[ \frac{\varepsilon_\eta(1, T/2; m)}{1 - \beta^T \varepsilon_\eta(1, T/2; m)} + \Delta_\eta(T; m) \right] \right\} \leq \frac{x^*}{u(x^*)}, \quad (4)$$

where  $\Delta_\eta(T; m) = \max_{t-1 \geq s_0 \geq T/2+1} [1 - F_{\eta, s_0}^t(T/2; m)]$ . Now notice that the right-hand side of the above inequality is increasing in  $\beta$ . Therefore, a sufficient condition for (4) is that

$$\underbrace{\frac{2^T}{N' - 1}}_A + \underbrace{\frac{T\varepsilon_\eta(1, T/2; m)}{1 - \varepsilon_\eta(1, T/2; m)}}_B + \underbrace{T[\Delta_\eta(T; m) - \Delta(T; m)]}_C + \underbrace{T\Delta(T; m)}_D \leq \beta^*,$$

where  $\Delta(T; m) = \lim_{\eta \rightarrow \infty} \Delta_\eta(T; m)$ .

By Lemma 3, there exists  $T' = T'(m)$  such that  $B \leq \beta^*/4$  if  $T \geq T'$ . Since

$$1 - F_{s_0}(T/2; m) = \frac{(2 - m)^{s_0 - \frac{T}{2}} - 1}{(2 - m)^{s_0 + 1} - 1}$$

is increasing in  $s_0$ , we then have that

$$T\Delta(T; m) = \frac{T(2 - m)^{\frac{T}{2} - 1}}{(2 - m)^T - 1}, \quad (5)$$

which converges to zero as  $T$  increases to infinity. So, there exists  $T'' = T''(m)$  such that  $D \leq \beta^*/4$  if  $T \geq T''$ . Let  $T_2 = \max\{T', T''\}$ . Now observe, by Lemma 6, that for each  $T \geq T_2$  there exists  $\eta' = \eta'(T)$  such that  $\max\{A, B\} \leq \beta^*/4$  if  $\eta \geq \eta'$ . We have thus established the following result:

**(II<sub>0</sub>)** There exists  $T_2$  such that if  $T \geq T_2$ , then there exists  $\eta'$  such that a private agent in state D in  $t \in \{0, \dots, T\}$  has no profitable one-shot deviation for all  $\beta \in (0, 1)$  when  $\eta \geq \eta'$ .

Consider now a private agent in state D in  $t \in \{jT + 1, \dots, (j + 1)T\}$  for some  $j \geq 1$  and once more let  $s_0$  be the first period in which he observed a private defection. If  $s_0 \geq jT$ , we can apply Lemma 4 to reduce this case to the case where  $t \leq T$ . If  $s_0 \leq jT - 1$ , then there are at least two private agents in state D in period  $jT$ , and so we can also apply the reasoning used when  $t \leq T$ . Now observe that if a private agent is in state D<sub>0</sub>, then either  $t \in \{jT + 1, \dots, (j + 1)T\}$  for some  $j \geq 1$ , or  $t \in \{1, \dots, T\}$  and the agent defected against a public agent in some previous period. In the first case, the analysis is the same as if the agent were in state D and  $s_0 \leq jT - 1$ . In the second case, the only benefit from a one-shot deviation is to increase the chance that from  $t + 1$  to  $T$  the agent meets with agents who do not defect. We then have the following result:

(II) There exists  $T_2$  such that if  $T \geq T_2$ , then there exists  $\eta'$  with the property that a private agent in states D or  $D_0$  has no profitable one-shot deviation for all  $\beta \in (0, 1)$  when  $\eta \geq \eta'$ .

