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COLLATERAL AVOIDS PONZI SCHEMES IN INCOMPLETE MARKETS

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Without introducing neither debt constraints nor transversality conditions to avoid the possibility of Ponzi schemes, we show existence of equilibrium in an incomplete markets economy with a collateral structure.

KEYWORDS: Exogenous Collateral, Incomplete Markets, Infinite Horizon, Equilibrium, Ponzi Schemes.

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1. INTRODUCTION

IN INCOMPLETE MARKETS ECONOMIES WITH INFINITE HORIZON, agents can make use of the so-called Ponzi schemes, which consist in the successive postponement, ad eternum, of their commitments through the successive appeal to new credit. These economies only have equilibrium if there is some mechanism that avoids the existence of those schemes. Along these lines, the existence of equilibrium has been established under either debt constraints or transversality conditions (see Araujo, Monteiro, and Páscoa (1996), Florenzano and Gourdel (1996), Hernandez and Santos (1996), Levine and Zame (1996), Magill and Quinzii (1994, 1996)).

These debt constraints or transversality conditions are added in a somewhat ad-hoc manner to the budget set. Besides, the state prices that are used as a present value process in the transversality conditions, although personalized, are chosen, jointly with the other equilibrium variables, without a clear objective criterium among the continuum of possibilities compatible with the absence of arbitrage.

In this paper we show that it is possible to eliminate the exogenous nature of both the debt constraints and the transversality condition. We study an incomplete market model with assets protected by collateral, where the lack of payment causes the seizure of the collateral by the lender. This structure of default was modelled in the two periods case in a pioneering work by Dubey, Geanakoplos, and Zame (1995).

In order to simplify, assets are real and short lived. There is a finite number of goods that are negotiated in each node of the economy. We allow for the existence of non-perishable goods, which suffer depreciation dependent on the node of the economy and on whether such goods are consumed or stored. We consider, like in the papers refered to above (except Araujo, Monteiro, and Páscoa (1996), which work with a continuum of states), countably many periods and a finite number of branches at each node. We assume that all assets are protected by collateral coefficients different from zero and that the structure of depreciation is sufficiently strong.

INSERTION IN THE LITERATURE AND CONTRIBUTION. The analysis of infinite horizon economies has been developed within two classes of models: those of overlapping generations and those of consumer with infinite life. This project extends the second class of models, although it is also possible to explore its implications for the first class. When markets are incomplete and consumers heterogeneous, new difficulties appear that did not occur in models with representative infinite-life consumers, as in Lucas (1978).

As in models with infinite horizon and complete markets, the problem of the consumer does

not have a solution if the successive postponement of the debt payment is allowed. In fact, the consumer would have all the interest in running into debt, along the time, in an increasing manner, using new credit to pay debt interests. These Ponzi schemes were avoided in the literature by the introduction of debt constrains or transversality conditions (see Blanchard and Fischer (1989, chapter 2)). Debt constraints put a uniform limit on the debt in all periods. Transversality conditions, in the deterministic case, require that the debt increases asymptotically slower than the interest rate (see Kehoe (1989), for instance). Although the approach through debts constraints is simplest, it can be considered more particular and ad-hoc.

When markets are incomplete, the choice of which deflator to use in the transversality condition becomes troublesome. In fact, there is no longer one unique vector of present values of one unit of returns in the future. The equivalent martingale measure, which serves as a deflator of asset prices in the absence of arbitrage, was unique under complete market hypothesis, but turns out to be indeterminate in the case of incomplete market. Magill and Quinzii (1994, 1996) proposed a solution that consists in adding to the budget restriction a transversality condition, requiring that the present value of an agent's debt tends asymptotically to zero, where the deflator would be personalized but determined in equilibrium. Hernandez and Santos (1996) proposed the use of the most punishing deflator among the continuum of possibilities compatible with the absence of arbitrage (or, in others words, that the budget set be defined as the intersection of all the sets that satisfy one condition of transversality for a certain equivalent martingale measure). In this way, the arbitrary imposition of a transversality condition (that must be verified in equilibrium, but that is not implied either from budgetary considerations or from consumer rationality) is strengthened by the arbitrary choice of the deflator.

In this project, we explore the structure of the model with default and exogenous collateral to show existence of equilibrium in incomplete markets with infinite horizon, without imposing, a priori, either restrictions on asset short sales or exogenous transversality conditions. The obligation of constituting collateral whenever an asset is sold must limit the asymptotic explosion of the debt, since the collateral is required to be different from zero, for all assets negotiated in the economy. A recursive argument shows that feasible short sales allocations are bounded, at each node, by the sum of aggregate depreciated endowments along the relevant path divided by the collateral coefficient at this node. However, we do not require collateral coefficients to be uniformly bounded away from zero along the entire infinite tree and therefore there is not a uniform upper bound on asset short sales that could be derived from the feasibility equations.

The collateral model used in this paper is an extension of Dubey, Geanakoplos, and Zame (1995) to an infinite tree, with a finite number of branches at each node. We show the existence

of equilibrium for the infinite horizon economy under hypotheses that place a bound on the structure of the depreciation throughout the event-tree.

METHODOLOGY AND RESULTS. To reach our objectives, we construct a model of a tree with a countable set of periods. The assets traded at each node are real and, for simplicity, give returns only at the nodes that immediately follow the one where trade took place. We suppose that the initial endowments at each node are uniformly bounded from above and that the exogenous collateral requirements are positive for all assets traded in the event-tree. Preferences are described by additive utilities in time and in states of nature. Each agent can fail in its promises, but this default gives place to the collateral seizure by the lenders. Thus each agent, whether an asset seller or buyer, turns out to have as payoff coefficient, at each state, the minimum between the value of real return and the collateral value.

We actually start by considering economies with finite horizon, where it is relatively easy to prove existence of equilibrium, since the equations of feasibility imply that aggregate assets sale are bounded from above at each node. The technique of proof follows the generalized game method, introducing an artificial auctioneer at each period and state. In this way, we extend the Dubey, Geanakoplos, and Zame (1995) results to the multi-periods case.

Then, in order to study the infinite horizon case, we analyze the equilibrium sequence associated with the truncated economies with increasing terminal periods. We prove that the sequence of marginal utilities of income, of each agent at each node, evaluated in this equilibrium, is uniformly bounded. This crucial step will allow us to get cluster points for the Lagrange multipliers. We extract cluster points, node by node, from all the equilibrium variables, using the fact that feasible short-sales have an upper bound, node by node, due to the required purchase of durable collateral. There might not exist a uniform upper bound on short sales along the infinite tree, but the countability of nodes allows us to use a diagonalization argument to extract the desired cluster points. We prove the feasibility of the vector of cluster points and then it only remains to verify individual optimality, which will follow from an argument by contradiction that uses the Kuhn-Tucker conditions of the truncated problems.

Finally, we make a remark about the welfare properties of equilibrium in the exogenous collateral economy: we prove that a constrained efficiency property is achieved, which is a result that had already been obtained by Dubey, Geanakoplos, and Zame (1995) in the case of two period models.

2. THE ECONOMY

As was mentioned in the introduction, we study a model of exogenous collateral in which the participants of the economy have the possibility of going on default in their promises.

The collateral obligations, which are owned by the borrowers, are given exogenously and are equal for all traders in the economy. The collateral bundles of commodities are distributed for consumption between borrowers and lenders, with the possibility that part of it be stored.

Due to the facts that the omission of the payments gives rise to collateral seizure by the lender, and that there doesn't exist any enforcement, such as utility loss or credit restrictions, in case of default, it follows that the borrowers deliver, at each node of the economy, the minimum between the claim and the value of collateral that is established at the moment of the negotiation.[2]

We now specify the model of uncertainty and the different structures of the economy.

2.1. *Model*

UNCERTAINTY: The model of uncertainty is essentially the one developed by Magill and Quinzii (1994, 1996). We consider an economy in which $\tau \in \mathbb{N} \cup \{\infty\}$ denotes the length of the time horizon. The set of periods is $\mathcal{T} = \{0, 1, \ldots, \tau - 1\}$ when τ is finite and $\mathcal{T} = \mathbb{N}$ when τ is infinite. Let S denotes the set of states of the nature. In the case of finite horizon, we suppose that the cardinality of S is finite.

The available information at period t in $\mathcal{T}$ is the same for every agent (symmetric information) and is given by a partition $\mathcal{F}_t$ of S, where the state of nature lies. We suppose that there is no information at $t = 0$, that is, $\mathcal{F}_0 = S$. When $\tau < \infty$, we consider $\mathcal{F}_{\tau-1} = \big\{\{s\} : s \in S\big\}$. So the information structure in the economy is given by a family: $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{\tau-1}\}$.

Our model reveals information along time, so if $t < t'$ then $\mathcal{F}_{t'}$ is finer than $\mathcal{F}_t$. The number of sets in $\mathcal{F}_t$ is finite for all t in $\mathcal{T}$.

Every pair $\xi = (t, \sigma)$, with t in $\mathcal{T}$ and σ in $\mathcal{F}_t$, is called a node of our economy. The set of all nodes, $\mathcal{D}^\tau$, is the *event-tree* induced by $\mathcal{F}$: $\mathcal{D}^\tau = \big\{(t, \sigma) \ : \ t \in T \ , \sigma \in \mathcal{F}_t\big\}$.

If $\xi = (t, \sigma)$ then $\tilde{t}(\xi) = t$ is the period associated with the node. We say that $\xi' = (t', \sigma')$ is a *successor* of $\xi = (t, \sigma)$ if $t' \geq t$ and $\sigma' \subseteq \sigma$; we use the notation $\xi' \geq \xi$. The set

[2] In their two periods model of exogenous collateral, Dubey, Geanakoplos, and Zame (1995) admit the possibility that the agents in the economy suffer a default penalty in their utility functions. Thus, in the default case, the seller delivers at least the collateral value but may choose to deliver more than this. For more details see also Dubey, Geanakoplos, and Shubik (1987). In this work we suppose, for simplicity, the absence of this type of penalties.

$\xi^+ = \{\xi' \in \mathcal{D}^\tau \; : \; \xi' \geq \xi \; , \; \tilde{t}(\xi') = \tilde{t}(\xi) + 1\}$ is the set of *immediate successors* of ξ in $\mathcal{D}^\tau$. We denote by b^τ_ξ the cardinality of ξ^+ and suppose that it is *finite* for all ξ in $\mathcal{D}^\tau$. Because $\mathcal{F}_t$ is finer than $\mathcal{F}_{t-1}$ for all t in $\mathcal{T}$, there is only one predecessor for each $\xi \in \mathcal{D}^\tau$. We denote this node by ξ^-. We denote by ξ_0 the node at $t = 0$.

Agents and Commodities: There exists a finite set of commodities, $\mathcal{L}$, at each node of the event-tree $\mathcal{D}^\tau$. So the set of goods in the economy is given by $\mathcal{D}^\tau \times \mathcal{L} = \big\{(\xi, l) \; : \xi \in \mathcal{D}^\tau \; , l \in \mathcal{L}\big\}$ and we suppose that they suffer a partial depreciation at the node branches.

