FUNDAÇÃO
GETULIO VARGAS

EPGE
Escola de Pós-Graduação
em Economia

Ensaios Econômicos

Escola de
Pós-Graduação
em Economia
da Fundação
Getulio Vargas

N° 363 ISSN 0104-8910

Bailey's Rule For The Welfare Of Inflation: A Theoretical Foundation

Samuel de Abreu Pessoa

Dezembro de 1999

URL: http://hdl.handle.net/10438/658

de Abreu Pessoa, Samuel
Bailey's Rule For The Welfare Of Inflation: A Theoretical Foundation/ Samuel de Abreu Pessoa - Rio de Janeiro : FGV,EPGE, 2010
(Ensaios Econômicos; 363)

Inclui bibliografia.

CDD-330

Bailey's Rule For The Welfare Of Inflation: A Theoretical Foundation[*]

Samuel de Abreu Pessoa[y]

E-mail: pessoa@fgv.br, pessoa@ssc.upenn.edu

Working Paper

Abstract

This paper demonstrates that for a very general class of monetary models (the Sidrauski type models and the cash-in-advance models), Bailey's rule to evaluate the welfare effect of inflation is indeed accurate. The result applies for any technology or preference, if the long-run capital stock does not depend on the inflation rate. In general, a dynamic version of Bailey's rule is established. In particular, the result extends to models in which there is a banking sector that supplies money substitutes services. Additionally, it is argued that the relevant money demand concept for this issue - the impact of inflation under welfare - is the monetary base.

[*]This paper benefits from conversations with Marcos de Barros Lisboa. Evidently remaining errors are the responsibility of the author. I acknowledge the superb guidance of Ms. Elizabeth Darby to the writing and her tolerance and understanding of my 'weird Latin-language-speaker' stile of writing in English. The author thanks the Brazilian Government's research assistance agencies - CAPES and CNPq - for financial support.

[y]Graduate School of Economics, Fundação Getulio Vargas - Rio de Janeiro, Brazil. Currently visiting University of Pennsylvania.

1 Introduction

Since Bailey's (1956) classic paper, we have been accustomed to measuring the welfare cost of the perfectly foreseen inflation by the area under the inverse money demand. Notwithstanding, there has not been much effort in trying to gather a more solid theoretical foundation for this approach. The aim of this paper is to show that it is quite simple to find that theoretical foundation which is lacking. It is demonstrated that for any model affiliated to Sidrauski or to the cash-and-advance families of monetary models, which present a stationary-state capital stock that is not sensitive to inflation, "Bailey's rule[1]" provides the accurate measurement of the impact of inflation under welfare. When inflation affects the stationary-state capital stock, it is possible to derive a dynamic version of Bailey's rule. In particular, this result applies to models in which it is taken into consideration that there is a second sector, called the banking sector, which provides services that are substitutes for money services. This last class of models presents the observable phenomenon of the increase of the share in the product of the banking sector, along with the inflation rate.

The second contribution of the paper is to establish that the relevant concept of money, as far as the impact of inflation upon welfare is concerned, is the narrow monetary aggregate, the monetary base. To the best of my knowledge, it seems that this point has not been attracting the deserved attention by the scholars. Bailey's discussion is not very clear in this respect. He begins his paper

[1] Due to the generality of the result and of it being a consequence of a very general property of monetary models, I decided to keep the expression "Bailey's rule," which I have unconsciously employed in the first draft of this paper.

supposing that banks are not present. Afterwards, he introduces the banks[2]. According to him, if the bank works rationally, then the correct concept is the monetary base; otherwise, the M1 demand should be considered, although it is not very clear what he means by a bank not "behaving absolutely rationally." Lucas (1981), Cooley and Hansen (1989) and (1990) and Lucas (1997) employ M1; Barro (1972), Fischer (1981) and Aiyagari, Braun and Eckstein (1998) use M0.

The importance of finding a theoretical foundation for Bailey's rule is that the alternative approach, to calibrate a dynamic general equilibrium monetary model to evaluate it[3], is not robust to parameters calibration[4]. The area under the inverse money demand function, a directly observable function, does not present a lack of robustness. Although, for very low inflation rates this measure could be inexpressive, it can assume quite high values for high inflations[5], being a reliable lower-bound estimation of the impact of inflation under welfare. Additionally, abstracting from capital accumulation effects, this measure is a true general equilibrium one, and the specific role played by money or the alternatives which are open to the economy in order to adjust to a higher inflation rate are not a very important issue.

Lucas' (1997) and Aiyagari, Braun and Eckstein's (1998) are the most related work to this one. The main difference between the formulation accomplished in this paper with Lucas' paper is the specific way the impact of the inflation

[2] See Bailey (1956), pgs. 103 and 104.
[3] Cooley and Hansen (1989) and (1991).
[4] As an example, see Benabu (1991).
[5] For the Brazilian case see, Pastore (1994).

on welfare is calculated. Lucas evaluated it by the proportional increase in consumption, which makes the household indifferent between the two situations - in the presence of or without inflation. In this paper two concepts are adopted. Firstly, the marginal impact of inflation under welfare, measured in terms of goods, is evaluated. The total impact of the inflation under welfare proceeds from the integration of this marginal effect. Secondly, the compensate income that should be given to the household in order to keep it in the same utility level as under Friedman rule it is considered. Additionally, Lucas does not consider the existence of a banking sector which supplies money substitutes services.

Aiyagari, Braun, and Eckstein (1998) examine a cash-in-advance economy in presence of credit goods. There is a continuum of goods, which could be acquired in the market in exchange for money or a credit service. Under this second possibility, the price of a good is the money price plus a cost which varies, depending on the good. The higher the inflation rate, the larger the range of goods acquired by credit and, consequently, the higher the money velocity is[6]. Similar to the present work, their model contemplates that the provision of this money substitutes services by the banking sector requires the employment of production factors, which have been diverted from the real sector. The cash-in-advance model which is investigated in this paper is a generalization of theirs. It is argued, in disagreement with them, that generally the share in the product of the banking sector is not the precise measure of the allocation impact of inflation under welfare, although the area under the inverse money demand function is.

[6] This manner of producing a variable money velocity in cash-in-advance models was introduced by Gillman (1993).

On the other hand, this paper generalizes their findings in many dimensions. It is shown that the results depend neither on the specific monetary model taken into consideration, nor on the intratemporal elasticity of substitution if the model considers a continuum of goods. Furthermore, if capital accumulation takes place in succession of an increase of inflation, there is a simple close expression between the marginal impact of inflation under welfare and the marginal impact of inflation under the money demand path.

In this paper it is supposed, as it is standard in this literature, that the economy works under the monetary regime: the unique role of the government is to print money, and, consequently, the seigniorage is rebated to the household in a lump-sum fashion. For this kind of economy the Friedman rule is satisfied. Although it is an open question[7] whether, in presence of other imperfections, to inflate the price index is a second-best policy or not, the monetary regime provides a benchmark and, as it will be seen, an analytical workable solution.

The first step in finding a theoretical foundation for Bailey's rule is to work with models that present a well-behaved, long-run-money demand. The idea behind Bailey's rule, which is a standard preference revelation argument, is that the reduction in the consumption surplus caused by the inflation is the correct measure of its impact under welfare. Consequently, the main ingredient for Bailey's rule is the idea of a stable money demand function. The difficulty is that usually the monetary models are dynamic in nature, and normally the economy begins a dynamic path in succession of an alteration of the inflation, in

[7] See Guidotti and Vegh (1993).

such a way that there is not a stable money demand but a stable money path. To be fair to Bailey's rule, it is necessary to work with models which do not exhibit a transitory dynamic, subsequently an alteration of the inflation rate from a long-run equilibrium. Fortunately, there is quite a large set of models which possess this property. Specific to this class of monetary models, after a change in the increase rate of the nominal quantity of money, the capital stock does not change, the real quantity of money and the consumption flow jump, and, a new long-run equilibrium is instantaneously attained. Under this condition, the path integral of the welfare function trivially becomes a standard one, and it is possible to calculate it without any consideration with respect to the specific path taken by the increase-rate of the nominal quantity of money.

The main result is that for a very general class of monetary models, the impact on welfare of an alteration on the rate of the increase of the nominal quantity of money is expressed by

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t} \lambda \frac{dm}{d\sigma} dt; \qquad (1)$$

in which

W - Welfare function;

σ - increase rate of the nominal quantity of money;

ρ - intertemporal discount rate;

λ - shadow price of the real quantity of money;

m — real quantity of money.

This result, a dynamic version of Bailey's rule, is essentially a consequence of Samuelson's envelop theorem. In this model the social benefit is equal to the social cost for every choice variable besides money. Consequently, as will be clear later, the impact upon welfare of a changing in σ, stemmed from the alterations of the choices variables, cancels out. What remains is the term that depends on the variation of the money demand, which is the variable that has private cost but does not have a social one. As a result, the amount expressed by (1) is left.

