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Uncertainty Aversion and the Optimal Choise of Fortfolio

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Abstract

In this paper we apply the theory of decision making with expected utility and non-additive priors to the choice of optimal portfolio. This theory describes the behavior of a rational agent who is averse to pure 'uncertainty' (as well as, possibly, to 'risk'). We study the agent's optimal allocation of wealth between a safe and an uncertain asset. We show that there is a range of prices at which the agent neither buys not sells short the uncertain asset. In contrast the standard theory of expected utility predicts that there is exactly one such price. We also provide a definition of an increase in uncertainty aversion and show that it causes the range of prices to increase.

1. Introduction

In this paper we describe some implications for economic analysis of a model of decision making under uncertainty which generalizes the expected-utility model accepted by most economists as a representation of rational behavior. The model we use is the model of expected utility under a non-additive probability measure, which seeks to distinguish between quantifiable 'risks' and unknown 'uncertainties'. An axiomatic treatment of the model may be found in Schmeidler (1984, 1986), Gilboa (1987) and Gilboa and Schmeidler (1986).

The focus of this paper is the problem of optimal investment decisions. Under the standard theory of expected utility, an agent who must allocate his wealth between a safe and a risky asset will buy some of the asset if the price is less than the (present value of) the expected return. The amount of the asset that is bought depends on the agent's attitude to risk. Conversely the agent will sell the asset short when the price is greater than the expected return. Our main theorem is a generalization of this result to the case of uncertainty. We also provide a definition of an increase in perceived uncertainty, and analyse the effects of such an increase on the investment decision.

The problem of making decisions under uncertainty has been of central importance to economics and statistics throughout the development of these disciplines. The expected utility theory, which owes its axiomatic development to von Neumann and Morgenstern (1947), initiates from the work of Bernoulli (1730). Savage (1954) has made a persuasive case that rational behavior necessarily entails actions represented by such a utility function and by a prior subjective probability distribution over possible events.

Suppose I am to gamble on the toss of a coin and must form a plan to decide which bets to take and which bets to refuse. To do this I need to decide whether

the coin is biased and if so, by how much. How can I form an estimate of the chance of heads? If I do not know the chance of heads, perhaps I can form a prior over possible values of this chance and find the expected value of the chance under this prior. If I have a model of how bias may arise, then this model will provide the prior. But suppose that I do not have such a model: then I could envisage a range of possible models and assign a prior probability to each one and, taking averages over these possible models and the chances of bias implied by each one, I could eventually find my belief of the chance of heads. But how do I know that the models considered were applicable ones, or that my beliefs about the likelihood of each were correct? Of course I do not know this, but I can form beliefs over the different possible relative likelihoods of models within a range of possible models, and over different possible ranges of possible models. This process may be carried on ad infinitum. The point that we wish to make here is that at each step in the process it becomes increasingly difficult to construct models and to form priors and that, therefore, a reasonable person might perhaps treat the final estimate of the chance of heads differently from the same chance estimated from a coin 'known' as a result of intensive study and numerous experiments. According to Savage, this distinction would be unreasonable.

This distinction between 'risk' and 'uncertainty' is proposed by Knight (1921), Ellsberg (1961), and Bewley (1986) among many others. In the series of papers referred to above, Schmeidler and Gilboa have given an axiomatic development of a model which incorporates this distinction. This model entails maximizing expected utility with a non-additive probability measure. With a non-additive probability measure, the 'probability' that either of two mutually exclusive events will occur is not necessarily equal to the sum of their two

'probabilities'. If it is less than the sum, then expected-utility calculations using this probability measure will reflect uncertainty-aversion as well as (possibly) risk-aversion. (In Section two we give an alternative "maximin" interpretation of the model which avoids the use of non-additive probabilities.)

Although there are good reasons, empirical and theoretical, for questioning the premises of the expected utility model, there is one factor which is strongly in its favor. While the theory of consumer behavior under certainty has only the most pedestrian empirical implications (homogeneity of degree zero and continuity of the demand function, and symmetry and negative semi-definiteness of the Slutsky matrix where demand is differentiable), the theory of expected utility yields some strong predictions, in particular the results on local risk neutrality and on complete insurance with actuarially fair policies. A generalization of the theory which eliminated the independence axiom completely would also lead to the loss of these useful predictions. The purpose of this paper is to show that the model of expected utility maximization with non-additive probabilities reflecting uncertainty-aversion preserves strong results which are analogous to these. We focus particularly on the local risk-neutrality theorem.

