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On choice of technique in the Robinson-Solow-Srinivasan model

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Abstract: We report results on the optimal “choice of technique” in a model originally formulated by Robinson, Solow and Srinivasan (henceforth, the RSS model) and further discussed by Okishio and Stiglitz. By viewing this vintage-capital model without discounting as a specific instance of the general theory of intertemporal resource allocation associated with Brock, Gale and McKenzie, we resolve long-standing conjectures in the form of theorems on the existence and price support of optimal paths, and of conditions sufficient for the optimality of a policy first identified by Stiglitz. We dispose of the necessity of these conditions in surprisingly simple examples of economies in which (i) an optimal path is periodic, (ii) a path following Stiglitz’ policy is bad, and (iii) there is optimal investment in different vintages at different times. (129 words)

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1 Introduction

In the late sixties and early seventies, under the general heading of “technical choice under full employment in a socialist economy,” Robinson (1960, 1969), Okishio (1966) and Stiglitz (1968, 1970, 1973) studied the problem of optimal economic growth in a model of an economy originally formulated by Robinson (1960, pp. 38-56), Solow (1962b) and Srinivasan (1962a) (henceforth, the RSS model).¹ The work generated controversy. Stiglitz argued, with justification, that the Robinson-Okishio assumption of a fixed labor allocation between the consumption and investment sectors had no place in an exercise that sought to determine the optimal growth path, and thereby an optimum time-path of the allocation of labor.² He identified a development policy, henceforth Stiglitz’ policy,³ under which there is investment only in the type of machine σ that minimizes effective labor costs and simultaneously maximizes the steady state consumption, and a utilization of only those types of machines whose output per man ratios are higher than the effective labor cost of producing σ . Stiglitz observed that the “number of workers working in the consumption-goods sector increases monotonically (capital ‘widening’ occurs in a smooth way), output of consumption goods need not be monotonically increasing”,⁴ and prescribed for the economy at any point in time an optimal choice of techniques, both to use and to produce, and thereby the (instantaneous) optimal levels of technological obsolescence – prescriptions which are all independent of the felicity function. Robinson commented on Stiglitz’ solution by criticizing his assumption of a fixed positive discount rate, continuous time and the linearity assumption in the specification of the planner’s felicity function.⁵

Robinson’s objections were explicitly acknowledged by Stiglitz,⁶ and as a first approximate step,⁷ he extended his earlier analysis to the case of a minimum consumption constraint in a setting with continuous time and a positive rate of discount. However, he emphasized that the important modifications concerned transition, rather than long-run, dynamics.⁸

Even if there is a minimum consumption constraint and a finite gestation period, the path of development will, after an initial “adjustment” period, look exactly as I have described it. [Unlike] long-run neoclassical models with malleable capital [where] the optimal policy

¹In Khan (2000, p.3), the model is referred to as the Solow-Srinivasan model; also see Solow (1962, concluding three paragraphs) and Khan (2000, Footnote 12) for the way it is seen in earlier work.

²See (1968, Paragraph 1; 1970, p. 421). Stiglitz (1970, p. 421) writes “There may be some special situations ... where the employment allocation is the same for all steady-state paths, but even then, in going from one steady-state path to another, one cannot infer that the employment allocation is unchanged – and it is this dynamic problem that we are discussing.”

³This is formalized in Definition 8 below. As we shall see in the sequel, Stiglitz’ policy can be usefully compared to Faustmann’s solution to the forestry problem, as formalized in Mitra-Wan (1986).

⁴See Stiglitz (1973, pp. 143-144). In the discussion of his policy, Stiglitz also drew attention to preliminary investigations of Bruno (1967).

⁵We restate Robinson’s criticisms in our own terminology; she phrases them in terms of a “discount rate chosen once and for all, ... negligible gestation periods, ...[and] ceasing to consume and living on air during the first phase of the plan.”

⁶See Stiglitz (1973, p. 144) and also Cass-Stiglitz (1969, p. 614).

⁷Thus Cass-Stiglitz (1969, Footnote 19) saw the instantaneous utility function $U(C) = -\infty$ for $C < \bar{C}$ and $U(C) = C$ for $C \geq \bar{C}$, \bar{C} a minimum consumption level, as “one approximation to the general instantaneous utility function satisfying $U'(C) > 0$ with $\lim_{C \rightarrow 0} U'(C) = \infty$ and $U''(C) \leq 0$.”

⁸Stiglitz’s (1970) response is important for the record; this essay can also be seen as a further investigation into the substance of this response.

is always of the so-called bang-bang variety – if the initial capital labor ratio is less than its long-run equilibrium value there is always a period of zero consumption, after which consumption jumps to its long-run equilibrium value, whereas in our *ex-post* fixed coefficients model consumption increases steadily to its long-run value.

Given the primary interest in the undiscounted case, Stiglitz interpreted the undiscounted case as a situation when the discount rate is “negligible”; he developed the intuitive ideas in discrete-time and then chose to translate them to the continuous-time framework.⁹

In his recent revisit of Srinivasan (1962a), Solow (2000, p. 7) asks for a solution to the “Ramsey problem for this model.” Since Stiglitz had already provided a solution with a “linear utility and positive time preference”, the open questions concern a rigorous treatment of the undiscounted case and of the discounted case with a “strictly concave social utility function for current per capita consumption.” Like Robinson, Solow also mentions that an “adoption of this [linear utility] criterion can indeed lead to unjustifiable neglect of early consumption,” and if one was to share “Ramsey’s belief that the only ethically defensible social rate of time preference is zero, a sufficiently sharply-concave utility function would enforce a closer approach to intergenerational equality.” In short, Solow’s question remains unanswered, and the generalization of Stiglitz’s work in the directions it prompts remains yet to be accomplished.¹⁰

In this paper, we address this general question, and most importantly from a methodological point of view, do so in the setting of the modern theory of optimal intertemporal allocation initiated originally by Ramsey and von Neumann and brought to completion at the hands of Brock, Gale and McKenzie.¹¹ Since this theory was being developed at the same time as the “capital controversy” between the two Cambridges,¹² it has not been brought to bear on the fundamental issues. Such an application has the advantage of illustrating the power and flexibility of the modern theory: it is ideally suited to deal with this problem, and the general results of this theory can be readily applied to this particular context, using extremely elementary methods. As such it is perhaps overdue. A secondary benefit of this application concerns the theory itself; it offers insights into its scope and suggests directions along which it may find fruitful extension.¹³ It also points clearly to issues to which the general theory has (and by its very nature, will have) very little to offer, thereby indicating that even after much theoretical progress has been made, some of the questions that were asked fifty years ago about the appropriate choice of technique remain hard unanswered problems that need to be approached case by case.

In terms of the specifics of this paper, we truly treat the Ramsey problem; that is, consider a

⁹See Footnotes 1 and 3 on page 608 and the discussion of the “correct” pricing system on page 606 in Stiglitz (1968).

¹⁰For some partial attempts at solution, and for a numerical example, see Stiglitz (1973); also see Stiglitz (1968, Footnote 2, p. 608) and Cass-Stiglitz (1969). However, in Khan (2000, p. 15), the situation is expressed as “The loose end remains loose.”

¹¹The relevant papers are Gale (1967), McKenzie (1968) and Brock (1970). In the sequel, when we refer to the “general theory of optimal intertemporal allocation”, we shall be having these papers in mind.

¹²It is of course not our intention to revisit this debate here – the interested reader may want to see Birner (2002) and his references.

¹³Thus, a distinction has to be drawn between applying a theorem from applying its methods of proof. As the reader will see in the sequel, the hypotheses of Brock’s existence theorem and of McKenzie’s price-support property are not literally fulfilled by the RSS model but their methods of proof *do* apply.

formulation in which there is no discounting of future utilities, and thus no appeal to the assumption of structural stability of the model at the zero discount rate, an assumption at best roundabout and at worst dubious. We are by this time very familiar with the overtaking criterion of Atsumi (1965) and von Weizsäcker (1965), and under this criterion, an optimal path in the undiscounted case can be shown to exist and its properties can be rigorously studied. Our treatment of time is discrete: the general theory of intertemporal allocation is developed in the simplest and most elegant way in such a setting (see McKenzie (1986) for a masterly presentation), and we can work with a reduced form of the RSS model¹⁴ in which the technological possibilities are given by a transition possibility set, and the objective function by a (reduced-form) utility function defined on this set (that is, defined on beginning and end of period capital stock vectors). We appeal to the methods of Brock (1970) and McKenzie (1968) to show the existence of an optimal program, and furthermore, to establish that, starting from an arbitrary initial stock, it converges asymptotically to a subset of the transition set, the so-called von Neumann facet, consisting of all plans which have “zero value-loss” at the golden-rule support prices. In the case of a strictly concave felicity function, the von Neumann facet shrinks to a point, and so we have asymptotic convergence to the golden-rule stock. These results furnish a complete resolution of the problem of the long-run choice of technique and thereby illustrate the power and elegance of the general theory.

It is important to appreciate the methodological significance of this reformulation of the RSS model. The standard treatment is in terms of Pontryagin’s principle,¹⁵ and in the application of this principle, as in the work of Stiglitz (1968), Sen (1968) and others, one appeals to the transversality conditions in the study of the differential equations pertaining to the state and auxiliary variables obtained by substituting the values of the controls that maximize the instantaneous Hamiltonian.¹⁶ Thus the relevance of the rest points is established only towards the end of the analysis. Here, we begin with the rest points, the golden-rule stock and the golden-rule prices, and use the value-loss function and the so-called average turnpike property of good programs to yield the optimal program.¹⁷ Stiglitz investigates the convergence (turnpike property) of a path that follows his (derived) policy prescriptions as to the choice of techniques (referred to earlier and formalized as a Stiglitz’ program in Definition 8 below),¹⁸ while we need to investigate whether the optimal path and its turnpike property is sustained by these prescriptions. To anticipate, this cannot be established, in general, for either a linear or a strictly concave felicity function, but only in special (identified) cases of either formulation. Thus,

¹⁴We choose to work with the version presented in Stiglitz (1968) rather than that in Solow (1962b) or Srinivasan (1962a). All of these variants can be viewed as special cases of the models considered in Bruno (1967) or the more general treatment in Koopmans (1971) and Koopmans-Hansen (1972). We leave the analysis of these papers as a task for future research; also see the third paragraph of the concluding Section 12 below.

¹⁵Thus Dixit (1990, p. 5) writes, “Nowadays Hamiltonians and phase diagrams are everyday stuff for the typical second-year graduate student”, and quotes Frank Hahn’s reference to the “unseemly haste to get down to the Hamiltonian.”

¹⁶In the context of Stiglitz (1968), see his Footnote 2 both on page 605 and on page 607.

¹⁷As the reader will see below, we appeal to McKenzie’s (1983, 1986) price-support property only to establish our final result pertaining to transition dynamics in the case of a strictly concave felicity function (Theorem 7 below).

¹⁸As alluded to in Footnote 3, we see the Stiglitz policy as the analogue of the Faustmann solution in the economics of forestry, and it would be interesting to pursue the analytics of this analogy; see the fourth paragraph of the concluding remarks in Section 12 below.

through the introduction of a new conceptual vocabulary, a difficult step in one perspective is rendered straightforward in another.¹⁹

In terms of substantive results, our methods yield surprises even for the case of a linear felicity function. We present simple examples of an economy consisting of only a single type of machine and thereby posing no issue as to the choice of technique in production,²⁰ in which consumption and capital stock exhibit a period-two cycle along an optimal program or a period-four cycle along a Stiglitz program which is thereby shown to be bad.²¹ The first shows the optimality of periodic over-building and over-consuming relative to the golden-rule levels, and the second, the non-optimality of a no excess capacity policy – phenomena that Stiglitz (1968) apparently does not encounter in his continuous-time formulation of the model.²² In the case of a general concave felicity function, we establish the non-optimality of a Stiglitz policy – an affirmative answer to the question as to whether there is a compelling reason to ever produce a machine (other than the golden-rule machine) which we know would eventually be depreciated to zero? We present a concrete example of an economy consisting of two types of machines in which this phenomenon can occur. Stiglitz (1973, pp. 146-148) had earlier presented an example of a four machine economy where such a phenomenon can occur, but it is with discounting and a minimum consumption constraint.²³

All of these exhibit in a dramatic way both the strength and the weakness of the general theory of intertemporal allocation alluded to earlier, and thereby reveal exactly why the choice of an appropriate technique is such a difficult and multi-faceted problem. Since we have a complete resolution of the problem of the long-run choice of technique, the natural question arises as to the choice of technique in *transition* to the steady state, the issue of transition dynamics: a determination of the type and amounts of machines that are produced and used in the short run. Unfortunately, it is on this hard problem that the general theory has little to offer, and it is fair to say that the literature has no concrete results of any generality on this issue.²⁴ However, we can offer a (novel) set of sufficient conditions under which the Stiglitz’ policy is indeed optimal for an economy with a general concave felicity function, and *a fortiori*, for an economy with a linear utility function and a minimum consumption constraint.

These sufficient conditions, in pointing to an interesting distinction between choice of technique that is appropriate in the short-run from that which is appropriate in the long-run, also connects to the

¹⁹Stiglitz (1968, Footnote 2, p. 608) recognizes that the results for the linear utility function might not carry over to the general concave case, and in a subsequent analysis of the problem, one with a minimum consumption constraint, he refers to the difficulty of showing that there is only one type of machine that maximizes $p(x, t)$, see Stiglitz (1973; p. 145). He also refers in this connection to Bliss (1968) and Cass-Stiglitz (1969). For this scepticism concerning the linear case, also see Solow (2000) in addition to Robinson (1969).

²⁰The question of the choice of technique in terms of use of course remains: should all of the stocks of the machine be utilized until production is undertaken?

²¹As is well-known to students of the general theory of intertemporal resource allocation associated with Brock, Gale and McKenzie, this is a technical term; see Definition 6 below.

²²Part of the reason for this undoubtedly lies in the procedure of applying results for the discounted case by setting the discount rate to be zero (negligible). However, it is worth re-emphasizing in this connection that Brock’s 1970 results were not available to Stiglitz in 1968.

²³In this work, Stiglitz is primarily interested in what he calls the phenomena of “recurrence” – a situation when a machine once in service is put out of service to be brought back into service again.

²⁴See the third paragraph of the concluding remarks in Section 12 below.

literature of the sixties on planning in India (and elsewhere). This literature²⁵ comes as close to stating the problem as precisely as can be expected in the pre-Pontryagin period.

