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***First-Price auction symmetric equilibria with a general
distribution***

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First-Price Auction Symmetric Equilibria with a General Distribution

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Abstract

In this paper I obtain the mixed strategy symmetric equilibria of the first-price auction for any distribution. The equilibrium is unique. The solution turns out to be a combination of absolutely continuous distributions case and the discrete distributions case.

1 Introduction

In this paper I consider first-price auctions in the symmetric independent private values model. In this model the set of types is usually an interval $[a, b]$ and the distribution of types $F : [a, b] \rightarrow [0, 1]$ has a strictly positive density. The discrete distribution case is considered mainly for examples and counter-examples. If the distribution has a density the symmetric equilibrium is a continuous strictly increasing function. If the distribution is discrete the equilibrium is in mixed strategies. There is also a kind of monotonicity in that a bidder with a higher valuation always bids higher than bidders with lower valuations. The need for a mixed strategy arises only from competition with bidders having the same valuation. For example if there are two bidders and $f > 0$ is the density of F the equilibrium bidding function is $b(x) = \frac{1}{F(x)} \int_0^x y f(y) dy$. This function is the expected valuation of the second highest bid conditional of the highest valuation being x . And if we consider a discrete distribution, say each type $v \in \{0, 1\}$ occurs with probability $\frac{1}{2}$ then a bidder with null valuation bids 0 and a bidder with a valuation of 1 bids in the interval $[0, b]$ with probability $G(b) = \frac{b}{1-b}$, $b \in [0, \frac{1}{2}]$. I will find the equilibrium bidding strategies for any distribution. The general case is a mixture of both previous cases. There is a pure strategy part¹ with bids given by a function analogous to the b function above. And the mixed strategy part consist of a distribution quite similar to G . The generalized “bidding function” b is monotonic and maybe discontinuous.

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¹This happens at the continuity points of the distribution.

The discontinuities of b are gaps and the gaps determine the range of the mixed strategy part.

The second result of the paper is the unicity of the symmetric equilibrium. It is possible that the equilibrium be unique not only amongst the symmetric ones. If the distribution has only a finite number of discontinuities the differential equation techniques of Maskin and Riley(2003) might be useful here. The last result in the paper is an application of the general result to the multi-dimensional set of types case. This will be easy. Its main interest being to show that neither monotonicity nor continuity plays a role in the general case.

There are a few results in the literature concerning existence or unicity. For example Maskin and Riley(2000, 2003) have both existence and unicity results. The present paper differ from this literature in that being restricted to private values an equilibrium is found which is easy to calculate, is unique and valid for any distribution.

2 Preliminaries

In this section I collect some basic definitions and auxiliary results. I begin recalling the definition of a distribution.

Definition 1 *A function $F : \mathbb{R} \rightarrow [0, 1]$ is a distribution if*

1. *F is increasing: $x < y \Rightarrow F(x) \leq F(y)$;*
2. *F is right-continuous: $F(x) = \lim_{y \uparrow x} F(y)$ and*
3. *$\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.*

A distribution $F : \mathbb{R} \rightarrow [0, 1]$ may be discontinuous at some points. Define \mathcal{C} as the set of continuity of F and \mathcal{D} the set of discontinuities of F . Then \mathcal{D} is countable.² Moreover

$$\mathcal{D} = \{x \in \mathbb{R}; F(x) > F(x-)\}.$$

As usual $F(x-) := \sup \{F(y); y < x\} = \lim_{y \uparrow x} F(y)$. If F is a distribution so is F^m for every $m > 0$. If F is a distribution define

$$\begin{aligned} \underline{v} &:= \inf \{x : F(x) > 0\} \text{ and} \\ \bar{v} &:= \sup \{x; F(x) < 1\}. \end{aligned}$$

Throughout this paper we suppose that \underline{v} and \bar{v} are finite. Abusing notation, I denote by F the restriction $F|_{[\underline{v}, \bar{v}]}$. This entails no confusion. It is immediate that $\mathcal{D} \subset [\underline{v}, \bar{v}]$. If F is a distribution we denote by μ_F the Lebesgue-Stieltjes measure associated to F . Thus μ_F is a Borelean measure such that for every real numbers u, v :

$$\begin{aligned} \mu_F((u, v]) &= F(v) - F(u), \\ \mu_F([u, v]) &= F(v) - F(u-) \end{aligned}$$