The structure of depreciation in the event-tree is given by a function $Y : \mathcal{D}^\tau \times \mathcal{L} \times \{c, s\} \to \Re^{\mathcal{L}}_+$, where $(Y^c_{\xi,l})_{l'}$ denotes the amount of the good l that is obtained at the node ξ if one unit of the good l' was *consumed* at the node ξ^-. Thus, for example, if the commodity l' is perishable then $(Y^c_{\xi,l})_{l'} = 0$ for all pairs (ξ, l). Otherwise, there exist commodities such that $(Y^c_{\xi,l})_{l'} \neq 0$. This is, for instance, the case of perfectly durable goods, simply one period older goods or goods whose consumption causes its partition into many pieces. Analogously, $(Y^s_{\xi,l})_{l'}$ denotes the amount of commodity l that is obtained at the node ξ if one unit of the good l' was *stored* at the node ξ^-. It is clearly important to differentiate the depreciation in the event-tree in the case of a commodity such as tobacco, which is perfectly storable but not durable.

At each node there are spot markets for the negotiations of the commodities. Let $p = \big(p_{\xi,l}\big)$ in $\Re^{\mathcal{D}^\tau \times \mathcal{L}}_+$ be the spot price process and $p_\xi = \big(p_{\xi,l} \; : l \in \mathcal{L}\big)$ the spot price vector at the node $\xi \in \mathcal{D}^\tau$.

There exits a finite set, $\mathcal{I}$, of infinite-life agents in the economy. They demand commodities (for consumption and storage) in spots markets and negotiate assets at every node of $\mathcal{D}^\tau$.

We characterize each agent i in $\mathcal{I}$ by an endowment process $w^i = \big(w^i(\xi, l) \; : (\xi, l) \in \mathcal{D}^\tau \times \mathcal{L}\big)$ that belongs to the non-negative orthant of $\Re^{\mathcal{D}^\tau \times \mathcal{L}}$, which we denote by X^i. So the endowment of the agent i at the node $\xi \in \mathcal{D}^\tau$ is $w^i(\xi) = \big(w^i(\xi, l) \; : l \in \mathcal{L}\big) \in \Re^{\mathcal{L}}_+$.

Each agent in the economy chooses a consumption plan free of collateral $x^i = \big(x^i(\xi, l) \; : (\xi, l) \in \mathcal{D}^\tau \times \mathcal{L}\big) \in X^i$ in the event-tree [3] and a storage plan free of collateral y^i in $\tilde{X}^i = \{y \in X^i : y(\xi) = 0 \;\; \forall \xi \in \mathcal{D}^\tau, \; b^\tau_\xi = 0\}$.

The utility function $U^i : X^i \to \Re_+$ represents the preferences of the agent i.

Assets and Collateral: We work with a structure of real assets that live only one period.

Let $\mathcal{J}$ denote the set of securities in the economy and $\mathcal{J}(\xi) \subset \mathcal{J}$ the set of real assets

[3] It is not necessary to restrict the consumption bundles of the agent because the collateral structure guarantees the uniform limitation of the consumption allocations in equilibrium (see Lemma 1 and Remark 4.).

negotiated at the node ξ. So the set of assets in the event-tree is given by $\mathcal{D}^\tau(\mathcal{J}) = \{(\xi, j) \ : \xi \in \mathcal{D}^\tau \ , j \in \mathcal{J}(\xi)\}$. We suppose that the cardinality of $\mathcal{J}(\xi)$ is finite for all $\xi \in \mathcal{D}^\tau$ and $\mathcal{J}(\xi) = \emptyset$ for all *terminal node* in $\mathcal{D}^\tau$ - that is, the nodes in $\mathcal{D}^\tau$ such that $b_\xi^\tau = 0$.

The function $A : \mathcal{D}^\tau \times \mathcal{J} \to \Re_+^{\mathcal{L}}$ characterizes the promises of the asset's structure, so $A(\xi, j)$ describes the bundles yielded by the asset j at the immediate successors nodes of ξ^-. We suppose that $A(\xi, j) = 0$ if $j \notin \mathcal{J}(\xi^-)$ and, for each asset j in $\mathcal{J}(\xi)$, that there exists a node μ in ξ^+ such that $A(\mu, j)$ is different from zero, that is, there are no trivial securities in the economy.

If $j \in \mathcal{J}(\xi)$ then denote by $q_{\xi,j}$ the unit price of the asset j at the node ξ. At each node ξ of the event-tree, denote by $\theta^i(\xi, j)$ the number of units of the asset $j \in \mathcal{J}(\xi)$ bought by the agent i at the node and by $\varphi^i(\xi, j)$ the number of units of the asset sold. Let $Z^i(\xi) = \theta^i(\xi) - \varphi^i(\xi)$ be the portfolio of agent i.

In this model we suppose that, for every unit of the asset j in $\mathcal{J}(\xi)$ sold by agent i at the node ξ, he should establish a collateral $C_j^\xi \in \Re_+^{\mathcal{L}}$, which is given *exogenously* and has the purpose of protecting the buyer when the sellers don't honor their commitments.

The collateral established by the seller of the asset j at the node ξ, C_j^ξ, can be decomposed, as in Dubey, Geanakoplos, and Zame (1995), as $C_j^\xi = C_j^{W,\xi} + C_j^{B,\xi} + C_j^{L,\xi}$, where $C_j^{W,\xi}$ denotes the part that is stored, such as commodities that are fragile, or that can be easily stolen; $C_j^{B,\xi}$ is the part of the collateral that is held by the borrower, as a house or a car; and $C_j^{L,\xi}$ is the part that is held by the lender, such as a painting.

At each immediate successor ξ' of ξ, the collateral established in ξ is depreciated. In this way, the seller of asset j at the node ξ delivers at each immediate successor ξ' of ξ the amount $D_j^{\xi'} \equiv \min \left\{ p_{\xi'} A(\xi', j) \ , p_{\xi'} \left[Y_{\xi'}^c (C_j^{B,\xi} + C_j^{L,\xi}) + Y_{\xi'}^s C_j^{W,\xi} \right] \right\}$ to the asset's buyer. This means that, since we are not considering penalties, each debtor decides to deliver the minimum between his debt and the depreciated value of the collateral in this state. Similarly, each lender expects to receive only the minimum between the claim and the market value of the collateral.

To prevent the existence of securities negotiated in ξ that do not deliver returns in any states $\mu \in \xi^+$, we suppose that, for all j in $\mathcal{J}(\xi)$, if $C_j^{L,\xi} = 0$ then there is a node $\mu \in \xi^+$ such that both $A(\mu, j) > 0$ and $\left[Y_\mu^c C_j^{B,\xi} + Y_\mu^s C_j^{W,\xi} \right] > 0$.

For convenience of notations, let us define $C^\xi \equiv \left(C_j^\xi \right)_{j \in \mathcal{J}(\xi)}$, $D^\xi \equiv \left(D_j^\xi \right)_{j \in \mathcal{J}(\xi^-)}$, $Y_\xi C^{\xi^-} \equiv Y_\xi^c \left(C^{B,\xi^-} + C^{L,\xi^-} \right) + Y_\xi^s C^{W,\xi^-}$, and for all terminal node in the event-tree, $C^\xi \equiv 0$. Moreover, given a vector $z = (z_1, z_2, \ldots, z_n)$ in the euclidean space $\Re^n$, we denote by $||z||_\Sigma$ the norm of the sum of z, that is, the value of $\sum_{k=1}^n |z_k|$.

Therefore, the economy with *exogenous collateral*, $\mathcal{E}_{ex}^{\tau}$, is characterized by the event-tree $\mathcal{D}^{\tau}$, the utility functions that represent the agents' preferences $\mathcal{U} = (U^i)_{i\in\mathcal{I}}$, the agents' endowment processes $\mathcal{W} = (w^i)_{i\in\mathcal{I}}$, the assets structure $\mathcal{A} = \Big(\mathcal{J}, \big(\mathcal{J}(\xi)\big)_{\xi\in\mathcal{D}^{\tau}}, \big(A(\xi,j)\big)_{(\xi,j)\in\mathcal{D}^{\tau}\times\mathcal{J}}, C^{\xi}_{\{\xi\in\mathcal{D}^{\tau}\}}\Big)$ and the depreciation $\mathcal{Y} = \big(Y_{\xi}^{s}; Y_{\xi}^{c}\big)_{\xi\in\mathcal{D}^{\tau}}$.

2.2. *Equilibrium in* $\mathcal{E}_{ex}^{\tau}\Big(\mathcal{D}^{\tau}, \mathcal{U}, \mathcal{W}, \mathcal{A}, \mathcal{Y}\Big)$

Given $p \in \Re_{+}^{\mathcal{D}^{\tau}\times\mathcal{L}}$ a spot price process and $q \in \Re_{+}^{\mathcal{D}^{\tau}(\mathcal{J})}$ an asset price process, the agent i in $\mathcal{I}$ can choose an allocation $\big(x^i, y^i, \theta^i, \varphi^i\big)$ in the state space $\mathbb{E}^{\tau} = X^i \times \tilde{X}^i \times \Re_{+}^{\mathcal{D}^{\tau}(\mathcal{J})} \times \Re_{+}^{\mathcal{D}^{\tau}(\mathcal{J})}$, subject to the budgetary restrictions :

$$p_{\xi_0}\Big[x^i(\xi_0) + y^i(\xi_0)\Big] + p_{\xi_0} C^{\xi_0} \varphi^i(\xi_0) + q_{\xi_0} Z^i(\xi_0) \leq p_{\xi_0} w^i(\xi_0); \tag{1}$$

$$\begin{aligned} p_{\xi}\Big[x^i(\xi) + y^i(\xi)\Big] &+ p_{\xi} C^{\xi} \varphi^i(\xi) + q_{\xi} Z^i(\xi) \\ &\leq p_{\xi} w^i(\xi) + p_{\xi}\Big[Y_{\xi}^{c} x^i(\xi^-) + Y_{\xi} C^{\xi^-} \varphi^i(\xi^-) + Y_{\xi}^{s} y^i(\xi^-)\Big] + D^{\xi} Z^i(\xi^-), \end{aligned} \tag{2}$$

$\forall \xi \in \mathcal{D}^{\tau} : \xi > \xi_0$.

The agent's *process of consumption* in the event-tree is $\big(x^i(\xi) + C^{B,\xi}\varphi^i(\xi) + C^{L,\xi}\theta^i(\xi)\big)_{\xi\in\mathcal{D}^{\tau}}$ and $(\theta^i(\xi,j), \varphi^i(\xi,j))_{(\xi,j)\in\mathcal{D}^{\tau}(\mathcal{J})}$ are the asset's buying and selling processes of the agent i.

So the budget set for agent i is $\mathcal{B}_{ex}^{\tau,i}(p,q) = \big\{(x,y,\theta,\varphi) \in \mathbb{E}^{\tau}$ s.t. equations (1) and (2) hold$\big\}$.

The homogeneity of the equations (1) and (2) in (p_{ξ}, q_{ξ}) implies that this pair can be normalized. So we consider the pair in $\Delta_{+}^{\mathcal{L}+\mathcal{J}(\xi)-1} = \{v \in \Re_{+}^{\mathcal{L}+\mathcal{J}(\xi)} : \sum_k v_k = 1\}$. Therefore, the space of prices will be $\mathcal{P}^{\tau} = \big\{(p,q) = (p_{\xi}, q_{\xi})_{\xi\in\mathcal{D}^{\tau}}$ such that $(p_{\xi}, q_{\xi}) \in \Delta_{+}^{\mathcal{L}+\mathcal{J}(\xi)-1}\big\}$.