[T]he whole point of the exercise is to get alternative time-paths of consumption, out of which choice can be made according to circumstances. ... So, the choice, in this system, should not be put so much as one between a faster and a slower growth rate, but as a choice between constant consumption, or its constant rate of growth, or its constant rate of growth of rate of growth, and so on. It should be noted that from the long-run point of view, each case is better than the earlier one, though the position is the reverse in the short period. Which type of policy we choose will depend very much on the time necessary for the long-run effects to compensate the short-run results.²⁶

However, this literature and earlier work notwithstanding, it will generally be recognized today that whether we are interested in this issue from a planning perspective or from the modern perspective of a competitive representative agent, the problem of an appropriate choice of technique should really be viewed as part of the general theory of economic growth. A subsidiary motivation of this paper is to facilitate this re-orientation.²⁷

We conclude this introduction with a schematic outline of the paper. In section 2 we present the model and show how it can be converted to its Gale-McKenzie reduced form. In Section 3, under a standing hypothesis on the finite set of parameters that define the RSS model, we show the existence and uniqueness of the golden-rule stock, and its values to be independent of the felicity function; we also identify a set of golden-rule prices. In Section 4, we show the existence of a program that is optimal starting from any given initial stock of machines. In Section 5, we consider the question of the correct choice of technique for the long-run, and through the identification of the von Neumann facet, present results for both linear and strictly concave felicity functions. This material applies the standard theory to the RSS model and suggests the example of cyclical optimal paths (presented in Section 6) and that of bad Stiglitz programs (presented in Section 7) in a simple one-machine setting with a linear felicity function. These examples also suggest a sufficient parameterization under which a Stiglitz program is optimal and uniquely optimal in the case of a linear felicity function. We present these results in Section 8. In Section 9, we turn to programs, labeled σ -programs, in which there is no specification as to use but only to the construction of the machine of type σ . We present an example that shows in a simple two-machine setting with a non-linear felicity function that an optimal program is not a σ -program. This, along with Sections 6 and 7, is concerned with transition dynamics. In Sections 10 and 11, we consider the setting of a general felicity function, and after a development of the price-support and other

²⁵In addition to Raj-Sen (1961) and Sen (1960) in particular, also see Dobb (1956, 1960, 1961, 1967), Halevi (1987), Mirrlees (1962), Naqvi (1963), Solow (1962a); and for an open-economy perspective, Bardhan (1971).

²⁶See Raj-Sen (1961, pp. 48, 51 and concluding section). We remind the reader that an earlier version of the Raj-Sen paper (with a different title) was published in *Arthaniti* in 1959, and that Naqvi (1963) is a follow-up to the Raj-Sen paper as his leading footnote clearly indicates.

²⁷For an early emphasis on this, see Mirrlees (1962) and Srinivasan (1962b); Okishio (1987) represents the alternative perspective. This point was independently underscored to Khan by Debraj Ray as a comment on Khan (2000).

properties of the optimal path, present sufficient conditions under which a σ -program is optimal. The concluding Section 12 lists the salient results and identifies problems that remain open. The technical and computational details of some of the proofs are collected in an Appendix.

2 The Model and its Reduced Form

We begin with some preliminary notation. Let \mathbb{N} (\mathbb{N}_+) be the set of non-negative (positive) integers, \mathbb{R} (\mathbb{R}_+) the set of real (non-negative) numbers. We shall work in finite-dimensional Euclidean space \mathbb{R}^n with non-negative orthant $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$. For any x, y in \mathbb{R}^n , let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x \gg y$, $x > y$, $x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the i^{th} unit vector in \mathbb{R}^n , and e be an element of \mathbb{R}_+^n all of whose coordinates are unity. For any $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean norm of x . The empty set is denoted by \emptyset and set-theoretic subtraction between A and B by A/B .

2.1 Technology

Our choice of \mathbb{R}^n is dictated by the consideration of an economy capable of producing a finite number n of alternative types of machines. For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Thus, the production possibilities of the economy can be represented by an (labor) input-coefficients vector, $a = (a_1, \dots, a_n) \gg 0$ and an output-coefficients vector, $b = (b_1, \dots, b_n) \gg 0$. Without loss of generality²⁸ we shall assume that the types of machines are numbered such that $b_1 \geq b_2 \geq \dots \geq b_n$.

We shall assume that all machines depreciate at a rate $d \in (0, 1)$. Thus the effective labor cost of producing a unit of output on a machine of type i is given by $(1 + da_i)/b_i$: the direct labor cost of producing unit output, and the indirect cost of replacing the depreciation of the machine in this production.²⁹ We shall work with the reciprocal of the effective labor cost, the effective output that takes the depreciation into account, and denote³⁰ it by c_i for the machine of type i . Throughout this paper, we shall assume that there is a unique machine type σ at which this effective labor cost $(1 + da_i)/b_i$ is minimized, or at which the effective output per man $b_i/(1 + da_i)$ is maximized. Thus, we shall assume:

$$\text{There exists } \sigma \in \{1, \dots, n\} \text{ such that for all } i = 1, \dots, n, i \neq \sigma, c_\sigma > c_i. \quad (1)$$

2.2 Programs

For each date $t \in \mathbb{N}$, let $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$ denote the amounts of the n types of machines that are available in time-period t , and let $z(t+1) = (z_1(t+1), \dots, z_n(t+1)) \geq 0$ be the *gross investments*

²⁸Note that Stiglitz (1968) assumes that $b_i > b_j$ implies that $a_i > a_j$; whereas this is a natural hypothesis, we make no such assumption.

²⁹See Stiglitz (1968, pp. 608-609) on a "labour theory of value" interpretation.

³⁰As we shall see below, c_i is the value of the steady-state consumption per man if only machines of type i are used and produced, a consideration that governs our choice of notation.

in the n types of machines during period $(t + 1)$. Hence, $z(t + 1) = (x(t + 1) - x(t)) + dx(t)$, the sum of net investment and of depreciation. Let $y(t) = (y_1(t), \dots, y_n(t))$ be the amounts of the n types of machines used for production of the consumption good, $by(t)$, during period $(t + 1)$.³¹ Let the total labor force of the economy be stationary and positive. We shall normalize it to be unity. Clearly, gross investment, $z(t + 1)$ representing the production of new machines of the various types, will require $az(t + 1)$ units of labor in period t . Also, $y(t)$ representing the use of available machines for manufacture of the consumption good, will require $ey(t)$ units of labor in period t . Thus, the availability of labor constrains employment in the consumption and investment sectors by $az(t + 1) + ey(t) \leq 1$. Note that both the flow of consumption and of investment (new machines) are in gestation during the period and available at the end of it. We now give a formal summary of this technological structure.

Definition 1 A program from x_o in \mathbb{R}_+^n is a sequence³² $\{x(t), y(t)\}$ with $(x(t), y(t)) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$,

$$x(t + 1) \geq (1 - d)x(t), \quad 0 \leq y(t) \leq x(t), \quad a(x(t + 1) - (1 - d)x(t)) + ey(t) \leq 1.$$

A program $\{x(t), y(t)\}$ is simply a program from $x(0)$.

Definition 2 Associated with any program $\{x(t), y(t)\}$ is a gross investment sequence $\{z(t + 1)\}$ with $z(t + 1) \in \mathbb{R}_+^n$, and a consumption sequence $\{by(t)\}$ such that for all $t \in \mathbb{N}$,

$$z(t + 1) = x(t + 1) - (1 - d)x(t).$$

Definition 3 A program $\{x(t), y(t)\}$ is called stationary if for all $t \in \mathbb{N}$, $(x(t), y(t)) = (x(t + 1), y(t + 1))$.

2.3 Preferences

The preferences of the planner are represented by a felicity function, $w : \mathbb{R}_+ \rightarrow \mathbb{R}$, which is assumed to be continuous, strictly increasing and concave, and differentiable when strictly concave. We suppose, as in the literature taking its lead from Ramsey (1928), that future welfare levels are treated like current ones in the planner's objective function. The precise criterion of optimality that we work with is due to Atsumi (1965) and von Weizsäcker (1965).

Definition 4 A program $\{x^*(t), y^*(t)\}$ from x_o is called optimal if

$$\liminf_{T \rightarrow \infty} \sum_{t=1}^T [w(by(t)) - w(by^*(t))] \leq 0$$

for every program $\{x(t), y(t)\}$ from x_o . It is called a stationary optimal program if it is stationary and optimal.

³¹The reader may choose to think of the consumption in period t as the scalar $c(t + 1)$, with c_i reserved for $b_i/(1 + da_i)$, we avoid this notation in the text to prevent any ambiguity.

³²Note $\{x(t), y(t)\}$ is an abbreviation of $\{x(t), y(t)\}_{t \in \mathbb{N}}$; we use it for notational simplicity.

Note that the optimality notion can be restated to say that there does not exist any other program $\{x(t), y(t)\}$, $x(0) = x_o$, a number $\varepsilon > 0$ and a time period t_ε such that $\sum_{t=1}^T [w(by(t)) - w(by^*(t))] > \varepsilon$ for all $T \geq t_\varepsilon$. Thus an optimal program is one in comparison to which no other program from the same initial stock is eventually significantly better, for any given level of significance.

2.4 Conversion to the Reduced Form

Following McKenzie (1968), we convert the above model into its “reduced form”, and as emphasized in the introduction, thereby exploit as far as possible the results of the general theory of intertemporal allocation for our particular case.

Define the *transition possibility set* Ω as a collection of pairs (x, x') , such that it is possible to obtain the amounts of the n types of machines x' in the next period (tomorrow) from the amounts of the n types of machines x available in the current period (today). Formally,

$$\Omega = \{(x, x') \in \mathbb{R}_+^n \times \mathbb{R}_+^n : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}.$$

For any $(x, x') \in \Omega$, one can consider the amounts y of the n types of machines available for the production of the consumption good. Formally, we have a correspondence $\Lambda : \Omega \longrightarrow \mathbb{R}_+^n$ given by

$$\Lambda(x, x') = \{y \in \mathbb{R}_+^n : 0 \leq y \leq x \text{ and } ey \leq 1 - a(x' - (1-d)x)\}.$$

For any $(x, x') \in \Omega$, we shall denote the number of machines that are produced in the period $(x' - (1-d)x)$ by z . Note that $z \geq 0$. Finally, the *reduced form utility function*, $u : \Omega \longrightarrow \mathbb{R}_+$, is defined on Ω such that

$$u(x, x') = \max\{w(by) : y \in \Lambda(x, x')\}.$$

We leave it to the reader to check for herself that our assumptions on w imply that the reduced form utility function, u , is upper semicontinuous³³ and concave function on Ω , and that it is increasing in its first argument and decreasing in its second argument.

Given the description of the transition possibility set Ω , and of the reduced form utility function, u , it is clear that for any program $\{x(t), y(t)\}$ from x_o , $(x(t), x(t+1)) \in \Omega$ and $y(t) \in \Lambda(x(t), x(t+1))$ for all $t \in \mathbb{N}$. Also, for any optimal program $\{x^*(t), y^*(t)\}$ from x_o , $w(by^*(t)) = u(x^*(t), x^*(t+1))$ for all $t \in \mathbb{N}$, and for every program $\{x(t), y(t)\}$ from x_o ,

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T [u(x(t), x(t+1)) - u(x^*(t), x^*(t+1))] \leq 0.$$

In summary, the basic data of the model denoted by the triple $(w, (a_i, b_i)_{i=1}^n, d)$ summarizing the felicity function w , the technology $(a_i, b_i)_{i=1}^n$, and the depreciation rate d , is converted to the pair (u, Ω) summarizing the reduced-form utility function u and the transition possibility set Ω .

³³It is now well understood that continuity of w does not necessarily imply the continuity of u ; see Dutta-Mitra (1989) for details.

3 The Existence and Uniqueness of a Stationary Optimal Stock

A stationary optimal program is of special significance, and in this section we take the first step in establishing the existence of such a program. We show the existence and uniqueness of a stationary optimal stock (synonymously, golden-rule stock), and simultaneously, provide a “price support” property of such a stock. This constitutes a stationary price-support property for the stationary optimal program. We invoke techniques from the general theory of optimal intertemporal allocation, but rather than an appeal to the Kuhn-Tucker theorem, we exploit the concrete structure of the RSS model to provide a purely constructive proof of our claims. This has the additional advantage that we can identify the shadow prices in terms of the basic data of the model.

We begin with a definition.

Definition 5 A stationary optimal stock (synonymously, a golden-rule stock) is $\hat{x} \in \mathbb{R}_+^n$ such that (\hat{x}, \hat{x}) is a solution to the problem: maximize $u(x, x')$ subject to (i) $x' \geq x$, (ii) $(x, x') \in \Omega$.

If we limit ourselves to a stationary program in which only a machine of type i is used and produced, the constraint of labor allows us to maintain the stock³⁴ $(1/(1 + da_i))$ and obtain a stationary consumption stream in the amount $b_i/(1 + da_i) = c_i$. Since we have assumed (in (1) above) that a machine of type σ is the one that uniquely minimizes effective labor costs, we see that it is also the type that uniquely maximizes the consumption per unit of labor.³⁵ Denote $\hat{y} = (1/(1 + da_\sigma))e(\sigma)$, and note that if we are in such a stationary state, $b\hat{y} = (b_\sigma/(1 + da_\sigma))$ and $w'(b\hat{y})$ the marginal utility of output produced. Furthermore, since the labor cost of a machine of type i is a_i , and a unit of labor is worth $((1 + da_i)/b_i)^{-1}$ units of output, a machine is worth $a_i \times (b_i/(1 + da_i))$ in terms of output, and $w'(b\hat{y})(a_i \times (b_i/(1 + da_i)))$ in terms of utils. We can then identify a stationary price system (\hat{q} in terms of the consumption good and \hat{p} in terms of utils)³⁶ for the various types of machines as $\hat{q}_i = (a_i b_i/(1 + da_i))$ and $\hat{p}_i = w'(b\hat{y})\hat{q}_i$ for each $i = 1, \dots, n$.

We can now present a simple but important result.

Lemma 1 $w(b\hat{y}) \geq w(by) + \hat{p}x' - \hat{p}x$ for any $(x, x') \in \Omega$, and for any $y \in \Lambda(x, x')$.

Proof: Now for any $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$, we have³⁷

$$\begin{aligned}
 b\hat{y} - by - \hat{q}(x' - x) &= c_\sigma - by - \hat{q}(x' - x) \\
 &= c_\sigma - by - q(x' - (1 - d)x) + dqx \\
 &= c_\sigma(1 - ey - az) + c_\sigma ey + c_\sigma az - by - qz + dqx \\
 &= c_\sigma(1 - ey - az) + \sum_{i=1}^n (c_\sigma - b_i)y_i + \sum_{i=1}^n (c_\sigma - c_i)a_i z_i + dqx
 \end{aligned} \tag{2}$$

³⁴The labor requirements of the consumption sector in the amount $(1/(1 + da_i))$ plus those of the investment sector arising from replacement for depreciation in the amount $da_i/(1 + da_i)$ add up to the total labor available.