According to this result, an agent who starts from a position of certainty will invest in an asset if, and only if, the expected return on the asset (to be precise: the present value of the expected return) exceeds the price. This result holds in the absence of transactions costs whenever it is possible to buy small quantities of an asset. Conversely, if the expected return is lower than the price of the agent will wish to sell the asset short. Consequently an agent's demand for an asset should be positive below a certain price, negative above that price and zero at exactly that price.

With a non-additive subjective probability distribution over returns on the asset, we show that this result has a straightforward analog which is intuitively plausible and is compatible with observed investment behavior. There is an interval of prices within which the agent neither buys nor sells short the asset. At prices below the lower limit of this interval, the agent is willing to buy this asset. At prices above the upper end of this interval the agent is willing to sell the asset short.

The highest price at which the agent will buy the asset is to the expected value of the return on the asset under the non-additive probability measure. The lowest price at which the agent sells the asset is the expected value of selling the asset short. This reservation price is larger than the other one if the beliefs reflect uncertainty aversion: with a non-additive probability measure, the expectation of a random variable is less than the negative of the expectation of the negative of the random variable.

These two reservation prices, then, depend only on the beliefs and aversion to uncertainty incorporated in the agent's prior, and not on attitudes to risk. This result is the non-additive analog of the local risk-neutrality result.

We have suggested above that a reasonable person may not act according to Savage's model. Maximizing utility with a non-additive prior may be a reasonable model of rational behavior in some circumstances. However, we do not argue that this model is the only way, nor necessarily the best way, to represent genuine uncertainty. What we show here is that provides a tractable framework for economic analysis of the types of problems which expected utility theory itself is useful for.

We have outlined a rationale for non-additive probability models as a normative theory of behavior under uncertainty. However, we also wish to suggest

that the model provides a description of actual behavior. Most people do not invest in most assets, partly of course because of transactions costs and informational asymmetries (i.e., the price reflects information of other traders, and hence demand need not be downward sloping), but partly, perhaps, because they simply do not 'know' enough about the uncertainty of possible returns on the asset. It seems perfectly natural - even in the absence of transactions costs and informational asymmetries - to wish neither to buy nor to sell short an asset. Yet this is incompatible with the standard theory of expected utility. In the last section of the paper, we mention some observations of behavior of financial institutions which would appear to be of this type.

The organization of the rest of the paper is as follows. In section 2 we present a simple example which illustrates the basic features of the model. In section 3 we present a definition of an increase in uncertainty aversion and results on expectation of a random variable with a non-additive distribution. In section 4 we give our main theorem on asset choice under uncertainty. Section 5 contains a discussion and conclusion. The Appendix contains mathematical results for reference.

2. An Example.

In this section we present an example which illustrates the principles of how an agent with a non-additive subjective probability distribution should make portfolio choiçes (strictly speaking, we should regard the utility function of the agent and the non-additive probability distribution as jointly implicit in the decisions taken by the agent: the wording of the previous sentence - we use similar wording elsewhere too - is a convenient abuse of language. Maximization

of expected utility with a non-additive subjective probability distribution reflects both the presence of uncertainty, and aversion to it).

Among the axioms that imply the existence of an expected utility function and an additive subjective probability distribution is the independence axiom. It is this axiom which is usually questioned by those who doubt the validity of the model. A non-additive probability distribution results from a weakened version of this axiom. According to the independence axiom, if lottery A is preferred to lottery B, a compound lottery giving lottery A or lottery C is preferred to a compound lottery giving lottery B or lottery C, for every other lottery C so long as the chance of getting lottery C is the same in both compound lotteries. The weakened version of the axiom requires this to hold only in case C is a certain outcome. (See Gilboa and Schmeidler (1986)).

Let us suppose that an agent must choose to allocate wealth W between a safe and a risky asset. The risky asset may pay either of two possible returns, high H or low L. Throughout we use the convention that these numbers represent the present values of the returns discounted at the safe interest rate.

Since the risky asset may take only two values its distribution is characterized by two numbers: the probabilities q of a high return and q' of a low return. If $q+q' < 1$ then the decisions taken by the agent reflect uncertainty-aversion.

We consider the expected return from buying one unit of the asset at price p. The return can at worst be $(L-p)$, net of the price. With probability q, however, the return is $(H-p)$, i.e. an improvement of $(H-L)$ over the worst outcome. This possible improvement is uncertain and its evaluation reflects that: thus the expected return from buying a unit of the asset is $[L+q(H-L)]-$

p]. If the price p is less than $L+q(H-L)$, a risk neutral investor will buy the asset.