²Since F is monotonic.

and so on. The Lebesgue-Stieltjes integral $\int \phi(x) d\mu_F(x)$ will be denoted $\int \phi(x) dF(x)$. Let $\chi_A : \mathbb{R} \rightarrow \{0, 1\}$ denote the indicator function of the set $A \subset \mathbb{R}$. If F is a distribution we define

$$\begin{aligned} \int_{\underline{v}}^v \phi(x) dF(x) &= \int \chi_{(\underline{v}, v]}(x) \phi(x) dF(x) \text{ and} \\ \int_{\underline{v}-}^v \phi(x) dF(x) &= \int \chi_{[\underline{v}, v]}(x) \phi(x) dF(x). \end{aligned}$$

Thus $\int_{\underline{v}}^v dF(x) = F(v) - F(\underline{v}) = F(v)$ and $\int_{\underline{v}-}^v dF(x) = F(v)$. The following function is the main ingredient of the equilibrium strategy:

Definition 2 Define $b_F : [\underline{v}, \bar{v}] \rightarrow \mathbb{R}$ by $b_F(\underline{v}) = \underline{v}$ and

$$b_F(v) = \frac{\int_{\underline{v}-}^v y dF^{n-1}(y)}{F^{n-1}(v)}, v \in (\underline{v}, \bar{v}]. \quad (1)$$

In the next lemma I prove the basic properties of b_F .

Lemma 1 *The following properties are true:*

- (i) $b_F(v) < v$ if $v > \underline{v}$;
- (ii) $F(v') < F(v'')$ if and only if $b_F(v') < b_F(v'')$;
- (iii) b_F is right-continuous, increasing and

$$\{x \in [\underline{v}, \bar{v}]; b_F \text{ is continuous at } x\} = \mathcal{C}.$$

Proof: (i) Suppose $v > \underline{v}$. Then since the set $\{y; v - y > 0\} = [\underline{v}, v)$ has measure $F(v-) > 0$,

$$vF^{n-1}(v) - \int_{\underline{v}-}^v y dF^{n-1}(y) = \int_{\underline{v}-}^v (v - y) dF^{n-1}(y) > 0.$$

Hence $b_F(v) < v$.

(ii) Suppose $F(v') < F(v'')$. Then $v' < v''$ and writing $b = b_F$:

$$\begin{aligned} b(v'') &= \frac{\int_{\underline{v}-}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')} = \frac{\int_{\underline{v}-}^{v'} y dF^{n-1}(y) + \int_{v'}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')} = \\ &= \frac{b(v') F^{n-1}(v') + \int_{v'}^{v''} y dF^{n-1}(y)}{F^{n-1}(v'')}. \end{aligned}$$

Thus $\int_{v'}^{v''} y dF^{n-1}(y) \geq v' (F^{n-1}(v'') - F^{n-1}(v')) > b(v') (F^{n-1}(v'') - F^{n-1}(v'))$ and $b(v'') > b(v')$. Now if $F(v'') = F(v')$ then $\int_{v'}^{v''} y dF^{n-1}(y) = 0$ and $b(v'') = b(v')$.

(iii) If v_k decreases to v , we have that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\underline{v}-}^{v_k} y dF^{n-1}(y) &= \lim_{k \rightarrow \infty} \int \chi_{[\underline{v}, v_k]}(y) y dF^{n-1}(y) = \\ &= \int \chi_{[\underline{v}, v]}(y) y dF^{n-1}(y) = \int_{\underline{v}-}^v y dF^{n-1}(y). \end{aligned}$$

Thus b is right-continuous. Now if v_k increases to v then $F(v_k) \rightarrow F(v-)$. Moreover

$$\begin{aligned} \int_{\underline{v}}^{v_k} y dF^{n-1}(y) &\rightarrow \int \chi_{[\underline{v}, v)}(y) dF^{n-1}(y) = \\ &\int_{\underline{v}}^v y dF^{n-1}(y) - v(F^{n-1}(v) - F^{n-1}(v-)). \end{aligned}$$