If additionally, it is supposed that the stationary-state capital stock does not depends on σ, even if the other real variables (for example consumption, labor supply, banking service demand, etc.) do, then, following a changing in σ, there is no dynamic, and it makes sense to talk about a stable money demand function. Under this conditions it follows from (1) that

$$\rho \frac{dW}{d\sigma} = \lambda^{*} \frac{dm^{*}}{d\sigma};$$

which means that the flow measure of the marginal impact on welfare of a changing in σ, in units of capital is

$$\frac{\rho}{1^{*}} \frac{dW}{d\sigma} = \frac{\lambda^{*}}{1^{*}} \frac{dm^{*}}{d\sigma} \qquad (2)$$

in which

1 - shadow price of capital.

Remembering that the relative price of the real quantity of money in units of capital is the nominal interest rate, Bailey's rule follows from the integration of (2)

$$\rho \Delta W_{\text{Units of Capital}} = \int_{-\rho}^{\sigma} R^{*} \frac{dm^{*}}{d\sigma'} d\sigma' = -\int_{m^{*}(-\rho)}^{m^{*}(\sigma)} R(m) dm \qquad (3)$$

in which

R - nominal interest rate.

Furthermore, for the same class of monetary models which (3) applies, this paper shows that, if the impact of inflation under welfare is measured by the compensate income which should be given to the household to keep his in the same utility level, than the integral in (3) should be taken along the compensate money demand.

In the final part of the paper it is argued that the relevant concept of money for this issue - the impact of inflation under welfare - is the monetary base and not M1. The reason is that the demand deposit is a service which belongs to the bundle of services that are offered by the banking sector. The result follows because the area under the inverse money demand grasp all the general equilibrium effects of a increase of inflation on the economy, including the increase of the share of the banking sector. To put in another way, as far as the effects of inflation under welfare are concerned, money is the good which has private cost but does not have a social one. In this specific sense the demand deposit

should be excluded from the concept of money. It is offered by the banking institutions, and, consequently, has a positive social cost. To make this point clear, a model in which inside money takes place can be found in the seventh Section of the paper.

The paper has the following organization. In the subsequent Section to this introduction, the setup of the model is exposed, and in the third Section the generality of Bailey's rule is demonstrated. The fourth Section deals with the situation in which unbound growth is present, and the validity of Bailey's rule for the cash-in-advance class of monetary models is discussed in the fifth Section. The ensuing Section presents the validity of Bailey's rule when the compensate demand concept to measure welfare variations is applied and the seventh Section discusses the correct concept of money for this subject - welfare effects of inflation. The conclusion follows.

2 The General Model

Usually money can be incorporated in an otherwise standard macroeconomic dynamic real model in two ways: directly into the preference, the standard Sidrauski (1967) model or as an argument of a transaction cost function into the budget constraint, the way popularized by McCallum and Goodfriend (1987) in their entry in Palgrave's dictionary[8]. In order to keep the model exposed here

[8] It seems to me that Drazen (1979) was the first to suggest this manner of building a dynamic general equilibrium monetary model. McCalum (1983) introduced it as a specific shopping time restriction; Feenstra (1986) derived it from some traditional approaches to money microfoundations models and demonstrated the equivalence between this formulation and the money-into-utility approach.

as general as possible, it will be supposed that both possibilities are present. In addition, it is considered that there is another good, along with the traditional good which could be consumed and stocked as capital, called banking service which helps the household in reducing transaction costs, wherever it appears. Bassole and Pessoa (1999) elsewhere treated this model in detail. The most interesting feature is that it displays the phenomenon of the increase of the banking sector for providing the public goods and services, which are substitutes for money.

Households

The choice problem of the household is the following

$$\max \int_0^\infty e^{-\rho t} u(c_{1t}, l(c_{1t}, m_{1t}, c_{21t}))dt, \tag{4}$$

subject to

$$\dot{a}_t = r_t a_t + w_t + \tau_t - c_{1t} - p_t c_{2t} - g(c_{1t}, m_{2t}, c_{22t}) - (\pi_t + r_t) m_t, \tag{5}$$

$$m_t \equiv \frac{M_t}{P_{1t}}, \text{ and } p_t \equiv \frac{P_{2t}}{P_{1t}}, \tag{6}$$

$$a_t \equiv k_t + m_t, \tag{7}$$

$$m_t \equiv m_{1t} + m_{2t}, \tag{8}$$

$$c_{2t} \equiv c_{21t} + c_{22t}. \tag{9}$$

in which

u-... instant utility;

l - ... leisure;

g - ... transaction-cost function;

M - ... nominal per capita money stock;

P_1 - ... nominal price of the first good (which could be consumed or accumulated as capital);

P_2 - ... nominal price of the banking service;

k_t - ... per capita capital stock;

r - ... real interest rate or remuneration of capital services;

π - ... inflation rate;

w - ... remuneration of the labor services;

c_1 - ... flow of consumption good;

m_1 - ... services of the real monetary stock allocated for saving time;

m_2 - ... services of the real monetary stock allocated for saving transaction cost;

c_{21} - ... flow of banking services employed for saving time;

c_{22} - ... flow of banking services employed for saving transaction costs;

τ - ... transfer from the government.

This is quite a general model[9]. For example, if it is supposed that leisure depends only on the quantity of money and if there is no banking sector and transaction costs, we are back to Sidrauski model. On the other hand, if it is assumed that the instant utility depends only on consumption and that the banking sector does not exist, then we are back to McCallum and Goodfriend model. Finally, if it is supposed that leisure depends only on money and banking services and that there are no transaction costs, the model becomes a simple two-sector model which could rationalize the idea of a banking sector. It is possible to imagine any combination of these three models. The existence problem is not the main concern of this paper. It is supposed that the solution exists and is well-behaved. If leisure and the transaction function do not depend on the consumption flow, it is easy to see that it is possible to suppose that both functions are strictly concave and, consequently, that existence and uniqueness is guaranteed[10].

First-Order Conditions

Let μ_t represents the costate variable associated with the restriction (5), which is obviously the shadow price of the capital good. The maximization problem of the household is a standard one. The control variables are[11]: c_1,

[9] The standard assumptions are: $u_i > 0$, $l_1 < 0$, $l_i > 0$, $g_1 > 0$, $g_i < 0$; $u_{ii} < 0$, $|u_{ij}| > 0$, $g_{11} > 0$, $g_{ii} < 0$, $|g_{ij}| > 0$ and conditions that establishes that money and banking services are substitutes: $l_{23} < 0$ and $g_{23} < 0$.

[10] Evidently, ruling out monetary bubbles.

[11] The time subscript will be omitted whenever the understanding is clear.

m_1, c_{21}, m_2 and c_{22}. It follows the first-order conditions

$$u_1 + u_2 l_1 = \mu(1 + g_1); \tag{10}$$

$$u_2 l_2 = \mu(\pi + r); \tag{11}$$

$$u_2 l_3 = \mu p; \tag{12}$$

$$-g_2 = \pi + r; \tag{13}$$

$$-g_3 = p. \tag{14}$$

For the household state variable (assets), the Euler equation follows

$$\frac{\dot{\mu}}{\mu} = \rho - r.$$

Firms

This economy is a two-sector economy. The first sector, applying a first-order degree homogenous production function and employing capital and labor, produces a good which could be consumed or accumulated as capital. The second sector, applying an equivalent technology, produces a service called banking services, which could be acquired by the household in the market. It is assumed that the factors market clears continuously; factors are perfectly mobile across sectors and are supplied ineslaticly. Under these conditions the equilibrium of the supply side of the economy could be represented by the following two

supplies functions (one for each sector)

$$y_1 = y_1(p;k) \text{ and } y_2 = y_2(p;k);$$

in which

y_i-... per capita production of the i-esimo good.

From the inclination of the possibilities production frontier it is known that

$$y_{11} + py_{21} = 0; \tag{15}$$

and from the social marginal impact of capital it is known that

$$\frac{d}{dk}(y_1 + py_2) = y_{12} + py_{22} = f_1^0(k_1(p)) = pf_2^0(k_2(p)) = r \tag{16}$$

in which

f_i-... i-esimo sector product per worker;

k_i-... i-esimo sector capital per worker ratio.

Government

The role of the Government in this economy is to print money, which is a standard assumption in this literature. Evidently, if it has been assumed that there has been government consumption which should be financed by distorted taxes, the calculation of the inflation impact under welfare would had been changed. However, numerical calculations by Lucas (1997) showed that

quantitatively this effect is not very large. Under the monetary regime, the government transference to the public is the seigniorage which is equal to the inflationary tax plus the increase in the real quantity of money. That is

$$\chi = \dot{m} + \pi m.$$

Short Run Equilibrium and Dynamics

The market for banking services clears continuously, which means that its relative price (p) adjusts to accomplish this equilibrium. Due to Walras' law, this equilibrium condition, plus the equilibrium in the money market, implies the equilibrium of the goods market. The condition for the equilibrium in the banking services market

$$y_2(p; k) - c_2 = 0;$$

along with the equations (8), (9), (10)-(14), determine c_1, m_1, c_{21}, m_2, c_{22}, p, c_2 and π as function of the state variable k, the costate variable μ and the state-like variable m. This establishes the momentary equilibrium for this economy.