As the last step before the proof of Proposition 2, consider a public agent in state D in some period  $t \geq 1$  and let  $s_0 \leq t - 1$  be the first period in which he observed a private defection. We know from above that if  $\beta \geq \beta^*$ , then he has no profitable one-shot deviation if  $t \neq jT$  for all  $j \geq 1$ . Consider then the case where  $t = jT$  and let  $\varepsilon^j$  be the probability that the agent assigns to the event that there is no other public agent in state D in this period. By Lemma 2, a one-shot deviation is not profitable if,

$$(1 - \beta)\alpha x^* \geq \frac{\varepsilon^j}{1 - \beta^T \varepsilon^j} (1 - \beta^T)\alpha u(x^*),$$

and a sufficient condition for this to hold for all  $\beta \in (0, 1)$  is that  $T\varepsilon^j \leq (1 - \varepsilon^j)\beta^*$ . As before, assume that  $T$  is even. If either  $j \geq 2$  and  $s_0 \leq jT - 1$  or  $j = 1$  and  $s_0 \leq T/2$ , Lemma 2 implies that  $\varepsilon^j \leq \varepsilon_\eta(1, T/2; m)$ . In the other cases, Lemmas 4 and 5 imply that

$$\begin{aligned} \varepsilon^j &\leq F_{\eta, s_0}^T(T/2; m)\varepsilon_\eta(1, T/2; m) + 1 - F_{\eta, s_0}^T(T/2; m) \\ &\leq \varepsilon_\eta(1, T/2; m) + \Delta(T; m) + F_{s_0}(T/2; m) - F_{\eta, s_0}^T(T/2; m). \end{aligned}$$

So, by Lemma 6, we have the following result:

(III) There exists  $T_3$  such that if  $T \geq T_3$ , then there exists  $\eta'$  with the property that a public agent in state D has no profitable one-shot deviation for all  $\beta \geq \beta^*$  when  $\eta \geq \eta'$ .

**Proof of Proposition 2:** From (I) to (III), we know that there exists  $T' = T'(m)$  such that if we set  $T$  in  $\sigma^*(T)$  equal to  $T'$ , then there exist  $\beta' \in (0, 1)$  and  $\eta' \geq 1$  with the property that no agent in states C, A, D, and  $D_0$  has a profitable one-shot deviation when  $\beta \geq \beta'$  as long as  $\eta \geq \eta'$ . Notice that this holds regardless of how a private agent behaves in states  $D_1$  to  $D_K$ . Thus, for each  $\beta \geq \beta'$  and  $\eta \geq \eta'$ , we just need to specify the behavior of a private agent in states  $D_1$  to  $D_K$  in a way that it is sequentially rational (given the behavior of the agents in the other states). The details of how this can be done are in the Appendix.  $\square$ .

Notice that the strategy profile we consider is not “efficient” when  $m$  is close to one. The reason is that the value of  $T$  needed to make (5) small enough for (4) to be satisfied

increases with  $m$ . It is, of course, not necessary to set  $T$  large, which requires a high  $\beta$ , when  $m$  is close to one, since in this case a defection process quickly reaches a public agent. The advantage of using the strategy profile  $\sigma^*(T)$  is that it works for all  $m \in (0, 1)$ .

## 5 Robustness

There are other notions of memory that we could have used in our analysis. We discuss three of them. The first, simpler, would be to assume that for every meeting that takes place there is a probability  $m \in (0, 1)$  that the announcements in the meeting become public. In this case, the first-best can be sustained by a grim-trigger profile as long as  $\beta$  is sufficiently large. Assuming that the agents don't know whether their decisions will be observed by everyone else greatly simplifies the analysis.

The second notion of memory generalizes the one we use by assuming that the announcements of the public agents are observed with a lag, either deterministic or stochastic.<sup>11</sup> The effect of an increase in the lag is potentially ambiguous. On the one hand, it increases the amount of time it takes until a defection by a public agent is observed, which requires a larger  $\beta$  to sustain cooperation on the path of play. On the other hand, a larger lag makes it easier to provide a public agent with the incentive to communicate a private defection to the rest of the population. If the lag is large enough, the first effect dominates the second, though. It is straightforward to adapt the proof of Proposition 2 to cover this type of memory. The lower bound on  $\beta$  will now depend on both  $m$  and the size of the lag.