Therefore

$$\begin{aligned} b(v-) &= \frac{\int_{\underline{v}}^v y dF^{n-1}(y) - v(F^{n-1}(v) - F^{n-1}(v-))}{F^{n-1}(v-)} = \\ &= \frac{b(v)F^{n-1}(v) - v(F^{n-1}(v) - F^{n-1}(v-))}{F^{n-1}(v-)}. \end{aligned} \quad (2)$$

Thus

$$b(v)F^{n-1}(v) - b(v-)F^{n-1}(v-) = v(F^{n-1}(v) - F^{n-1}(v-)). \quad (3)$$

Since $b(v) < v$ it follows that $F(v) \neq F(v-)$ implies $b(v) \neq b(v-)$. Reciprocally if $v \in \mathcal{C}$ then $b(v) = b(v-)$. QED

For later use note that it follows from (3) that

$$(v - b(v))F^{n-1}(v) = (v - b(v-))F^{n-1}(v-). \quad (4)$$

The next lemma finish our preliminary work.

Lemma 2 *Suppose $v \in \mathcal{D}$. Then the function*

$$G_v(x) = \frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{v - b(v-)}{v - x} \right)^{\frac{1}{n-1}} \right), x \in [b(v-), b(v)]. \quad (5)$$

is a continuous, strictly increasing distribution.

Proof: It is immediate that $G_v(b(v-)) = 0$ and that G_v is continuous. Moreover since $v - b(v-) > 0$ the function G_v is strictly increasing. Finally using (4) we have that

$$\begin{aligned} G_v(b(v)) &= \frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{v - b(v-)}{v - b(v)} \right)^{\frac{1}{n-1}} \right) = \\ &= \frac{F(v-)}{F(v) - F(v-)} \left(-1 + \left(\frac{F^{n-1}(v)}{F^{n-1}(v-)} \right)^{\frac{1}{n-1}} \right) = 1. \end{aligned}$$

QED

3 The equilibrium

There are n bidders participating in a first-price auction. Values are private and bidders types are independent identically distributed according to the distribution $F : [\underline{v}, \bar{v}] \rightarrow [0, 1]$. The equilibrium is in mixed strategies. However it is not

very wild. The mixed part occurs only at the discontinuities of F (which are countable). Moreover the support of the mixed strategies are non-intersecting and monotonic.

The equilibrium strategy is composed of two parts. First if $v \in [\underline{v}, \bar{v}] \cap \mathcal{C}$ the bidder bids $b(v)$ where $b = b_F$ is defined by (1). If $v \in \mathcal{D}$ the bidder bids the mixed strategy μ_{G_v} . Thus for every $x \in [b(v-), b(v)]$ he bids in the interval $[b(v-), x]$ with probability $G_v(x)$. Define $\mathbf{M} = (\mu_v)_{v \in [\underline{v}, \bar{v}]}$ where

$$\mu_v = \begin{cases} \text{pure strategy } b(v) & \text{if } v \in \mathcal{C}, \\ \text{mixed strategy } G_v & \text{if } v \in \mathcal{D}. \end{cases}$$

Thus the pure strategy $b(v)$ is played at the continuity points of the distribution and a the mixed strategy G_v is played if the distribution is discontinuous at v . The next two lemmas show that the distribution of bids is continuous. I begin with the pure strategy part.

Lemma 3 *For every $x \geq 0$, $\Pr(\{\omega \in \mathcal{C}; b(\omega) = x\}) = 0$.*

Proof: Suppose $\omega^0 \in \mathcal{C}$ and $b(\omega^0) = x$. Define

$$\begin{aligned} \omega^+ &= \sup \{\omega; b(\omega) = x\} \text{ and} \\ \omega^- &= \inf \{\omega; b(\omega) = x\}. \end{aligned}$$

If $\omega^+ = \omega^-$ then $\omega^+ = \omega^- = \omega^0$ and $\Pr(\omega^0) = 0$. If $\omega^+ > \omega^-$ then

$$\Pr(\omega^-, \omega^+) = \lim_{n \rightarrow \infty} \Pr((\omega^- + 1/n, \omega^+ - 1/n]) = 0$$

since $F(\omega^- + 1/n) = F(\omega^+ - 1/n)$. QED

The mixed part is even easier.

Lemma 4 *For every $x \geq 0$, $\Pr(\omega \in \mathcal{D}; b(\omega) = x) = 0$.*

Proof: This is immediate since G_ω is continuous and \mathcal{D} is countable.