The dynamic is given by the following equations

$$\dot{k} = y_1(p; k) - c_1 - g(c_1; m_2; c_{22}); \quad (17)$$

$$\dot{\mu} = \mu(\rho - f_1'(k_1(p))); \quad (18)$$

$$\dot{m} = m(\sigma - \pi) \quad (19)$$

in which

$$\sigma \equiv \frac{\dot{M}}{M}.$$

By looking at this system of equations, the peculiar role played by the parameter σ is highlighted. Although it displaces the equilibrium position, it does not directly change any first-order condition. This property will be essential later.

A very important case that will be dealt with later is the situation in which the technology is the same across sectors. If this is true, although from the demand point of view the two goods are distinct, from the supply point of view they are equal. Under this condition, the economy works as if it was an one-sector economy, which means that the relative price of the banking service is constant and that the interest rate is determined as usual by

$$r = f'(k).$$

It follows in this situation, from this last equation and (18), evaluated in the stationary-state, that the long-run capital stock is fixed and independent of σ. That is, after an alteration of the increase rate of the nominal quantity of money, the economy will not present any dynamics. The following variables - the control variable, the state-like variable, and the costate variable - jump, and a new long-run equilibrium is immediately attained. Only under this situation does the very concept of a money demand make sense.

3 Impact On Welfare

In this representative agent economy, welfare is equal to the intertemporal utility of the household, expression (4). Then, it is possible to directly calculate the impact upon welfare of a marginal increase of σ.

$$\begin{aligned}\frac{dW}{d\sigma} &= \int_0^\infty e^{-\rho t}\frac{d}{d\sigma}u(c_1, l(c_1, m_1, c_{21}))dt \\ &= \int_0^\infty e^{-\rho t}\left[(u_1 + u_2 l_1)\frac{dc_1}{d\sigma} + u_2 l_2\frac{dm_1}{d\sigma} + u_2 l_3\frac{dc_{21}}{d\sigma}\right]dt. \qquad (20)\end{aligned}$$

Substituting in this last equation the first-order conditions (10)-(12), it follows that

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t}\lambda\left[(1 + g_1)\frac{dc_1}{d\sigma} + (\pi + r)\frac{dm_1}{d\sigma} + p\frac{dc_{21}}{d\sigma}\right]dt.$$

From the equilibrium in the market for goods, it is known that

$$\int_0^\infty e^{-\rho t}\lambda\frac{d}{d\sigma}\left(y_1(p, k) - c_1 - g(c_1, m_2, c_{22}) - \dot{k}\right)dt = 0$$

and for the banking services market

$$\int_0^\infty e^{-\rho t}\lambda p\frac{d}{d\sigma}(y_2(p, k) - c_2)dt = 0;$$

which could respectively be written as

$$\int_0^\infty e^{-\rho t}\left(y_{11}\frac{dp}{d\sigma}+y_{12}\frac{dk}{d\sigma}-(1+\tau)\frac{dc_1}{d\sigma}-\tau_2\frac{dm_2}{d\sigma}-\tau_3\frac{dc_{22}}{d\sigma}-\frac{d\dot{k}}{d\sigma}\right)dt=0 \tag{21}$$

and

$$\int_0^\infty e^{-\rho t}p\left(y_{21}\frac{dp}{d\sigma}+y_{22}\frac{dk}{d\sigma}-\frac{dc_2}{d\sigma}\right)dt=0\,. \tag{22}$$

Integrating by parts the last term in (21) and recalling that capital does not jump and it is bounded, it follows that

$$\int_0^\infty e^{-\rho t}\frac{d}{dt}\frac{dk}{d\sigma}dt=-\int_0^\infty e^{-\rho t}\left(-\rho+\frac{\dot{\lambda}}{\lambda}\right)\frac{dk}{d\sigma}dt. \tag{23}$$

Substituting (23) in (21), adding the result and (22) to (20) it is left

$$\begin{aligned}\frac{dW}{d\sigma}=&\int_0^\infty e^{-\rho t}\Big\{(1+\tau)\frac{dc_1}{d\sigma}+(\pi+r)\frac{dm_1}{d\sigma}+p\frac{dc_{21}}{d\sigma}\\&+\left(y_{11}\frac{dp}{d\sigma}+y_{12}\frac{dk}{d\sigma}-(1+\tau)\frac{dc_1}{d\sigma}-\tau_2\frac{dm_2}{d\sigma}-\tau_3\frac{dc_{22}}{d\sigma}+\left(-\rho+\frac{\dot{\lambda}}{\lambda}\right)\frac{dk}{d\sigma}\right)\\&+p\left(y_{21}\frac{dp}{d\sigma}+y_{22}\frac{dk}{d\sigma}-\frac{dc_2}{d\sigma}\right)\Big\}dt.\end{aligned}$$

After recalling (13), (14), (15), (16), and (18), every term which is not multiplied by $\frac{dm_i}{d\sigma}$ cancels out. It remains

$$\frac{dW}{d\sigma}=\int_0^\infty e^{-\rho t}(\pi+r)\frac{dm}{d\sigma}dt. \tag{24}$$

This canceling out expresses that besides money, the others choice variables present a social benefit and a social cost, which by the choice mechanism are equal, although welfare theorems are not satisfied for monetary models[12]. In others words, this is a welfare maximizing economy restricted to the fact that the household is consuming less monetary services than the social optimum. That is, a Social Planer who could not avoid inflation, and who could not induce the households to increase their money holdings, would have done no better than the market. Consequently, because money has benefit but does not have cost, there is not this kind of canceling out; the amount expressed by (24) remains. It is important to note that there was not any supposition about the specific value of σ in deriving this result, which means that expression (24) applies to every value for σ. Then, regardless of its value, a further increase (or decrease) produces that canceling out, if it is taken into consideration that the decision-makers redo their optimum calculations[13]. This result, which is the one we are interested in[14], is amazingly general. It states that the marginal impact of σ on welfare is the present value, in units of utilities, of the marginal impact of σ on the money demand. The specific adjustment which takes place in succession a alteration on σ does not matter; the money demand reflects its. This result is a dynamic version of Bailey's rule.

In deriving (24) there was not made any hypothesis respect to the variable

[12] This derivation resembles Samuelson's envelop theorem; however, it is not quite the same. In deriving the envelop theorem for a restrict maximum, the restriction faced by the decision maker is added to the indirect utility function. Differently, in order to derive (24), the restriction seen by the social planer, which are the physical balance equation for the goods produced by the economy, was added to the indirect welfare function.

[13] Which is another way to say that this model is Pareto optimum restricted.

[14] Recall (1).

σ. That is to say, σ could be any exogenous variable. As an example, if it had been supposed that there was a purchase tax for any good, following the same route which leads us to (24), it would sent us to

$$\frac{dW}{d\sigma}\Big|_{\tau=0} = \int_0^\infty e^{-\rho t}(\pi + r)\frac{dm}{d\tau}dt$$

in which τ is the tax rate. The important distinction is that this derivative would apply in the neighborhood of the tax rate close to zero; in contrast, due to the particular role played by the parameter σ in monetary models[15], (24) is a global result.

To go further, the long-run capital stock should not be sensitive to the inflation rate. As it was seen, it is necessary to assume that the technology is the same among sectors; otherwise, the concept of a stable demand function is meaningless. In such a situation, it is possible to integrate (24) to get[16]

$$\rho\frac{dW}{d\sigma} = \mu^* R^* \frac{dm^*}{d\sigma}:$$

This means that in units of capital[17], it follows that

$$\frac{\rho}{\mu^*}\frac{dW}{d\sigma} = R^*\frac{dm^*}{d\sigma}:$$

[15] Generally, a parameter displaces some first-order conditions. That is not the case regarding to σ.

[16] The '$*$' is to remember that from this point on the results refer to a stationary state capital stock that does not change with inflation.

[17] Recall that μ is the shadow price of capital.

Integrating this last equation on σ we are back to (3)

$$\rho\Delta W_{\text{Units of Capital}} = \int_{-\rho}^{\sigma} R^{*}\frac{dm^{*}}{d\sigma'}d\sigma' = -\int_{m^{*}(\sigma)}^{m^{*}(-\rho)} R^{*}(m)dm. \qquad (25)$$

The equation (25) is the main result of the paper. For a very general class of monetary models, the area under the inverse money demand function is the accurate measurement of the impact of inflation under welfare. Said differently:

> "This conclusion, that the area under the observed demand curve for real cash balances during an inflation measures the welfare costs of the reduction of these balances, applies regardless of the particular manner in which these costs affect real income and leisure." (Bailey, (1956), pg.102, emphasis added.)

Usually, the welfare cost of inflation is measured in units of consumption goods and not in units of capital goods. Why, in general is Bailey's rules valid when welfare is measured in units of capital but not in units of consumption goods?

Looking at (11) or (13), it is clear that the relative price of money in units of capital is the nominal interest rate. On the other hand, it follows from (10) that the relative price of the consumption good in units of capital is $1+g$. Then, in terms of consumption good, it is possible to rewrite (25) as

$$-\rho\Delta W_{\text{Units of Cons. Goods}} = \int_{m^{*}(\sigma)}^{m^{*}(-\rho)} \frac{R(m)}{1+g(c(m);m)}dm \leq \int_{m^{*}(\sigma)}^{m^{*}(-\rho)} R(m)dm \qquad (26)$$

the equality occurring if $g_c = 0$. If this last condition applies, Bailey's rule is exact for welfare in units of consumption good. It means that for the standard Sidrauski model, the McCallum-Goodfriend model (if the transaction cost function does not depend on consumption) and for the two-sector model with a banking sector offering substitutes for money (if the technology is the same across sector), Bailey's rule is exact. The area under the inverse demand function overestimates the welfare cost of inflation in units of consumption goods if the transaction cost function is sensitive to the amount which has been consumed.