³⁵As alluded to in Footnote 30 above.

³⁶When the felicity function is linear, the magnitudes of \hat{p} and \hat{q} are identical, though their units remain different. Note also the identities $q_i = a_i c_i$ and $c_i + dq_i = b_i$ for all i .

³⁷Note that in the derivation of (2) and (3) below, we appeal to the identities referred to in Footnote 36.

$$= c_\sigma(1 - ey - az) + \sum_{i=1}^n (c_\sigma - c_i)y_i + \sum_{i=1}^n (c_\sigma - c_i)a_i z_i + dq(x - y) \quad (3)$$

Since $(x, x') \in \Omega$, $z \geq 0$. Since $y \in \Lambda(x, x')$, $x \geq y$ and $1 - ey - az \geq 0$. We can now appeal to our standing hypothesis as described in (1) to assert that

$$b\hat{y} - by - \hat{q}(x' - x) \geq 0 \implies by - b\hat{y} \leq \hat{q}x - \hat{q}x'. \quad (4)$$

Given our hypotheses on the felicity function w , we obtain as a consequence of (4),

$$w(by) - w(b\hat{y}) \leq w'(b\hat{y})(by - b\hat{y}) \leq w'(b\hat{y})(\hat{q}x - \hat{q}x') = (\hat{p}x - \hat{p}x')$$

A simple transposition of terms completes the proof. ■

We can now state the principal result of this section.³⁸

Theorem 1 *There exists a unique stationary optimal stock $\hat{x} = (1/(1 + da_\sigma))e(\sigma)$.*

Proof: Let $\hat{y} = \hat{x} = (1/(1 + da_\sigma))e(\sigma)$, and check that $(\hat{x}, \hat{x}) \in \Omega$, and $\hat{y} \in \Lambda(\hat{x}, \hat{x})$. Next, appeal to Lemma 1 to assert that (\hat{x}, \hat{x}) is a solution to the problem specified in Definition 5, and hence that \hat{x} is a golden-rule stock.

We can also show that it is a unique solution to this problem. Suppose to the contrary that (\tilde{x}, \tilde{x}') is another solution with a corresponding $\tilde{y} \in \Lambda(\tilde{x}, \tilde{x}')$ and $\tilde{z}' = \tilde{x}' - (1 - d)\tilde{x}$. Since $w(\cdot)$ is strictly increasing, $b\tilde{y} = b\hat{y} = c_\sigma$. On substituting \tilde{x}, \tilde{y} and \tilde{z} for x, y and z in (3) above, we obtain the fact that the right hand side of (3) equals zero, which implies that each of its four terms is zero. This implies that $\tilde{y}_i = 0 = \tilde{z}_i$ for $i \neq \sigma$, that $\tilde{x}_i = \tilde{y}_i$ for all i , and that $\tilde{y}_\sigma + a_\sigma \tilde{z}_\sigma = 1$. Coupling the first assertion with the equality $b\tilde{y} = c_\sigma$, we obtain that $\tilde{y}_\sigma = 1/(1 + da_\sigma)$, and hence from the third assertion that $\tilde{x} = 1/(1 + da_\sigma)e(\sigma)$. From the last assertion we can then obtain that $\tilde{z}_\sigma = d/(1 + da_\sigma)$ and hence that $\tilde{x}' = \tilde{z} - (1 - d)\tilde{x} = (d/(1 + da_\sigma) + (1 - d)/(1 + da_\sigma))e(\sigma) = (1/(1 + da_\sigma))e(\sigma)$. The demonstration is complete. ■

4 The Existence of an Optimal Program

In this section, we prove the existence of an optimal program from an arbitrarily given initial stock. We follow the methods of Brock (1970) which in turn build on those of Gale (1967) – this methodology relies on the concept of a *good* program and the exploits the assumption of a unique golden-rule stock to deduce the average turnpike property of such a program. We follow the same conceptual benchmarks in the context of the RSS model and present a unified treatment both to highlight certain steps that are crucial for subsequent argumentation and to avoid possibly confusing cross-referencing.³⁹

³⁸We remind the reader of our standing hypothesis as expressed in (1).

³⁹Note that we cannot directly apply the relevant theorems in Gale, Brock (1970) or McKenzie (1968, 1987) since the assumptions of these theorems are not directly satisfied; instead, the concrete structure of the RSS model allows a simplification of the arguments.

Definition 6 A program $\{x(t), y(t)\}$ is called good if there exists $G \in \mathbb{R}$ such that $\sum_{t=0}^T (w(by(t)) - w(b\hat{y})) \geq G$ for all $T \in \mathbb{N}$. A program is called bad if $\lim_{T \rightarrow \infty} \sum_{t=0}^T (w(by(t)) - w(b\hat{y})) = -\infty$.

Proposition 1 There exists a good program from any arbitrary initial stock $x_o \in \mathbb{R}_+^n$.

Proof: For each $t \in \mathbb{N}$, let $z(t+1) = d\hat{x}e(\sigma)$. Define $y(0) = 0$, and $y(t+1) = (1-d)y(t) + d\hat{x}e(\sigma)$ for $t \in \mathbb{N}$. Then, $y(t)$ is monotonically non-decreasing, and converges to \hat{x} as $t \rightarrow \infty$. Given an arbitrary initial stock, x_o , define $x(0) = x_o$, and for each $t \in \mathbb{N}$, $x(t+1) = (1-d)x(t) + z(t+1)$. Then, it is easy to check that $\{x(t), y(t)\}$ is a program from x_o . Given the definition of the sequence $\{y(t)\}$, we have $(by(t) - b\hat{x}) = (1-d)^t (by(1) - b\hat{x})$ for $t \geq 2$, and $by(t) \geq db\hat{x}$ for $t \in \mathbb{N}_+$. Thus, we have for $t \in \mathbb{N}_+$,

$$[w(b\hat{x}) - w(by(t))] \leq w'(by(t))(b\hat{x} - by(t)) \leq w'(db\hat{x})(b\hat{x} - by(1))(1-d)^{t-1}.$$

Thus, the sequence $\{w(b\hat{x}) - w(by(t))\}$ is summable, and so $\{x(t), y(t)\}$ is a good program from x_o . ■

Proposition 2 For any program $\{x(t), y(t)\}$, there exists $m(x(0)) \in \mathbb{R}_+$ such that $x(t) \leq m(x(0))e$ for any $t \in \mathbb{N}$.

Proof: The case $t = 0$ is a triviality. For $t \in \mathbb{N}_+$, $(x(t-1), x(t)) \in \Omega$ implies $ax(t) \leq 1 + (1-d)ax(t-1) \leq \sum_{\tau=0}^{t-1} (1-d)^\tau + (1-d)^t ax(0)$. Since $0 < d < 1$, we obtain $ax(t) \leq (1/d) + ax(0)$. Let $a_j = \min_{1 \leq i \leq n} a_i$. Since $a_i > 0$ for all $i = 1, 2, \dots, n$, we obtain $x_i(t) \leq (1/a_j)((1/d) + ax(0)) \equiv m(x(0))$ for all $i = 1, \dots, n$, and complete the proof. ■

Proposition 3 For any program $\{x(t), y(t)\}$, there exists $M(x(0)) \in \mathbb{R}_+$ such that for any $t_1 \in \mathbb{N}$, and any integer $t_2 \geq t_1$, $\sum_{t=t_1}^{t_2} (w(by(t)) - w(b\hat{y})) \leq M(x(0))$.

Proof: From Lemma 1, for any $t_2 \geq t_1$, $\sum_{t=t_1}^{t_2} w(by(t)) - w(b\hat{y}) \leq \hat{p}(x(t_1) - x(t_2 + 1)) \leq \hat{p}(x(t_1)) \leq m(x(0)) \sum_{j=1}^n \hat{p}_j$. Let $M(x(0)) = m(x(0))w'(b_\sigma/1 + da_\sigma) \sum_{j=1}^n a_i b_i / (1 + da_i)$ to complete the proof. ■

Proposition 4 Any program that is not good is bad.

Proof: For any program $\{x(t), y(t)\}$ that is not good, and for any $N \in \mathbb{R}$, there exists T_N such that $\sum_{\tau=0}^{T_N} (w(by(\tau)) - w(b\hat{y})) \leq N - M(x(0))$, $M(x(0))$ the real number whose existence is asserted in Proposition 3. By choosing $t_1 = T_N + 1$ and $t_2 = t > T_N + 1$ in Proposition 3, we obtain that $\sum_{\tau=T_N+1}^t (w(by(\tau)) - w(b\hat{y})) \leq M(x(0))$ for all $t > T_N + 1$. On adding these two expressions, we obtain that $\sum_{\tau=0}^t (w(by(\tau)) - w(b\hat{y})) \leq N$ for all $t > T_N$, and complete the proof. ■

Proposition 5 Any optimal program is good.

Proof: Let $\{x(t), y(t)\}$ be an optimal program, and suppose it is not good. By Proposition 1, there exists a good program $\{x'(t), y'(t)\}$ starting from $x(0)$. Hence there exists $G \in \mathbb{R}$ such that for all $T \in \mathbb{N}_+$, $\sum_{t=0}^T (w(by'(t)) - w(b\hat{y})) \geq G$. Pick any $\varepsilon > 0$, and appeal to Proposition 4 to guarantee the existence of t_ε such that $\sum_{t=0}^T (w(by(t)) - w(b\hat{y})) < G - \varepsilon$ for all $T \geq t_\varepsilon$. Putting these two expressions together, we obtain that $\sum_{t=0}^T (w(by'(t)) - w(by(t))) > \varepsilon$ for all $T \geq t_\varepsilon$, and hence a contradiction to the fact that $\{x(t), y(t)\}$ is an optimal program. ■

Definition 7 A program $\{x(t), y(t)\}$ exhibits the average turnpike property if $\lim_{T \rightarrow \infty} (\bar{x}(T), \bar{y}(T)) = (\hat{x}, \hat{y})$, where $\bar{x}(T) = (1/T) \sum_{t=0}^{T-1} x(t)$ and $\bar{y}(T) = (1/T) \sum_{t=0}^{T-1} y(t)$ for all $T \in \mathbb{N}_+$.

Proposition 6 If the golden-rule stock \hat{x} is unique, every good program exhibits the average turnpike property.

Proof: For any program $\{x(t), y(t)\}$, Proposition 2, and the fact that $y(t) \leq x(t)$ for all $t \in \mathbb{N}$, guarantee that the sequence $\{\bar{x}(T), \bar{y}(T)\}$ has at least one convergent subsequence. Let (x^∞, y^∞) be the limit of one such subsequence. We leave it to the reader to check that $(x^\infty, x^\infty) \in \Omega$ and that $y^\infty \in \Lambda(x^\infty, x^\infty)$.

Since $\{x(t), y(t)\}$ is a good program, there exists $G \in \mathbb{R}$ such that for all $T \in \mathbb{N}_+$,

$$G/T \leq (1/T) \sum_{t=0}^{T-1} (w(by(t)) - w(b\hat{y})) \leq w(b\bar{y}(T)) - w(b\hat{y}),$$

where the second assertion is a consequence of the concavity of $w(\cdot)$. On taking limits as $T \rightarrow \infty$ along the chosen subsequence, and on appealing to the continuity of w , we obtain $w(by^\infty) \geq w(b\hat{y})$. Hence x^∞ is a golden-rule stock. Since the golden rule stock is unique, $x^\infty = \hat{x}$, and since w is strictly increasing, $y^\infty = \hat{y}$. Since we worked with an arbitrary convergent subsequence, the argument is complete. ■

For any $y \in \Lambda(x, x')$ and any $(x, x') \in \Omega$, let

$$\delta_{(\hat{x}, \hat{p})}(x, x') = w(b\hat{y}) - w(by) - \hat{p}(x' - x) = \hat{p}(x - x') - (w(by) - w(b\hat{y})). \quad (5)$$

Whenever there is no possibility of confusion, we shall abbreviate $\delta_{(\hat{x}, \hat{p})}(x(t), x(t+1))$ by $\delta(t)$ for any program $\{x(t), y(t)\}$. We shall refer to $\{\delta(t)\}$ as the value-loss sequence associated with the program $\{x(t), y(t)\}$.

Proposition 7 The value-loss sequence $\{\delta(t)\}_{t \in \mathbb{N}}$ of any program $\{x(t), y(t)\}$ is non-negative, and

$$\sum_{t=0}^T (w(by(t)) - w(b\hat{y})) = \hat{p}(x(0) - x(T+1)) - \sum_{t=0}^T \delta(t) \text{ for all } T \in \mathbb{N}.$$

Proof: For each $t \in \mathbb{N}$, let $\delta(t) = \hat{p}(x(t) - x(t+1)) - (w(by(t)) - w(b\hat{y}))$. Since $\{x(t), y(t)\}$ is a program, we can appeal to Lemma 1 to assert that $\delta(t) \geq 0$ for all $t \in \mathbb{N}$. On summing over t , and rearranging, we complete the proof of the assertion. ■

We now define the aggregate value-loss associated with any program starting from x_o as

$$\Delta(x_o) = \inf \left\{ \sum_{t=0}^{\infty} \delta(t) : \{x(t), y(t)\} \text{ is a program from } x_o \right\}.$$

Our next two results assert that this infimum is a finite number and that it can be attained.

Proposition 8 The value-loss sequence $\{\delta(t)\}_{t \in \mathbb{N}}$ of any program $\{x(t), y(t)\}$ is summable if and only if it is good. Hence $\lim_{t \rightarrow \infty} \delta(t) = 0$ for any good program.

Proof: For any good program, Proposition 7 allows us to assert the existence of $G \in \mathbb{R}$ such that for all $t \in \mathbb{N}$,

$$\begin{aligned} \sum_{t=0}^T \delta(t) &= \hat{p}(x(0) - x(T+1)) - \sum_{t=0}^T (w(by(t)) - w(b\hat{y})) \\ &\leq \hat{p}(x(0) - x(T+1)) - G \leq \hat{p}(x(0)) - G. \end{aligned}$$

Since $\sum_{t=0}^{\infty} \delta(t)$ is a finite number, certainly $\lim_{t \rightarrow \infty} \delta(t) = 0$. On the other hand, the first equality and Proposition 2 allows us to assert that a program with a summable value-loss sequence is good. ■

Proposition 9 *There exists a program $\{x'(t), y'(t)\}$ from an arbitrary initial stock x_o such that its associated value-loss sequence $\{\delta'(t)\}$ satisfies $\sum_{t=0}^{\infty} \delta'(t) = \Delta(x_o)$ where $0 \leq \Delta(x_o) < \infty$.*

Proof: Proposition 1 guarantees that there exists a good program $\{x(t), y(t)\}$ from x_o , and hence from Proposition 7, we obtain that $\Delta(x_o) < \infty$. Since $\{\delta(t)\}$ is a non-negative sequence, certainly $0 \leq \Delta(x_o)$.