Let us define $V^i\big(x,y,\theta,\varphi\big) \equiv U^i\Big[\big(x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\big)_{b_{\xi}^{\tau}>0}; \big(x(\xi)\big)_{b_{\xi}^{\tau}=0}\Big]$.

Definition. An *equilibrium* for the economy $\mathcal{E}_{ex}^{\tau}$ is a vector $\big[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}); (\overline{p},\overline{q})\big]$ in $\big(\mathbb{E}^{\tau}\big)^{\mathcal{I}} \times \mathcal{P}^{\tau}$ with $\big(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}\big) = \big(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\big)_{i\in\mathcal{I}}$, such that:

- The allocation $(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i)$ solves:

$$\begin{aligned} &\max_{(x,y,\theta,\varphi)} && V^i\big(x,y,\theta,\varphi\big), \\ &\text{subject to} && (x,y,\theta,\varphi) \in \mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q}). \end{aligned} \tag{3}$$

- The following feasibility conditions are satisfied:

$$(4) \qquad \sum_{i\in\mathcal{I}} \left[\overline{x}^i(\xi_0) + \overline{y}^i(\xi_0) + C^{\xi_0}\overline{\varphi}^i(\xi_0)\right] = \sum_{i\in\mathcal{I}} w^i(\xi_0);$$

$$(5)\sum_{i\in\mathcal{I}} \left[\overline{x}^i(\xi) + \overline{y}^i(\xi) + C^{\xi}\overline{\varphi}^i(\xi)\right] = \sum_{i\in\mathcal{I}} \left[w^i(\xi) + Y^c_\xi\overline{x}^i(\xi^-) + Y_\xi C^{\xi^-}\overline{\varphi}^i(\xi^-) + Y^s_\xi\overline{y}^i(\xi^-)\right],$$

for all $\xi \in \mathcal{D}^T : \xi > \xi_0$.

- The pair $(\overline{\theta}, \overline{\varphi})$ satisfies:

$$(6) \qquad \sum_{i\in\mathcal{I}} \overline{\theta}^i = \sum_{i\in\mathcal{I}} \overline{\varphi}^i.$$

Remark 1. If the structure of depreciation were the same for commodities consumed or stored, that is $Y^s_\xi = Y^c_\xi$ for all ξ in $\mathcal{D}^T$, then the agents are not interested in storing commodities free of collateral. In this way, equations (4) and (5) in the equilibrium definition would be equivalent to the condition that, for each node in the economy, the total demand of commodities is equal to the total endowment accumulated until this date. That is,

$$(7) \qquad \sum_{i\in\mathcal{I}} \left[\overline{x}^i(\xi) + C^{\xi}\overline{\varphi}^i(\xi)\right] = \sum_{i\in\mathcal{I}}\sum_{k=0}^{t(\xi)} Y\big(\xi, (\xi^-)^k\big) w^i((\xi^-)^k),$$

where $(\xi^-)^k$ denotes the k-times predecessor of the node ξ, and $Y\big(\xi, (\xi^-)^{(k+1)}\big) = Y^c_\xi Y\big(\xi^-, ((\xi^-)^-)^k\big)$, with $Y(\xi, (\xi^-)^0) \equiv I$, is the accumulated depreciation factor.

Therefore, in the finite horizon case, this equivalence implies that, when securities markets are cleared (equation (6)), the consumption allocations are uniformly bounded in equilibrium. The proof of this fact, in the general case ($Y^c \neq Y^s$), is contained in the Lemma 1.

In the following section, we shall give sufficient conditions that guarantee the existence of equilibrium in the case of finite horizon. Then, we will show the existence of equilibrium in infinite horizon economies by using the existence for the finite case and a non arbitrage condition (Proposition 1) satisfied by prices in equilibrium.

3. EQUILIBRIUM EXISTENCE IN THE FINITE HORIZON CASE

The aim of this section is to prove the following theorem that characterizes the existence of equilibrium in finite horizon ($\tau < \infty$).

THEOREM 1. *For an economy* $\mathcal{E}_{ex}^{\tau}\left(\mathcal{D}^{\tau}, \mathcal{U}, \mathcal{W}, \mathcal{A}, \mathcal{Y}\right)$ *in which*

a. For all agents $i \in \mathcal{I}$, w^i *belongs to* $\Re_{++}^{\mathcal{D}^{\tau} \times \mathcal{L}}$;

b. The utility functions $U^i : X^i \to \Re_+$ *are continuous, strictly increasing, strictly quasi-concave and* $U^i(0) = 0$;

c. The collateral vector C_j^{ξ} *is different from zero, for all* (ξ, j) *in* $\mathcal{D}^{\tau}(\mathcal{J})$,

there exists an equilibrium.

Some commentaries about the hypothesis: Conditions a. and b. are classical in the finite horizon incomplete markets models. Condition c. guarantees that the requirements of collateral have nontrivial implications for the wealth of the agents.

However, it is possible to show the existence of equilibrium without imposing the third hypothesis. In fact, if $C_j^{\xi} \equiv 0$ for a node ξ and an asset $j \in \mathcal{J}(\xi)$, then the security does not deliver any return at the inmediate successors nodes $\mu \in \xi^+$, and the lenders (or the borrowers) cannot improve their utility through the consumption of the collateral bundle. Moreover, it follows from the non-arbitrage condition (Proposition 1) that, in equilibrium, the price of the asset j is zero.

Therefore the agents are indifferent to the amount of asset j negotiated.

In this way, to show existence of equilibrium, we may suppose that the asset j is not traded.

For the proof of the Theorem 1 we need some previous results.

LEMMA 1. *Under the hypotheses a. and c. in the theorem, an allocation* (x, y, θ, φ) *in* $\left(\mathbb{E}^{\tau}\right)^{\mathcal{I}}$ *that satisfies feasibility conditions of the equilibrium definition in finite horizon is bounded.*

PROOF: Let (x, y, θ, φ) be an allocation satisfying the feasibility conditions, then

$$\sum_{i \in \mathcal{I}} \left[x^i(\xi_0) + y^i(\xi_0) + \sum_{j \in \mathcal{J}(\xi_0)} C_j^{\xi_0} \varphi^i(\xi_0, j)\right] = \sum_{i \in \mathcal{I}} w^i(\xi_0). \tag{8}$$

Therefore,

$$(9)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \Big[x^i(\xi_0,l)+y^i(\xi_0,l)+\sum_{j\in\mathcal{J}(\xi_0)} C^{\xi_0}_{j,l}\varphi^i(\xi_0,j)\Big] = \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} w^i(\xi_0,l)\le W\mathcal{I},$$

where in the last inequality $W=\max_{\{\xi\in\mathcal{D}^\tau,i\in\mathcal{I}\}}||w^i(\xi)||_\Sigma$.

Let $\overline{Y}=\max\{Y^\alpha_{\xi,l,l'}:(\alpha,\xi,l,l')\in\{c,s\}\times\mathcal{D}^\tau\times\mathcal{L}\times\mathcal{L}\}$. Then it follows for the feasibility conditions that, for all $\xi\in\mathcal{D}^\tau$such that $\tilde{t}(\xi)>0$, the inequality below holds:

$$(10)\quad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)} C^{\xi}_{j,l}\varphi^i(\xi,j)\Big]$$

$$\le W\mathcal{I}+\overline{Y}\mathcal{L}\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \Big[x^i(\xi^-,l)+y^i(\xi^-,l)+\sum_{j\in\mathcal{J}(\xi^-)} C^{\xi^-}_{j,l}\varphi^i(\xi^-,j)\Big].$$

From the expression above and equation (9), we have that for ξ in $\mathcal{D}^\tau$ such that $\tilde{t}(\xi)=t$

$$(11)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} \Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)} C^{\xi}_{j,l}\varphi^i(\xi,j)\Big]\le W\mathcal{I}\sum_{k=0}^{t}(\overline{Y}\mathcal{L})^k.$$

Now, due to hypothesis c. in the theorem , we have that $m_\xi=\min_{j\in\mathcal{J}(\xi)}||C^\xi_j||_\Sigma>0$, therefore

$$(12)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} x^i(\xi,l) \;\le\; W\mathcal{I}\sum_{k=0}^{\tau}(\overline{Y}\mathcal{L})^k=\chi<\infty,$$

$$(13)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}} y^i(\xi,l) \;\le\; \chi$$

$$(14)\qquad \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(\xi)}\varphi^i(\xi,j) \;\le\; \frac{\chi}{m_\xi}=\Psi_\xi<\infty.$$

The feasibility condition in the portfolios implies that

$$(15)\qquad \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}(\xi)}\theta^i(\xi,j)\le\Psi_\xi,\ \forall\xi\in\mathcal{D}^\tau.$$

That ends the proof because $x^i,y^i,\theta^i,\varphi^i$ are positive streams. *Q.E.D.*

Remark 2.

i. In this model we don't have a uniform bounded short sales constraint in the event-tree. Nevertheless, there is an uniform borrowing constraint in equilibrium, that is, for each agent i in $\mathcal{I}$, the value of the short sales, $q_\xi\varphi^i(\xi)$, is uniformly bounded (See the discussion in Section 4).

ii. Because the securities markets feasibility conditions hold, it follows from the proof above that the aggregate consumption is uniformly bounded along the event-tree. That is,

$$\sum_{i\in\mathcal{I}}\left[x^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)}C^{B,\xi}_{j,l}\varphi^i(\xi,j)+\sum_{j\in\mathcal{J}(\xi)}C^{L,\xi}_{j,l}\theta^i(\xi,j)\right]\leq\chi. \tag{16}$$

for all (ξ,l) in $\mathcal{D}^\tau\times\mathcal{L}$. This is a direct consequence of equation (11).

The lemma proven above allows us to bound the consumption and the portfolios in the economy with the aim of proving Theorem 1.

We achieve this by establishing the existence of equilibrium in a generalized game with a finite set of utility maximizing consumers and auctioneers at each node maximizing the value of the excess demand in the markets.