If instead the investor sells a unit of the asset short, the return will be $(p-H)$ if the asset is worth H, and $(p-L)$ if it is worth L. At worst the return is $(p-H)$ and, with probability q', it amounts to $(p-L)$. The expected value is therefore $p - H + q'(H-L)$. Thus if p exceeds $H+q'(H-L)$, the investor will sell the asset short. Fig. 1 shows the expected return from buying and selling the asset as a function of p.

Note that because $q+q'< 1$, $H + q'(H-L) > L + q(H-L)$. At prices in between these two numbers the investor will not trade in the asset.

A closely related representation of decisions is to suppose that the decision maker evaluates expected utility for a set of possible prior distributions, and acts to maximize the minimum of expected utility over these possible priors (Gilboa and Schmeidler (1986)). At one extreme, the decision maker considers only one prior - a 'known' distribution - and acts according to expected utility. At the other extreme, if all prior distributions over outcomes are considered, the decision maker considers only the worst possible outcome. This "Maximin" rule has been proposed by Wald (1950) for situations of complete uncertainty, and Ellsberg (1961) and Rawls (1971) also suggest that this rule should be considered in such circumstances. Simonsen (1987) is a more recent application to the theory of inflationary inertia.

In the above example we could say that the agent believes the chance of a high return to lie between q (at least) and $(1-q')$ (at most), with a low return having the complementary probabilities between $(1-q)$ and q'. The payoff from buying a unit of the asset is then

$$\mathrm{Min}(L + \lambda(H-L) - p \mid \lambda \epsilon [q, 1-q']),$$

and from selling it short,

$$\mathrm{Min}(p - H + \lambda(H-L) \mid \lambda \epsilon [q', 1-q]).$$

Note that if all priors are considered 'possible' in this framework, the payoffs are $(L-p)$ and $(p-H)$: decisions are made so as to maximize the worst possible outcome. This formulation of the model may be misleading however: an agent who satisfies the axioms of Schmeidler (1984) takes actions which can be represented by a utility function and a probability distribution. If the probability distribution is non-additive this reflects both the presence of uncertainty and the agent's aversion to uncertainty. For instance, in the above example we could have $q = q' = 1/2$: this need not mean that the agent 'knows' the risk with certainty. It could mean that the agent thinks both outcomes are equally likely and is not averse to uncertainty.

The above example of decision-making with non-additive probabilities should make clear the following properties of expectation (See the Appendix). On the one hand, adding a constant to a random variable or multiplying it by a positive constant has the same effects on its expectation. This property is not true for negative constants: in general $-E(-X) > E(X)$. It is this inequality which gives rise to the price range with no trade. It should be clear that the purpose of using an example with two states was to illustrate this inequality and that, in general, $E(X)$ and $-E(-X)$ are the two reservation prices. The fact that expectation here retains its familiar properties with respect to addition of a constant and multiplication by a positive constant enables us to speak, as usual,

of present values of future incomes, of returns net of the price, and of expected return per unit.

3. Uncertainty Aversion

In the previous section we gave an example showing that uncertainty aversion is responsible for the phenomenon of "avoiding" risks: i.e., a person may want to stay out of an asset even though, according to the standard expected utility theory, he should want either to buy or to sell a little unless the price is exactly at the expectation.

The range of prices under which no trade occurs was shown to be the interval

$$[EX,\ -E(-X)]$$

which degenerates to EX in the case of an additive probability. This suggests that we should be able to show that in some sense, the more we deviate from additive behavior, the wider this interval of no trade becomes. In this section we prove a theorem (Theorem 3.5) which confirms this intuition.

We start by defining a measure of uncertainty aversion, following an idea of Schmeidler (1984), for the case of two states of nature. In what follows the reader should refer to the Appendix as necessary for the notation and for mathematical properties of non-additive probabilities.

3.1 Definition: Let P be a non-additive probability, and $A \subset \Omega$ an event. We say that

$$c(P,A) = 1-P(A)-P(A^c)$$

is the <u>uncertainty aversion of P at A</u>.

Notice that this number measures the amount of probability "lost" by the existence of uncertainty. It gives us the deviation of P from additivity at the event A. As one should expect, $c(P,A) = c(P,A^c)$. The next lemma states that this number is positive if and only if the probability is not finitely additive.

<u>3.2 Lemma</u>: $c(P,A) = 0$ for all events $A \subset \Omega$ if, and only if, P is finitely additive.

<u>Proof</u>: ($\Leftarrow$) This follows immediately from the fact that $A \cap A^c = \emptyset$, $A \cup A^c = \Omega$, $P(\Omega) = 1$.