For any $\nu \in \mathbb{N}_+$, there exists a program $\{x^\nu(t), y^\nu(t)\}$ from x_o such that its associated value-loss sequence $\{\delta^\nu(t)\}$ satisfies

$$\sum_{t=0}^{\infty} \delta^\nu(t) \leq \Delta(x_o) + (1/\nu). \quad (6)$$

From Proposition 2, the sequence $\{x^\nu(t)\}_{\nu=1}^{\infty}$ is bounded independently of ν (and of t but not of x_o). Since $y^\nu(t) \leq x^\nu(t)$, the same is true of the sequence $\{y^\nu(t)\}_{\nu=1}^{\infty}$. We can now appeal to a diagonalization argument (see Rudin (1967; Theorem 7.23) to extract a subsequence indexed by ν_k , $k \in \mathbb{N}_+$, that converges for all $t \in \mathbb{N}$. Let $\{x'(t), y'(t)\}$ be the limit of this subsequence. Let $\delta'(t) = \hat{p}(x'(t) - x'(t+1)) - (w(by'(t)) - w(b\hat{y}))$ for all $t \in \mathbb{N}$. Since $w(\cdot)$ is a continuous function, $\delta^{\nu_k}(t) \rightarrow \delta'(t)$ for all $t \in \mathbb{N}$.

It is easy to check that $\{x'(t), y'(t)\}$ is a program from \mathbf{x} . Since $\{\delta'(t)\}$ is its associated value-loss sequence, certainly $\sum_{t=0}^{\infty} \delta'(t) \geq \Delta(x_o)$. If we now consider the set of natural numbers \mathbb{N} as a measure space equipped with a counting measure, we can appeal to Fatou's lemma (see Rudin (1967; Theorem 11.31) via (6) to assert

$$\Delta(x_o) \geq \lim_{k \rightarrow \infty} \sum_{t=0}^{\infty} \delta^{\nu_k}(t) \geq \sum_{t=0}^{\infty} \liminf_{k \rightarrow \infty} \delta^{\nu_k}(t) = \sum_{t=0}^{\infty} \delta'(t).$$

This completes the proof of our claim.⁴⁰ ■

Proposition 10 *A program $\{x(t), y(t)\}$ whose associated value-loss sequence $\{\delta(t)\}$ satisfies $\sum_{t=0}^{\infty} \delta(t) = \Delta(x(0))$ is optimal if it exhibits the average turnpike property.*

Proof: We know from Proposition 8 that the program $\{x(t), y(t)\}$ is good. Now suppose that it is not optimal. Then there exists a program $\{x'(t), y'(t)\}$, $x'(0) = x(0)$, a number $\varepsilon > 0$ and a time period t_ε such that $\sum_{t=1}^T [w(by'(t)) - w(by(t))] > \varepsilon$ for all $T \geq t_\varepsilon$. Since $\{x(t), y(t)\}$ is good, $\{x'(t), y'(t)\}$ is good, and by Proposition 6 both programs satisfy the average turnpike property. Let $\{\delta'(t)\}$ be the value-loss

⁴⁰For a direct argument that does not appeal to integration and to Fatou's lemma, see Brock (1970, p. 278).

sequence associated with the latter program. We can now appeal to Proposition 7 to assert that for all $T \geq t_\varepsilon$,

$$\varepsilon < \sum_{t=0}^T (w(by'(t)) - w(by(t))) = \hat{p}(x(T+1) - x'(T+1)) + \sum_{t=0}^T \delta(t) - \sum_{t=0}^T \delta'(t).$$

From the minimality of $\sum_{t=0}^\infty \delta(t)$, there exists t'_ε such that for all $T \geq t'_\varepsilon$, $(\varepsilon/2) < \hat{p}(\bar{x}(T+1) - \bar{x}'(T+1)) = \hat{p}(\bar{x}(T+1) - \hat{x} + \hat{x} - \bar{x}'(T+1))$. On taking limits with respect to T , we obtain a contradiction. ■

Theorem 2 *For any arbitrary initial stock, $x_o \in \mathbb{R}_+^n$, there exists an optimal program from x_o . If the initial stock x_o equals $\hat{x} = \hat{y} = (1/(1 + da_\sigma))e(\sigma)$, then the stationary program $\{\hat{x}, \hat{y}\}$ is an optimal program from x_o .*

Proof: Consider the program whose existence is asserted in Proposition 9. From Proposition 8, it is certainly a good program, and hence from Proposition 6 satisfies the average turnpike property. All of the hypotheses of Proposition 10 are fulfilled, and we can appeal to it to complete the proof of the first claim. For the second claim, note that the aggregate value-loss of the stationary program is (trivially) zero and that it trivially satisfies the average turnpike property. Again, an appeal to Proposition 10 completes the argument. ■

5 Choice of Techniques in the Long-Run

We are now in a position to describe what the economy looks like in the long-run. Towards this end, we begin with a characterization of the von Neumann facet as described in McKenzie (1968, 1986). It is of interest that under our standing hypothesis as described in (1), this reduces to a line in the Euclidean space of dimension $2n$.

Lemma 2 *The von Neumann facet $\{(x, x') \in \Omega : \delta_{(\hat{p}, \hat{x})}(x, x') = 0\}$ is a subset of $\{(x, x') \in \Omega : x'_i = x_i = 0, i \neq \sigma, x'_\sigma = (1/a_\sigma) + \xi_\sigma x_\sigma\}$, $\xi_\sigma = 1 - d - (1/a_\sigma)$, with equality if the felicity function w is linear. If the felicity function is strictly concave, the facet is the singleton $\{(\hat{x}, \hat{x})\}$.*

Proof: Pick $(\tilde{x}, \tilde{x}') \in \Omega$ and $\tilde{y} \in \Lambda(\tilde{x}, \tilde{x}')$ such that $\delta_{(\hat{x}, \hat{p})}(\tilde{x}, \tilde{x}') = 0$. From (5) we obtain $w(b\tilde{y}) - w(b\hat{y}) + \hat{p}(\tilde{x}' - \hat{x}) = 0$. On appealing to the concavity of $w(\cdot)$, this reduces to

$$w(b\tilde{y}) - w(b\hat{y}) \leq w'(b\hat{y})(b\tilde{y} - b\hat{y}) \implies b\hat{y} - b\tilde{y} - q(\tilde{x}' - \hat{x}) \leq 0. \quad (7)$$

This combined with (4) and (3) yields

$$c_\sigma(1 - ey - az) + \sum_{i=1}^n (c_\sigma - c_i)y_i + \sum_{i=1}^n (c_\sigma - c_i)a_i z_i + dq(x - y) = 0.$$

This implies that $\tilde{z}_i = 0 = \tilde{y}_i = \tilde{x}_i = \tilde{x}'_i$ for all $i \neq \sigma$. Furthermore, that $\tilde{y}_\sigma = \tilde{x}_\sigma$ and that

$$\tilde{y}_\sigma - a_\sigma \tilde{z}_\sigma = 1 \implies \tilde{x}_\sigma + a_\sigma(\tilde{x}' - (1 - d)\tilde{x}) = 1 \implies \tilde{x}'_\sigma = (1/a_\sigma) + \xi_\sigma \tilde{x}_\sigma.$$

Now suppose that $w(\cdot)$ is strictly concave and that $b\tilde{y} \neq b\hat{y}$. We then obtain a strict inequality in (7) and thereby contradict (4). Thus $b\tilde{y} = b\hat{y} = c_\sigma$. On appealing to the computations above, we obtain that $\tilde{y}_\sigma = 1/(1 + da_\sigma) = \tilde{x}_\sigma$, and hence that $\tilde{x}'_\sigma = (1/a_\sigma) + \xi_\sigma \tilde{x}_\sigma = 1/(1 + da_\sigma)$.

For the reverse implication in the linear case, pick $(x, x') \in \Omega$ such that $x'_\sigma = (1/a_\sigma) + \xi_\sigma x_\sigma$, $x'_i = x_i = 0$, $i \neq \sigma$, and $y_\sigma = x_\sigma$. On substituting these values in the left hand side of (3), we see that it is equal to zero. But that is precisely $\delta_{(\hat{x}, \hat{p})}(x, x')$ in the linear case. ■

We can now present

Theorem 3 *Any optimal program $\{x(t), y(t)\}$ converges to the von Neumann facet, and thus $\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} y_i(t) = \lim_{t \rightarrow \infty} z_i(t) = 0$ for all $i \neq \sigma$. If the felicity function $w(\cdot)$ is strictly concave, $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = (1/1 + da_\sigma)e(\sigma)$ and $\lim_{t \rightarrow \infty} z(t) = (da_\sigma/1 + da_\sigma)e(\sigma)$.*

Proof: Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}_+$, there exists $t(k) \geq k$ such that $\sum_{i \neq \sigma} \|x(t(k))\| > \varepsilon$. We can then assert that for the value-loss sequence $\{\delta(t(k))\}_{k \in \mathbb{N}_+}$, there exists δ_o and $k_o \in \mathbb{N}_+$ such that for all $k \geq k_o$, $\delta(t(k)) \geq \delta_o$. If the assertion is valid, we obtain a contradiction to Proposition 8 and complete the proof of the first claim. Thus, suppose that the assertion is false. Then we can manufacture a sequence of integers $\{k_i\}_{i \in \mathbb{N}_+}$ such that $\lim_{i \rightarrow \infty} \delta(t(k_i)) = 0$. Now consider the sequence $\{(x(t(k_i)), x(t(k_i) + 1))\}_{i \in \mathbb{N}_+}$ and appeal to Proposition 2 to guarantee the existence of a subsequence that converges to a point (\tilde{x}, \tilde{x}') . Since Ω is closed, and $w(\cdot)$ is continuous, $\delta_{(\hat{p}, \hat{x})}(\tilde{x}, \tilde{x}') = 0$. We now appeal to Lemma 2 to obtain a contradiction to our initial hypothesis.

For the case of a strictly concave felicity function, repeat the argument above but with $\|x(t(k)) - \hat{x}\| + \|x(t(k+1)) - \hat{x}\| > \varepsilon$. In this case, $(\tilde{x}, \tilde{x}') = (\hat{x}, \hat{x})$, and we again appeal to Lemma 2 to obtain a contradiction to our initial hypothesis. ■

6 An Example of an Optimal Periodic Program

In this section, we present an example of a simple economy with a linear felicity function in which the optimal path cycles around the golden rule stock. The economy has available to it only one type of machine whose (labor) input and output coefficients (a_1, b_1) are given by $(2/3, 1)$, the depreciation rate d by $1/2$, the felicity function by $w(by) = y$, and the initial stock of the machine $x_o = 1/2$. The reduced form of the economy is given by:

$$\begin{aligned} \Omega &= \{(x, x') \in \mathbb{R}_+^2 : (1/2)x + (3/2) \geq x' \geq (1/2)x\}, \\ \Lambda(x, x') &= \{y \in \mathbb{R}_+ : y \leq x \text{ and } y \leq 1 + (1/3)(x - 2x')\} = \{y \in \mathbb{R}_+ : y \leq \min[1 + (1/3)(x - 2x'), x]\}, \\ u(x, x') &= \max\{w(by) : y \in \Lambda(x, x'), (x, x') \in \Omega\} = \min[1 + (1/3)(x - 2x'), x]. \end{aligned}$$

Consider the program $\{x(t), y(t)\}$ given by $x(t) = y(t) = 3/4$, with a gross investment of $z(t+1) = x(t+1) - (1-d)x(t) = 3/4 - (1/2)(3/4) = 3/8$, for all $t \in \mathbb{N}$. We claim that this is a stationary optimal program from $x(0) = 3/4$. Towards this end, we show that $(3/4, 3/4)$ is the unique solution to the problem delineated in Definition 5, and hence that $3/4$ is the unique golden-rule stock.

First observe that $u(3/4, 3/4) = \min[1 - (1/3)(3/4), 3/4] = 3/4$, and that $u(x, x') = \min[1 - (1/3)x - (2/3)(x' - x), x]$. Now if $0 \leq x < 3/4$, $u(x, x') < (3/4) = u(3/4, 3/4)$. Thus suppose $x > 3/4$. In this case, $x' \geq x$ implies $1 - (1/3)x - (2/3)(x' - x) \leq 1 - (1/3)x < 3/4$, and hence that $u(x, x') < (3/4) = u(3/4, 3/4)$. The argument is complete.

Next, consider a program such that $y(t) = x(t)$ for all $t \in \mathbb{N}$, $x(t) = 1/2$ for all even $t \in \mathbb{N}$, and $x(t) = 1$ for all odd $t \in \mathbb{N}$. It is easy to check that this is a program that starts from $1/2$ and oscillates around $3/4$. All that we need to show is that it is an optimal program starting from $1/2$. Towards this end, we note that $\hat{p} = \hat{q} = 1/2$, and that this program makes a zero value-loss in each period at these prices:

$$\delta(t) = \begin{cases} (1/2) + (1/2) - (1/2)(1/2) - 3/4 = 0 & \text{for } t = 0, 2, \dots \\ 1 + (1/2)(1/2) - (1/2) - 3/4 = 0 & \text{for } t = 1, 3, \dots \end{cases}$$

Since it also satisfies the average turnpike property, an appeal to Proposition 10 then completes the argument.