Theorem 1 *The mixed strategy \mathbf{M} is a symmetric equilibrium of the first-price auction.*

Proof: Suppose bidders $i = 2, \dots, n$ bid the mixed strategy \mathbf{M} . Suppose bidder 1 has valuation v . The two lemmas above show that if bidder 1 bids x the probability of a tie is null. Thus if he bids x his expected utility is

$$\phi = (v - x) (\Pr(\omega; b(\omega) \leq x) + \Pr(\omega, b(\omega-) \leq x < b(\omega)) G_\omega(x))^{n-1}.$$

Suppose first that $x = b(y-)$. Thus

$$\begin{aligned} \phi &= (v - b(y-)) (F(y-))^{n-1} = v F^{n-1}(y-) - \int \chi_{[\underline{v}, y)}(z) z dF^{n-1}(z) = \\ &= \int (v - z) \chi_{[\underline{v}, y)}(z) dF^{n-1}(z) \leq \int (v - z) \chi_{[\underline{v}, v)}(z) dF^{n-1}(z). \end{aligned}$$

Suppose now that $x \in (b(y-), b(y)]$. Then

$$\begin{aligned}\phi &= (v-x) \left(F(y-) + (F(y) - F(y-)) G_y(x) \right)^{n-1} = \\ &= (v-x) \left(F(y-) + F(y-) \left(-1 + \left(\frac{y-b(y-)}{y-x} \right)^{\frac{1}{n-1}} \right) \right)^{n-1} = \\ &= (v-x) F^{n-1}(y-) \left(\frac{y-b(y-)}{y-x} \right) = \\ &= \frac{v-x}{y-x} F^{n-1}(y) (y-b(y)) = \left(\frac{v-y}{y-x} + 1 \right) F^{n-1}(y) (y-b(y)).\end{aligned}$$

If $v > y$ the best x is $x = b(y)$. Then

$$\phi = (v-b(y)) F^{n-1}(y) \leq (v-b(v)) F^{n-1}(v).$$

If $v < y$ the best is $x = b(y-)$ and

$$\phi = (v-b(y-)) F^{n-1}(y-) \leq (v-b(v)) F^{n-1}(v).$$

If $v = y$ any x will do. In particular the mixed strategy G_v will do as well.

4 Unicity of the mixed strategy equilibrium

In this section I show that the mixed strategy equilibrium \mathbf{M} is unique.

Theorem 2 *Suppose $\Upsilon = (\tau_v)_v$ is a symmetric mixed strategy equilibrium. Then $\Upsilon = \mathbf{M}$.*

Define H as the distribution of bids when the mixed strategy Υ is played. Thus

$$H(x) = \int_{v-}^{\bar{v}} \tau_y[0, x] dF(y).$$

Denote by \mathcal{P} the set of Borelean probabilities measures on \mathbb{R} . The following lemma is basic.

Lemma 5 *Suppose $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is measurable, $\bar{v} \in \mathcal{P}$ and that*

$$\int \phi(z) d\bar{\nu}(z) = \sup_{\nu \in \mathcal{P}} \int \phi(z) d\nu(z).$$

Then $\phi^{\max} := \max_{z \in \mathbb{R}} \phi(z)$ exists and $\bar{\nu}(\{z; \phi(z) = \phi^{\max}\}^c) = 0$.

Proof: First let δ_x denote the Dirac measure at $x \in \mathbb{R}$. Then

$$\left\{ \int \phi(z) d\nu(z); \nu \in \mathcal{P} \right\} \supset \left\{ \int \phi(z) d\delta_x(z); x \in \mathbb{R} \right\} = \{\phi(x); x \in \mathbb{R}\}.$$

Thus

$$\sup \phi(\mathbb{R}) \geq \sup_{\nu \in \mathcal{P}} \int \phi(z) d\nu(z) \geq \sup \phi(\mathbb{R}).$$

Therefore

$$\int (\sup \phi(\mathbb{R}) - \phi(z)) d\bar{\nu}(z) = \sup \phi(\mathbb{R}) - \int \phi(z) d\bar{\nu}(z) = 0.$$