($\Rightarrow$) Let A and B be events such that $A \cap B = \emptyset$. By the definition of non-additive probability, one has $P(A \cup B) \geq P(A) + P(B)$. We have only to show that $P(A \cup B) \leq P(A) + P(B)$. In fact, $P((A \cup B)^c) = P(A^c \cap B^c) \geq P(A^c) + P(B^c) - P(A^c \cup B^c)$. But $A \cap B = \emptyset \Rightarrow A^c \cup B^c = \Omega$. Hence, $P(A^c \cap B^c) \geq 1-P(A) + 1-P(B)-1$ by the fact $c(P,X) = 0$ for all events X, which implies that $P((A \cup B)^c) \geq 1-P(A)-P(B)$. Similarly, by the fact that uncertainty aversion is zero, $1-P(A \cup B) \geq 1-P(A)-P(B) \Rightarrow P(A) + P(B) \geq P(A \cup B)$.

<u>Q.E.D.</u>

<u>3.3 Example</u>: Consider the case of non-additive P defined over a finite Ω, for which the event space is the power set of Ω, (Ω). Set $P(\{\omega\}) = \frac{1-c}{n}$, where n is the number of elements of Ω, c a number $0 \leq c \leq 1$, for all

$\omega \in \Omega$. Define, for $A \subset \Omega$, $A \neq \Omega$, $P(A) = \sum_{\omega \in A} P(\{\omega\})$. It is easy to check that $c(P,A) = c$, $\forall A \neq \Omega, \emptyset$. That is to say, this is a distribution with constant uncertainty aversion. In general a non-additive probability does not have to be so simple.

<u>3.4 Example</u>: <u>The case of maximin behavior</u>. As discussed in section 2, a person with extreme uncertainty aversion who is completely uninformed maximizes the payoff of the worst possible outcome. Suppose that $c(P,A) = 1$ for all events $A \neq \Omega$ or $\emptyset$. Then $P(A) = 0$ for all $A \neq \Omega$. Let $u: \Re \to \Re_+$ be the subjective utility function of the agent. Then:

$$Eu = \int_\Omega u dP = \int_0^\infty P(u \geq \alpha)\, d\alpha.$$

Let $\bar{u} = \inf_{x \in \Re} u(x)$. Then $\text{Prob}(u \geq \bar{u}) = 1$ and $\text{Prob}(u \geq \bar{u}+\varepsilon) = 0 \ \forall \varepsilon > 0$. Therefore

$$Eu = \int_0^{\bar{u}} 1 d\alpha = \bar{u} = \inf_{x \in \Re} u(x).$$

We now proceed to extend this "local" measure of uncertainty aversion to the whole range of two non-additive probabilities.

<u>3.4 Definition</u>: Given two non-additive probabilities P and Q, defined over the same space of events, we say that <u>P is at least as uncertainty averse as Q</u> if for all events $A \subset \Omega$, $c(P,A) \geq c(Q,A)$.

We are now ready to show that the range on which no trade occurrs increases as the uncertainty aversion increases. (The symbol $E_P X$ means: $\int_\Omega X dP$, etc.)

3.5 Theorem: The two statements below are equivalent:

(i) P is at least as uncertainty averse as Q;

(ii) For all random variables X for which $E_P X, E_Q X, E_P(-X), E_Q(-X) < +\infty$,

$-E_P(-X) - E_P X \geq -E_Q(-X) - E_Q X$.

Proof: (i) => (ii). By the integrability assumption, all the integrals which appear here make sense.

$$E_P X = \int_{-\infty}^{0} (P(X \geq \alpha)-1)d\alpha + \int_{0}^{\infty} P(X \geq \alpha)d\alpha.$$

Let $A(\alpha) = \{\omega \in \Omega;\ X(\omega) \geq \alpha\}$. Then

$$E_P X = \int_{-\infty}^{0} [P(A(\alpha))-1]d\alpha + \int_{0}^{\infty} P(A(\alpha))d\alpha.$$

Also,

$$E_P(-X) = \int_{-\infty}^{0} (P(-X \geq \alpha)-1)d\alpha + \int_{0}^{\infty} P(-X \geq \alpha)d\alpha =$$

$$= \int_{-\infty}^{0} (P(X \leq -\alpha)-1)d\alpha + \int_{0}^{\infty} P(X \leq -\alpha)d\alpha =$$

$$= -\int_{+\infty}^{0} (P(X < \alpha)-1)d\alpha - \int_{0}^{-\infty} P(X < \alpha)d\alpha =$$

$$= \int_{0}^{\infty} [P(A(\alpha)^c)-1]d\alpha + \int_{-\infty}^{0} P(A(\alpha)^c)d\alpha.$$