Therefore, we define the generalized game $\mathcal{G}^\tau_{ex}$ in the following way:

- Given (p,q) in $\mathcal{P}^\tau$, each agent i maximizes V^i in the truncated budget set $\mathcal{B}^{\tau,i}_{ex}(p,q,2\Psi,2\chi)$, where

$$\mathcal{B}^{\tau,i}_{ex}(p,q,2\Psi,2\chi)=\left\{\begin{array}{rl}(x,y,\theta,\varphi)\in\mathcal{B}^{\tau,i}_{ex}(p,q) & :\ x(\xi,l)\leq 2\chi,\ y(\xi,l)\leq 2\chi,\ \theta(\xi,j)\leq 2\Psi_\xi\\ & \varphi(\xi,j)\leq 2\Psi_\xi\ \ \forall(l,\xi,j)\in\mathcal{L}\times\mathcal{D}^\tau\times\mathcal{J}.\end{array}\right\}.$$

- Given an allocation $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$, the auctioneer at the node ξ_0 chooses (p_{ξ_0},q_{ξ_0}) in $\Delta_+^{\mathcal{L}+\mathcal{J}(\xi_0)-1}$ in order to maximize

$$p_{\xi_0}\sum_{i\in\mathcal{I}}\left[x^i(\xi_0)+y^i(\xi_0)+C^{\xi_0}\varphi^i(\xi_0)-w^i(\xi_0)\right]+q_{\xi_0}\sum_{i\in\mathcal{I}}Z^i(\xi_0).$$

- Given an allocation $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$, the auctioneer at the node ξ, with $\xi>\xi_0$ and $b^\tau_\xi>0$, chooses $(p_\xi,q_\xi)\in\Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1}$ in order to maximize

$$p_\xi\sum_{i\in\mathcal{I}}\left[x^i(\xi)+y^i(\xi)+C^\xi\varphi^i(\xi)-w^i(\xi)-Y^c_\xi x^i(\xi^-)-Y_\xi C^{\xi^-}\varphi^i(\xi^-)-Y^s_\xi y^i(\xi^-)\right]+q_\xi\sum_{i\in\mathcal{I}}Z^i(\xi).$$

- There is one auctioneer for each terminal node of $\mathcal{D}^\tau$ such that , given $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$, his objective is to maximize

$$p_\xi\sum_{i\in\mathcal{I}}\left[x^i(\xi)-w^i(\xi)-Y^c_\xi x^i(\xi^-)-Y_\xi C^{\xi^-}\varphi^i(\xi^-)-Y^s_\xi y^i(\xi^-)\right].$$

So an equilibrium for the generalized game $\mathcal{G}_{ex}^{\tau}$ is a vector $\left[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}),(\overline{p},\overline{q})\right]$ in $\left(\mathbb{E}^{\tau}\right)^{\mathcal{I}}\times\mathcal{P}^{\tau}$ that solves the four items stated above.

LEMMA 2. *If* $\left[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}),(\overline{p},\overline{q})\right]$ *is an equilibrium for the generalized game* $\mathcal{G}_{ex}^{\tau}$ *then it is an equilibrium for the economy* $\mathcal{E}_{ex}^{\tau}$.

PROOF: See the Appendix.

In order to show that there is equilibrium in $\mathcal{E}_{ex}^{\tau}$ it is enough to guarantee that the generalized game $\mathcal{G}_{ex}^{\tau}$ has equilibrium.

LEMMA 3. *Under the hypothesis of the Theorem 1 there exists a pure strategies equilibrium for the generalized game* $\mathcal{G}_{ex}^{\tau}$.

PROOF: This result follows from the equilibrium existence theorem in a generalized game of Debreu (1952). In fact, the objective functions of the agents are continuous and quasi-concave in their strategies. Furthermore, the objective functions of the auctioneers are continuous and linear in their own strategies, and therefore quasi-concave.

The correspondence of admissible strategies, for the agents and for the auctioneers, has compact domain and compact-, convex- and nonempty-values. Such correspondences are upper semi-continuous, because it has compact values and closed graph. The lower semi-continuity of interior correspondences follows from the hypothesis a. in the Theorem 1 (see Hildenbrand (1974, ch. II.1.2)). Because the closure of a lower semi-continuity correspondence is also lower semi-continuous, the continuity of these set functions is guaranteed. We can apply Kakutani's fixed point theorem to the correspondence of optimal strategies in order to find the equilibrium. *Q.E.D.*

We have shown the equilibrium existence theorem for the economy with finite horizon $\mathcal{E}_{ex}^{\tau}$. Now, we shall give a non arbitrage condition that holds when the commodities- and assets-prices are in equilibrium. This condition has already appeared in the literature in Dubey, Geanakoplos, and Zame (1995) and is very important for our proof of the equilibrium existence theorem in the infinite horizon case.

Proposition 1. *Let* $(\overline{p},\overline{q}) \in \mathcal{P}^{\tau}$ *be a price allocation equilibrium, then for all pairs* (ξ,j) *in the set* $\mathcal{D}^{\tau}(\mathcal{J})$

$$\overline{p}_{\xi} C_j^{\xi} - \overline{q}_{\xi,j} \geq 0.$$

Moreover, if $C_j^{B,\xi} \neq 0$ *then the inequality above is strict.*

Proof: See the Appendix.

4. EXISTENCE OF EQUILIBRIA IN $\mathcal{E}_{ex}^{\infty}$

In this section we establish the existence of equilibrium in infinite horizon economies by truncating $\mathcal{E}_{ex}^{\infty}$ to finite horizon and using the results already obtained.

Given an economy $\mathcal{E}_{ex}^{\infty}(\mathcal{D}^{\infty},\mathcal{U},\mathcal{W},\mathcal{A},\mathcal{Y})$, we define the *truncated economy* $\mathcal{E}_{ex}^{T}$ as an economy in finite horizon with T+1 periods such that:

$$\begin{aligned} &\mathcal{D}^T = \Big\{\xi \in \mathcal{D}^{\infty} \,:\, \tilde{t}(\xi) \leq T\Big\}, && U^{T,i}\Big((x(\xi))_{\{\xi\in\mathcal{D}^T\}}\Big) = U^i\Big((x(\xi))_{\xi\in\mathcal{D}^T};0\Big),\\ &w^{T,i} = \Big(w^i(\xi) \,:\, \xi \in \mathcal{D}^T\Big), && \mathcal{A}^T = \mathcal{A}|_{\mathcal{D}^{T-1}},\\ &\mathcal{Y}^T = \mathcal{Y}|_{\mathcal{D}^T}. \end{aligned}$$

Now, suppose that the economy $\mathcal{E}_{ex}^{\infty}$ satisfy

A. For any agent i in $\mathcal{I}$, w^i belongs to $X_+^i = \Re_{++}^{\mathcal{D}^{\infty}\times\mathcal{L}}$ and there exists $\overline{w}$ in $\Re_{++}$, such that $||w^i(\xi)||_{\Sigma} \leq \overline{w}$ for all $\xi \in \mathcal{D}^{\infty}$;

B. The utility functions $U^i : X^i \to \Re_+ \cup \{\infty\}$ are time- and state-separable in the setting

$$U^i(x) = \sum_{\xi\in\mathcal{D}^{\infty}} u_i\big(\xi, x(\xi)\big),$$

and are *finite* for all x in $l_+^{\infty}(\mathcal{D}^{\infty}\times\mathcal{L}) \subset X^i$. The function $u_i(\xi,*) : \Re_+^{\mathcal{L}} \to \Re_+$ satisfies $u_i(\xi,0) = 0$, is continuous, strictly increasing and concave for all ξ in $\mathcal{D}^{\infty}$;

C. The collateral vector C_j^{ξ} is different from zero for all (ξ,j) in $\mathcal{D}^{\infty}(\mathcal{J})$;

D. The structure of depreciation in the event-tree is $[Y_{\xi}^c, Y_{\xi}^s] \equiv \big[diag\big[a_l(\xi)\big], diag\big[b_l(\xi)\big]\big]_{l\in\mathcal{L}}$, and there is a scalar $\kappa \in (0,1)$ such that $\max_{l\in\mathcal{L}}\{a_l(\xi), b_l(\xi)\} < \kappa$ for all ξ in $\mathcal{D}^{\infty}$.

From the former section, we know that there exists an equilibrium $\big[(x^T,y^T,\theta^T,\varphi^T);(p^T,q^T)\big]$ for the truncated economy $\mathcal{E}_{ex}^{T}$ for all $T \in I\!N$.

Remark 3. Conditions A, B and C are analogous to the finite horizon hypotheses. Condition D guarantees a strongly depreciation along the event-tree. This implies that the total endowment accumulated until each node is uniformly bounded in the event-tree.

In the subsection below we prove, using a sequence of finite horizon equilibria with increasing terminal periods: $\left[(x^T, y^T, \theta^T, \varphi^T); (p^T, q^T)\right]_{T \in I\!N}$, the main result of this work that guarantees the equilibrium existence in $\mathcal{E}_{ex}^{\infty}$. That is:

Theorem 2. *For an economy $\mathcal{E}_{ex}^{\infty}$ satisfying hypotheses A, B, C and D there exists an equilibrium.*

Remark 4. on bounds on short sales. As already mentioned in Remark 2, we do not have a uniform bounded short sales constraint in the economy $\mathcal{E}_{ex}^{\infty}$. Thus, given equilibrium prices, agents have the possibility of allocating their wealth in portfolios that are not uniformly bounded along the event-tree.

In fact, consider for instance an economy $\mathcal{E}_{ex}^{\infty}$ characterized by an event-tree $\mathcal{D}^{\infty}$ in which there are only two branches at each node, that is, $\xi^+ = \{\xi_u, \xi_d\}$ for any node ξ; only one commodity $\mathcal{L} = \{l\}$ is negotiated at each node; and having two agents $\mathcal{I} = \{A, B\}$ with endowments $w^A(\xi) = w^B(\xi) = 0.5$ and utility functions $\{U^A, U^B\}$ satisfying the statements of Theorem 2. Given ξ in $\mathcal{D}^{\infty}$, the depreciation structure is characterized by $Y_{\xi_u} = \left(Y^s_{\xi_u}, Y^c_{\xi_u}\right) = (0, 0.5)$ and $Y_{\xi_d} = \left(Y^s_{\xi_d}, Y^c_{\xi_d}\right) = (0,0)$.
At each node ξ there is only one asset j_ξ for trading, with returns given by $A(\xi_u, j_\xi) = 2^{-(\tilde{t}(\xi)+2)}$, $A(\xi_d, j_\xi) = 1$. The collateral requeriments for such an asset are $C_j^{\xi} = C_j^{B,\xi} = 2^{-\tilde{t}(\xi)}$.

By Theorem 2, there exists an equilibrium. Therefore, given equilibrium prices $(\overline{p}, \overline{q}) \in \mathcal{P}^{\infty}$, both agents can choose the portfolio $(\tilde{\theta}, \tilde{\varphi}) = (0, \tilde{\varphi})$, where

$$\tilde{\varphi}(\xi) = \frac{\overline{p}_\xi}{\overline{p}_\xi C_j^{\xi} - \overline{q}_\xi} = \frac{1}{(1 + \frac{1}{2^{\tilde{t}(\xi)}}) - \frac{1}{\overline{p}_\xi}}. \tag{17}$$

Now, because $(\overline{p}_\xi, \overline{q}_\xi)$ belongs to $\Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1}$, Proposition 1 guarantees that $\overline{p}_\xi$ tends to one as $\tilde{t}(\xi)$ tends to infinity. This implies that

$$\tilde{\varphi}(\xi) \longrightarrow \infty \;\text{ as }\; \tilde{t}(\xi) \to \infty. \tag{18}$$

Thus we have given an example in which, at equilibrium prices $(\overline{p}, \overline{q})$, the agents can choose a portfolio allocation that is not uniformly bounded in the event-tree.