The third notion of memory also generalizes the one we use by assuming that for each public agent, there exists a fraction  $F < 1$  of the private agents that observe his actions. As with the second alternative notion of memory, a defection by a public agent no longer triggers an immediate reversion to global autarky. Nevertheless, a defection by a public agent eventually leads to global autarky (and the time elapsed does not depend on the population

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<sup>11</sup>When  $m = 1$ , this coincides with the notion of memory used in Kocherlakota and Wallace (1998). The lag is deterministic if for every period  $t$  the announcements of the public agents in  $t$  become public in the period  $t + L$ , with  $L \geq 0$ . The lag is stochastic if  $L$  is random.

size). Notice that a decrease in  $F$  has the same two effects on incentives described in the previous paragraph as an increase in the observation lag. It is straightforward to adapt the proof of Proposition 2 to cover this third type of memory. The lower bound on  $\beta$  will now depend on  $m$  and  $F$ .

We assume that the good is not perfectly divisible. This assumption is done purely for technical reasons: the notion of a sequential equilibrium is problematic for games with infinite action spaces. Nowhere in the proof of Proposition 2 does the indivisibility assumption play a role in sustaining cooperation. We do need to change Proposition 1 when the good is perfectly divisible, though. It is easy to see that the following is true when perfect divisibility holds.

**Proposition 1’.** *Suppose that  $m = 0$ . For each  $\beta$  and  $\tilde{x} > 0$ , there exists  $N'(\beta, \tilde{x})$  such that if  $N' \geq N'(\beta, \tilde{x})$ , then an agent does not announce more than  $\tilde{x}$  on the path of play in any Nash equilibrium.*

An important concern in the literature on social norms is the robustness of cooperation to trembles (deviations) in the population. This is especially important in our environment, given our focus on allocations (and not on ex-ante payoffs). In the strategy profile we use to prove Proposition 2, a deviation eventually leads to permanent autarky. We do this for simplicity. It is easy to see that if we modify this strategy profile so that agents in state A return to state C after sufficiently many periods, it still remains a sequential equilibrium. Thus, we can sustain the first-best with a sequential equilibrium where cooperation is eventually restored after a tremble.

To finish this section, notice that our results remain the same if agents make their production decisions only after they know whether they are producers or not. The only difference in this case is that now the defection process is slower: an agent in state D can only defect in a single-coincidence meeting where he is the producer. This amounts, in rough terms, to a reduction in the quantity of memory in the economy.<sup>12</sup>

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<sup>12</sup>Also notice that our results do not change if agents are not anonymous. Indeed, if the identities of the agents are observable, it still is an equilibrium to ignore these identities and follow the strategy described in the proof of Proposition 2.

## 6 Discussion

The notion of memory we use has two distinct elements. First, it is a record of the past actions of some agents. Second, it publicly discloses this information to everyone else in the population. The first feature constitutes a pure record-keeping device, while the second feature constitutes a coordinating device. Kocherlakota (1998) introduces a different notion of memory where the coordination element is absent: in every meeting that he participates, an agent observes his partner's past history as well as the past histories of all his partner's direct and indirect partners. Limited memory can be naturally defined in Kocherlakota's environment as a restriction on the amount of information an agent obtains about his partners upon meeting them. We show in Araujo and Camargo (2009) that the first-best can be achieved with this type of memory even when it is quite limited.

There is a relation between our work and the work of Cavalcanti and Wallace (1999a,b) (CW hereafter), on the co-existence of inside and outside money. They also assume that there exists a positive measure of agents, that they label as banks, whose histories are publicly known. Banks are able to issue inside money. Overissue does not happen because the trading histories of banks are public, and so they can be punished if they fail to redeem outstanding notes. Our paper shows that money (be it inside or outside) is not essential in CW's environment if the arrival rate of trading opportunities is large enough. Putting it differently, our work unveils a tension between the existence of an equilibrium in which endogenously issued money is valued as a medium of exchange and its essentiality. The same technology that sustains the acceptability of inside money may also allow the existence of self-enforcing credit arrangements that achieve the first-best.<sup>13</sup>

This paper shows that the emphasis on limited memory as a fundamental friction for

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<sup>13</sup>It is important to note that the results in CW crucially depend on the role of memory as a coordinating device. Indeed, if we eliminate the coordinating component of memory and assume that an agent must physically meet a bank in order to observe its past transactions, then there is no equilibrium in which banks redeem outstanding notes. The reason is that if the notes issued by a bank are valued as a medium of exchange, then an agent has no incentive to refuse production to this bank in exchange for a note, whether it has refused to redeem outstanding notes in the past or not.

money to be essential needs to be re-considered. It also suggests that the role of money goes beyond that of being a (primitive) record-keeping technology. A natural question to ask is what other frictions in the exchange process can play a key role on the essentiality of money. One candidate, which we are currently investigating, is private information about preferences and technology.