7 Choice of Techniques with a Linear Felicity Function: An Example

In this section, we turn to the question of which machines are optimally *used* and *produced* – the choice of techniques – when the felicity function is linear. In particular, we examine the optimality of a policy prescribed in Stiglitz (1968). Towards this end, let $D = \{i \in \{1, \dots, n\} : b_i \geq c_\sigma = b_\sigma / (1 + da_\sigma)\}$ be the set of machine-types whose output per unit labor ratios are not less than the effective output per unit labor ratio of machines of type σ .⁴¹ We shall refer to such types as *desirable* and to those not in D as *undesirable*. Under a Stiglitz policy, labor is allocated in each time period to a set of available desirable machines with a higher type of machine having a priority over a lower one,⁴² and any remaining labor allocated towards producing only *one* type of machine, that delineated by σ . More formally,

Definition 8 *A program $\{x(t), y(t)\}$ with an associated gross investment sequence $\{z(t+1)\}$ is said to be a Stiglitz program if for any $t \in \mathbb{N}$ the following policy prescriptions are followed.*

- (i) *If $D = \emptyset$ or $x_i(t) = 0$ for all $i \in D$, let $y(t) = 0$ and $z(t+1) = e(\sigma)$,*
- (ii) *If $0 \leq \sum_{i \in D} x_i(t) \leq 1$, let $y_i(t) = x_i(t)$ for all $i \in D$, $y_i(t) = 0$ for all $i \notin D$, and $z(t+1) = ((1 - \sum_{i \in D} x_i(t)) / a_\sigma) e(\sigma)$,*
- (iii) *If $\sum_{i \in D} x_i(t) > 1$ and $x_1(t) \geq 1$, let $y(t) = e(1)$ and $z(t+1) = 0$,*
- (iv) *If $\sum_{i \in D} x_i(t) > 1$ and $x_1(t) < 1$, let $y_i(t) = x_i(t)$ for all $i = 1, \dots, i_o - 1$, $y_{i_o}(t) = 1 - \sum_{i=1}^{i_o-1} x_i(t)$ and $z(t+1) = 0$, where $i_o \in D$ such that $\sum_{i=1}^{i_o-1} x_i(t) < 1$ and $\sum_{i=1}^{i_o} x_i(t) > 1$.*

We can now ask whether the set of optimal programs is identical to the set of Stiglitz programs, and it is perhaps surprising that this is decisively not the case. We present an example of a simple

⁴¹See Footnote 30 and the associated text.

⁴²Recall that without any loss of generality, the machine types have been numbered so that $b_i \geq b_{i+1}$ for all $i = 1, \dots, n$.

economy with a linear felicity function in which at a particular initial stock, the unique Stiglitz program is bad, leave alone optimal. We turn to the details.

The economy has available to it only one type of machine whose (labor) input and output coefficients (a_1, b_1) are given by $(2/5, 1)$, the depreciation rate d by $1/2$, the felicity function by $w(by) = y$, and the initial stock of the machine $x_o = 1/2$. The reduced form of the economy is given by:

$$\begin{aligned}\Omega &= \{(x, x') \in \mathbb{R}_+^2 : (1/2)x + (5/2) \geq x' \geq (1/2)x\}, \\ \Lambda(x, x') &= \{y \in \mathbb{R}_+ : y \leq x \text{ and } y \leq 1 + (1/5)(x - 2x')\} = \{y \in \mathbb{R}_+ : y \leq \min[1 + (1/5)(x - 2x'), x]\}, \\ u(x, x') &= \max\{w(by) : y \in \Lambda(x, x'), (x, x') \in \Omega\} = \min[1 + (1/5)(x - 2x'), x].\end{aligned}$$

Consider the program $\{x(t), y(t)\}$ given by $x(t) = y(t) = 5/6$, with a gross investment of $z(t+1) = x(t+1) - (1-d)x(t) = 5/6 - (1/2)(5/6) = 5/12$, for all $t \in \mathbb{N}$. We claim that this is a stationary optimal program from $x(0) = 5/6$. Towards this end, we show that $(5/6, 5/6)$ is the unique solution to the problem delineated in Definition 5, and hence that $3/4$ is the unique golden-rule stock.

First observe that $u(5/6, 5/6) = \min[1 - (1/5)(5/6), 5/6] = 5/6$, and that $u(x, x') = \min[1 - (1/5)x - (2/5)(x' - x), x]$. Now if $0 \leq x < 5/6$, $u(x, x') < (5/6) = u(5/6, 5/6)$. Thus suppose $x > 5/6$. In this case, $x' \geq x$ implies $1 - (1/5)x - (2/5)(x' - x) \leq 1 - (1/5)x < 5/6$, and hence that $u(x, x') < (5/6) = u(5/6, 5/6)$. The argument is complete.

Next, consider a program such that for all $t \in \mathbb{N}$, $x(4t) = 1 = y(4t)$, $x(4t+1) = 1/2 = y(4t+1)$, $x(4t+2) = 3/2, y(4t+2) = 1$, $x(4t+3) = 3/4 = y(4t+3)$. It is easy to check that this is a program that starts from 1 and returns to it after four periods. It is also easy to see that it is a unique Stiglitz program starting from 1. In terms of Definition 8, $D = \{1\}$, and in three of the four periods of the 4-period cycle, condition (ii) applies and usage and production levels are uniquely set to maintain full employment and no excess capacity. In other words, in these periods, all of the desirable machines are utilized, and all of the remaining labor (none in one of the 3 periods) is allocated to the construction of new machines. In the one remaining period, $4t+2$, there is full employment but also excess capacity.

Now suppose the Stiglitz program defined above is a good program. It is easy to check that $u(1, 1/2) = 1$, $u(1/2, 3/2) = 1/2$, $u(3/2, 3/4) = 1$, and $u(3/4, 1) = 3/4$. Hence for all $n \in \mathbb{N}_+$, $\sum_{t=0}^{4n} [u(x(t), x(t+1)) - (5/6) = -(1/12)]$, a contradiction. From Proposition 5, we can then conclude that the Stiglitz program is not optimal.

Since this is a unique Stiglitz program starting from $x(0) = 1$, the optimal program from $x(0) = 1$, which exists by virtue of Theorem 2, is not a Stiglitz program.

8 Choice of Techniques in Transition: A Linear Felicity Function

In the light of the example presented in Section 7, we can ask for conditions on the RSS model under which the set of Stiglitz programs coincide with the set of optimal programs. We turn to this question.

The point to be noted about each of the two examples presented above is the particular value of the parameter $1 - d - (1/a_1)$. We have already referred to this parameter in Section 5 as ξ_1 , and

it takes the value -1 in the example in Section 6, and the value -2 in the example in Section 7. The sufficient condition for the optimal choice of technique that we present in this section then⁴³ requires that $\xi_\sigma \geq -1$.

Theorem 4 *With a linear felicity function w , and with $1 > \xi_\sigma \geq -1$, any Stiglitz program is an optimal program.*⁴⁴

This theorem is a consequence of the following lemma.

Lemma 3 *With a linear felicity function w , and with $1 > \xi_\sigma \geq -1$, the aggregate value-losses of any Stiglitz program starting from $x(0)$ equal $\Delta(x(0))$.*

The proof of Lemma 3 relies crucially on the sources of value-loss already identified in the proof of Lemma 1. On rewriting (2), we obtain

$$\begin{aligned} \delta(t) &= b\hat{y} - by(t) - \hat{q}(x(t+1) - x(t)) \\ &= c_\sigma(1 - ey(t) - az(t+1)) + \sum_{i=1}^n (c_\sigma - b_i)y_i(t) + \sum_{i=1}^n (c_\sigma - c_i)a_iz_i(t+1) + dqx(t) \\ &= \alpha(t) + \sum_{i \in D} (c_\sigma - b_i)y_i(t) + \sum_{i \notin D} (c_\sigma - b_i)y_i(t) + \sum_{i=1}^n (c_\sigma - c_i)a_iz_i(t+1) + dqx(t) \end{aligned} \quad (8)$$

where $\alpha(t) = c_\sigma(1 - ey(t) - az(t+1))$ is the value-loss from unemployment.⁴⁵ This is a five-fold decomposition⁴⁶ of the value-loss at any time-period: the other four terms concern value losses from incorrect usage and incorrect investment. The proof can now be executed by the comparison, *period by period*, of the magnitudes of the value-losses of the Stiglitz program and those of any other candidate program. We relegate the details to the Appendix, and turn to a

Proof of Theorem 4: Let $\{x^s(t), y^s(t)\}$, with an associated value-loss sequence $\{\delta^s(t)\}$, be a Stiglitz program. Since only one type of machine σ is constructed under a Stiglitz' policy, and since $\xi_\sigma \geq -1$, we can assert that the Stiglitz program is a good program, and to Proposition 6 to assert that therefore it exhibits the average turnpike property. In order to complete the proof, we need to establish that the hypotheses of Proposition 10 are satisfied, which is to say that $\{\delta^s(t)\}$ satisfies $\sum_{t=0}^{\infty} \delta^s(t) = \Delta(x(0))$. An appeal to Lemma 3 then completes the proof. ■

Next we ask whether, under the conditions identified in Theorem 4, a Stiglitz program is uniquely optimal. Towards this end, we can present

Theorem 5 *With a linear felicity function w , and with $-1 < \xi_\sigma < 1$, any optimal program $\{x(t), y(t)\}$ is a Stiglitz program.*

⁴³Note that by default $D = \{1\} = \{\sigma\}$ in each of the one-machine examples considered above.

⁴⁴Note that $\xi_\sigma < 1$. If not, then $d < -(1/a_\sigma)$, a contradiction.

⁴⁵Introduced only for the typographical reason of reducing the length of the expression below.

⁴⁶For a general discussion of the importance of decomposition principles across time in problems of intertemporal resource allocation, see Bliss (1975, Chapter 6).

Before considering the proof of this theorem, we draw the reader's attention to the fact that unlike Theorem 4, Theorem 5 does not cover the case $\xi_\sigma = -1$. Indeed, Theorem 5 is false for this case. In the example presented in Section 6, there is an optimal program which is not a Stiglitz program.⁴⁷ Thus, we need to rule out the case when the optimal program stays in the von Neumann facet but does not converge to the golden-rule values. Towards this end we can present a result which strengthens the conclusions of Theorem 3 in the case of a linear felicity function and also shows them to hold for a Stiglitz program.

Proposition 11 *With a linear felicity function w , and with $-1 < \xi_\sigma < 1$, for a program $\{x(t), y(t)\}$ that is either an optimal or a Stiglitz program, $\lim_{t \rightarrow \infty} y_i(t) = \lim_{t \rightarrow \infty} z_i(t) = 0$ for all $i \neq \sigma$, and $\lim_{t \rightarrow \infty} y_\sigma(t) = \lim_{t \rightarrow \infty} x_\sigma(t) = \hat{x} = 1/(1 + da_\sigma)$, $\lim_{t \rightarrow \infty} z_\sigma(t) = d/(1 + da_\sigma)$.*

We relegate the computational details to the Appendix, and turn to a sharpening of Lemma 3.

Lemma 4 *In a setting with a linear felicity function w , and $-1 \leq \xi_\sigma < 1$, let $\{\delta(t)\}$ be the value-loss sequence of a program that is not a Stiglitz program and $\{\delta^s(t)\}$ the value-loss sequence of a Stiglitz program starting from the same initial stock. Then there exists $\varepsilon > 0$ such that $\sum_{t=0}^{\infty} \delta(t) - \sum_{t=0}^{\infty} \delta^s(t) > \varepsilon$.*

We indicate in the Appendix how the proof of Lemma 4 is a straightforward modification of the computations presented in the proof of Lemma 3. We can now present

Proof of Theorem 5: Suppose to the contrary that there exists an optimal program $\{x(t), y(t)\}$ with an associated value-loss sequence $\{\delta(t)\}$ that is not a Stiglitz' program. Let $\{x^s(t), y^s(t)\}$ be a Stiglitz program starting from $x(0)$ and with an associated value-loss sequence $\{\delta^s(t)\}$. An appeal to Lemma 4 and to Proposition 7 yields for all $T \in \mathbb{N}_+$,

$$\begin{aligned} \sum_{t=0}^T (by(t) - by^s(t)) &= \hat{p}(x^s(T+1) - x(T+1)) + \sum_{t=0}^T \delta^s(t) - \sum_{t=0}^T \delta(t) \\ &< \hat{p}(x^s(T+1) - x(T+1)) - \varepsilon. \end{aligned}$$

We can now assert that $\liminf_{T \rightarrow \infty} \sum_{t=0}^T (by(t) - by^s(t)) > 0$. If not, then we obtain $\limsup_{T \rightarrow \infty} \hat{p}(x^s(T+1) - x(T+1)) > \varepsilon$. Now an appeal to Proposition 11 leads us to conclude that $\hat{p}\varepsilon < 0$, a contradiction. This verifies the truth of the initial claim, and completes the argument that any optimal program is a Stiglitz program. ■

⁴⁷ A detailed verification of this claim would lead us outside the scope of an already long paper; see Khan-Mitra (2002) for details.

9 Choice of Techniques with a Non-Linear Felicity Function: An Example

In Sections 5 and 6, we considered optimal programs in the context of both the long and the short run when the felicity function is linear; here we seek to understand in what way this case is different from the one pertaining to a strictly concave felicity function. Essentially, the situation is this. When future utilities are not discounted, utilities are summed across time, and in the linear case this translates to summing consumption levels across time. Thus, there is no particular disadvantage in postponing part of today's consumption to a future period (or vice versa) so long as the *sum* of the consumption levels in the two periods remains the same and consumption in all other periods are unaffected. In the strictly concave case, in considering such swaps, one has to take into account the original levels of consumption in the two periods, because a *ceteris paribus* transfer of a unit of consumption from a lower consumption level period to the higher one will definitely reduce social welfare.

The scenario in which there *can* be a difference in the choice of techniques is one where the short-run consumption requirements are quite different from the long-run consumption requirements on an optimal program. As we have seen in Section 5, the unique golden-rule type of machine, σ , is the best machine to use for meeting the long-run consumption requirements, regardless of whether the social welfare function is linear or strictly concave. In the short-run, however, the important question is which machine built today will provide the most consumption *tomorrow*, given an available amount of labor for new machine production today, and without taking into account the fact that the machine depreciates. This is clearly qualitatively different from the long-run (golden-rule) problem and points to b_i/a_i rather than to $b_i/(1+da_i)$. When the orderings of these two magnitudes differ, one machine is best for the short-run problem and another machine is best for the long-run problem. This seems to suggest that, in transition, one may produce the machine that is best for the short-run, but asymptotically run it down via depreciation, so that in the long-run, only the golden-rule machine is produced and used.

While the basic intuition is clear, it requires considerable more work to show that such scenarios can indeed occur. First, we present the following

Definition 9 A program $\{x(t), y(t)\}$ with an associated gross investment sequence $\{z(t+1)\}$ is said to be a σ -program if for any $t \in \mathbb{N}$, $z_i(t+1) = 0$ for all $i \neq \sigma$.

By appealing to the example presented in Section 7, we can see that a σ -program is not in general an optimal program. Our question can now be phrased as to whether every optimal program is a σ -program. Towards this end, we present an example of a simple economy in which there are two types of machines ($n = 2$) whose input coefficients vector is given by $a = (2, 3)$, the output coefficients vector by $b = (4, 5)$ and the depreciation rate, d , by 0.45 ($m \equiv (1 - d) = 0.55$). Note that

$$\frac{b_1}{a_1} = 2 > (5/3) = \frac{b_2}{a_2} \text{ and } \frac{b_1}{(1+da_1)} \approx 2.1052 < 2.1276 \approx \frac{b_2}{(1+da_2)},$$

and thus $\sigma = 2$: machines of type 2 constitute the golden-rule stock. The social welfare function, w , is

defined as follows:⁴⁸

$$w(y) = \begin{cases} y - 2 & \text{for } y \geq 2 \\ 1000(y - 2) & \text{for } 0 \leq y < 2 \end{cases}$$

The initial stock of machines is specified as $x_o = (0.5, 0)$.