However, under hypotheses A, C and D, equation (9) implies that the *feasible allocations* (x,y,θ,φ) satisfy

$$(19)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)}C^{\xi}_{j,l}\varphi^i(\xi,j)\Big]$$
$$\leq \overline{w}\mathcal{I}+\kappa\sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi^-,l)+y^i(\xi^-,l)+\sum_{j\in\mathcal{J}(\xi^-)}C^{\xi^-}_{j,l}\varphi^i(\xi^-,j)\Big],$$

for all ξ in $\mathcal{D}^\infty$. Therefore, analogous to the finite horizon case, the aggregate demand of commodities is uniformly bounded in the event-tree. That is, for any node ξ in the economy, we have

$$(20)\qquad \sum_{(l,i)\in\mathcal{L}\times\mathcal{I}}\Big[x^i(\xi,l)+y^i(\xi,l)+\sum_{j\in\mathcal{J}(\xi)}C^{\xi}_{j,l}\varphi^i(\xi,j)\Big]\leq \overline{w}\mathcal{I}\sum_{s=0}^{\infty}\kappa^s=\frac{\overline{w}\mathcal{I}}{1-\kappa}<\infty.$$

It follows from the equation above that, node by node, there is a bounded short sales constraint

$$(21)\qquad \sum_{i\in\mathcal{I}}\varphi^i(\xi,j)\leq\frac{1}{||C^{\xi}_j||_\Sigma}\frac{\overline{w}\mathcal{I}}{(1-\kappa)}<\infty.$$

If collateral coefficients were uniformly bounded from below by a strictly positive scalar, we could find a uniform upper bound on feasible allocations. But we have no need to impose such an assumption.

Finally, note that, in equilibrium, the value of short sales, $\overline{q}_\xi\overline{\varphi}^i(\xi)$, is uniformly bounded. In fact, it follows the Proposition 1 that

$$(22)\qquad \overline{q}_\xi\overline{\varphi}^i(\xi)\leq\overline{p}_\xi C^\xi\overline{\varphi}^i(\xi)\leq||C^\xi\overline{\varphi}^i(\xi)||_\Sigma\leq\frac{\overline{w}\mathcal{I}}{(1-\kappa)},$$

where the last inequality is a consequence of equation (20).

Equation (22) and the market clear condition imply that the equilibrium allocation in an economy with exogenous collateral is also an equilibrium with an *implicit debt constraint*. That is, an equilibrium in an economy where besides the collateral structure there exists an constraint in the agent's budget set of the type: $(qZ)\in l^\infty(\mathcal{D}^\infty(\mathcal{J}))$.

In a seminal paper about existence of equilibrium in incomplete markets with infinite horizon, Magill and Quinzii (1994) show that the existence of equilibrium with a implicit debt constraint guarantees the existence of equilibrium with a *explicit debt constraint* of the type: $q_\xi(\theta(\xi)-\varphi(\xi))\geq -M$ for all $\xi\in\mathcal{D}^\infty$, that *never binds*.

In our context the same conclusion holds. In fact, every equilibrium with collateral satisfy the implicit debt constraint $(q(\theta-\varphi))\in l^\infty(\mathcal{D}^\infty(\mathcal{J}))$, so it is sufficient to choose M great to $||q(\theta-\varphi)||_\infty$.

4.1. *Proof of Theorem 2*

Consider a sequence of equilibriums with increasing terminal periods: $\left[(x^T, y^T, \theta^T, \varphi^T); (p^T, q^T)\right]_{T\in\mathbb{N}}$. To shorten the inequalities below, define

$$
\begin{aligned}
(23)\qquad M_\xi^T(x,y,\theta,\varphi) &\equiv p_\xi^T x(\xi) + p_\xi^T y(\xi) + p_\xi^T C^\xi \varphi(\xi) + q_\xi^T Z(\xi),\\
(24)\qquad L_\xi^{T,i}(x,y,\theta,\varphi) &\equiv M_\xi^T(x^{T,i}, y^{T,i}, \theta^{T,i}, \varphi^{T,i}) - M_\xi^T(x,y,\theta,\varphi),\\
(25)\qquad L_\xi^{T,i} &\equiv L_\xi^{T,i}(0,0,0,0),\\
(26)\qquad S_\xi^T(x,y,\theta,\varphi) &\equiv p_\xi^T\left\{Y_\xi^c x(\xi^-) + Y_\xi C^{\xi^-}\varphi(\xi^-) + Y_\xi^s y(\xi^-)\right\} + D^\xi Z(\xi^-),\\
(27)\qquad R_\xi^{T,i}(x,y,\theta,\varphi) &\equiv S_\xi^T(x^{T,i}, y^{T,i}, \theta^{T,i}, \varphi^{T,i}) - S_\xi^T(x,y,\theta,\varphi),\\
(28)\qquad R_\xi^{T,i} &\equiv R_\xi^{T,i}(0,0,0,0),\\
(29)\qquad V^{T,i}(x,y,\theta,\varphi) &\equiv U^{T,i}\left[\left(x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\right)_{b_\xi^T>0}, \left(x(\xi)\right)_{b_\xi^T=0}\right],\\
(30)\qquad V^i(x,y,\theta,\varphi) &\equiv U^i\left[\left(x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\right)_{\xi\in\mathcal{D}^\infty}\right].
\end{aligned}
$$

The next result follows from Kuhn-Tucker's theorem and Slater's Condition for the agent's maximization problem [see Avriel (1976)]:

Lemma 4. *Given an equilibrium $\left[(x^T, y^T, \theta^T, \varphi^T), (p^T, q^T)\right]$ for the truncated economy $\mathcal{E}_{ex}^T$, there exist Lagrange's multipliers $\left(\mu_\xi^{Ti}\right)_{\{\xi\in\mathcal{D}^T\}} \in \Re_+^{\mathcal{D}^T}$ for each i in $\mathcal{I}$ such that*

$$(31)\qquad \mu_{\xi_0}^{Ti} L_{\xi_0}^{T,i} = \mu_{\xi_0}^{Ti} p_{\xi_0}^T w^i(\xi_0)\ ;$$

$$(32)\qquad \mu_\xi^{Ti}\{L_\xi^{T,i} - R_\xi^{T,i}\} = \mu_\xi^{Ti} p_\xi^T w^i(\xi) \quad \forall \xi \in \mathcal{D}^T : \xi > \xi_0.$$

Moreover, for every (x,y,θ,φ) in the state-space $\mathbb{E}^T$, we have:

$$V^{T,i}(x,y,\theta,\varphi) - V^{T,i}(x^{T,i}, y^{T,i}, \theta^{T,i}, \varphi^{T,i}) \le -\sum_{\xi\ge\xi_0} \mu_\xi^{Ti} L_\xi^{T,i}(x,y,\theta,\varphi) + \sum_{\xi>\xi_0} \mu_\xi^{Ti} R_\xi^{T,i}(x,y,\theta,\varphi).$$

Observation: In a strict sense, the Kuhn-Tucker conditions for the consumer problem in the economy $\mathcal{E}_{ex}^T$ includes the Lagrange's multipliers associated to sign restrictions ($x(\xi,l) \ge 0$, $y(\xi,l) \ge 0$, $\theta(\xi,j) \ge 0$, $\varphi(\xi,j) \ge 0$) in the budget set.

Nevertheless, this does not affect the validity of equations above because we work with vectors in the state-space $\mathbb{E}^T$.

Note that Lemma 4 holds without any hypothesis about *differentiability* of the utility functions (See Theorem 4.41 in Avriel(1976))

The following result is a direct consequence of the above lemma and its proof is in the Appendix.

LEMMA 5. *For all ξ in $\mathcal{D}^T$ such that $b_\xi^T = 0$, we have*

$$u^i(\xi, x(\xi)) - u^i(\xi, x^{T,i}(\xi)) \leq \mu_\xi^{Ti} p_\xi^T (x(\xi) - x^{T,i}(\xi)). \tag{33}$$

Moreover, if $b_\xi^T > 0$ then

$$\mu_\xi^{Ti} L_\xi^{T,i} \leq u^i\big(\xi, x^{T,i}(\xi) + C^{B,\xi}\varphi^{T,i}(\xi) + C^{L,\xi}\theta^{T,i}(\xi)\big) + \sum_{\eta \in \xi^+} \mu_\eta^{Ti} \{L_\eta^{T,i} - p_\eta^T w^i(\eta)\}. \tag{34}$$

Because $(x^{T,i}, y^{T,i}, \theta^{T,i}, \varphi^{T,i})$ is an equilibrium allocation for the agent i in $\mathcal{E}_{ex}^T$, it is bounded. Then there exists β in $\Re_{++}^{\mathcal{L}}$ such that $x^{T,i}(\xi) + C^{B,\xi}\varphi^{T,i}(\xi) + C^{L,\xi}\theta^{T,i}(\xi) \leq \beta$, for all ξ in $\mathcal{D}^T$. This bound does not depend on T or i because of the hypotheses C and D.[4] So it follows from Lemma 5 and hypothesis B that, for all ξ in $\mathcal{D}^T$,

$$\mu_\xi^{Ti} L_\xi^{T,i} \leq \sum_{\{\xi' \in \mathcal{D}^T ; \xi' \geq \xi\}} u^i(\xi', \beta). \tag{35}$$

From Lemma 4 it follows that

$$\mu_\xi^{Ti} L_\xi^{T,i} = \mu_\xi^{Ti} p_\xi^T w^i(\xi) + \mu_\xi^{Ti} R_\xi^{T,i},$$

where $\mu_\xi^{Ti} R_\xi^{T,i} \geq 0$. Thus, from hypothesis A and hypothesis B, we conclude that

$$\mu_\xi^{Ti} ||p_\xi^T||_\Sigma \min_{(l,i) \in \mathcal{L} \times \mathcal{I}} w^i(\xi, l) \leq \sum_{\{\xi' \in \mathcal{D}^\infty : \xi' \geq \xi\}} u^i(\xi', \beta). \tag{36}$$

Because the non-arbitrage condition [Proposition 1] holds, we have

$$\Big(1 + \max_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}(\xi)} C_{j,l}^\xi\Big) ||p_\xi^T||_\Sigma \geq ||p_\xi^T||_\Sigma + \sum_{j \in \mathcal{J}(\xi)} p_\xi^T C_j^\xi \geq ||p_\xi^T||_\Sigma + \sum_{j \in \mathcal{J}(\xi)} q_{\xi,j}^T = 1. \tag{37}$$

So for all ξ in $\mathcal{D}^T$ we have

$$\mu_\xi^{Ti} \leq \frac{\left(1 + \max_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}(\xi)} C_{j,l}^\xi\right) \max_{i \in \mathcal{I}} \sum_{\{\xi' \in \mathcal{D}^\infty : \xi' \geq \xi\}} u^i(\xi', \beta)}{\min_{(l,i) \in \mathcal{L} \times \mathcal{I}} w^i(\xi, l)} < \infty. \tag{38}$$

[4] This fact is a consequence of equation (20) and market clear conditions.