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## Appendix: Omitted Details and Proofs

STATE TRANSITIONS IN  $D_1$  to  $D_K$ .

An agent in state  $D_k$  in period  $t$ , with  $k \in \{1, \dots, K\}$ , moves to state  $A$  if there is a public defection and moves to state  $D_0$  if there is no public defection, but either he defects against a public agent or  $t$  is a multiple of  $T$ .

For each  $k \in \{1, \dots, T\}$ , let  $S_k = \{0, 1_{cc}, 1_{cd}, 1_{dc}, 1_{dd}\}^{k-1}$  and denote a typical element of this set by  $s = (s_1, \dots, s_{k-1})$ . Consider a private agent and suppose that  $t = jT + s$ , with  $j \geq 0$  and  $s \in \{1, \dots, T\}$ . The agent can be in states  $D_1$  to  $D_K$  only if he started a defection process in some  $t_0 \in \{jT, \dots, t-1\}$  and so far has not defected against a public agent. Suppose this is the case and let  $k = t - t_0 \in \{1, \dots, T\}$  be the number of periods elapsed since the agent started the defection process. We can describe his experience since  $t_0$  by an element  $s$  of  $S_k$ : if  $s_i$  denotes his experience in  $k_0 + i$ , then  $s_i = 0$  if he meets a public agent,  $s_i = 1_{cc}$  if he meets a private agent and both agents cooperate,  $s_i = 1_{cd}$  if he meets a private agent and cooperates, but his partner does not, and so on.

For each  $k \in \{1, \dots, T\}$ , let  $\geq_k$  be a total order on  $S_k$ . Now let  $S = \bigcup_k S_k$  and introduce a total order  $\geq$  on this set as follows. For each  $s, s' \in S$ ,

$$s > s' \quad \text{if} \quad \begin{cases} s \in S^k \text{ and } s' \in S^{k'} \text{ with } k > k' \\ s, s' \in S^k \text{ and } s >_k s' \end{cases}.$$

Order the elements of  $S$  from lowest to highest according to  $\geq$  and let  $s(n)$  denote its  $n$ th element. To finish, let  $N : S \rightarrow \mathbb{N}$  be such that  $N(s(n)) = n$ . A private agent who starts a defection process is in state  $D_{N(s)}$  if his experience since he started the process is described by  $s \in S$ . Notice, by construction, that  $K$  is the number of elements of  $S$  and that  $D_{N(s(1))} = D_1$ .

## PROOF OF LEMMA 1

We start with an auxiliary result. Let  $q_\eta(n, s; m)$  be the probability that if there are  $n$  private agents in state D and the remaining agents are in state C, then  $s$  of the private agents in state D meet with a public agent and the remaining  $n - s$  ones meet with a private agent in state C. Now let

$$q_\infty(n, s; m) = \binom{n}{s} m^s (1 - m)^{n-s}.$$

We claim that for each  $n \in \mathbb{N}$  and  $s \leq n$ ,  $\lim_{\eta \rightarrow \infty} q_\eta(n, s; m) = q_\infty(n, s; m)$ . Indeed, just notice that if  $2n \leq N'_1$ , then

$$\begin{aligned} q_\eta(n, s; m) &= \binom{n}{s} \frac{N'_0}{N' - 1} \cdots \frac{N'_0 - (s - 1)}{N' - 2(s - 1) - 1} \cdot \frac{N'_1 - n}{N' - 2(s + 1) - 1} \cdots \frac{N'_1 - (2n - s - 1)}{N' - 2(n - 1) - 1} \\ &= \binom{n}{s} \frac{m}{1 - \frac{1}{N'}} \cdots \frac{m - \frac{s-1}{N'}}{1 - \frac{2(s-1)+1}{N'}} \cdot \frac{1 - m - \frac{n}{N'}}{1 - \frac{2s+1}{N'}} \cdots \frac{1 - m - \frac{2n-s-1}{N'}}{1 - \frac{2(n-1)+1}{N'}}. \end{aligned}$$