In the last paragraph it was seen that Bailey's rule overestimates the welfare cost of inflation measured in units of consumption goods if $g_c > 0$, for the McCallum-Goodfriend model. Once Bailey's rule is not exact in this context the question remains: is there another interpretation for the inflation impact on welfare? A possible route is to calculate welfare in terms of the consumption demand instead of the money demand. It is useful to work on a simplified version of the general model of the second Section to accomplish this. If it is supposed that the momentary utility function depends only on consumption, and that there is only one sector, the second Section model turns into the McCallum-Goodfriend model. The unique departure of this model from the standard Ramsey-Cass-Koopmans model is the transaction cost $g(c; m)$, which subtract resources from the household budget constraint. As we know, expression (25) applies to this economy. It is allowed to start from there. From the market equilibrium condition for this simplified version of the general model, it

follows that

$$f(k^*) = c^* + g(c^*; m^*)$$

which means that

$$(1 + g_1)\frac{dc^*}{d\sigma} = -g_2\frac{dm^*}{d\sigma} = (\sigma + r)\frac{dm^*}{d\sigma}. \tag{27}$$

The last equality follows from (13). Substituting (27) in (26), it follows that

$$-\rho\Delta W_{\text{Units of Cons. Goods}} = \Delta c^*.$$

As expected, the welfare cost of inflation for the McCallum-Goodfriend framework[18] is the reduction of consumption which takes place due to the increase of the inflation rate. It is clear now why the welfare cost of inflation measured in units of consumption is lower than in units of income, for the McCallum-Goodfriend model; to produce one unit of consumption good it is necessary $1 + g_1$ units of income.

Summing up and remembering that $r^* = \rho$, it is possible to write

$$r^*\Delta W_{\text{Units of Assets}} = -\int_{m^*(\sigma)}^{m^*(-\rho)} R(m)dm. \tag{28}$$

[18] This result is valid if this model augmented with banking services in the transaction function is considered. This is true because the term $-dc_2$ in the budget constraint is canceled by $g_3(c_1; m; c_2)dc_2$.

It is important to emphasize here that the marginal transformation rate between money and assets, for the household, is the nominal interest rate[19]. If the capital relative price in units of assets is constant, Bailey's rule applies in units of capital; if the consumption relative price in units of assets is constant, Bailey's rule applies in units of consumption goods.

4 Money Demand and Growth

The model that was discussed in Section 2 does not present growth. It is known that at first approximation the income elasticity of money demand is roughly one[20]. It would be interesting to know how the result that we have so far gotten would change, or not, in a model which exhibits a stationary solution when there is a long-run trend in income. The standard Sidrauski model augmented with exogenous technological progress presents this property of constancy in the long-run of the income money velocity[21]. On the other hand it will be possible to compare the result in this paper with Lucas (1997), which is qualitatively different[22]. It will be shown that the introduction of a trend in income does not change the result that Bailey's rules is exact in the standard Sidrauski model.

Household

[19] It follows directly from the private budget restriction (5).

[20] In fact, it is known that this elasticity is lower than one (see for example Lucas (1997), figure 1).

[21] The models which contemplate a banking sector generally do not present a long-run-growth stationary solution. The exogenous technological change continuously reduces the relative price of the banking services, if the inflation rate does not present a trend, and, consequently, the long-run money demand presents a lessening tendency. It seems to me that Aiyagari et alii (1998) did not notice this fact (see their discussion on pg.1289).

[22] See Lucas (1997) section 3.

The household solves

$$\max \int_0^\infty e^{-\rho t} \frac{(u(\tilde{c}; \tilde{m}))^{1-\alpha} - 1}{1-\alpha} dt$$

subject to

$$\frac{d\tilde{a}}{dt} = \tilde{w} + \tilde{a}r - \tilde{c} - (\pi + r)\tilde{m} + \tilde{\psi};$$

in which $u(\cdot, \cdot)$ is first-order degree homogenous. It is possible to write[23] $u(\tilde{c}; \tilde{m}) = \tilde{c}\varphi(\frac{m}{c})$, in which the variables without '~' are detrended ones. Let $\tilde{\mu}$ be the shadow price of $\tilde{a}$ and $\mu \equiv \tilde{\mu} e^{\alpha g t}$ the detrended price; then the first-order conditions for the control variables follows

$$\begin{aligned} (c\varphi(\frac{m}{c}))^{-\alpha}\left[\varphi(\frac{m}{c}) - \frac{m}{c}\varphi'(\frac{m}{c})\right] &= \mu; \\ (c\varphi(\frac{m}{c}))^{-\alpha}\varphi'(\frac{m}{c}) &= \mu(\pi + r) \end{aligned}$$

and for the state variable

$$\dot{\mu} = \mu(\rho + \alpha g - r):$$

Firms

[23] Lucas' (1997) notation is followed here.

The first-order conditions for the firms lead to

$$w \equiv \frac{\tilde{w}}{e^{gt}} = f(k) - kf'(k); \tag{29}$$

$$r = f'(k). \tag{30}$$

General Equilibrium and Dynamics

Remembering that the Government detrended transfer satisfies

$$\chi = \dot{m} + (\pi + g)m; \tag{31}$$

and substituting (29)-(31) into the detrended household budget restriction, we are left with the following dynamic system

$$\begin{aligned} \dot{k} &= f(k) - c - gk; \quad (32)\\ \dot{m} &= m(\sigma - (\pi + g)); \\ \dot{\mu} &= \mu(\rho + \alpha g - f'(k)). \end{aligned}$$

Welfare

The impact on Welfare of the inflation could be calculated from

$$\begin{aligned} \frac{dW}{d\sigma} &= \int_0^\infty e^{-\rho t}(u(\tilde{c};\tilde{m}))^{-\alpha}\left[u_1(\tilde{c};\tilde{m})\frac{d\tilde{c}}{d\sigma} + u_2(\tilde{c};\tilde{m})\frac{d\tilde{m}}{d\sigma}\right]dt; \\ &= \int_0^\infty e^{-(\rho - g(1-\alpha))t}\mu\left[\frac{dc}{d\sigma} + (\pi + r)\frac{dm}{d\sigma}\right]dt \end{aligned}$$

in which, as usual, it is supposed that $\rho - g(1-\alpha) > 0$, and the second equality

follows after the substitution of the first-order condition for consumption and money services. After differentiating (32) against α, and redoing the steps of the last Section, it follows that

$$\frac{dW}{d\sigma} = \int_0^{\infty} e^{-(\rho - g(1-\alpha))t} \left[(\sigma + r)\frac{dm}{d\sigma} + (\alpha - 1)\frac{dk}{d\sigma} \right] dt.$$

Recalling that in the stationary-state the capital stock does not vary with σ; it is possible to solve the integral

$$(\rho - g(1-\alpha))\frac{dW}{d\sigma} = (r^{*} - g)\frac{dW}{d\sigma} = \mu^{*} R^{*} \frac{dm^{*}}{d\sigma};$$

which means that

$$(r^{*} - g) \cdot W_{\text{Units of Goods}} = -\int_{m^{*}(\sigma)}^{m^{*}(-(\rho+\alpha g))} R(m)dm. \qquad (33)$$

Comparing (33) with (28), the differences in the measurement of the welfare cost of inflation when technological exogenous progress takes place are twofold. Firstly, to calculate the impact of inflation under welfare, the area under the inverse money demand function should be divided by the interest rate net of the growth rate. Therefore, the presence of unbounded growth strengthens or weakens the case against monetary finance whether the intertemporal elasticity is lower or higher than one. Secondly, the superior limit of the integral in

(33), m¤(¾), is higher under growth[24,25], strengthened the case against in‡ation …nance in this context[26]. The next Section shows that the validity of (1) and (3) are not an artifact of the Sidrauski model or the McCallum-Goodfriend version of it.

5 A Cash-in-Advance Economy

From the point of view of getting a deeper acquaintance of the monetary phenomenon, the models that were investigated until the last Section belong to the family of Sidrauski models. The next category of monetary models in increasing order of understanding of the monetary phenomenon are the cash-in-advance models. The aim of this Section is to demonstrate that the results which were derived for the Sidrauski-type models are valid to this family of monetary models. The same route will be followed: for a very general cash-in-advance model, which could encompass many models as a particular case, (1) and (3) will be established.

The drawback of the standard[27] cash-in-advance model is the constancy in income velocity. The manner which has been suggested to cope with this

---

[24] The nominal interest rate, when the increase rate of the nominal quantity of money is ¾, is ¾ + ½ + ®g, which is higher than ¾ + ½ whenever g > 0. On the other hand, the inverse money demand functions are iqual, once it is recalled that the marginal conditions that bring them about are equal.

[25] In this growing economy, Friedman's rule requires a de‡ation rate equal to ½ + ®g, which is higher than the usual ½.