Hence: $-E_P(-X) - E_P(X) = \int_{-\infty}^{\infty} [1-P(A(\alpha)) - P(A(\alpha)^c)]d\alpha.$

By the same argument,

$$-E_Q(-X) - E_Q(X) = \int_{-\infty}^{\infty} [1-Q(A(\alpha)) - Q(A(\alpha)^c)]d\alpha.$$

As P is at least as uncertainty averse as Q, it follows that

$$1-P(A(\alpha)) - P(A(\alpha)^c) \geq 1-Q(A(\alpha)) - Q(A(\alpha)^c), \ \forall\ \alpha \in \Re.$$

Thus, we have (ii). <u>Q.E.D.</u>

(ii) => (i). Let A be an event. Define a random variable $X = 1A$ (the characteristic function of the set A: $1A(\omega) = 1$ if, and only if, $\omega \in A$). By the definition of the integral:

$$E_P X = P(A),\ E_P(-X) = P(A^c)-1,$$

$$E_Q X = Q(A),\ E_Q(-X) = Q(A^c)-1.$$

Applying (ii) to X, we get (i). <u>Q.E.D.</u>

The example below illustrates the effect of uncertainty aversion on the dispersion between $-E(-X)$ and EX.

<u>3.6 Example</u>: Let $X \geq 0$ be a random variable such that $\underline{X} = \inf_\omega X(\omega)$ and $\infty > \bar{X} = \sup_\omega X(\omega)$. Let P be an additive probability. We define a non-additive probability which is obtained by uniformly increasing the uncertainty aversion

from P: let $P_c(\Omega) = 1$, and $P_c(A) = (1-c)P(A)$ for $0 \leq c \leq 1$, and for all $A \neq \Omega$. It is easy to check that $c(P_c, A) = c$ for all $A \neq \Omega, \emptyset$, and that

$$E_{P_c} X = c\underline{X} + (1-c)\, E_P X \quad \text{and} \quad -E_{P_c}(-X) = c\bar{X} + (1-c)\, E_P X.$$

It is clear that $-E_{P_c}(-X) - E_{P_c} X = c(\bar{X} - \underline{X})$, which is increasing in the uncertainty aversion c, as the theorem says. Ellsberg (1961) suggested this as an ad hoc decision rule; this example provides a rationale for the rule.

4. The Choice of Portfolio

The aim of this section is to show that the results obtained for a risk neutral investor in section two can be generalized for a risk averse individual. This should be expected a priori, because the theorem of section three shows that the spread between $-E(-X)$ and EX is due only to deviations from additivity, that is to say, due only to uncertainty aversion. Thus one should expect the range of no trade to depend not on the risk aversion, but only on the uncertainty aversion.

Before proving the main result, we need a lemma. Let $W > 0$ be the wealth of the individual, $u \geq 0$ the utility function, n a real number, and aZ a random variable with non-additive distribution P.

4.1 Lemma: Suppose $EZ < +\infty$ and $-E(-Z^2) < +\infty$. Assume also that u is C^2, $u' > 0$, and $u'' \leq 0$. Let $f(n) = Eu(W+nZ)$. Then:

(i) f is right differentiable at $n = 0$;

(ii) $f'_+(0) = u'(W).EZ$.

Proof: When $f'_+(0)$ exists,

$$f'_+(0) = \lim_{n\to 0^+} \frac{Eu(W+nZ) - u(W)}{n} .$$

We are going to show that, under the assumptions of the lemma, this limit exists and is equal to $u'(W).EZ$.

$$\frac{Eu(W+nZ)-u(W)}{n} = \frac{\int_0^\infty P(u(W+nZ) \geq \alpha)d\alpha - u(W)}{n} ,$$

because $u \geq 0$. Let us assume $u(\infty) = \infty$, with no loss of generality. Then, by changing variables $\beta = (u^{-1}(\alpha)-W)/n$:

$$Eu(W+nZ) = \int_{\frac{u^{-1}(0)-W}{n}}^{\infty} P(Z \geq \beta)\ u'(n\beta+W)nd\beta =$$

$$= \int_{\frac{u^{-1}(0)-W}{n}}^{0} P(Z \geq \beta)\ u'(n\beta+W)nd\beta + \int_0^\infty P(Z \geq \beta)\ u'(u\beta+W)nd\beta$$