For the proof of Lemma 1, fix  $\epsilon > 0$  and assume that  $\eta \geq T$ , so that  $N'_1 \geq T$ . Now let  $t \in \{1, \dots, T - 1\}$  and suppose that  $n \leq 2^t$ . First notice that  $\sum_{s=0}^n (2n - s) q_\infty(n, s; m) = (2 - m)n$  and that  $E[N_{\eta,t}(n, T; m) | N_{\eta,t-1}(n, T; m) = n] = \sum_{s=0}^n (n + n - s) q_\eta(n, s; m)$ . So, by the previous paragraph, there exists  $\underline{\eta}$  such that

$$|E[N_{\eta,t}(n, T; m) | N_{\eta,t-1}(n, T; m) = n] - (2 - m)n| < \frac{\epsilon}{2^{T-1}}.$$

for all  $n \in \{1, \dots, 2^{T-1}\}$  and all  $t \in \{1, \dots, T - 1\}$  if  $\eta \geq \underline{\eta}$ . Now observe that there are at most  $2^{t-1}$  private agents in state D in  $t - 1$ . Hence, by the triangle inequality,

$$|E[N_{\eta,t}(n, T; m)] - (2 - m)E[N_{\eta,t-1}(n, T; m)]| < \frac{\epsilon}{2^{T-1}}$$

for all  $t \in \{1, \dots, T - 1\}$  if  $\eta \geq \underline{\eta}$ . Thus, using the triangle inequality again, we obtain that

$$\begin{aligned} &|E[N_{\eta,t}(n, T; m)] - (2 - m)^{t-1}| \\ &\leq \sum_{s=1}^t (2 - m)^{t-s} |E[N_{\eta,s}(n, T; m)] - (2 - m)E[N_{\eta,s-1}(n, T; m)]| < \epsilon. \end{aligned}$$

if  $\eta \geq \underline{\eta}$ , which implies the desired result.  $\square$

### PROOF OF LEMMA 3

First notice that

$$\varepsilon_\eta(1, T; m) = \frac{N_1 - 1}{N' - 1} \cdot \varepsilon_\eta(2, T - 1; m) = \frac{(1 - m) - \frac{1}{N'}}{1 - \frac{1}{N'}} \cdot \varepsilon_\eta(2, T - 1; m).$$

Now observe that for each  $t \geq 1$ ,

$$\begin{aligned} \varepsilon_\eta(2, t; m) &= \frac{1}{N' - 1} \cdot \varepsilon_\eta(2, t - 1; m) + \frac{N'_1 - 2}{N' - 1} \cdot \frac{N'_1 - 3}{N' - 3} \cdot \varepsilon_\eta(4, t - 1; m) \\ &\leq \left( \frac{1}{N'} + \frac{(1 - m) - \frac{2}{N'}}{1 - \frac{1}{N'}} \cdot \frac{(1 - m) - \frac{3}{N'}}{1 - \frac{1}{N'}} \right) \varepsilon_\eta(2, t - 1; m), \end{aligned}$$

where the inequality follows from Lemma 2. Therefore,

$$\begin{aligned} \varepsilon_\eta(1, T; m) &\leq \frac{(1 - m) - \frac{1}{N'}}{1 - \frac{1}{N'}} \left( \frac{1}{N'} + \frac{(1 - m) - \frac{2}{N'}}{1 - \frac{1}{N'}} \cdot \frac{(1 - m) - \frac{3}{N'}}{1 - \frac{1}{N'}} \right)^{T-1} \\ &\leq (1 - m) \left( \frac{1}{N'} + (1 - m)^2 \right)^{T-1}, \end{aligned}$$

from which the desired result holds (recall that  $1/N' < m$  by construction).  $\square$