[26] The quali…cations on the measure of the impact of in‡ation under welfare when growth takes place was quite an important issue in the sixties and seventies. See Tower (1971), Marty (1973) and (1976), Cathcart (1974), Tatom (1976), and Chappell (1981). However, these works address this issue under a diverse set of hypotheses, and, consequently, are not appropriate for comparison with this paper.

[27] For example, Lucas (1980).

limitation is to add goods that can be purchased by credit[28]. As put forth by Gillman (1993), it is possible to consider a continuum of goods, which, from the preference point of view possesses symmetric roles, although not from the transaction technology point of view. Under this formulation, every good can be purchased by money or credit. The distinction is that there is a credit cost attached to each good which varies across goods, in such a way that as inflation increases, the range of goods which are credit goods increases. If it is considered that these credit services are offered by a sector of the economy which employes production factors in order to produce it, we are in the Aiyagari, Braun, and Eckestein (1998) framework.

The model that will be study in this Section is a generalization of their model in one direction: the aggregator function, which defines the consumption good and the investment good, presents elasticity of substitution across types of goods larger than zero. There are two main reasons for this choice. Firstly, it is intended to work in a more general set up, which can deliver other models as a particular case. Secondly, the situation in which the elasticity across types of goods is higher than zero produces another impact of inflation under welfare. Due to the symmetric role played by the goods in preference, the household prefers to smoothe the consumption across types. Notwithstanding this, among the goods acquired as credit goods, the relative price - the credit cost relative to the nominal interest rate - varies in such a way that following an increase in inflation rate, the variability of consumption across types increases. This is a

[28] Lucas and Stokey (1983).

relatively rich description of a monetary economy under certainty. Following an increase in inflation, the range of cash goods decreases, the consumption profile of the household twists, the banking sector absorbs production factors to offer transaction services, and the accumulation of capital is hindered. However, it will be shown that (1) represents the marginal impact under welfare of the inflation. Moreover, if it is supposed that capital accumulation is not affected by inflation, Bailey's rules is again valid.

5.1 The Model

There is a continuum of goods index by $z \in [0;1]$. They are identical goods from the supply point of view, which means that the producer price P_t is the same, regardless of the type. There is another sector in this economy, the banking sector, which produces a service. Each good could be acquired as cash good or credit good. In the first case, the household pays P_t, but has to have it as cash, which means that the cost it faces is $(1 + R_t)P_t$, in which R is the nominal interest rate. When buying a good as credit good the household pays P_t to the good's producer plus the intermediation services cost. Following Aiyagari et alii, it is supposed that to acquire a unit of good of any quality as credit good, it is necessary to buy $R(z)$ units of banking services, which cost $q_s R(z)$ in units of goods. Consequently, the effective cost of a credit good to the household is $P_t(1 + q_s R(z))$. It is supposed that the production function for goods and transaction services are the same, which means that it is possible to normalize $q_s = 1$. The total per capita production of goods and services is $f(k_t; n_t)$, in

which n_t is the per capita supply of labor services. Moreover, the transaction services cost function is increase in the index z and $R(0) = 0$. At any moment there is a cut-off index, z_t, such that any good whose index is lower than the cut-off is bought as credit good, and the others are bought as cash.

Household Choice

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t; 1 - n_t) \tag{34}$$

in which

$$c_t = \left(\int_0^1 c_t^{\frac{\theta - 1}{\theta}}(z)\, dz \right)^{\frac{\theta}{\theta - 1}} \tag{35}$$

is an aggregator function that defines the unit of consumption.

The household faces two sorts of restrictions. One is the cash-in-advance and the other is the budget constraint. Before going to the good market, it is possible to go to the credit market, in order to take cash. This operation is costless. Let $M_t; B_t$ and X_t be, respectively, the nominal quantity of money, of bonds in the household portfolio, and the nominal value of government transfer. The cash-in-advance restriction is

$$\frac{M_t + X_t}{P_t} + \frac{B_t}{P_t} - \frac{B_{t+1}}{P_t(1 + R_t)} \geq \frac{1}{P_t} \int_{z_t}^1 P_t(z)(c_t(z) + i_t(z))\, dz: \tag{36}$$

The left side of (36) is the amount of cash carried by the consumption before

going to the goods market in the instant t, and the right side is the nominal cost of cash goods. The budget constraint is

$$\frac{M_t + X_t}{P_t} + \frac{B_t}{P_t} + w_t n_t + r_t k_t \geq \frac{1}{P_t}\int_0^1 P_t(z)(c_t(z) + i_t(z))dz + \frac{M_{t+1}}{P_t} + \frac{B_{t+1}}{P_t(1+R_t)}: \tag{37}$$

The movement equation for capital is

$$k_{t+1} = i_t + (1 - \delta)k_t; \tag{38}$$

in which, i_t is an aggregator function that defines the investment good

$$i_t = \left(\int_0^1 i_t^{\frac{\theta-1}{\theta}}(z)dz\right)^{\frac{\theta}{\theta-1}}: \tag{39}$$

Taking the limit $\theta \to 0$ this model deliver Aiyagari et alii model; the limit $\theta \to \infty$ reproduces Gillman model if an economy without capital is considered. If the cut-off index, z_t; is fixed and if there are neither banking services nor transaction services, the model reproduces Lucas and Stokey's (1983) economy, and if there are no credit goods, the model generates Stockman's (1981) model. Additionally, if capital is costless credit good, Lucas's (1980) model is obtained.

First-Order Conditions

For this constant-substitution-elasticity aggregator is known that

$$\left(\frac{c_t(z)}{c_t}\right)^{-\frac{1}{\theta}} = \left(\frac{i_t(z)}{i_t}\right)^{-\frac{1}{\theta}} = \frac{1+R(z)}{Q_t} \text{ if } z \leq \bar{z}_t \tag{40}$$

$$\text{and } \left(\frac{c_t(z)}{c_t}\right)^{-\frac{1}{\theta}} = \left(\frac{i_t(z)}{i_t}\right)^{-\frac{1}{\theta}} = \frac{1+R_t}{Q_t} \text{ if } z > \bar{z}_t; \tag{41}$$

in which

$$Q_t \equiv P_t(1+\tau_t) \equiv P_t\left[\int_0^{\bar{z}_t} (1+R(z))^{1-\theta} dz + (1-\bar{z}_t)(1+R_t)^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{42}$$

is the effective price index faced by the household.

Let $\beta^t\lambda_t\mu_t$, $\beta^t\lambda_t$, and $\beta^t\lambda_t q_t$ be the Langranger multipliers of (36), (37), and (38). The first-order conditions for the flows variables, consumption and investment, are

$$\begin{aligned} u_1(c_t, 1-n_t)c_t^{\frac{1}{\theta}} c_t^{-\frac{1}{\theta}}(z) &= \lambda_t(1+\mu_t)\frac{P_t(z)}{P_t} \\ \text{and } i_t^{\frac{1}{\theta}} i_t^{-\frac{1}{\theta}}(z) &= (1+\mu_t)\frac{P_t(z)}{P_t} \text{ if } z > \bar{z}_t; \end{aligned} \tag{43}$$

$$\begin{aligned} \text{and } u_1(c_t, 1-n_t)c_t^{\frac{1}{\theta}} c_t^{-\frac{1}{\theta}}(z) &= \lambda_t(1+R(z))\frac{P_t(z)}{P_t} \\ \text{and } i_t^{\frac{1}{\theta}} i_t^{-\frac{1}{\theta}}(z) &= (1+R(z))\frac{P_t(z)}{P_t} \text{ if } z \leq \bar{z}_t. \end{aligned} \tag{44}$$

The first-order conditions for the labor supply and the cut-off index are

$$u(c_t, 1-n_t) = \lambda_t w_t \tag{45}$$

and

$$1+\mu_t = 1 + R(\zeta_t).$$

This last condition states that the relative price of money in units of bonds is equal to the credit cost of the cut-off good. This relative price should be equal to the nominal interest rate in order to keep the Budget restriction bounded; otherwise it would be possible to gain money selling (or buying) cash the ζ_t good, and buying (or selling) it as credit good. At each instant the cut-off good is determined with the aim of meeting this non-arbitrage condition. That is

$$\mu_t = R_t = R(\zeta_t). \tag{46}$$

As Gillman (1993) stressed, (46) is a Baumol-type condition which equates the marginal cost of hold money with the marginal transaction cost.