We can write $u(W) = Eu(W)$

$= \int_0^\infty P(u(W) \geq \alpha)d\alpha$. Changing the variable α to $\beta = (u^{-1}(\alpha)-W)/n$, we have, $d\alpha = u'(n\beta+W)nd\beta$, and hence

$$u(W) = \int_{\frac{u^{-1}(0)-W}{n}}^{\infty} P(0 \geq \beta)\ u'(n\beta+W)nd\beta,$$ so that

$$\frac{f(n)-f(0)}{n} = \int_{\frac{u^{-1}(0)-W}{n}}^{0} (P(Z \geq \beta)-1)\ u'(n\beta+W)d\beta + \int_0^{\infty} P(Z \geq \beta)\ u'(n\beta+W)d\beta$$

$$\left| \frac{f(n)-f(0)}{n} - u'(W)EZ \right| \leq$$

$$\left| \int_{\frac{u^{-1}(0)-W}{n}}^{0} (P(Z \geq \beta)-1)u'(n\beta+W)d\beta - \int_{-\infty}^{0} (P(Z \geq \beta)-1)u'(W)d\beta \right| +$$

$$\left| \int_0^{\infty} P(Z \geq \beta)u'(n\beta+W)d\beta - \int_0^{\infty} P(Z \geq \beta)u'(W)d\beta \right| .$$

One can easily see that the second part of the sum converges to zero as n goes to zero, by the monotone convergence theorem (the function $u'(n\beta+W)$ is less than or equal to $u'(n'\beta+W)$ where $0 < n' \leq n$, $\beta \geq 0$, and $u'' \leq 0$).

What is left to show is that the first part goes to zero as well, when n tends to zero. We can rewrite the first part as:

$$\left| \int_{\frac{u^{-1}(0)-W}{n}}^{0} [P(Z \geq \beta)-1][u'(W)-u'(n\beta+W)]d\beta \right. +$$

$$\left. \int_{-\infty}^{\frac{u^{-1}(0)-W}{n}} (P(Z \geq \beta)-1)u'(W)d\beta \right| \leq$$

$$\left| \int_{\frac{u^{-1}(0)-W}{n}}^{0} [P(Z \geq \beta)-1][u'(W)-u'(n\beta+W))d\beta \right| +$$

$$u'(W)\left|\int_{-\infty}^{\frac{u^{-1}(0)-W}{n}} (P(Z \geq \beta)-1)d\beta\right| .$$

The second part of this inequality tends to zero as n approaches zero, by the definition of the improper integral. Thus, we are reduced to showing that:

$$\lim_{n\to 0}\left|\int_{\frac{u^{-1}(0)-W}{n}}^{0} [P(Z \geq \beta)-1)][u'(W)-u'(W+n\beta)]d\beta\right| = 0.$$

Recall the fact that $u \in C^2$, and the mean value theorem:

$$\frac{u'(W)-u'(W+n\beta)}{-n\beta} = u''(\theta) \quad \text{for some} \quad \theta \in [W+n\beta, W].$$

But $\frac{u^{-1}(0)-W}{n} \leq \beta \leq 0$. Hence $\theta \in [u^{-1}(0), W]$, for all n. Let $M = \max_{\theta \in [u^{-1}(0),W]} |u''(\theta)|$.

M is well defined because $u \in C^2$. We can see that

$$|u'(W)-u'(W+n\beta)| \leq nM|\beta|.$$

Thus

$$\left| \int_{\frac{u^{-1}(0)-W}{n}}^{0} [P(Z \geq \beta)-1][u'(W)-u'(W+n\beta)]d\beta \right| \leq$$

$$\int_{\frac{u^{-1}(0)-W}{n}}^{0} \left| (P(Z \geq \beta)-1) \right| \cdot Mn \cdot |\beta| d\beta.$$

By the fact that $-E(-Z^2) < +\infty$, it is easy to check that $\int_{-\infty}^{0} \left|P(Z \geq \beta)-1\right| |\beta| d\beta$ $= I < +\infty$.

Therefore: $$\left| \int_{\frac{u^{-1}(0)-W}{n}}^{0} (P(Z \geq \beta)-1)(u'(W)-u'(W+n\beta))d\beta \right| \leq$$

$\leq MnI \to 0$ as $n \to 0$. Q.E.D.

We now derive our main result, namely the behavior of the risk averse or risk neutral consumer under uncertainty aversion. Suppose the investor is faced with the problem of choosing the sum of money S he/she will invest in asset. Suppose X (which has non-additive distribution P) is the present value of the random return on one unit of the asset next period. Let p be the price of the asset.