### PROOF OF LEMMA 4

Let  $r \in \{jT, \dots, s\}$ . Then,

$$F_{\eta, s_1, \dots, s_k}^t(r; m) = \frac{\sum_{q=0}^r P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t)}{\sum_{q=0}^s P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t)} \geq \frac{\sum_{q=jT}^r P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t)}{\sum_{q=jT}^s P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t)}.$$

Now observe that if  $q \in \{jT, \dots, s\}$ , then

$$P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t) = P_\eta(\mathcal{O}^{t-jT}(s_1 - jT, \dots, s_k - jT) | \mathcal{D}_{q-jT}^{t-jT}),$$

and so the right-hand side of the above inequality is equal to  $F_{\eta, s_1-jT, \dots, s_k-jT}^{t-jT}(r - jT; m)$ .  $\square$

## PROOF OF LEMMA 5

Let  $\succsim_r$  be the monotone likelihood ratio order and define  $g_s$  and  $g_{s,s_1,\dots,s_k}$  to be the probability density functions on  $\{0, \dots, s\}$  such that  $g_s(q) = P_\eta(\mathcal{D}_q^t | \mathcal{O}_s^t)$  and  $g_{s,s_1,\dots,s_k}(q) = P_\eta(\mathcal{D}_q^t | \mathcal{O}^t(s, s_1, \dots, s_k))$ . The desired result holds if  $g_s \succsim_r g_{s,s_1,\dots,s_k}$ . We know from the main text that for each  $r \leq s$ ,

$$P_\eta(\mathcal{D}_r^t | \mathcal{O}^t(s, s_1, \dots, s_k)) = \frac{P_\eta(\mathcal{O}^t(s, s_1, \dots, s_k) | \mathcal{D}_r^t)}{\sum_{q=0}^s P_\eta(\mathcal{O}^t(s, s_1, \dots, s_k) | \mathcal{D}_q^t)}.$$

Hence,  $g_s \succsim_r g_{s,s_1,\dots,s_k}$  if, and only if,

$$P_\eta(\mathcal{O}_s^t | \mathcal{D}_r^t) P_\eta(\mathcal{O}^t(s, s_1, \dots, s_k) | \mathcal{D}_{r'}^t) \geq P_\eta(\mathcal{O}_s^t | \mathcal{D}_{r'}^t) P_\eta(\mathcal{O}^t(s, s_1, \dots, s_k) | \mathcal{D}_r^t) \quad (6)$$

for all  $r' < r \leq s$ . Now observe that if  $q \leq s$ , then

$$P_\eta(\mathcal{O}^t(s, s_1, \dots, s_k) | \mathcal{D}_q^t) = P_\eta(\mathcal{O}_s^t | \mathcal{D}_q^t) P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_q^t, \mathcal{O}_s^t).$$

Thus, (6) holds if, and only if,  $P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_{r'}^t, \mathcal{O}_s^t) \geq P_\eta(\mathcal{O}^t(s_1, \dots, s_k) | \mathcal{D}_r^t, \mathcal{O}_s^t)$  for all  $r' < r \leq s$ , which is the case when  $t \leq T - 1$ .  $\square$

## BEHAVIOR IN STATES $D_1$ to $D_K$ .

Let  $t = jT + s$ , with  $j \geq 0$  and  $s \in \{1, \dots, T\}$ , and consider a private agent with state in the set  $\{D_1, \dots, D_K\}$ . Notice that: (i) the agent's state can be in  $\{D_1, \dots, D_K\}$  only if he initiated a defection process in some  $t_0 \in \{jT, \dots, t - 1\}$ ; (ii) the only payoff relevant information for such an agent is his experience from  $t_0 + 1$  to  $t - 1$ , which determines his belief about the number of private and public agents in  $D$ , and the number  $(j + 1)T - t$  of periods left before a public defection can occur.<sup>14</sup> Therefore, we only need to determine behavior in the states  $D_1$  to  $D_K$  for  $t \in \{1, \dots, T\}$ , which we can accomplish by a backward induction argument starting at  $T$ —recall that from  $T + 1$  on the private agent is either in state  $A$  or in state  $D_0$ .

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<sup>14</sup>Notice that under  $\mu^*$ , a private agent who starts a defection process and later observes a private defection assigns zero probability to the event that some other private agent started a defection process.