Hence $\phi(z) = \sup \phi(\mathbb{R})$ for almost every z with respect to $\bar{\nu}$ and therefore the supremum is achieved. Moreover $\bar{\nu}(\{z; \phi(z) = \phi^{\max}\}^c) = 0$. QED

Let us now consider a bidder with valuation v . Since Υ is an equilibrium the best reply is τ_v . If there is a tie we suppose that the tie is solved with equal probability amongst the winners. Thus if a bidder bids b he wins with probability

$$\tilde{H}(b) = \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} H^{n-1-j}(b_-) \cdot \frac{(H(b) - H(b_-))^j}{j+1}.$$

Thus

$$\int (v-b) \tilde{H}(b) d\tau_v(b) = \sup_{\tau \in \mathcal{P}} \int (v-b) \tilde{H}(b) d\tau(b).$$

The lemma above implies that

$$A_v = \left\{ \bar{b} \geq 0; (v - \bar{b}) \tilde{H}(\bar{b}) = \max_{b \geq 0} (v - b) \tilde{H}(b) \right\} \neq \emptyset, \quad (6)$$

and $\tau_v(A_v^c) = 0$. The next lemma implies that $v \notin A_v$ since

$$\left(v - \frac{v + \underline{v}}{2} \right) \tilde{H}\left(\frac{v + \underline{v}}{2} \right) \geq \frac{v - \underline{v}}{2} H^{n-1}\left(\frac{v + 3\underline{v}}{4} \right) > 0.$$

Lemma 6 *For every $v > \underline{v}$, $H(v) > 0$.*

Proof: Let $\tilde{v} = \inf \{v; H(v) > 0\}$. For every $v < \tilde{v}$, $A_v \subset [\underline{v}, v) \subset [\underline{v}, \tilde{v})$. Thus $\tau_v([\underline{v}, \tilde{v})) = 1$ for every $v < \tilde{v}$. Now

$$0 = H(\tilde{v}) = \int \tau_y[0, \tilde{v}] dF(y) \geq F(\tilde{v}).$$

Hence $\tilde{v} = \underline{v}$.

QED

Lemma 7 *For any v the distribution H is continuous on A_v .*

Proof: If $\bar{b} \in A_v$ and $b_n \downarrow \bar{b}$ through points of continuity of H then

$$\tilde{H}(b_n) = H^{n-1}(b_n) \rightarrow H^{n-1}(\bar{b})$$

and therefore

$$(v - \bar{b}) H^{n-1}(\bar{b}) = (v - \bar{b}) \tilde{H}(\bar{b}).$$

Since $\bar{b} < v$, $H^{n-1}(\bar{b}) = \tilde{H}(\bar{b})$. Therefore $H(\bar{b}) = H(\bar{b}-)$. Thus H is continuous at A_v . QED

Lemma 8 *If $v' < v''$ then $\sup A_{v'} \leq \inf A_{v''}$.*

Proof: Suppose $v' < v''$ and that there exist $b' \in A_{v'}, b'' \in A_{v''}$ such that $b'' \leq b'$. It is always true that

$$(v' - b') \tilde{H}(b') \geq (v' - b'') \tilde{H}(b''), \text{ and} \\ (v'' - b'') \tilde{H}(b'') \geq (v'' - b') \tilde{H}(b').$$

Adding and collecting terms $(v'' - v') (\tilde{H}(b'') - \tilde{H}(b')) \geq 0$. Hence $H^{n-1}(b'') = \tilde{H}(b'') \geq \tilde{H}(b') = H^{n-1}(b')$. Thus $H(b') = H(b'') > 0$. This implies

$$v' - b' \geq v' - b''$$

and therefore $b' \leq b''$. Thus $b' = b''$. QED

Lemma 9 Suppose now that $\#A_v > 1$. Then $v \in \mathcal{D}$.

Proof: Take $b', b'' \in A_v, b' < b''$. Therefore

$$H^{n-1}(b'') = \tilde{H}(b'') > \tilde{H}(b') = H^{n-1}(b').$$

Thus $H(b'') > H(b')$. Now if ϵ is sufficiently small,

$$0 < H(b'' - \epsilon) - H(b') = \int (\tau_y([0, b'' - \epsilon]) - \tau_y([0, b'])) dF(y) = \\ \int \tau_y(b', b'' - \epsilon) dF(y) = \tau_v(b', b'' - \epsilon) (F(v) - F(v-)). \quad (7)$$

In (7) I used Lemma 8. Thus $F(v) - F(v-) > 0$. QED

Lemma 10 For every $b \geq 0, v \in \mathcal{D}$ we have that $\tau_v(b) = 0$. In particular H is continuous in $(\inf A_v, \sup A_v)$.