4.2 Theorem: A risk neutral or risk averse investor, who is faced with an asset that yields X per unit, whose price is $p > 0$ per unit, will buy the asset if,

and only if $p < EX$ (the case $p = EX$ may be of indifference). Moreover, he/she will sell the asset if, and only if, $p > -E(-X)$ (again the case of $p = -E(-X)$ may be of in difference).

__Proof__: The problem of the investor is to choose $S \leq W$ to maximize:

$$Eu(W-S + \frac{S}{p} X) = Eu(W + \frac{S}{p} (X-p)).$$

By Jensen's inequality (see the Appendix):

$$Eu(W-S + \frac{S}{p} X) \leq u(E(W-S + \frac{S}{p} X)).$$

By the example of section two, if $EX \leq p \leq -E(-X)$, it follows that $E(W-S + \frac{S}{p} X) \leq W$. Thus the investor is at least as well off not trading, because in this case the expected utility is $u(W)$. (Observe that $EX < p < -E(-X)$ implies strict inequalities, so that, if they hold, no trade is in fact best.)

What we are going to show now is that if $p < EX$ the investor will want to buy some of the asset, and if $p > -E(-X)$ he will want to sell some of the asset, so that the theorem will be proved. Assume, first, $EX > p > 0$. Then, the investor has to choose to maximize $g(S) = E\,u(W + \frac{S}{p} (X-p))$. By lemma 4.1 we have that $g'_+(0)$ exists, and:

$$g'_+(0) = \frac{d}{d(S/p)} \Big|_{S=0+} Eu(W + \frac{S}{p} (X-p)) \cdot \frac{1}{p} = \frac{u'(W)}{p} \cdot E(X-p) > 0$$

since $EX > p$. Thus the investor will always want to buy some of the asset. Suppose, now, that $p > -E(-X)$. Then the investor again has to choose S to maximize $g(S)$. Changing variables, so that $-S = T$, we have $h(T) = g(-T) = Eu(W + \frac{T}{p}(p-X))$. Again, by the lemma:

$$h'_+(0) = \frac{u'(W)}{p} \cdot E(p-X) = \frac{u'(W)}{p} \cdot (p + E(-X)) > 0.$$

Hence the investor will prefer to have $T > 0$, which means $S < 0$, which means preference for selling the asset. Q.E.D.

Remark: If u is not differentiable at some point W then, even with an additive probability measure, one obtains a range of prices with no trade. If u is concave, the set of W where u is not differentiable has measure zero.

5. Summary and Concluding Remarks

We have shown how the theory of uncertainty aversion may be used to explain an agent's choice of portfolio. In particular, the theory predicts that there is an interval of prices at which demand is zero, which increases if there is an increase in uncertainty aversion.

This type of behavior is also caused by transaction costs and by informational asymmetries. However, it seems possible that these two reasons are not sufficient to explain all observed absences from trading. Although transactions costs are relatively small for large institutions, they are still observed not to have positions in many assets. While private information may be used to explain bid-ask spreads, equilibrium models with private information predict that no trade should ever occur, except because of exogenous noise traders.

One example of how uncertainty may cause traders to drop out of a market is provided by the trading behavior of foreign currency speculators in early January 1988. At this time there was uncertainty in the financial community over whether or not the U.S., Western European and Japanese monetary authorities had agreed to support the U.S. Dollar. Many of the normally-active smaller banks refrained from speculating on the value of the dollar. One could argue that this is precisely the sort of uncertainty that is hard to quantify exactly, and that this decline in trade was the result of uncertainty aversion.

In this paper we have set out the simplest investment decision to analyze, namely that where there is only one asset whose return is uncertain. In case there are several assets the analysis becomes more complex because one must consider the issue of statistical dependence and independence of the returns. We hope to pursue this issue in the future.

Appendix

The mathematical treatment of non-additive probabilities can be found in Schmeidler (1982,1984,1986), Choquet (1955), Dellacherie (1970), Gilboa (1987), Gilboa and Schmeidler (1986), Shafer (1976) and Dempster (1967). The interested reader is referred to these. In particular, Schmeidler (1986) contains only material related to the mathematical aspects of the theory.