Observe that the latter bound depends neither on T nor on $i \in \mathcal{I}$. This proves the following:

Lemma 6. *The sequence* $\left\{\mu_\xi^{Ti},\ T \in I\!N,\ T \geq \tilde{t}(\xi),\ i \in \mathcal{I}\right\}$ *of marginal utilities of income at node* ξ *of the event tree is uniformly bounded.*

Define the set $I\!F(\xi) = \Re_+^{\mathcal{L}\mathcal{I}} \times \Re_+^{\mathcal{L}\mathcal{I}} \times \Re_+^{\mathcal{J}(\xi)\mathcal{I}} \times \Re_+^{\mathcal{J}(\xi)\mathcal{I}} \times \Delta_+^{\mathcal{L}+\mathcal{J}(\xi)-1} \times \Re_+^{\mathcal{I}}$.
It follows from equation (20), hypotheses C, D and the lemma above that the sequence

$$\left\{\left(x^T(\xi), y^T(\xi), \theta^T(\xi), \varphi^T(\xi), p_\xi^T, q_\xi^T, \mu_\xi^T\right)\right\}_{T > \tilde{t}(\xi)} \subset I\!F(\xi)$$

is uniformly bounded for each ξ in $\mathcal{D}^\infty$. Countability of $\mathcal{D}^\infty$ implies that there exists an order in its nodes $\{\xi_1, \xi_2, \ldots\}$. So we know that there is a subsequence $\{T_k^1\}_{k \in I\!N} \subset I\!N$ such that

$$\lim_{k \to \infty} \left\{\left(x^{T_k^1}(\xi_1), y^{T_k^1}(\xi_1), \theta^{T_k^1}(\xi_1), \varphi^{T_k^1}(\xi_1), p_{\xi_1}^{T_k^1}, q_{\xi_1}^{T_k^1}, \mu_{\xi_1}^{T_k^1}\right)\right\}$$

exists in $I\!F(\xi_1)$. In the same way, there exists a subsequence $\{T_k^2\}_{k \in I\!N} \subset \{T_k^1\}_{k \in I\!N}$ such that the sequence is convergent in ξ_2. Repeating this process throughout $\mathcal{D}^\infty = \{\xi_1, \xi_2, \ldots\}$ we obtain a sequence of families $\{T_k^1\}_{k \in I\!N} \supseteq \{T_k^2\}_{k \in I\!N} \supseteq \ldots$ such that

$$\lim_{k \to \infty} \left\{\left(x^{T_k^s}(\xi_s), y^{T_k^s}(\xi_s), \theta^{T_k^s}(\xi_s), \varphi^{T_k^s}(\xi_s), p_{\xi_s}^{T_k^s}, q_{\xi_s}^{T_k^s}, \mu_{\xi_s}^{T_k^s}\right)\right\}.$$

exists in $I\!F(\xi_s)$.

Define the sequence $\{T_k\}_{k \in I\!N}$ as $T_k = T_k^k$. Then, for a natural number s, we have $\{T_k\}_{k \geq s} \subseteq \{T_k^s\}_{k \in I\!N}$ and it is an infinite set. We obtain that

$$\lim_{k \to \infty} \left\{\left(x^{T_k}(\xi_s), y^{T_k}(\xi_s), \theta^{T_k}(\xi_s), \varphi^{T_k}(\xi_s), p_{\xi_s}^{T_k}, q_{\xi_s}^{T_k}, \mu_{\xi_s}^{T_k}\right)\right\}$$

exists for all s $\in I\!N$. We have found a sequence $\{T_k\}_{k \in I\!N}$ such that

$$\left\{\left(x^{T_k}, y^{T_k}, \theta^{T_k}, \varphi^{T_k}, p^{T_k}, q^{T_k}, \mu^{T_k}\right)\right\} \in I\!F,$$

where $I\!F = l_+^\infty(\mathcal{D}^\infty \times \mathcal{L} \times \mathcal{I}) \times l_+^\infty(\mathcal{D}^\infty \times \mathcal{L} \times \mathcal{I}) \times \Re_+^{\mathcal{D}^\infty(\mathcal{J}) \times \mathcal{I}} \times \Re_+^{\mathcal{D}^\infty(\mathcal{J}) \times \mathcal{I}} \times \mathcal{P}^\infty \times \Re_+^{\mathcal{D}^\infty \times \mathcal{I}}$, converges when k goes to infinity to some allocation $(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}, \overline{p}, \overline{q}, \overline{\mu}) \in I\!F$.

LEMMA 7. *The allocation* $\big((\overline{x},\overline{y},\overline{\theta},\overline{\varphi}),(\overline{p},\overline{q})\big)$ *is an equilibrium for* $\mathcal{E}_{ex}^{\infty}$.

PROOF: The feasibility conditions follow directly from the fact that $\big(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\big)$ is the limit of $\big(x^{T_k,i},y^{T_k,i},\theta^{T_k,i},\varphi^{T_k,i}\big)$ as k goes to infinity. Because $\big(x^{T_k,i},y^{T_k,i},\theta^{T_k,i},\varphi^{T_k,i}\big)$ is $\mathcal{E}_{ex}^{T_k}$-equilibrium it satisfies the feasibility conditions, and so does its limit. We have to show the optimality of $\big(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\big)$ in $\mathcal{B}_{ex}^{\infty,i}(\overline{p},\overline{q})$.

Suppose, by contradiction, that there is $\big(x,y,\theta,\varphi\big)$ in $\mathcal{B}_{ex}^{\infty,i}(\overline{p},\overline{q})$ and $\delta\in\Re_{++}$ such that

$$V^i\big(x,y,\theta,\varphi\big)-V^i\big(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\big)\geq\delta>0. \tag{39}$$

We claim that for all $\xi\in\mathcal{D}^\infty$ we have:

$$u^i\Big(\xi,x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\Big)-u^i\Big(\xi,\overline{x}^i(\xi)+C^{B,\xi}\overline{\varphi}^i(\xi)+C^{L,\xi}\overline{\theta}^i(\xi)\Big)$$
$$\leq\lim_{k\to\infty}\Big\{-\mu_\xi^{T_k,i}L_\xi^{T_k,i}(x,y,\theta,\varphi)+\sum_{\eta\in\xi^+}\mu_\eta^{T_k,i}R_\eta^{T_k,i}(x,y,\theta,\varphi)\Big\}.$$

In fact, given $\xi\in\mathcal{D}^\infty$ there exits a natural number k such that $\tilde{t}(\xi)<T_k$, then applying Lemma 4 to $\mathcal{E}_{ex}^{T_k}$ the claim follows from taking the pointwise limit as k tends to infinity.
From this inequality, given $N\in\mathbb{N}$, we have

$$\sum_{\{\xi,0\leq\tilde{t}(\xi)\leq N\}}\left(u^i\Big(\xi,x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\Big)-u^i\Big(\xi,\overline{x}^i(\xi)+C^{B,\xi}\overline{\varphi}^i(\xi)+C^{L,\xi}\overline{\theta}^i(\xi)\Big)\right)$$
$$\leq\lim_{k\to\infty}\left\{-\sum_{\{\xi,0\leq\tilde{t}(\xi)\leq N\}}\mu_\xi^{T_k,i}L_\xi^{T_k,i}(x,y,\theta,\varphi)+\sum_{\{\xi,1\leq\tilde{t}(\xi)\leq N+1\}}\mu_\xi^{T_k,i}R_\xi^{T_k,i}(x,y,\theta,\varphi)\right\}.$$

By taking the pointwise limit in equations (31) and (32) of Lemma 4, we obtain:

$$\lim_{k\to\infty}\mu_{\xi_0}^{T_k,i}L_{\xi_0}^{T_k,i}=\overline{\mu}_0^i\overline{p}_{\xi_0}w^i(\xi_0), \tag{40}$$

$$\lim_{k\to\infty}\mu_\xi^{T_k,i}\{L_\xi^{T_k,i}-R_\xi^{T_k,i}\}=\overline{\mu}_\xi^i\overline{p}_\xi w^i(\xi). \tag{41}$$

So from $(x,y,\theta,\varphi)\in\mathcal{B}_{ex}^{\infty,i}(\overline{p},\overline{q})$ it follows that:

$$\sum_{\{\xi,0\leq\tilde{t}(\xi)\leq N\}}\left(u^i\Big(\xi,x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\Big)-u^i\Big(\xi,\overline{x}^i(\xi)+C^{B,\xi}\overline{\varphi}^i(\xi)+C^{L,\xi}\overline{\theta}^i(\xi)\Big)\right)$$

$$
\begin{aligned}
&\leq \lim_{k\to\infty} \sum_{\{\xi,\tilde{t}(\xi)=N+1\}} \mu_\xi^{T_k,i} R_\xi^{T_k,i}(x,y,\theta,\varphi) \\
&\leq \lim_{k\to\infty} \sum_{\{\xi,\tilde{t}(\xi)=N+1\}} \mu_\xi^{T_k,i} R_\xi^{T_k,i} \\
&\leq \lim_{k\to\infty} \sum_{\{\xi,\tilde{t}(\xi)=N+1\}} \mu_\xi^{T_k,i} L_\xi^{T_k,i} \\
&\leq \sum_{\{\xi,\tilde{t}(\xi)\geq N+1\}} u^i(\xi,\beta),
\end{aligned}
$$

where the last inequality follows by taking the pointwise limit in the inequality (35).

Because $V^i(x,y,\theta,\varphi) = \sum_{\xi\in\mathcal{D}^\infty} u^i\Big(\xi, x(\xi) + C^{B,\xi}\varphi(\xi) + C^{L,\xi}\theta(\xi)\Big)$ there is a N^* in $I\!N$ such that

$$
\sum_{\{\xi,\tilde{t}(\xi)\leq N\}} u^i\Big(\xi, x(\xi)+C^{B,\xi}\varphi(\xi)+C^{L,\xi}\theta(\xi)\Big) - V^i\big(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\big) \geq \frac{\delta}{2}, \qquad \text{for all } N\geq N^*.
$$

Therefore,

$$
\frac{\delta}{2} \leq \sum_{\{\xi,\tilde{t}(\xi)\geq N+1\}} u^i(\xi,\beta), \qquad \text{for all } N\geq N^*. \tag{42}
$$

We obtain a contradiction with equation (39), because hypothesis B guarantees that

$$
\lim_{N\to\infty} \sum_{\{\xi,\tilde{t}(\xi)\geq N+1\}} u^i(\xi,\beta) = 0. \tag{43}
$$

We conclude that $\big(\overline{x}^i,\overline{y}^i,\overline{\varphi}^i,\overline{\theta}^i\big)$ is optimal in $\mathcal{B}_{ex}^{\infty,i}(\overline{p},\overline{q})$. *Q.E.D.*

Remark 5. on transversality conditions. In the incomplete markets literature, the existence of equilibrium in infinite horizon economies has been guaranteed through debt constraints or transversality conditions in the agent's budget set, in order to limit either the value or the amount of short sales and hence to prevent Ponzi schemes. In our model, it is not necessary to restrict the agent's budget set because the collateral structure limits by itself the assimptotic explosion of the debt.

Nevertheless, if we suppose hypotheses stronger than those in Theorem 2, we can guarantee the validity, in equilibrium, of transversality conditions like those assumed by Magill and Quinzii (1994, 1996).

In fact, if we assume that there are positive scalars $\overline{c}$, $\overline{w}$ such that $\sum_{j\in\mathcal{J}(\xi)} C^{\xi}_{j,l} \leq \overline{c}$ and $w^i(\xi,l) \geq \underline{w}$, for all (ξ,l,i) in $\mathcal{D}^{\infty}\times\mathcal{L}\times\mathcal{I}$, then

$$\lim_{T\to\infty} \sum_{\{\xi'\geq\xi\,;\,\overline{t}(\xi')=T\}} \overline{\mu}^i_{\xi'}\overline{q}_{\xi'}\overline{Z}^i(\xi') = 0 \tag{44}$$

for all ξ in $\mathcal{D}^{\infty}$.[5]

5. A SOCIAL WELFARE PROPERTY

We now present a social welfare property fulfilled by equilibrium allocations in $\mathcal{E}^{\tau}_{ex}$. The properties of welfare in models with collateral have been studied, in the case of two periods and a finite number of states of the nature, by Dubey, Geanakoplos, and Zame (1995). They show that the equilibrium allocations dominate, in the Pareto sense, all those feasible allocations that respect the budget restrictions of the agents, in all states of nature in period 1.