After substituting (40) and (41) into (43) and (44), recalling (42) and (46) it follows that

$$u_1(c_t, 1-n_t) = \lambda_t(1+\tau_t) \text{ and } q_t = 1+\tau_t. \tag{47}$$

The Euler equations for the capital stock and bonds are respectively

$$\lambda_t(1+\tau_t) = \beta\lambda_{t+1}(1+\tau_{t+1})(1-\delta+\frac{r_{t+1}}{1+\tau_{t+1}}) \tag{48}$$

and,

$$\lambda_t = \beta \lambda_{t+1}(1 + R_{t+1})\frac{P_t}{P_{t+1}}:$$

5.2 Impact Under Welfare

From (34), after substituting the first-order conditions (43) and (44), recalling (40) and (41), it follows that

$$\frac{dW}{d\sigma} = \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_0^{z_t} (1 + R(z))\frac{dc_t(z)}{d\sigma}dz + \int_{z_t}^{1} (1 + R_t)\frac{dc_t(z)}{d\sigma}dz - w_t\frac{dn_t}{d\sigma} \right]: \tag{49}$$

The material balance equation for this economy is

$$f(k_t; n_t) - \int_0^{z_t} (1 + R(z))(c_t(z) + i_t(z))dz - \int_{z_t}^{1} (c_t(z) + i_t(z))dz = 0;$$

which means that

$$\begin{aligned} 0 = {} & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t\frac{dk_t}{d\sigma} + w_t\frac{dn_t}{d\sigma} - \int_0^{z_t} (1 + R(z))\left(\frac{dc_t(z)}{d\sigma} + \frac{di_t(z)}{d\sigma}\right)dz \right] \qquad (50) \\ & - \sum_{t=0}^{\infty} \beta^t \lambda_t \left[R(z_t)(c_t(z_t) + i_t(z_t))\frac{dz_t}{d\sigma} + \int_{z_t}^{1} \left(\frac{dc_t(z)}{d\sigma} + \frac{di_t(z)}{d\sigma}\right)dz \right]: \end{aligned}$$

Adding (50) to (49) it follows that

$$
\begin{aligned}
\frac{dW}{d\sigma} = {} & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_{z_t}^{\infty} R_t \frac{dc_t(z)}{d\sigma} dz - R(z_t) c_t(z_t) \frac{dz_t}{d\sigma} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - \int_0^{z_t} (1 + R(z)) \frac{di_t(z)}{d\sigma} dz - \int_{z_t}^{\infty} \frac{di_t(z)}{d\sigma} dz - R(z_t) i_t(z_t) \frac{dz_t}{d\sigma} \right] :
\end{aligned} \tag{51}
$$

From the first-order condition for the investment, it follows that

$$
(1 + \tau_t) i_t = \int_0^{z_t} (1 + R(z)) i_t(z) dz + (1 + R_t) \int_{z_t}^{\infty} i_t(z) dz;
$$

which means that

$$
\begin{aligned}
0 = {} & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_0^{z_t} (1 + R(z)) \frac{di_t(z)}{d\sigma} dz + (1 + R_t) \int_{z_t}^{\infty} \frac{di_t(z)}{d\sigma} dz \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\frac{dR_t}{d\sigma} \int_{z_t}^{\infty} i_t(z) dz - i_t \frac{d(1 + \tau_t)}{d\sigma} - (1 + \tau_t) \frac{di_t}{d\sigma} \right] :
\end{aligned} \tag{52}
$$

Adding (52) to (51), recalling that

$$
\frac{dR_t}{d\sigma} \int_{z_t}^{\infty} i_t(z) dz - i_t \frac{d(1 + \tau_t)}{d\sigma} = 0 ;
$$

it follows that

$$
\begin{aligned}
\frac{dW}{d\sigma} = {} & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_{z_t}^{\infty} R_t \frac{d}{d\sigma} (c_t(z) + i_t(z)) dz - R(z_t)(c_t(z_t) + i_t(z_t)) \frac{dz_t}{d\sigma} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - (1 + \tau_t) \frac{di_t}{d\sigma} \right] :
\end{aligned} \tag{53}
$$

From the capital accumulation equation it is possible to rewrite the second

line in (53) as

$$
\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - (1+\tau_t)\left(\frac{dk_{t+1}}{d\sigma} - (1-\delta)\frac{dk_t}{d\sigma}\right)\right] \\
&= \sum_{t=0}^{\infty} \left[\beta^t \lambda_t r_t + \beta^t \lambda_t (1+\tau_t)(1-\delta) - \beta^{t-1}\lambda_{t-1}(1+\tau_{t-1})\right]\frac{dk_t}{d\sigma} \\
&= 0; \qquad (54)
\end{aligned}
$$

in which the first equality follows because the initial capital stock is an exogenous variable, and the second equality follows from the first-order condition for capital accumulation, equation (48). Substituting (54) into (53) it remains

$$
\begin{aligned}
\frac{dW}{d\sigma} &= \sum_{t=0}^{\infty} \beta^t \lambda_t R_t \frac{d}{d\sigma}\left(\int_{z_t}^{1} (c_t(z) + i_t(z))dz\right) \\
&= \sum_{t=0}^{\infty} \beta^t \lambda_t R_t \frac{dm_t^{Demand}}{d\sigma}. \qquad (55)
\end{aligned}
$$

The second equality follows firstly from (36) and secondly form the fact that the cash-in-advance restriction is binding. Equation (55) is equivalent to (1). Continuing along the same path that was taken in the first part of the paper, let's suppose that the economy presents a long-run capital stock that does not vary with σ. Integrating (55), Bailey's rule follows

$$(1-\beta)\cdot W_{Units\ of\ Assest} = -\int_{m^*(\sigma)}^{m^*(-\rho)} R(m)dm.$$

For this economy, Bailey's rule is the measure, in units of assets, of the impact under welfare of inflation. The area under the inverse money demand function

takes into consideration firstly the diversion of production factors to the banking sector and the reduction of labor supply[29], which results in the decrease of the average consumption level, and, secondly, the increase in the variability of consumption across types of consumption goods.

Let's suppose that labor supply does not change. From (47) and (55), it follows that

$$
\begin{aligned}
(1-\beta)\left.\frac{dW}{d\sigma}\right|_{\text{Units of Consumption Basket}} &= \frac{R_t}{1+\tau_t}\frac{d}{d\sigma}\int_{z_t}^{1} c_t(z)dz \\
&= \frac{dc}{d\sigma},
\end{aligned}
$$

in which the second equality follows from (35) and from the fact that the income

$$
y^* = \int_0^1 c(z)dz + \int_0^{z^*} R(z)c(z)dz + \delta k^*
$$

is constant under these hypotheses. As it was shown for the McCallum-Goodfriend model[30], the welfare cost of inflation measured in units of consumption goods is smaller than in units of assets or income[31]. The reason is the same. When calculating welfare in units of consumption bundle, the transaction cost associated with the consumption is not taken into consideration.

[29] In the models of the first part of this paper it was supposed that the labor supply was inelastic.

[30] See discussion in the third section.

[31] Aiyagari et alii (1998) found that both are equal (see page. 1290). In fact, the demand concept employed by Aiyagari et alii is different from the one used in this paper; additionally, their money-demand concept is not observable. The money demand which they define does not take into consideration that when the inflation rate increases, the public disposable income is reduced, due to the factor inflow into the banking sector.

6 Compensate Income

In this paper, the total impact of inflation under welfare has been defined as the integration of the marginal impact, in units of assets. This is a direct measure of the variation in welfare in units of assets or income, and, as was seen, provides a general theoretical foundation for Bailey's rule. The compensate income which should be given to the household, in order to keep it indifferent to the situation in the presence of inflation as compared to an initial position without inflation, is another measure of the welfare cost of inflation. This concept seems more natural when the researcher is considering a specific model, which could be calibrated to a real economy to deliver numerical calculations. Bailey's rule is more appropriate when the researcher has only an empirical estimation of the money demand function. To establish the link between those two different definitions of the impact of inflation under welfare, let's solve the dual problem of the general model of the second Section. For the stationary-state, it follows that[32]

$$\begin{aligned} \min y^{Private} &= c_1 + c_2 + g(c_1; m_2; c_{22}) + (\sigma + \rho)m + \chi \\ \text{subject to} &: \quad u(c_1; l(c_1; m_1; c_{21})) = \text{const.} \end{aligned}$$

After adding to the derivative of the income against σ the derivative of the restriction, recalling the first-order conditions and the government restriction,

[32] The price of the banking services was normalized to one.

it follows that

$$\frac{dy}{d\sigma} = -(\sigma + \rho)\frac{dm}{d\sigma};$$

in which

$$y \equiv c_1 + c_2 + g(c_1; m_2; c_{22}):$$

Then, the income that should be given to the household to compensate it for the harm of inflation is

$$\Delta y = \int_{\bar{m}(\sigma)}^{\bar{m}(-\rho)} R(\bar{m})d\bar{m};$$

in which the bar over the money demand is to remind us that this is the compensate demand. This is the social income that should be given to the household. The variation of income that the household observes is

$$\Delta y^{Private} = \int_{\bar{m}(\sigma)}^{\bar{m}(-\rho)} R(\bar{m})d\bar{m} + (\sigma + \rho)(\bar{m}(\sigma) - m(\sigma)):$$

The following thought experiment helps to understand the distinction between these two concepts of income. Suppose an economy, in which the increase rate of the nominal quantity of money is σ. Suddenly, a stock of mineral resources, valued at $\frac{1}{r}\Delta y$, is discovered. With this additional income, the money demand would increase by $\bar{m}(\sigma) - m(\sigma)$ and a new equilibrium, at the same

level of utility as it is possible under Friedman rule, would be attained. The additional quantity of money could be provided by the economy without cost.