We know from the analysis of Section 5 that in the long-run only machines of type 2 will be produced and used along an optimal program. We are interested in demonstrating that machines of type 1 will nevertheless be produced in some time period along an optimal program from x . Our method of demonstrating this is to suppose, on the contrary, that machines of type 1 are not produced in *period 1* along an optimal program. We show that a consequence of this is that an optimal program will suffer a large disutility (negative utility, large in absolute value) in either the first or the second period of consumption, which results in a large disutility even in the long-run. We construct a program from x_o which produces machines of type 1 initially, and reaches the golden-rule stock in a finite number (specifically, eight) of periods; it has non-negative utility in all periods, and, of course, (positive) golden-rule utility from period nine onwards. This shows that our hypothesis that machines of type 1 are not produced on the optimal program in period 1 must be false, and completes the demonstration. We relegate the computational details to the Appendix.

In conclusion to this section, note that the felicity w used in the above example is non-linear, but not strictly concave. We can check that all the calculations shown in the Appendix below remain valid with a strictly concave w defined as follows:

$$w(y) = \begin{cases} (53/47)2(y - 2)/(y - 1) & \text{for } y \geq 2 \\ 1000(y - 2) - 0.5(y - 2)^2 & \text{for } 0 \leq y < 2 \end{cases}$$

10 The Price-Support and Other Properties of an Optimal Program

So far we have worked only with the golden-rule price system, and in this section we use McKenzie's price support property presented as Theorem 6 below to make a variety of observations regarding an optimal program: there is a positive investment of machines of type σ for an infinity of time periods, machines valuable in the present were valuable in the past, there is investment in machines of type σ only if they are valuable, that such machines are always valuable, an equation for the prices of produced machines, the price system is bounded and finally, a property of a price-system using the machines of type σ as a numeraire.⁴⁹ Note that none of these results rely on strict concavity of the felicity function; they exploit the average-turnpike property of good programs, and as such on the uniqueness of the golden-rule stock. Thus the standing hypothesis presented as (1) above continues to be the driving force.

We begin the formalities with a price-support property for the RSS model. Since we do not exclude the situation where the economy has no stock of machines, $x_o = 0$, the result is a direct consequence of methods available in McKenzie (1986; Proof of Lemma 1), rather than a corollary. We

⁴⁸Notice the sharp non-linearity in the welfare function near the consumption level of 2. Higher consumption levels are rewarded at a marginal rate of 1, while lower consumption levels are penalized at the marginal rate of 1000.

⁴⁹These observations constitute the content of Propositions 11 to 17 presented below.

relegate to an Appendix the (straightforward) details of how McKenzie's interiority assumptions are fulfilled in our context, and allow his proof to work.

Theorem 6 *Let $\{x(t), y(t)\}$ be an optimal program starting from an arbitrary initial stock, $x_o \in \mathbb{R}_+^n$. Then there exists a sequence $\{p(t)\}_{t=0}^\infty$, $p(t) \in \mathbb{R}_+^n$, such that for all $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$,*

$$w(by(t)) + p(t+1)x(t+1) - p(t)x(t) \geq w(by) + p(t+1)x' - p(t)x.$$

Next we turn to the derivation of properties that can be used to show that an optimal program is a σ -program but are also of independent interest.

Proposition 12 *There exists a sequence $\{t_i\}_{i \in \mathbb{N}_+}$ such that $z_\sigma(t_i) > 0$ for all $i \in \mathbb{N}_+$.*

Proof: Suppose this not to be the case. Then there exists $T \in \mathbb{N}$ such that for all $t \geq T$, $z_\sigma(t) = 0$. Since all machines depreciate at the rate $d \in (0, 1)$, this implies that $x_\sigma(t) \rightarrow 0$ as $t \rightarrow \infty$, and therefore that the time-average of $x_\sigma(t)$, $\bar{x}_\sigma(t) \rightarrow 0$ as $t \rightarrow \infty$. An appeal to Propositions 5 and 6 furnishes a contradiction and completes the argument. ■

Proposition 13 *For any $t \in \mathbb{N}$, and any $i = 1, \dots, n$, $p_i(t+1) > 0 \implies p_i(t) > 0$.*

Proof: For any time-period t and any machine of type i , let $x = x(t) + e_i\varepsilon$, $z = z(t+1)$, $x' = (1-d)x + z$, $y = y(t)$ where $\varepsilon > 0$. Then, $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$. Then from Theorem 6, $p_i(t+1)(1-d)\varepsilon - p_i(t)\varepsilon \leq 0 \implies (1-d)p_i(t+1) \leq p_i(t)$. In particular, if for some $t \in \mathbb{N}$, and $i = (1, \dots, n)$, $p_i(t+1) > 0$, then we must have $p_i(t) > 0$. ■

Proposition 14 *For any $t \in \mathbb{N}$, $z_\sigma(t+1) > 0$ implies $p_\sigma(t) > 0$.*

Proof: Suppose that for some time-period t , $z_\sigma(t+1) > 0$ and $p_\sigma(t) = 0$. Then Proposition 13 implies that $p_\sigma(t+1) = 0$. Pick ε such that $0 < \varepsilon < a_\sigma z_\sigma(t+1)$, and define $x = x(t) + \varepsilon e(\sigma)$, $x' = x(t+1) + ((1-d)x_\sigma - x_\sigma(t+1))e(\sigma)$, and $y = y(t) + \varepsilon e(\sigma)$. Then $ey = ey(t) + \varepsilon$ and $a(x' - (1-d)x) < a(x(t+1) - (1-d)x(t)) - a_\sigma z_\sigma(t+1) < az(t+1) - \varepsilon$. Thus $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$, and from Theorem 6, $w(by(t)) + p(t+1)x(t+1) - p(t)x(t) \geq w(by(t) + b_\sigma\varepsilon) + p(t+1)x' - p(t)x$. This yields $w(by(t)) \geq w(by(t) + b_\sigma\varepsilon)$, a contradiction to the fact that w is strictly increasing. ■

Proposition 15 *For any $t \in \mathbb{N}$, $p_\sigma(t) > 0$.*

Proof: Suppose that there exists $t \in \mathbb{N}$ such that $p_\sigma(t) = 0$. From Proposition 12, there exists a time-period $t_i > t$ such that $z_\sigma(t_i) > 0$. From Proposition 14, this implies that $p_\sigma(t_i - 1) > 0$. If $t_i = t + 1$, we obtain a contradiction. If $t_i > t + 1$, we make as many (finite) appeals to Proposition 13 as is necessary to obtain a contradiction. ■

Proposition 16 *For any $t \in \mathbb{N}$, and any $i = 1, \dots, n$, $x_i(t) > y(t) \implies p_i(t+1)(1-d) = p_i(t)$.*

Proof: Let $\varepsilon = x_i(t) > 0$, and define $x = x(t) - \varepsilon e_i$, $y = y(t)$, and $x' = x(t+1) - (1-d)\varepsilon e(i)$. Then, it can be easily checked that $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$. Then from Theorem 6, we obtain $(1-d)p_i(t+1)\varepsilon \geq p_i(t)\varepsilon$. We can now complete the proof of the claim as in the proof of Proposition 13. ■

Proposition 17 $\liminf_{t \rightarrow \infty} \|p(t)\| < \infty$.

Proof: For any $t \in \mathbb{N}$, and any $i = 1, \dots, n$, define $x = x(t)$, $z = z(t+1) + (ey(t)/a_i)e^i$, $x' = z + (1-d)x$, $y = 0$. Then $az + ey = az(t+1) + ey(t)$, and hence $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$. We can now appeal to Theorem 6 to obtain $p_i(t+1)(ey(t)/a_i) \leq w(by(t)) - w(0) \leq w(be) - w(0)$. Thus, there is $M > 0$ such that for all $t \in \mathbb{N}$, $\|p(t+1)\|(ey(t)) \leq M$. If $\liminf_{t \rightarrow \infty} \|p(t)\| = \infty$, then we must have $ey(t) \rightarrow 0$ as $t \rightarrow \infty$. But, then, we contradict Proposition 5 and establish the claim. ■

Proposition 18 For any $i \in 1, \dots, n$, $z_i(t+1) > 0 \implies p_i(t+1)/p_\sigma(t+1) \geq a_i/a_\sigma$.

Proof: Suppose that for any $t \in \mathbb{N}$, and any $i = 1, \dots, n$, $z_i(t+1) > 0$. Define $x = x(t)$, $y = y(t)$, $z = z(t+1) - z_i(t+1)e(i) + (z_i(t+1)(a_i/a_\sigma))e(\sigma)$, $x' = (1-d)x + z$. Since $az + ey = az(t+1) + ey(t) \leq 1$, $(x, x') \in \Omega$ and $y \in \Lambda(x, x')$. We can now appeal to Theorem 6 to obtain $p_i(t+1)z_i(t+1) \geq p_\sigma(t+1)z_i(t+1)(a_i/a_\sigma)$. Since $z_i(t+1) > 0$, the proof of the claim is complete. ■

11 Choice of Techniques in Transition: A Sufficient Condition

In keeping with our discussion of the short run versus the long in Section 9, we assume that a unique type of machine σ is best irrespective of the time horizon under which the planning exercise is being conducted, and furthermore, that it requires for its production more labour than a machine of any other type.⁵⁰

Assumption 1 $(b_\sigma/a_\sigma) > \max_{i \neq \sigma} (b_i/a_i)$ and $a_\sigma > \max_{i \neq \sigma} a_i$.

We can now show that under this congruence, the golden-rule machine σ is the only type that is produced.

Theorem 7 Under Assumption 1, an optimal program is a σ -program

Before presenting a formal proof, we provide a heuristic argument. Suppose the theorem is false, which is to say that a type of machine other than the σ , say the first, is constructed at a time period, again say the first. We can show that this is an impossibility by constructing a feasible program from the same initial stock of machines which gives higher welfare, and thereby contradicts, the optimality of the original program.

As the first step of the argument, consider the case in which the first machine fully used in the (next) second period. Consider an alternative program in which a slightly lesser amount, say ε , where

⁵⁰In these requirements we go beyond the assumptions of Stiglitz (1968); also see Footnote 28 above.

$z_1^*(1) > \varepsilon > 0$, of this type of machine is constructed in the first period. Use the labour thereby saved to produce an additional amount of the machine of type σ . In terms of accounting, the labour saving in the amount εa_1 is translated to obtain an additional amount of $\varepsilon(a_1/a_\sigma)$ machines of type σ . Since all of the machines of the first type were being used in the consumption sector in the next period, there is a loss of second-period output in the amount εb_1 . On the other hand, by using the additional machines of type σ now made available in the second period, there is a gain of second-period output in the amount $(\varepsilon(a_1/a_\sigma))b_\sigma$. We now appeal to Assumption 2 to observe that $(b_1/a_1) < (b_\sigma/a_\sigma) \implies \varepsilon b_1 < \varepsilon(a_1/a_\sigma)b_\sigma$. Hence, in our alternative program, simply by a “marginal” reallocation of labor in the investment sector in the first period, we obtain (strictly) more of the consumption good in the second period without changing the availability of the consumption in any other period.

However, for the completion of the argument even in this first step, some additional accounting is necessary. How do we know that there is enough labour to run the additional machines of type σ that have been constructed in the comparison program? It is precisely here that we utilize the standing hypothesis (expressed as (1)) and the simplification that all of the machines of the first type are being used in the second period. Since we now have fewer machines of this type in the amount ε , the labour saving is also of the amount ε . Furthermore, the labour requirements for the machines of type σ , are $\varepsilon(a_1/a_\sigma)$ – recall that the capital-labour ratio for all types is unity. We now appeal to Assumption 1 to observe that $a_1 < a_\sigma \implies \varepsilon(a_1/a_\sigma) < \varepsilon$. Thus the comparison program is indeed feasible, and it contradicts the optimality of the original program. The first step of the argument is complete.

Now let us suppose that even though machines of the first type are constructed in the first period, not all of them are used in the second period. The relevant observation is that in this case, there is no construction of only the machines of type σ in the second period. If there was such construction, one would consider another comparison program in which again a slightly lesser amount, say ε , where $z_1^*(1) > \varepsilon > 0$, of this type of machine is constructed in the first period, and the labor thereby saved is used to produce an additional amount of the machine of type σ . We now have an extra (albeit depreciated) amount of machines of type σ coming into the second period, freeing the labor used for their construction in the second period. This labor can be used to produce additional amount of the consumption good, thereby again contradicting the optimality of the original program.

Indeed, in the case under consideration of machines of the first type being constructed in the first period and not all of them being used in the second, there is no construction of the machines of type σ in the third period. The logic is the same: under an alternative regime of less machines of the first type and more of type σ in the first period, and the use of the latter in the third period to save on construction costs yields more of the consumption good in that period.

The general claim is now clear: if machines of a type other than σ are produced in a period and not all used in a subsequent period, there is no construction of machines of type σ in *any* subsequent period. The direction of the argument is also clear. The only difference between the second, third or n^{th} period relates to the changes in the quantitative magnitudes affected by depreciation – less of machines of type σ survive to the thousandth period as do to the second. Since all we need to produce

a contradiction is to show that the comparison program is better than the optimal one, and not how much better, the precise amount of the consumption good that can be obtained is of no significance whatsoever.

But with the general claim established, we again have a contradiction to the optimality of the original program. We know that the optimal program converges asymptotically to the golden rule levels of the machine of type σ ; and if there is no production of these types of machines along an optimal program, the existence of a positive rate of depreciation contradicts its optimality. The heuristic argument is complete.

We now turn to a (succinct) formal expression of this basic intuition. First, a preliminary result.

Proposition 19 *For any time-period $t \in \mathbb{N}$, and any $i \in 1, \dots, n$,*

$$x_i(t) > 0 \implies (1-d)[(b_i/b_\sigma)p_\sigma(t+1) - p_i(t+1)] \leq [(b_i/b_\sigma)p_\sigma(t) - p_i(t)].$$

Proof: We consider two cases: (i) $y_i(t) < x_i(t)$, and (ii) $y_i(t) = x_i(t)$.