In particular, this shows that it is not possible to improve the social welfare by means of a governmental intervention that generates taxes and subsidizes markets in the initial period and maintains the markets cleared, at the equilibrium prices, in each state of the nature, in the second period. It is not difficult to prove an extension of this result for the economy $\mathcal{E}^{\tau}_{ex}$.

We will show that the equilibrium allocations cannot be dominated in the Pareto sense by the feasible allocations that satisfy the budget restrictions of the agents, at the equilibrium prices, at all nodes of the economy except at a node ξ where there may be transferences subsidizing or taxing agents. Thus,

PROPOSITION 3. *Given a node ξ in $\mathcal{D}^{\tau}$, the equilibrium allocation $\left(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i\right)_{i\in\mathcal{I}}$ for the economy $\mathcal{E}^{\tau}_{ex}$ dominates, in the Pareto sense, any feasible allocation $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$ that satisfies the budget restrictions of the agents at each node $\mu\neq\xi$ at the original equilibrium prices.*

PROOF: Suppose by contradiction that there is an allocation $\left(x^i,y^i,\theta^i,\varphi^i\right)_{i\in\mathcal{I}}$ that satisfies the markets clear conditions, belongs in the agent's budget set for all nodes $\mu\neq\xi$ and $V^i(x^i,y^i,\theta^i,\varphi^i) > V^i(\overline{x}^i,\overline{y}^i,\overline{\theta}^i,\overline{\varphi}^i)$ for all i in $\mathcal{I}$, then it follows from the individual optimality of the equilibrium allocation that at the node ξ,

[5] The proof follows from equation (35) and the non arbitrage condition (see Proposition 1). The proof is contained in the Appendix.

$$p_\xi\left(x^i(\xi)+y^i(\xi)+C^\xi\varphi^i(\xi)-w^i(\xi)\right)+q_\xi Z^i(\xi) > p_\xi\left[Y_\xi^c x^i(\xi^-)+Y_\xi C^{\xi^-}\varphi^i(\xi^-)+Y_\xi^s y^i(\xi^-)\right]+D^\xi Z^i(\xi^-),$$

Adding in $i \in \mathcal{I}$ we contradict the fact that $\left(x^i, y^i, \theta^i, \varphi^i\right)_{i\in\mathcal{I}}$ is feasible. *Q.E.D.*

6. CONCLUDING REMARKS

In this paper we showed that it is not necessary to impose exogenous conditions, in the form of debt constraints or transversality conditions, to avoid Ponzi schemes in incomplete markets, provided that there is a structure of collateral that protects agents in case of default. Several extensions of this analysis appear to be interesting and we intend to address some of them in the future.

As mentioned in the introduction, the analysis of infinite horizon economies may be carried out not just in the context of infinite-life consumers, but also in the context of overlapping generations. In the latter and, in particular, for the case of the simplest models where money is not introduced, it is possible to find a natural extension of our results. In fact, one only needs a structure of incomplete participation of agents in the event-tree. Theorem 1 would still hold since the finite horizon economy would still have a finite number of agents, as long as at each node the set of agents allowed to trade remains finite. Therefore, it is not hard to establish a version of Theorem 2 in this context, using the fact that both truncated equilibrium allocations and Lagrange multipliers are still uniformly bounded at each node.

However, in order to address existence of equilibria in overlapping generations models with money we would have to introduce infinitely lived assets, such as fiat money, as in the incomplete markets model without default by Santos and Woodford (1997). Our approach seems to be extendable to multiperiod assets and actually two interesting situations that were absent in the model without default would now occur and deserve special attention. One is the case where assets die as default occurs and the other is the case where there is the possibility of automatic renegotiation when assets default. Besides, it becomes particularly important to identify the secondary market as only those agents that issue assets should constitute collateral.

Another interesting line of research deals with endogenizing the collateral structure. Araujo, Orrillo, and Páscoa (2000) allowed agents to choose the collateral coefficients backing their short-sales, provided that these sellers purchase at the same time a default insurance (or equivalently, as long as a spread penalizing defaulters is deducted from the asset sale price). Existence was

established in a two-periods economy where the agents set was supposed to be a continuum (in order to overcome nonconvexities in the budget sets) under the constraint that the value of the collateral coefficients chosen should exceed the asset price by some arbitrarily small amount, common to all debtors. We intend to pursue this issue in the context of infinite horizon economies and examine whether the above protective lower bound on the difference between the value of the collateral (at the node where it is constituted) and the asset price can be allowed to vary across nodes, possibly decreasing along the infinite tree.

APPENDIX

Proof of Lemma 2. *Feasibility.* Being $\left[(\overline{x},\overline{y},\overline{\theta},\overline{\varphi}),(\overline{p},\overline{q})\right]$ an equilibrium for the generalized game, we have that

$$\overline{p}_{\xi_0}\left[\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)\right]+\overline{p}_{\xi_0}C^{\xi_0}\overline{\varphi}^i(\xi_0)+\overline{q}_{\xi_0}\left[\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\right]\leq\overline{p}_{\xi_0}w^i(\xi_0), \tag{45}$$

$$\begin{aligned}\overline{p}_{\xi}\left[\overline{x}^i(\xi)+\overline{y}^i(\xi)\right]+\overline{p}_{\xi}C^{\xi}\overline{\varphi}^i(\xi)+\overline{q}_{\xi}\overline{Z}^i(\xi)&\\ \leq\overline{p}_{\xi}w^i(\xi)+\overline{p}_{\xi}\left[Y^c_{\xi}\overline{x}^i(\xi^-)+Y_{\xi}C^{\xi^-}\overline{\varphi}^i(\xi^-)+Y^s_{\xi}\overline{y}^i(\xi^-)\right]&+D^{\xi}\overline{Z}^i(\xi^-),\end{aligned} \tag{46}$$

$\forall\xi\in\mathcal{D}^{\tau}:\xi>\xi_0$.
Adding in i gives:

$$\overline{p}_{\xi_0}\Big[\sum_{i\in\mathcal{I}}\Big(\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)-w^i(\xi_0)+C^{\xi_0}\overline{\varphi}^i(\xi_0)\Big)\Big]+\overline{q}_{\xi_0}\sum_{i\in\mathcal{I}}\Big(\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\Big)\leq 0, \tag{47}$$

$$\begin{aligned}\overline{p}_{\xi}\Big[\sum_{i\in\mathcal{I}}\left(\overline{x}^i(\xi)+\overline{y}^i(\xi)+C^{\xi}\overline{\varphi}^i(\xi)\right)\Big]+\overline{q}_{\xi}\sum_{i\in\mathcal{I}}\overline{Z}^i(\xi)&\\ \leq\overline{p}_{\xi}\left[\sum_{i\in\mathcal{I}}\left[w^i(\xi)+Y^c_{\xi}\overline{x}^i(\xi^-)+Y_{\xi}C^{\xi^-}\overline{\varphi}^i(\xi^-)+Y^s_{\xi}\overline{y}^i(\xi^-)\right]\right]&+D^{\xi}\sum_{i\in\mathcal{I}}\overline{Z}^i(\xi^-),\end{aligned} \tag{48}$$

$\forall\xi\in\mathcal{D}^{\tau}:\xi>\xi_0$.

From the fact that $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ solves the auctioneer's problem, we have

$$\sum_{i\in\mathcal{I}}\Big(\overline{x}^i(\xi_0)+\overline{y}^i(\xi_0)-w^i(\xi_0)+\sum_{j\in\mathcal{J}(\xi_0)}C^{\xi_0}_j\overline{\varphi}^i(\xi_0,j)\Big)\leq 0, \tag{49}$$

$$\sum_{i\in\mathcal{I}}\Big(\overline{\theta}^i(\xi_0)-\overline{\varphi}^i(\xi_0)\Big)\leq 0. \tag{50}$$

Given $\xi \in (\xi_0)^+$, equations (50), (48) and the fact that $(\overline{p}_\xi, \overline{q}_\xi)$ solves the auctioneer's problem at the node ξ imply that

$$\sum_{i\in\mathcal{I}} \Big(\overline{x}^i(\xi) + \overline{y}^i(\xi) - w^i(\xi) + C^\xi \overline{\varphi}^i(\xi)\Big) \leq \sum_{i\in\mathcal{I}} \Big(Y^c_\xi \overline{x}^i(\xi_0) + Y_\xi C^{\xi_0} \overline{\varphi}^i(\xi_0) + Y^s_\xi \overline{y}^i(\xi_0)\Big), \tag{51}$$

$$\sum_{i\in\mathcal{I}} \Big(\overline{\theta}^i(\xi) - \overline{\varphi}^i(\xi)\Big) \leq 0. \tag{52}$$

Repeating the arguments made at the nodes $\xi \in (\xi_0)^+$, we have that inequalities (51) and (52) hold for the nodes with period $t = 2$. In fact, with the same argument across time shows that inequalities (51) and (52) hold for all successors ξ from ξ_0 in $\mathcal{D}^\tau$.

Equations (49), (50), (51) and (52) imply that $\overline{x}^i(\xi, l) \leq \chi$, $\overline{y}^i(\xi, l) \leq \chi$, $\overline{\varphi}^i(\xi, l) \leq \Psi_\xi$ and $\overline{\theta}^i(\xi, l) \leq \Psi_\xi$ for all nodes ξ in $\mathcal{D}^\tau$.

Then, it follows from the fact that the allocation $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ belongs to $\mathcal{B}^{\tau,i}_{ex}(\overline{p}, \overline{q}, 2\Psi, 2\chi)$ and the monotonicity of the utility functions that the inequalities (45), (46), (47) and (48) are in fact equalities.

We claim that inequality (49) is in fact an equality too. To prove this, suppose that there exists $l \in \mathcal{L}$ such that inequality (49) is strict

$$\sum_{i\in\mathcal{I}} \left[\overline{x}^i(\xi_0, l) + \overline{y}^i(\xi_0, l) - w^i(\xi_0, l) + \sum_{j\in\mathcal{J}(\xi_0)} C^{\xi_0}_{j,l} \overline{\varphi}^i(\xi_0, j)\right] < 0, \tag{53}$$

then $\overline{p}(\xi_0, l) = 0$, implying that at the node ξ_0 the agents' consumption of the commodity l is the maximum available because it represents no cost and the preferences are monotone. In this situation $\overline{x}^i(\xi_0, l) = 2\chi$, which contradicts the consumption bounds already obtained. So, for all $l \in \mathcal{L}$

$$\sum_{i\in\mathcal{I}} \left[\overline{x}^i(\xi_0, l) + \overline{y}^i(\xi_0, l) - w^i(\xi_0, l) + \sum_{j\in\mathcal{J}(\xi_0)} C^{\xi_0}_{j,l} \overline{\varphi}^i(\xi_0, j)\right] = 0. \tag{54}$$

We affirm that inequality (50) is an equality too. Suppose that there exists an asset $j \in \mathcal{J}(\xi_0)$ such that it is a strict inequality. Then the price of this asset is zero, i.e $\overline{q}(\xi_0, j) = 0$. Therefore the agent is motivated to buy the greatest amount possible of units of this asset, so $\overline{\theta}^i(\xi_0, l) = 2\Psi_{\xi_0}$, which contradicts the bounds already obtained. We have shown that inequalities (49) and (50) are in fact equalities. At ξ_0 the feasibility conditions hold.