For the cash-in-advance model the dual problem is

$$\min y = \int_0^1 c(z)dz + \int_0^{z^*} R(z)c(z)dz - n$$

subject to[33]

$$u\left(\left(\int_0^1 c^{\frac{\theta-1}{\theta}}(z)dz\right)^{\frac{\theta}{\theta-1}}; 1 - n\right) = \text{const.}$$

It is straightforward to show that

$$\frac{dy}{dR} = -R\frac{d\bar{m}}{dR}, \text{ in which } \bar{m} = \int_0^z c(z)dz:$$

Because $\bar{m}(\sigma) \geq m(\sigma)$ and $\bar{m}(-\rho) = m(-\rho)$; it is not possible to compare the areas under the two inverse money demand functions. They should be quantitatively very close, but whenever income effect is present, it is not possible to compare them[34]. The other common employed measure is the consumption which leaves the household in the same utility level. For the standard Sidrauski model, this measure overstates the welfare cost of inflation because it does not consider that the decision-maker will increase her money demand if her consumption level is augmented. Applying it to the McCallum-Goodfriend model

[33] The wage rate was normalized to one.

[34] It is a microeconomic text-book result that the consumer surplus is a perfect measure of welfare when the utility is quasi-linear.

and the cash-in-advance model investigated in the last Section, this measure underestimates because it does not consider the increase in the transaction cost due to the additional quantity of consumption good.

The general conclusion of the paper is that the money demand caries with it a lot of information. But what is meant by 'money'? The next Section argues that the relevant concept of money for this subject - the impact under welfare of inflation - is the narrow monetary aggregate, the monetary base.

7 A Model with Inside Money

At this point the message of this paper should be very clear. Abstracting from impacts of inflation under long-run capital, Bailey's rule is the accurate measure of the reduction on welfare caused by a perfectly predicted inflation. This conclusion is quite general and does not depend on the specific role played by money in this economy or the specific kind of adjustment faced by the real sector in order to avoid or to help the public to cope with inflation. But what is meant exactly by 'money demand'? What is money? Whenever the researcher is studying the short-run equilibrium of the economy, money is the asset which possesses the property of liquidity. It is usually cash out of the banking sector plus demand deposits. But, that is not what is meant by money in this context. Here money is that good which has benefit but does not have social

cost[35;36].

When inflation increases, the public demand for demand deposits decreases, which could be considered a welfare cost of inflation. However, because this service - demand deposit - requires capital and work force to be supplied, the reduction in the public demand for demand deposit is not a cost, from the social point of view. What occurs is that the increase of inflation decreases the demand-deposit demand, but it increases the demand for the other bank services, in such a way that the demand for an aggregated bundle of banking services increases. Those effects were taken into consideration in the models studied in this paper. Saying it differently, the demand-deposit is just another service which is supplied by the banking sector to help the public to cope with inflation. The variant of the second Section model sketched below is intended to clarify this issue.

Household

There are three liquidity instruments: cash, demand deposits and another banking service. The household solves

$$\max \int_0^{\infty} e^{-\rho t} u(c_{1t}; l(m_{1t}; m_{2t}; c_{2t}))dt; \tag{56}$$

[35] This concept of money applies to Friedman's rule. The asset whose consumption should be pushed to satiation is the monetary base.

[36] Differently, Lucas (1981) pg. 44, defines money, as far as the welfare impact of inflation is concerned, as any

> "noninterest-bearing assets or to assets the interest on which is restricted to below-market rates."

In the same Section he offers a discussion of the money concept and its role as a liquidity instrument. The point here is that the precise way that money takes place in the economy - if it provides liquidity or if there are restrictions and regulation in its usage - is not the heart of the question, which is that money has social value and does not have social cost.

subject to

$$a'_t = r_t a_t + w_t + \chi_{H,t} + \Pi - c_{1t} - p_t c_{2t} - p^d_t m_{1t} - (\pi_t + r_t)(m_{1t} + m_{2t}) \quad (57)$$

in which[37]

$a \equiv k + m_1 + m_2$;

m_{1t} – stock of cash in household's portfolio;

m_{2t} – stock of demand deposits in household's portfolio;

$\chi_{H,t}$ – Government transfers to the household;

Π – bank's profits;

p^d_t – demand deposit price.

For simplicity, the other banking services are treated as flow of services and not as assets. Because of the possibility of very low inflation rates the banks charge a fee to held demand deposits. It is possible, if inflation is sufficiently high, that this price could be zero. The first-order conditions for this standard

[37] Nothing would change if this model had been constructed as general as the model in the section two.

problem is

$$
\begin{aligned}
u_1 &= \mu; && (58)\\
u_2 l_1 &= \mu(\pi + r); &&\\
u_2 l_2 &= \mu(\pi + r + p^d); &&\\
u_2 l_3 &= \mu p; &&\\
\frac{\dot{\mu}}{\mu} &= \rho - r. && (59)
\end{aligned}
$$

The Banks

This is a two-sector economy. The real sector produces a good, which can be consumed and accumulated as capital. The second sector, banks, in this Section are multiproduct firms. They employ capital and work force to produce a service, (called banking services, which help the household in saving transaction time) and to produce another liquidity service, named demand deposit. As usual, it is supposed that the demand deposits are denominated in nominal units; consequently, the income of the banking in offering this services is the price that it could charges plus the nominal interest rates. Therefore, the per capita profit function for the banks, in units of the good are

$$\Pi = pc_2 + (p^d + (\pi + r)(1 - \zeta))m_2 - (rk_2 + w)l_2 + \chi_B \qquad (60)$$

in which

ζ – Reserves requirement ratio;

k_2 - capital-labor ratio in the banking sector;

l_2 - ratio of the work force employed by the banking sector;

χ_B - Govern's transfer to the Banks.

It is supposed that the issue of a new demand deposit is a costless activity, as it is to the government to issue base, such that the seigniorage is an income appropriated by the banking institution. The banks maximize (60), subject to the technological restriction[38]

$$y_2 = l_2 f_2(k_2) = g(c_2; m_2). \tag{61}$$

It states that the per capita production of this industry can be distributed across the two products according to the transformation function g. This function is concave and first-order-degree homogeneous. Let q be the Lagrange multiplier for (61). The first-order conditions for the maximization problem for the banks are as follows

$$\begin{aligned} p &= qg_1; & (62)\\ p^d + (\pi + r)(1 - \zeta) &= qg_2; & (63)\\ r &= qf_2'(k_2); & (64)\\ w &= q(f_2 - k_2 f_2'(k_2)). & (65) \end{aligned}$$

[38] This modeling of a multiproduct firm was taken from Drazen (1979).

Due to the homogeneity of g, it follows from (62) and (63) that

$$pc_2 + (p^d + (\pi + r)(1 - \gamma))m_2 = qy_2; \qquad (66)$$

which means that the total per capita production of the Banks, evaluated in units of goods, is equal to the production of services, priced at p, and the production of demand deposits, priced at $p^d + (\pi + r)(1 - \gamma)$. The price q is the price, in units of goods, of a optimum bundle of transaction services and demand deposits. This is the relevant price for the allocation decision for the production factors[39]. At each instant the price q determines the relative rentability across the sectors, and, accordingly, the allocation of factors between the real sector and the banking sector[40]. Consequently, the sector's offers function can be written as follows

$$y_1(q; k) \text{ and } y_2(q; k):$$

Similar to the other sections, proprieties (15) and (16) are satisfied. Given an amount of banking output, y_2, the relative price between services and demand deposits determines at which point of the transformation function g the banking sector will be positioned. On the other hand, equation (66) could be seen as an equilibrium equation for the banking sector. Totally differentiating (66) after

[39] From (64) and (65) it is possible to verify it directly.

[40] It apparent that this economy does not satifices Friedman's rule for demand deposit. If inflation decrease, to offer this service the banks will charge the fee p^d, in order to pay for the cost of this provision. See footnote 35.

substituting

$$dy_2 = q^{-1}(p\,dc_2 + (p^d + (\pi + r)(1 - \zeta))dm_2)$$

it follows that

$$y_2\,dq = c_2\,dp + m_2\,d(p^d + (\pi + r)(1 - \zeta)): \tag{67}$$

This last result will be useful later.