Proof: If $b \notin A_v$ then $0 \leq \tau_v(b) \leq \tau_v(A_v^c) = 0$. If $b \in A_v$ then

$$0 = H(b) - H(b-) = \int \tau_y\{b\} dF(y).$$

Therefore $\tau_y\{b\} = 0$ for almost every y with respect to F . Hence $\tau_v\{b\} = 0$. QED

Lemma 11 Suppose $b' < b''$ are elements of $A_v, v \in \mathcal{D}$. Then $(b', b'') \cap A_v \neq \emptyset$.

Proof: Since $(v - b') H^{n-1}(b') = (v - b'') H^{n-1}(b'')$ and $b' < b''$ it follows that $H(b'') > H(b')$. Now

$$0 < H(b'') - H(b') = \int \tau_y(b', b'') dF(y) = \tau_v(b', b'') (F(v) - F(v-))$$

and therefore $\tau_v(b', b'') = \tau_v(b', b'') > 0$ ending the proof.

Lemma 12 For every $v \in \mathcal{D}, A_v \supset (\inf A_v, \sup A_v)$.

Proof: Suppose $v \in \mathcal{D}$. For any $x \in (\inf A_v, \sup A_v)$ define

$$\bar{x} = \inf \{b \in A_v; b > x\}.$$

If $\bar{x} \notin A_v$ there exist $b_l \in A_v$ $b_l \downarrow \bar{x}$. Then if we define $\phi^{\max} = \max \{(v - b) \tilde{H}(b); b \geq 0\}$ it is true that

$$\phi^{\max} = (v - b_l) H^{n-1}(b_l) \rightarrow (v - \bar{x}) H^{n-1}(\bar{x}-) = (v - \bar{x}) \tilde{H}(\bar{x}).$$

Thus $\bar{x} \in A_v$. Analogously we define \underline{x} :

$$\underline{x} = \sup \{b \in A_v; b < x\}.$$

Thus $\underline{x} \in A_v$. Now if $\underline{x} < \bar{x}$ then $(\underline{x}, \bar{x}) \cap A_v = \emptyset$ a contradiction. Hence $\underline{x} = \bar{x} = x$. QED

Define $b(v)$ as the pure strategy played when $v \in \mathcal{C}$. And if $v \in \mathcal{D}$ define $b(v) = \inf_{\omega > v} b(\omega)$. Thus b is increasing and right-continuous. We have that

$$H(b(v)) = \int \tau_y([0, b(v)]) dF(y) = F(v).$$

Theorem 3 For every v , $b(v) = b_F(v)$.

Proof: Suppose $v \in \mathcal{C}$. Then for every $\omega \in \mathcal{C}$,

$$(v - b(v)) F^{n-1}(v) \geq (v - b(\omega)) F^{n-1}(\omega).$$

By the right-continuity of b and F this is also true for every ω and for every v . The inequality above is equivalent to

$$v(F^{n-1}(v) - F^{n-1}(\omega)) \geq b(v) F^{n-1}(v) - b(\omega) F^{n-1}(\omega).$$

Interchanging v with ω we get:

$$\omega(F^{n-1}(\omega) - F^{n-1}(v)) \geq b(\omega) F^{n-1}(\omega) - b(v) F^{n-1}(v).$$

Thus for every v and ω :

$$\begin{aligned} v(F^{n-1}(v) - F^{n-1}(\omega)) &\geq b(v) F^{n-1}(v) - b(\omega) F^{n-1}(\omega), \\ b(v) F^{n-1}(v) - b(\omega) F^{n-1}(\omega) &\geq \omega(F^{n-1}(v) - F^{n-1}(\omega)). \end{aligned} \quad (\#)$$