Let's consider a node ξ in $(\xi_0)^+$. By arguments analogous to the one made for the node ξ_0,we show that the equality holds for inequalities (51) and (52). By applying these results to the trees's nodes with period $t = 2$ and repeating the process along time, we have that the feasibility conditions hold for $\Big[(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}), (\overline{p}, \overline{q})\Big]$.

Optimality.

We want to prove that $\left(\overline{x}, \overline{y}, \overline{\theta}, \overline{\varphi}\right)$ solves the consumer's problem in $\mathcal{E}^{\tau}_{ex}$ at prices $(\overline{p}, \overline{q})$. This means $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ is solution of:

$$\begin{array}{ll} \max_{(x,y,\theta,\varphi)} & V^i(x,y,\theta,\varphi) \\ \text{subject to} & (x,y,\theta,\varphi) \in \mathcal{B}^{\tau,i}_{ex}. \end{array} \tag{55}$$

We know that $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ solves:

$$\begin{array}{ll} \max_{(x,y,\theta,\varphi)} & V^i(x,y,\theta,\varphi) \\ \text{subject to} & (x,y,\theta,\varphi) \in \mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q},2\Psi,2\chi) \subset \mathcal{B}^{\tau,i}_{ex}. \end{array} \tag{56}$$

Suppose that $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ does not solve (55). Then there exists $(x,y,\theta,\varphi) \in \mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q})$ such that:

$$V^i(x,y,\theta,\varphi) > V^i(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i). \tag{57}$$

Because $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ satisfies the feasibility conditions, $\overline{x}^i(\xi,l) \leq \chi$, $\overline{y}^i(\xi,l) \leq \chi$, $\overline{\varphi}^i(\xi,j) \leq \Psi_\xi$ and $\overline{\theta}^i(\xi,j) \leq \Psi_\xi$ for all (l,ξ,j) in $\mathcal{L} \times \mathcal{D}^\tau \times \mathcal{J}$, it is an interior point of $\mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q},2\chi,2\Psi)$. Therefore, due to the finite number of nodes in our tree, there is $\lambda \in (0,1)$, λ near zero, such that:

$$(1-\lambda)(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i) + \lambda(x,y,\theta,\varphi) \in \mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q},2\chi,2\Psi), \tag{58}$$

and from U^i strict quasi-concavity:

$$V^i\left[(1-\lambda)(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i) + \lambda(x,y,\theta,\varphi)\right] > V^i\left[\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right]. \tag{59}$$

This contradicts the fact that $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ solves the problem (56). *Q.E.D.*

Proof of Proposition 1. Let $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ be the equilibrium's allocation associated to equilibrium prices $(\overline{p}, \overline{q})$ of the agent i. Suppose, by contradiction, that there is a node ξ' and $j' \in \mathcal{J}(\xi')$ such that $\overline{p}(\xi')C^{\xi'}_{j'} - \overline{q}_{\xi',j'} < 0$.

Let $l' \in \mathcal{L}$, $y = \overline{y}^i$, $\theta = \overline{\theta}^i$, $\varphi(\xi,j) = \overline{\varphi}^i(\xi,j)$, for all $(\xi,j) \neq (\xi',j')$, and $x(\xi,l) = \overline{x}^i(\xi,l)$ for all $(\xi,l) \neq (\xi',l')$.

Because the equilibrium price $\overline{p}_\xi$ is strictly positive, we can take

$$\varphi(\xi',j') \quad > \quad \overline{\varphi}^i(\xi',j'), \tag{60}$$

$$x(\xi',l') \quad = \quad \overline{x}^i(\xi',l') + \frac{\overline{p}_{\xi'}C^{\xi'}_{j'} - \overline{q}_{\xi',j'}}{\overline{p}_{\xi',l'}}\left(\overline{\varphi}^i(\xi',j') - \varphi(\xi',j')\right) > \overline{x}^i(\xi',l'). \tag{61}$$

Then because of the strict monotonicity of the utility function of the agent i, we can find an allocation $\left(x,y,\theta,\varphi\right)$ that improves utility at prices $(\overline{p},\overline{q})$ and is in $\mathcal{B}^{\tau,i}_{ex}(\overline{p},\overline{q})$. This contradicts the fact that $\left(\overline{x}^i, \overline{y}^i, \overline{\theta}^i, \overline{\varphi}^i\right)$ is optimal.

In the case where $C_{j'}^{B,\xi'} \neq 0$, if $\overline{p}_{\xi'} C_{j'}^{\xi'} - \overline{q}_{\xi',j'} = 0$ then we define $\varphi(\xi',j') > \overline{\varphi}^i(\xi',j')$ and keep the other allocations. Therefore, we can find an allocation $\big(x,y,\theta,\varphi\big)$ in $\mathcal{B}_{ex}^{\tau,i}(\overline{p},\overline{q})$ that improves utility. *Q.E.D.*

PROOF OF CLAIM IN REMARK 5 : Given ξ in $\mathcal{D}^\infty$ and $K \in \mathbb{N}$, it follows from the equation (35) that

$$\sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{Z}^i(\xi') = \lim_{T\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \mu_{\xi'}^{Ti} q_{\xi'}^T Z^{T,i}(\xi'), \tag{62}$$

$$\leq \lim_{T\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \mu_{\xi'}^{Ti} L_{\xi'}^{T,i}, \tag{63}$$

$$\leq \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')\geq K\}} u^i(\xi',\beta). \tag{64}$$

Taking the limit as K goes to infinity in the last inequalities, we conclude that

$$\lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{Z}^i(\xi') \leq 0. \tag{65}$$

Now, because $\sum_{j\in\mathcal{J}(\xi)} C_{j,l}^{\xi} \leq \overline{c}$ and $w^i(\xi,l) \geq \underline{w}$, it follows from equations (36), (37) and the non arbitrage condition [Proposition 1] that

$$\lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{\varphi}^i(\xi') \leq \lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{p}_{\xi'} C^{\xi'} \overline{\varphi}^i(\xi'), \tag{66}$$

$$\leq \lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i ||C^{\xi'} \overline{\varphi}^i(\xi')||_\Sigma, \tag{67}$$

$$\leq \frac{\overline{w}\mathcal{I}}{(1-\kappa)} \lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i, \tag{68}$$

$$\leq \frac{\overline{w}\mathcal{I}}{(1-\kappa)} \frac{1+\overline{c}}{\underline{w}} \lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')\geq K\}} u^i(\xi',\beta) = 0. \tag{69}$$

Finally, we have

$$0 \leq -\lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{\varphi}^i(\xi') \leq \lim_{K\to\infty} \sum_{\{\xi'\geq\xi:\tilde{t}(\xi')=K\}} \overline{\mu}_{\xi'}^i \overline{q}_{\xi'} \overline{Z}^i(\xi') \leq 0. \tag{70}$$

Q.E.D.

PROOF OF LEMMA 5. Let ξ' be such that $b^T_{\xi'} = 0$. Define $\theta(\xi) = \theta^{T,i}(\xi)$, $\varphi(\xi) = \varphi^{T,i}(\xi)$, $y(\xi) = y^{T,i}(\xi)$ for all $\xi \in \mathcal{D}^T$ and $x(\xi) = x^{T,i}(\xi)$ for all $\xi \in \mathcal{D}^T,\ \xi \neq \xi'$. It follows from item ii. in Lemma 4 that

$$u^i(\xi', x(\xi')) - u^i(\xi', x^{T,i}(\xi')) \leq \mu^{Ti}_{\xi'} p^T_{\xi'}(x(\xi') - x^{T,i}(\xi')). \tag{71}$$

In the case where $b^T_\xi > 0$, remember that $q^T_\mu \equiv 0$ and $C^\mu \equiv 0$ for all $\mu \in \mathcal{D}^T$ such that $b^T_\mu = 0$, and consider the allocation (x, y, θ, φ) in $\mathbb{E}^T$ given by

$$x(\xi) = \begin{cases} x^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{72}$$

$$y(\xi) = \begin{cases} y^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{73}$$

$$\theta(\xi) = \begin{cases} \theta^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{74}$$

$$\varphi(\xi) = \begin{cases} \varphi^{T,i}(\xi) & \xi \in \mathcal{D}^T, \xi \neq \xi', \\ 0 & \xi = \xi'. \end{cases} \tag{75}$$

It follows from Lemma 4 and hypothesis B that

$$-u^i\left(\xi', x^{T,i}(\xi') + C^{B,\xi'}\varphi^{T,i}(\xi') + C^{L,\xi'}\theta^{T,i}(\xi')\right) \leq -\mu^{Ti}_{\xi'} L^{T,i}_{\xi'} + \sum_{\eta \in (\xi')^+} \mu^{Ti}_\eta R^{T,i}_\eta. \tag{76}$$

Applying equations (31) and (32) we obtain the result. *Q.E.D.*

REFERENCES

ARAUJO, A., P.K.MONTEIRO, AND M.R. PÁSCOA (1996): "Infinite Horizon Incomplete Markets," *Mathematical Finance*, 6, 119-132.

ARAUJO, A., J. ORRILLO, AND M.R. PÁSCOA (2000): "Equilibrium with Default and Endogenous Collateral," *Mathematical Finance*, 10, No. 1.

AVRIEL, M. (1976): *Nonlinear Programming - Analysis and Methods,* Prentice-Hall, Inc.

BLANCHARD, O., AND S. FISCHER (1989): *Lectures in Macroeconomics,* MIT Press.

DEBREU, G. (1952): "A Social Equilibrium Existence Theorem," *Proceedings of The National Academy of Sciences*, 38, 886-893.

DUBEY, PR., J. GEANAKOPLOS, AND W.R. ZAME (1995): "Default, Collateral and Derivatives," Yale University, mimeo.

DUBEY, PR., J. GEANAKOPLOS, AND M. SHUBIK (1987): "The Revelation of Information in Strategic Market Games: A Critique of Rational Expectation Equilibrium," *Journal of Mathematical Economics*, 16, 105-137.

FLORENZANO, M., AND P. GOURDEL (1996): "Incomplete Markets in Infinite Horizon: Debt Constraints versus Node Prices," *Mathematical Finance*, 6, 167-196.

HERNANDEZ, A., AND M. SANTOS (1996): "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102-130.

HILDENBRAND, W. (1974): *Core and Equilibria of a Large Economy,* Princeton University Press. Princeton. New Jersey.

KEHOE, T. (1989): "Intertemporal General Equilibrium Models," in *The Economics of Missing Markets, Information and Games,* ed. by F. Hahn. Oxford University Press.

LEVINE, D.K., AND W.R. ZAME (1996): "Debts Constraints and Equilibrium in Infinite Horizon Economies with Incomplete Markets," *Journal of Mathematical Economics*, 26, 103-131.

LUCAS, R. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

MAGILL, M., AND M. QUINZII (1994): "Infinite Horizon Incomplete Markets," *Econometrica*, 62, 853-880.

——————(1996): "Incomplete Markets over an Infinite Horizon: Long-lived Securities and Speculative Bubbles," *Journal of Mathematical Economics*, 26, 133-170.

SANTOS, M., AND M. WOODFORD (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19-57.