Under case (i), let $\varepsilon = x_i(t) - y_i(t)$, $x = x(t) + \varepsilon e(i)$, $z = z(t+1)$, $y = y(t)$ and $x' = (1-d)x + z$. Then $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$, and using Theorem 6, we obtain $(1-d)p_i(t+1)\varepsilon \leq p_i(t)\varepsilon$. An appeal Proposition 16 yields $(1-d)p_i(t+1) = p_i(t)$, and

$$(b_i/b_\sigma)[(1-d)p_\sigma(t+1) - p_\sigma(t)] \leq [(1-d)p_i(t+1) - p_i(t)].$$

Next, consider case (ii) where $y_i(t) = x_i(t) > 0$. Let $0 < \varepsilon < x_i(t)$, $\nu = (b_i/b_\sigma)\varepsilon$, and note that from Assumption 1 that $(b_i/b_\sigma) \leq (a_i/a_\sigma) \leq 1$, which implies that $\nu \leq \varepsilon$. Define $x = x(t) - \varepsilon e_i + \nu e_\sigma$, $y = y(t) - \varepsilon e(i) + \nu e(\sigma)$, $z = z(t+1)$, $x' = (1-d)x + z$, and note that $ey \leq ey(t)$, $az = az(t+1)$ and $0 \leq y \leq x$. Thus, $(x, x') \in \Omega$, and $y \in \Lambda(x, x')$. Then from Theorem 6, we obtain $p_\sigma(t+1)(1-d)\nu - p_i(t+1)(1-d)\varepsilon - p_\sigma(t)\nu + p_i(t)\varepsilon \leq 0$. This establishes the claim after transposing terms. ■

We can now furnish a proof of Theorem 7.

Proof: Suppose that for some time-period T , and any $i = 1, \dots, n$, $z_i(T+1) > 0$. Then by Propositions 12, 18 and Assumption 1, we obtain

$$p_i(T+1) \geq p_\sigma(T+1)(a_i/a_\sigma) > p_\sigma(T+1)(b_i/b_\sigma).$$

Also, we must have $x_i(t+1) > 0$ for all $t \geq T$. Then, iterating on the result presented as Proposition 18, we obtain $p_i(t+1) \rightarrow \infty$ as $t \rightarrow \infty$. But, this contradicts Proposition 17. ■

12 Concluding Remarks

If we leave aside the methodological reformulation of the RSS model in the vocabulary of the Gale-McKenzie reduced form, we see the principal contribution of this work in its delineation of the relationship among optimal, Stiglitz and σ -programs of the RSS model. For this, the three simple examples are

of decisive importance, but they may also be of independent interest for future investigations of related issues that remain open. In conclusion, we briefly single out four of these.

Throughout this paper, we have emphasized the sharp and surprising differences that arise between our results and those of Stiglitz: in particular, the parameters ξ_i do not appear in his paper. It is of some importance to settle the issue as to whether this is a consequence of the different treatment of time in the two papers, discrete versus continuous, or to the more primitive methods that Stiglitz had to work with in 1968.⁵¹

In his retrospective, Stiglitz (1990, p. 61) observes the “greatest challenge facing growth theory”:

We now need to understand better the relationship between the short-run behavior of the economy – in which imperfect information and imperfect competition in financial, labor, and product markets will play a central role – and its long-run dynamics.

It is interesting that this remains a challenge even for a planning framework without uncertainty and the stark simplicity of the specifications of the RSS model, technological and otherwise. The complete characterization of the optimal path in the short-run remains an open problem when the planners’ felicity function is linear but $\xi_\sigma < -1$, and when it is strictly concave.

We have drawn attention to the conceptual similarities between our work and that of Mitra-Wan (1987) on the economics of forestry.⁵² It would be of interest if the analogy is analytically explored in a synthesis based on the multi-sectoral setting of Koopmans (1971) and Koopmans-Hansen (1972). This work also gives a singular prominence to Kuhn-Tucker theory.

Finally, the results reported in this paper are a testimony to the strength of the standing hypothesis that there is a unique type of machine that minimizes effective labor costs and simultaneously maximizes the steady state consumption; see (1) above. It would be of interest to examine how the results are modified without this hypothesis.

13 Appendix

We begin with a

Proof of Lemma 3: Let $\{x^s(t), y^s(t)\}$ be a Stiglitz program with an associated gross investment sequence $\{z^s(t+1)\}$ and an associated value-loss sequence $\{\delta^s(t)\}$. We shall denote corresponding values of any other (candidate) program starting from $x^s(0)$ by $\{x(t), y(t)\}$, $\{z^s(t+1)\}$, and $\{\delta(t)\}$. We shall consider three different ranges for the value of ξ_σ and make repeated use of (8) and of Definition 8.

Case (i) $0 < \xi_\sigma < 1$:

Suppose that for any $t \in \mathbb{N}$, $0 \leq \sum_{i \in D} x_i^s(t) \leq 1$. In this case, we see from (ii) of Definition 8 that $y_i^s(t) = x_i^s(t)$ for all $i \in D$, $y_i^s(t) = 0$ for all $i \notin D$, and $z^s(t+1) = (1/a_\sigma) \left(1 - \sum_{i \in D} x_i^s(t)\right) e(\sigma)$. We leave it to the reader to check that $(x^s(t), x^s(t+1)) \in \Omega$ and that $y^s(t) \in \Lambda(x^s(t), x^s(t+1))$. On substituting these values

⁵¹See Footnotes 9 and 22 above, and for preliminary work on this question, Khan-Mitra (2003).

⁵²See Footnotes 3 and 18.

in (8), we obtain that⁵³

$$\begin{aligned}\delta^s(t) &= \sum_{i \in D} (c_\sigma - b_i) x_i^s(t) + dq x^s(t) = \sum_{i \in D} (c_\sigma - b_i + dq_i) x_i^s(t) + d \sum_{i \notin D} q_i x_i^s(t) \\ &= \sum_{i \in D} (c_\sigma - c_i) x_i^s(t) + d \sum_{i \notin D} q_i x_i^s(t).\end{aligned}$$

Again from (8) we obtain⁵⁴

$$\begin{aligned}\delta(t) &\geq \sum_{i \in D} (c_\sigma - b_i) y_i(t) + dq x(t) \geq \sum_{i \in D} (c_\sigma - b_i) x_i(t) + dq x(t) \\ &= \sum_{i \in D} (c_\sigma - b_i + dq_i) x_i(t) + d \sum_{i \notin D} q_i x_i(t) = \sum_{i \in D} (c_\sigma - c_i) x_i(t) + d \sum_{i \notin D} q_i x_i(t).\end{aligned}$$

Next, we claim that for all $t \in \mathbb{N}$, $x_i(t) \geq x_i^s(t)$ for all $i \neq \sigma$. Since the candidate program starts from the same initial stock as the Stiglitz program, the claim holds for $t = 0$. Suppose it to be true for any $t \in \mathbb{N}$, in keeping with the induction hypothesis. Then

$$x_i^s(t+1) = (1-d)x_i^s(t) \leq x_i^s(t) \leq x_i(t) \leq x_i(t) + z_i(t+1) = x_i(t+1).$$

Given the standing hypothesis, it is clear that for all $t \in \mathbb{N}$, $\delta^s(t) \leq \delta(t)$. Thus, we need only to verify that the Stiglitz program is feasible in the sense that once in the range $0 \leq \sum_{i \in D} x_i^s(t) \leq 1$, the program always remains in it. We proceed by induction. For any $t \in \mathbb{N}$, note that

$$x_i^s(t+1) = \begin{cases} (1-d)x_i^s(t) & \text{for all } i \neq \sigma \\ (1-d)x_\sigma^s(t) + (1/a_\sigma)(1 - \sum_{i \in D} x_i^s(t)) & \text{for } i = \sigma \end{cases} \quad (9)$$

Since $0 \leq \sum_{i \in D} x_i^s(t) \leq 1$, we obtain from (9) that $z^\sigma(t+1) = x_\sigma^s(t+1) - (1-d)x_\sigma^s(t) \geq 0$, and that

$$\begin{aligned}\sum_{i \in D} x_i^s(t+1) &= (1-d) \sum_{i \in D} x_i^s(t) + \frac{1}{a_\sigma} \left(1 - \sum_{i \in D} x_i^s(t)\right) \\ &= \left(1-d - \frac{1}{a_\sigma}\right) \sum_{i \in D} x_i^s(t) + \frac{1}{a_\sigma} = \xi_\sigma \sum_{i \in D} x_i^s(t) + \frac{1}{a_\sigma}.\end{aligned} \quad (10)$$

Given the possible values of ξ_σ , we obtain $0 < \sum_{i \in D} x_i(t+1) < 1$.

We can now collect these steps to assert that for all $t \in \mathbb{N}$, $\delta^s(t) \leq \delta(t)$, and hence that $\sum_{t=0}^\infty \delta^s(t) = \Delta(x^s(0))$.

Next we turn to the case when for any $t \in \mathbb{N}$, $\sum_{i \in D} x_i^s(t) > 1$, $x_1^s(t) < 1$. In this case, we see from (iv) of Definition 8 that $y_i^s(t) = x_i^s(t)$ for all $i < i_o$, $y_{i_o}^s(t) = 1 - \sum_{i=1}^{i_o-1} x_i^s(t)$, $y_i^s(t) = 0$ for all $i > i_o$, and that $z_i^s(t+1) = 0$ for all i . We leave it to the reader to check that $(x^s(t), x^s(t+1)) \in \Omega$ and that $y^s(t) \in \Lambda(x^s(t), x^s(t+1))$. On substituting these values in (8), we obtain that

$$\begin{aligned}\delta^s(t) &= \sum_{i=1}^{i_o-1} (c_\sigma - b_i) x_i^s(t) + (c_\sigma - b_{i_o}) \left(1 - \sum_{i=1}^{i_o-1} x_i^s(t)\right) + dq x^s(t) \\ &= \sum_{i=1}^{i_o-1} (c_\sigma - c_i) x_i^s(t) + (c_\sigma - b_{i_o}) \left(1 - \sum_{i=1}^{i_o-1} x_i^s(t)\right) + d \sum_{i \geq i_o} q_i x_i^s(t),\end{aligned}$$

⁵³Note that in the third equality we use the identity referred to in Footnote 36 above. We shall not draw attention to this in the sequel.

⁵⁴We rely on the standing hypothesis (1) and on the definition of desirable machines, in addition to the feasibility of the program.

and for any other (candidate) program with $D_o = D/\{1, \dots, i_o\}$ that

$$\begin{aligned}
\delta(t) &\geq \sum_{i=1}^{i_o-1} (c_\sigma - b_i) y_i(t) + (c_\sigma - b_{i_o}) y_{i_o}(t) + \sum_{i \in D_o} (c_\sigma - b_i) y_i(t) + dqx(t) \\
&\geq \sum_{i=1}^{i_o-1} (c_\sigma - b_i) x_i(t) + (c_\sigma - b_{i_o}) y_{i_o}(t) + \sum_{i \in D_o} (c_\sigma - b_i) y_i(t) + dqx(t) \\
&= \sum_{i=1}^{i_o-1} (c_\sigma - c_i) x_i(t) + (c_\sigma - b_{i_o}) y_{i_o}(t) + \sum_{i \in D_o} (c_\sigma - b_i) y_i(t) + d \sum_{i \geq i_o} q_i x_i(t).
\end{aligned}$$

Now by hypothesis, for all $i \in D_o$,

$$b_i \leq b_{i_o} \implies (c_\sigma - b_{i_o}) \leq (c_\sigma - b_i) \implies \sum_{i \in D_o} (c_\sigma - b_i) y_i(t) \geq (c_\sigma - b_{i_o}) \sum_{i \in D_o} y_i(t).$$

Hence we obtain

$$\begin{aligned}
(c_\sigma - b_{i_o}) y_{i_o}(t) + \sum_{i \in D_o} (c_\sigma - b_i) y_i(t) &\geq (c_\sigma - b_{i_o}) \sum_{i \in D_o \cup i_o} y_i(t) \geq (c_\sigma - b_i) \sum_{i \in D_o \cup i_o} x_i(t). \\
\sum_{i \in D_o \cup i_o} x_i(t) &\geq \sum_{i \in D_o \cup i_o} x_i^s(t) \geq (1 - \sum_{i=1}^{i_o-1} x_i^s(t)).
\end{aligned}$$

Now $\sum_{i \in D} x_i^s(0) > 1$ implies that there exists a first $t_1 \in T$ such that $\sum_{i \in D} x_i^s(t_1) \leq 1$. For the Stiglitz program, we know that for all $t \in \mathbb{N}, t < t_1$, $z_i^s(t+1) = 0$ for all i , and hence that $x^s(t) \leq x(t)$. In particular, $x_1^s(t) < 1$ for all $t \in \mathbb{N}, t < t_1$. This implies that for all $t \in \mathbb{N}, t < t_1$, $\delta^s(t) \leq \delta(t)$. [Note that i_o may vary with t , but given our period-by-period verification, it is of no consequence.] But for $t \geq t_1$, we are in the case considered earlier, and hence we can assert that for all $t \in \mathbb{N}$, $\delta^s(t) \leq \delta(t)$, and hence that $\sum_{t=0}^{\infty} \delta^s(t) = \Delta(x^s(0))$.

Next we turn to the case when for any $t \in \mathbb{N}$, $\sum_{i \in D} x_i^s(t) > 1$, $x_1^s(t) > 1$. In this case, we see from (iii) of Definition 8 that $y_1^s(t) = 1, y_i^s(t) = 0$ for all $i \neq 1$, and that $z_i^s(t+1) = 0$ for all i . We leave it to the reader to check that $(x^s(t), x^s(t+1)) \in \Omega$ and that $y^s(t) \in \Lambda(x^s(t), x^s(t+1))$. On substituting these values in (8), we obtain that

$$\delta^s(t) = (c_\sigma - b_1) + dqx^s(t),$$

and for any other (candidate) program that

$$\delta(t) \geq \sum_{i \in D} (c_\sigma - b_i) y_i(t) + dqx(t) \geq \sum_{i \in D} (c_\sigma - b_i) x_i(t) + dqx(t).$$

Now by hypothesis

$$b_i \leq b_1 \implies (c_\sigma - b_1) \leq (c_\sigma - b_i) \implies \sum_{i \in D} (c_\sigma - b_i) y_i(t) \geq (c_\sigma - b_1) \sum_{i \in D} y_i(t) \geq (c_\sigma - b_1) \sum_{i \in D} x_i(t).$$

Since both programs start from the same initial stock and $x_1^s(t) > 1$ implies $x_i^s(t-r) > 1$ for all $r = 0, \dots, t-1$, $x_i(t) \geq x_i^s(t)$ for all i . Hence $(c_\sigma - b_i) \sum_{i \in D} x_i(t) \geq (c_\sigma - b_1) \sum_{i \in D} x_i^s(t) > (c_\sigma - b_1)$. Thus it is clear that if $x_1^s(0) > 1$, there exists $t_1 \in T$ such that $x_1^s(t_1) \leq 1$. We have already seen that $\delta^s(t) \leq \delta(t)$ for all $t < t_1$. For all $t \geq t_1$, either $\sum_{i \in D} x_i^s(t_1) \leq 1$, in which case we appeal to the first case, or $\sum_{i \in D} x_i^s(t_1) > 1$, in which case we appeal to the second case and complete the demonstration that for all $t \in \mathbb{N}$, $\delta^s(t) \leq \delta(t)$. Hence $\sum_{t=0}^{\infty} \delta^s(t) = \Delta(x^s(0))$.