Take $\omega_0 = \underline{v} < \omega_1 < \dots < \omega_N = v$ a partition of $[\underline{v}, v]$ such that $\max_j |\omega_{j+1} - \omega_j| < \frac{1}{N}$. We have that

$$\omega_{j+1}(F^{n-1}(\omega_{j+1}) - F^{n-1}(\omega_j)) \geq b(\omega_{j+1}) F^{n-1}(\omega_{j+1}) - b(\omega_j) F^{n-1}(\omega_j)$$

$$\begin{aligned} \int \sum_{j=0}^{N-1} \omega_{j+1} \chi_{(\omega_j, \omega_{j+1}]}(y) dF^{n-1}(y) &= \sum_{j=0}^{N-1} \omega_{j+1} (F^{n-1}(\omega_{j+1}) - F^{n-1}(\omega_j)) \geq \\ \sum_{j=0}^{N-1} (b(\omega_{j+1}) F^{n-1}(\omega_{j+1}) - b(\omega_j) F^{n-1}(\omega_j)) &= b(v) F^{n-1}(v). \end{aligned}$$

Since

$$\sup_{v \leq y \leq \bar{v}} \left| \sum_{j=1}^N \omega_{j+1} \chi_{(\omega_j, \omega_{j+1}]}(y) - y \right| \leq \max_j |\omega_{j+1} - \omega_j| < \frac{1}{N}$$

by making $N \rightarrow \infty$ we get:

$$\int_v^v y dF^{n-1}(y) \geq b(v) F^{n-1}(v). \quad (8)$$

The other inequality is obtained from the inequality in (#). QED

Thus the pure strategy part is unique. The unicity of the mixed strategy is proved in an analogous manner.

Theorem 4 *The mixed strategy τ_v is unique for each $v \in \mathcal{D}$.*

Proof: Let us consider $v \in \mathcal{D}$. Suppose $b \in A_v, b < \sup A_v$. Then

$$H(b) - H(b(v-)) = \int \tau_y(b(v-), b] dF(y) = \tau_v(b(v-), b] (F(v) - F(v-)).$$

Therefore

$$H(b) = F(v-) + \tau_v(b(v-), b] (F(v) - F(v-)).$$

Since τ_v cannot have mass points,

$$(v - b) (F(v-) + \tau_v(b(v-), b] (F(v) - F(v-)))^{n-1} = (v - b(v-)) F^{n-1}(v-).$$

Therefore

$$\tau_v(b(v-), b] = \left(-1 + \left(\frac{v - b(v-)}{v - b} \right)^{\frac{1}{n-1}} \right) \frac{F(v-)}{F(v) - F(v-)}.$$

QED

5 Example and application.

I now show how the general multi-dimensional set of types case is reduced to a one dimensional case in complete generality. Suppose the set of types is the probability space (T, \mathcal{T}, P) . A bidder with type $t \in T$ has a utility $U(t)$ when receiving the object. The function $U : T \rightarrow \mathbb{R}$ is bounded and measurable. Define

$$F(x) = \Pr(U(t) \leq x), x \in U(T)$$

the distribution of U . Define $\underline{v} = \inf U(T)$ and $\bar{v} = \sup U(T)$. Define also $b_U = b_F \circ U$ and $G_t^U = G_{U(t)}$. The equilibrium is then to bid $b_U(t)$ if F is continuous at $U(t)$ and the mixed strategy G_t^U if F is discontinuous at $U(t)$. I finish with an example showing how to calculate the mixed strategies support.

Example 1 *Suppose there are two bidders and three possible valuations $v \in \{0, 1, 2\}$. And*

$$\begin{cases} \Pr(v = 0) = a, \\ \Pr(v = 1) = b \\ \Pr(v = 2) = 1 - a - b \end{cases}, a > 0, b > 0, a + b < 1.$$

The bidders with a zero valuation bids 0. A bidder with valuation 1 bids in the interval $[b(1-), b(1)]$,

$$b(1-) = 0, b(1) = \frac{\int_{0-}^1 y dF(y)}{F(1)} = \frac{b}{a+b}.$$

A bidder with valuation 2 bids in the interval $[b(2-), b(2)]$:

$$b(2-) = \frac{\int_{0-}^{2-} y dF(y)}{F(2-)} = \frac{b}{a+b}, b(2) = 2(1-a-b) + b = 2 - 2a + b.$$

If $a = b = 1/2$ we recover the example in the introduction.

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