Let Ω be a set, and Σ an algebra i.e., a set of subsets of Ω such that (i) $\Omega \in \Sigma$, (ii) $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$ and (iii) $A \in \Sigma \Rightarrow A^c \in \Sigma$ (here A^c means the set of elements of Ω not in A). The elements of Σ are called <u>events</u>. A function $P: \Sigma \to [0,1]$ is a <u>non-additive probability</u> if (i) $P(\emptyset) = 0$, (ii) $P(\Omega) = 1$, and (iii) $P(A) \leq P(B)$ if $A \subset B$. We impose an additional restriction (see Gilboa and Schmeidler (1986), Schmeidler (1986) and Shafer (1976)): (iv) $\forall\, A, B \in \Omega$, $P(A \cup B) + P(A \cap B) \geq P(A) + P(B)$. In section three of the paper we show that this amounts to saying that an individual is uncertainty averse. We do not think that it is reasonable to have the opposite behavior, that is to say, uncertainty preference.

A real valued function $X: \Omega \to \Re$ is said to be a <u>random variable</u> if for all open sets O of $\Re$, $X^{-1}(O) \in \Sigma$.

The <u>expected value</u> of a random variable X is defined as: $EX = \int_\Omega X dp = \int_{-\infty}^{0} (P(X \geq \alpha)-1)d\alpha + \int_0^\infty P(X \geq \alpha)d\alpha$, whenever these integrals exist (in the improper Riemann sense) and are finite.

The following results are true and, if not proved in one of the papers referred to (e.g. the last result), their proofs are immediate:

(i) $X \geq Y \Rightarrow EX \geq EY$;

(ii) $E(X+Y) \geqslant EX + EY$;

(iii) $-E(-X) \geqslant EX$;

(iv) $\forall\ a \geqslant 0$ and $b \in \mathbb{R}$, $E(aX + b) = aEX+b$;

(v) Let $u: \mathbb{R} \to \mathbb{R}$ be a concave non decreasing function. Then $Eu(X) \leqslant u(EX)$ (Jensen's inequality).

Other properties of the integral with respect to a non-additive probability measure (i. e. the expected value) can be found in Schmeidler (1986).

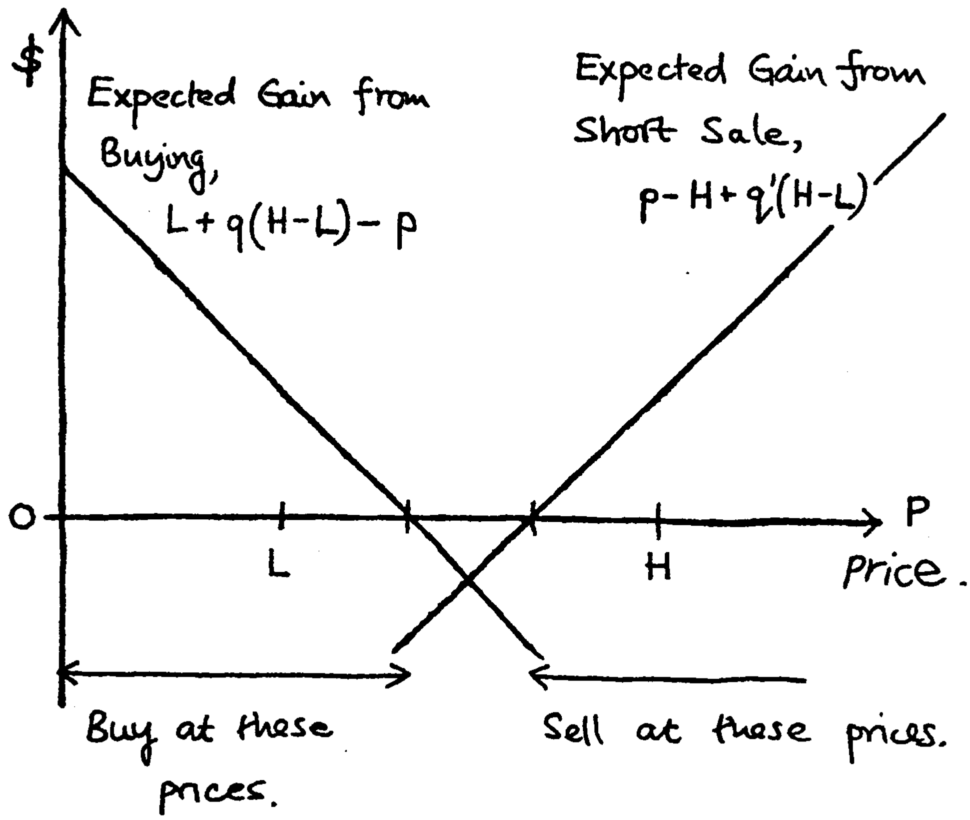

FIGURE ONE

Expected Gains from Buying and Selling Short One Unit of The Asset.

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