General Equilibrium and Welfare

Because the transformation frontier for the Banks is first-order-degree homogeneous the payment of factor by its marginal productivity is equal to the production of liquidity services - $pc_2 + (p^d + (\pi + r)(1 - \zeta))m_2$. Consequently, the bank's profit is the bank's seigniorage - $(1 - \zeta)\dot{m}_2$ - plus the government transfer - χ_B. After substituting the liquidity services equilibrium equation (66), remembering that the per capita income - $rk + w$ - is equal to the per capita output - $y_1 + qy_2$ - and that the total government transfer is equal to the seigniorage of the monetary base, the good's market equilibrium equation follows from (57)

$$\dot{k} = y_1(q; k) - c_1: \tag{68}$$

It is possible now to evaluate the impact under welfare of inflation. After

substituting the first-order conditions (58), it follows from (56) that

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t}\left[\frac{dc_1}{d\sigma} + (\pi + r)\frac{dm_1}{d\sigma} + (\pi + r + p^d)\frac{dm_2}{d\sigma} + p\frac{dc_2}{d\sigma}\right] dt. \qquad (69)$$

From the goods market equilibrium equation, it follows that

$$\int_0^\infty e^{-\rho t}\left(y_{11}\frac{dp}{d\sigma} + y_{12}\frac{dk}{d\sigma} - \frac{dc_1}{d\sigma} - \frac{d\dot{k}}{d\sigma}\right) dt = 0; \qquad (70)$$

and from the liquidity services equilibrium, equation (66), it follows that

$$\int_0^\infty e^{-\rho t}\left[q\left(y_{21}\frac{dp}{d\sigma} + y_{22}\frac{dk}{d\sigma}\right) - p\frac{dc_2}{d\sigma} - (p^d + (\pi + r)(1-\gamma))\frac{dm_2}{d\sigma}\right] dt$$
$$+ \int_0^\infty e^{-\rho t}\left[y_2\frac{d\theta}{d\sigma} - c_2\frac{dp}{d\sigma} - m_2\frac{d(p^d + (\pi + r)(1-\gamma))}{d\sigma}\right] dt = 0. \qquad (71)$$

Adding (70) and (71) to (69), recalling (67), it follows that

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t}(\pi + r)\frac{d(m_1 + \alpha m_2)}{d\sigma} dt. \qquad (72)$$

Defining $b = m_1 + \alpha m_2$, in which b stands for the monetary base, the result follows. Again, if the capital intensity across sectors is the same, it is possible to integrate (72) to get

$$\Delta W_{\text{Units of Consuption Goods}} = \int_{-\rho}^{\sigma} R^*\frac{db^*}{d\sigma'}d\sigma' = -\int_{b^*(\sigma)}^{b^*(-\rho)} R^*(b)db. \qquad (73)$$

It is straightforward to follow the steps of the last Section to calculate the

impact of inflation under welfare for the other concept of welfare's alterations - the compensate income. Nothing would change, except that the monetary-base demand would be the compensate demand.

Until this point the model has not been accurate with respect to demand deposits. In addition to being a service provided by the banking institution, its price is regulated. Usually the banking institutions can charge a fee for this service but can not pay to customers, in order to stimulate them to keep their money there. In other words, the relative price of demand deposit can not be lower than the relative price of currency, which means that

$$p^d \geq 0. \tag{74}$$

When inflation increases, the banks reduce p^d in order to attract customers. But, if inflation increases further, the inflationary income skyrockets, and the market solution of this model will reduce the price of demand deposit below the nominal interest rate. Under (74) this is not possible. However, because (74) does not introduce an edge between the price seen by the customers and the price seen by the banks, the calculations that lead to (73) are still valid[41]. The restriction (74) will totally change the general equilibrium solution of the model - particularly, the demand for monetary base, for demand deposit, and for the other transactions-saving services will be displaced. But the main result

[41] The benefit seen by the household in carrying a demand deposit in its portfolio is exactly matched by the cost for the banking institution in providing this service. This basic fact is not altered by (74).

expressed by (73) will be analytically valid.

Another issue is the impact under welfare of (74). If inflation is sufficiently low it is nil - (74) is not binding. Under high inflations levels the answer is ambiguous. Because welfare theorems are not satisfied for monetary models the impact under welfare of a distortion it is not clear. It is necessary to ask for computational methods to know it. The source of the ambiguity is that on one hand (74) reduces welfare because it induces a misallocation of factors towards transaction-saving services and out of demand deposits[42]; on the other hand, it stimulates the demand for monetary base, which improves welfare.

8 Conclusion

Firstly, it has been shown that the use of Bailey's rules to evaluate the impact of inflation on welfare is indeed exact for many monetary models, among others the standard Sidrauski model, the McCallum-Goodfriend model, and the cash-in-advance family of models. In particular, the result applies if the existence of a banking sector that provides services which are substitutes for money is taken into consideration. Although the banking sector helps the public to cope with the inflation, extracts production factors which has a positive social value in the good market. Notwithstanding these effects, the measure of the impact on welfare of inflation is the usual one: the area under the inverse demand curve for money. That does not mean that the increase of the banking sector

[42] The competitions among banks will decrease the price of the transaction-saving services.

is harmless. Due to the general equilibrium nature of the problem, if by any reason the banking share in the product had not been increased, the stationary-state money demand would be different. The point here is that all these general equilibrium effects[43] that follow an elevation of inflation rate, have the very same analytical expression for the impact on welfare of an increase in the inflation rate, which is exactly expressed by Bailey's formula. Therefore, when one calculates the welfare impact of inflation applying Bailey's rule, the researcher has already taken into consideration the fact that the banking sector is taken real resources from the other sector to provide banking services to the public. And this result is robust whether a Sidrauski-type model or a cash-in-advance model is taken into consideration.

Secondly, it has been argued that the relevant demand function for evaluating the impact of inflation under welfare is the narrow monetary aggregate, the monetary base. When it is recalled that the demand deposit is a service provided by the banking sector, and consequently, requires the employment of production factors in order to be offered, this observation is straightforward.

References

[1] Aiyagari, S. Rao, R. Anton Braun, and Zvi Eckstein 1998. "Transaction Services, Inflation, and Welfare." Journal of Political Economy, 106 1274-1301.

[43] The increase in the share of the banking sector, the reducing in the consumption level, the reduction in the quantity of money, and the elevation of the consumption of banking services.

[2] Barro, Robert J. 1972. "Inflationary Finance and the Welfare Cost of Inflation" Journal Political Economy, 80: 978-1001.

[3] Bassole, A. and Samuel de A. Pessoa 1999. "Tobin effect and Banking Sector," memo.

[4] Bailey, Martin J. 1956. "The Welfare Cost of Inflationary Finance." Journal of Political Economy, 64: 93-110.

[5] Baumol, William J. 1952. "The Transactions Demand For Cash: An Inventory Theoretical Approach." Quarterly Journal of Economics, 66: 545-556.

[6] Benabou, Roland 1991. Comment on The Welfare Costs of Moderate Inflations, Journal of Money, Credit, and Banking 23 (3): 504-513.

[7] Cathcart, Charles D. 1974. "Monetary Dynamics, Growth and the Efficiency of Inflationary Finance," Journal of Money, Credit, and Banking 6: 169-190.

[8] Cooley, Thomas F. and Gary D. Hansen 1989. "The Inflation Tax in a Real Business Cycle Model" American Economic Review, 79 (4): 733-748.

[9] Cooley, Thomas F. and Gary D. Hansen 1991. "The Welfare Costs of Moderate Inflations" Journal of Money, Credit, and Banking 23 (3): 483-503.

[10] Drazen, Allan 1979. "The Optimal Rate of Inflation Revisited," Journal of Monetary Economics 5: 231-248.

[11] Feenstra, Robert C. 1986. "Functional Equivalence Between Liquidity Cost and the Utility of Money," Journal of Monetary Economics 17: 271-291.

[12] Fischer, Stanley 1981. "Towards an Understanding of the Cost of Inflation II" Cornegie-Rochester Conference series in Public Policy 15: 5-42.

[13] Guidotti, Pablo E. and Carlos A. Vegh 1993. "The Optimal Inflation Tax When Money Reduces Transactions Costs," Journal of Monetary Economics, 31: 189-205.

[14] Lucas, Robert E., Jr. 1981a. "Equilibrium in a Pure Currency Economy," Economic Inquiry, 18: 203-220.

[15] Lucas, Robert E., Jr. 1881b Discussion of Stanley Fischer "Towards an Understanding of the Cost of Inflation II" Cornegie-Rochester Conference series in Public Policy 15: 43-52.

[16] Lucas, Robert E., Jr. 1997. "Inflation and Welfare." memo.

[17] Lucas, Robert E., Jr. and Nancy L. Stokey 1983. "Optimal Fiscal Policy in a Economy Without Capital," Journal of Monetary Economics, 12: 55-93.

[18] Marty, Alvin L. 1976. "A Note on The Welfare Cost of Money Creation," Journal of Monetary Economics 2: 121-124.

[19] – – – – – – 1965. "Growth and the Welfare Cost of Inflationary Finance," Journal of Political Economy," 73: 97-109.

[20] McCallum, Bennett T. 1983. "The Role of Overlapping Generations Models in Monetary Economics" Carniegie-Rochester Conference Series in Public Policy 18 Spring.

[21] McCallum, Bennett T., and Marvin S. Goodfriend 1987. "Demand For Money: Theoretical Studies." The New Palgrave: A Dictionary of Economics, John Eatwell, Murray Milgate, and Peter Newman, eds. London: Macmillan; Ney York: Stockton Press, 775-781.

[22] Pastore, Affonso C. 1994. "Déficit público, a sustentabilidade do crescimento das dívidas interna e externa, senhoriagem e inflação: uma análise do regime monetário brasileiro." Revista de Econometria Ano XIV, número 2, novembro de 1994/março de 1995.

[23] Sidrauski, Miguel 1967. "Inflation and Economic Growth." Journal of Political Economy, 75 (6): 798-810.

[24] Stockaman, Alan C. 1981. "Anticipated Inflation and the Capital Stock in a Cash-in-Advance Economy," Journal of Monetary Economics, 8: 387-393.

[25] Tatom, John A. 1976. "The Welfare Cost of Inflation," Federal Reserve Bank of St. Louis Review, 53: 9-22.

[26] Tower, Edwar 1971. "More on the Welfare Cost of Inflationary Finance, a Comment," Journal of Money, Credit, and Banking 3: 850-860.