Case (ii) $0 > \xi_\sigma > -1$:

Suppose that for any $t \in \mathbb{N}$, $(1-d) \leq \sum_{i \in D} x_i^s(t) \leq 1$. On examining the argument for this subcase within case (i) above, we see that the value of ξ_σ is used only to verify the feasibility of the Stiglitz program in Equation (10). However, with $0 > \xi_\sigma > -1$, $\sum_{i \in D} x_i^s(t) \leq 1$ implies $\xi_\sigma \sum_{i \in D} x_i^s(t) \geq \xi_\sigma$, and therefore

$$\sum_{i \in D} x_i^s(t+1) = \xi_\sigma^s \sum_{i \in D} x_i^s(t) + (1/a_\sigma) \geq \xi_\sigma + (1/a_\sigma) = (1-d).$$

Furthermore, $\sum_{i \in D} x_i^s(t) \geq (1-d)$ implies $\xi_\sigma \sum_{i \in D} x_i^s(t) \leq \xi_\sigma(1-d)$, and therefore

$$\sum_{i \in D} x_i^s(t+1) = \xi_\sigma \sum_{i \in D} x_i^s(t) + (1/a_\sigma) \leq \xi_\sigma(1-d) + (1/a_\sigma) = (1-d)^2 + (d/a_\sigma).$$

Since $\xi_\sigma = 1-d-(1/a_\sigma) > -1$, $(1/a_\sigma) < 2-d$, which implies that $(d/a_\sigma) < (2-d)d$ and hence that $(1-d)^2 + (d/a_\sigma) < (1-d)^2 + (2-d)d = 1$. We have thus shown that once in the range $(1-d) \leq \sum_{i \in D} x_i^s(t) \leq 1$, the program always remains in it.

For the other two subcases in the argument within case (i) above, we note that everything hinges on the value of d and ξ_σ plays no role. Thus the only case to be considered is when $0 \leq \sum_{i \in D} x_i^s(t) < (1-d)$. Here there are two possibilities: either $\sum_{i \in D} x_i^s(t+1) \leq 1$ or greater than 1. Since we have already shown that $\sum_{i \in D} x_i^s(t+1) \geq (1-d)$, under the first possibility there is nothing further to be shown. Under the second, there exists a first $t_1 \in \mathbb{N}$, $t_1 > t$ such that $\sum_{i \in D} x_i^s(t_1) \leq 1$. Since $\sum_{i \in D} x_i^s(t_1) = (1-d) \sum_{i \in D} x_i^s(t_1-1) > (1-d)$. The demonstration is complete.

Case (iii) $\xi_\sigma = 0$:

This is a trivial case where $(1/a) = 1/(1+ad) = \hat{x}$. Suppose that for any $t \in \mathbb{N}$, $0 \leq \sum_{i \in D} x_i^s(t) \leq 1$. Then we see from Equation (10) that $\sum_{i \in D} x_i^s(t) = 1/a$. For the other subcases, the argument is identical to that presented under case (i).

We have now covered all possible cases, and the proof of the lemma is complete. \blacksquare

Proof of Proposition 11: We shall prove the proposition for an optimal program; identical computations hold for a Stiglitz program. By virtue of Theorem 3, we know that eventually machines of type other than σ are never built in any optimal program, and hence for all $i \neq \sigma$, $\lim_{t \rightarrow \infty} x_i(t) = 0$. Since $y_i(t) \leq x_i(t)$, the proof of the first claim is complete. In the case of machines of type σ , we appeal to Lemma 2 to characterize the behavior of paths *on* the von Neumann facet. A subsequent appeal to Theorem 3 then proves the second claim. Towards the first requirement, note that

$$\begin{aligned} x_\sigma(t+1) &= (1-d)x_\sigma(t) + z_\sigma(t+1) = (1-d)x_\sigma(t) + (1/a_\sigma)(1 - \sum_{i \in D} x_i(t)) \\ &= \xi_\sigma x_\sigma(t) - (1/a_\sigma)(1 - \sum_{i \in D/\{\sigma\}} x_i(t)) = \xi_\sigma x_\sigma(t) - (1/a_\sigma)(1 - (1-d)^t \sum_{i \in D/\{\sigma\}} x_i(0)). \end{aligned}$$

Let $g(t)$ denote $(1/a_\sigma)(1 - (1-d)^t \sum_{i \in D/\{\sigma\}} x_i(0))$. We can then appeal to Elaydi (1999; p.4) to obtain

$$\begin{aligned} x_\sigma(t) &= \xi_\sigma^t x_\sigma(0) + \sum_{\tau=0}^{t-1} \xi_\sigma^{t-\tau-1} g(\tau) = \xi_\sigma^t x_\sigma(0) + (1/a_\sigma) \sum_{\tau=0}^{t-1} \xi_\sigma^{t-\tau-1} (1 - (1-d)^\tau \sum_{i \in D/\sigma} x_i(0)) \\ &= (1/a_\sigma) \sum_{\tau=0}^{t-1} \xi_\sigma^\tau + \xi_\sigma^t x_\sigma(0) - (1/a_\sigma) \sum_{\tau=0}^{t-1} \xi_\sigma^{t-\tau-1} (1-d)^\tau \sum_{i \in D/\{\sigma\}} x_i(0). \end{aligned}$$

Since $-1 < \xi_\sigma < 1$, this yields $\lim_{t \rightarrow \infty} x_\sigma(t) = 1/(1+da_\sigma)$, and hence that $\lim_{t \rightarrow \infty} y_\sigma(t) = 1/(1+da_\sigma)$ and $\lim_{t \rightarrow \infty} z_\sigma(t) = da_\sigma/(1+da_\sigma)$. \blacksquare

Proof of Lemma 4: Since the program $\{(x(t), y(t))\}$ starts from the same initial stock as a Stiglitz' program, and there exists a first time period $t_1 \in \mathbb{N}$, at which it departs from the Stiglitz program, we can appeal to the

computations of $\delta(t_1)$ and $\delta^s(t_1)$ in each of the relevant cases delineated in the proof of Lemma 3 to show that they will hold with strict inequality. This furnishes a proof of the claim. \blacksquare

Computations of the example in Section 9: First we define a program $\{x(t), z(t)\}$ from x_o which reaches the golden-rule stock in a finite number of periods.

$$\begin{aligned}
x(0) &= (0.5, 0) = y(0), \quad az(1) = (0.2, 0.3), \quad z(1) = (0.1, 0.1); \\
x(1) &= (0.375, 0.1) = y(1), \quad az(2) = (0.09, 0.435), \quad z(2) = (0.045, 0.145); \\
x(2) &= (0.25125, 0.2) \equiv (\alpha, 0.2) = y(2), \quad az(3) = (0, 0.54), \quad z(3) = (0, 0.18); \\
x(3) &= (m\alpha, 0.29), \quad y(3) = (0.1375, 0.29), \quad az(4) = (0, 0.57), \quad z(4) = (0, 0.19); \\
x(4) &= (m\alpha^2, 0.3495), \quad y(4) = (0.075, 0.34), \quad az(5) = (0, 0.57), \quad z(4) = (0, 0.19); \\
x(5) &= (m\alpha^3, 0.382225), \quad y(5) = (0.04, 0.38), \quad az(6) = (0, 0.579), \quad z(6) = (0, 0.193); \\
x(6) &= (m\alpha^4, 0.40322375) = (m\alpha^4, b), \quad y(6) = (0, 0.4), \quad az(7) = (0, 0.6), \quad z(7) = (0, 0.2); \\
x(7) &= (m\alpha^5, mb + 0.2), \quad y(7) = (0, 0.4), \quad az(8) = (0, 0.6), \quad z(8) = (0, 0.2); \\
x(8) &= (m\alpha^6, 1/2.35) \equiv (m\alpha^6, \beta), \quad y(8) = (0, \beta), \quad az(9) = (0, 1 - \beta), \quad z(9) = (0, 0.45\beta);
\end{aligned}$$

and for all $t \geq 9$,

$$x(t) = (m\alpha^{t-2}, 1/2.35), \quad y(t) = (0, 1/2.35), \quad az(t+1) = (0, 1.35/2.35), \quad z(t+1) = (0, 0.45/2.35).$$

It can be checked (although it is tedious to do so) that the above sequence defines a program from x_o . Note that for all $t \in \mathbb{N}$, the consumption in period $(t+1)$ is given $by(t)$. Hence $by(0) = 2$, $by(1) = 2$, $by(2) = 2.005$, $by(3) = 2$, $by(4) = 2$, $by(5) = 2.06$, $by(6) = 2$, $by(7) = 2$, $by(t) = 5/2.35 \approx 2.1276$ for all $t \geq 8$. Thus, the utility along this program is non-negative at all dates, and $u(x(t), x(t+1)) = u(x^*, x^*) > 0$ for all $t \geq 8$. Using this, we have for all $T \geq 8$,

$$\sum_{t=0}^T [u(x(t), x(t+1)) - u(x^*, x^*)] \geq -8u(x^*, x^*) \geq -8. \quad (11)$$

Now, consider the optimal program $\{x'(t), x'(t+1)\}$ from x_o , and suppose that machines of type 1 are not produced at all in period 1; that is, $z'_1(1) = 0$. Denote $y'_1(0)$ by λ , and note that $\lambda \leq 0.5$. We split up our analysis into two cases: (i) $\lambda \leq 0.49$; (ii) $\lambda > 0.49$.

In case (i), $c'(1) = by'(0) = 4\lambda \leq 1.96$, so that $u(x'(0), x'(1)) \leq -40$. We also have $az'(1) \leq (0, 1)$, and $z'(1) \leq (0, 1/3)$. Thus, $y'(1) \leq x'(1) \leq (0.275, 1/3)$, and $c(1) = by'(1) < 3$, so that $u(x'(1), x'(2)) < 1$. Thus, in this case, we have:

$$\sum_{t=0}^1 [u(x'(t), x'(t+1)) - u(x^*, x^*)] \leq -39. \quad (12)$$

In case (ii), $c'(1) = by'(0) \leq 4(0.5) = 2$, so that $u(x'(0), x'(1)) \leq 0$. We also have $az'(1) \leq (0, 0.51)$, and so $z'(1) \leq (0, 0.17)$. Thus, $y'(1) \leq x'(1) \leq (0.275, 0.17)$, and $c'(1) = by'(1) \leq 1.95$, so that $u(x'(1), x'(2)) \leq -50$. Thus, in this case, we have:

$$\sum_{t=0}^1 [u(x'(t), x'(t+1)) - u(x^*, x^*)] \leq -50. \quad (13)$$

We now obtain a bound for the sum of utilities (minus the golden rule utility) on this program from time period 2 onwards. Towards this end, appeal to Lemma 1 to obtain for $T \geq 2$,

$$\sum_{t=2}^T [u(x'(t), x'(t+1)) - u(x^*, x^*)] \leq p^* x'(2) + p^* x^*. \quad (14)$$

From the labor constraint, we know that $ax'(2) \leq 1 + (1-d)ax'(1)$, and in both cases (i) and (ii), we have $(1 + (1-d)ax'(1)) \leq 1.8525 < 2$, so that $ax'(2) < 2$, and $x'(2) \leq (1, 2/3)$. Also, $p^* \leq (5, 7)$, so $p^* x' \leq 10$, and

$p^* x^* \leq 3$. Using this in (14), we obtain

$$\sum_{t=2}^T [u(x'(t), x'(t+1)) - u(x^*, x^*)] \leq 13. \quad (15)$$

Using (12) and (15) in case (i), and (13) and (15) in case (ii), we obtain in either case for $T \geq 2$

$$\sum_{t=0}^T [u(x'(t), x'(t+1)) - u(x^*, x^*)] \leq -26. \quad (16)$$

Using (11) and (16), we get for $T \geq 8$ (in both cases):

$$\sum_{t=0}^T [u(x(t), x(t+1)) - u(x'(t), x'(t+1))] \geq 18,$$

which contradicts the optimality of $\{x'(t), y'(t)\}$ from x_o . This establishes that $z'_1(1) > 0$; that is, some machines of type 1 are produced in period 1 along the optimal program. ■

Proof of Theorem 6: The result is (part of) the assertion of Lemma 1 of McKenzie (1986). McKenzie works with a general setting in which the transition probability set and the reduced form utility function are allowed to change over time. He utilizes assumptions I, II and III to derive his assertions. Assumption I is a requirement that $u(\cdot, \cdot)$ is concave and closed and that Ω is convex. Assumption II is the requirement that there exists $\zeta_o \in \mathbb{R}_+$ such that for all $(x, x') \in \Omega$, $x < \zeta_o$, there exists $\zeta \in \mathbb{R}_+$ such that $x' < \zeta$. The specifications on $u(\cdot, \cdot)$ and on Ω presented in Section 2.4 and Proposition 2 above guarantee that I and II are fulfilled in our setting.

Assumption III is also a hypothesis on what McKenzie terms the “maximal path of capital accumulation” and on what we are referring to here as an optimal program. He works with the sets P_t and K_t for all $t \in \mathbb{N}$. In our stationary setting these sets are independent of t . P_t concerns only Ω and in our setting, it is \mathbb{R}_+^n for all $t \in \mathbb{N}$. K_t concerns also the optimal program, and since Theorem 2 above guarantees that there is an optimal program from any initial stock, it is also \mathbb{R}_+^n for all $t \in \mathbb{N}$. Now Assumption III requires that for any optimal program $\{x(t), y(t)\}$, $x(t)$ is in the relative interior of \mathbb{R}_+^n for all $t \in \mathbb{N}$. If $x(o) > 0$, this assumption is fulfilled for any program, leave alone an optimal one. However, $x(0) = 0$ is not in the relative interior of \mathbb{R}_+^n , and III is not fulfilled. The relevant observation here is to apply McKenzie’s Lemma 1 for the case $x(0) > 0$, and then to observe that for any optimal path starting from 0, $u(0, x(1)) = 0$, and hence that $V(x(0)) = V(x(1))$. (Note that McKenzie’s $V(k_t)$ is independent of the value of the golden-rule stock and is defined solely with respect to the optimal program.) It is clear that $x(1) > 0$ by virtue of it being a part on an optimal program. The price-support for $V(x(1))$ also works for $V(x(0))$, and allows the induction to proceed. The demonstration is then complete. ■

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