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HYPERSTABILITY OF NASH EQUILIBRIA *

Carlos Ivan Simonsen Leal

(*) Primeiro Capítulo da Tese de Doutorado de Carlos Ivan Simonsen Leal na "Princeton University.

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Chapter I: Hyperstability of Nash Equilibria

Chapter 1: Hyperstability of Nash Equilibria

1. Introduction

The primary solution concept for noncooperative game theory is the Nash equilibrium. Yet, it has been observed that certain Nash equilibria are not intuitively satisfactory. Substantial effort has been directed to defining refinements of Nash equilibrium that yield more acceptable solutions. With this goal in mind, E. Kohlberg and J. F. Mertens (1982) have introduced two refinements of Nash equilibrium that they called hyperstability and full stability.

Hyperstability concerns the robustness of Nash equilibria to pertubations in the pure strategy payoffs. The first reason why hyperstability is important is because in many games the payoffs are observed with an error. Equilibria should not be destroyed by small perturbations in the data of the problem. Full stability concerns the robustness of an equilibrium against small pertubations in rationality, that is, in equilibrium a small change in the players' actions should not change very much one's optimal response. A fully stable equilibrium is a proper equilibrium, as defined by Myerson (1978), therefore it is a sequential equilibrium (Kreps and Wilson 1982) . A fully stable equilibrium is not destroyed when a dominated strategy is deleted. Also, one can prove that a hyperstable equilibrium is fully stable.

One of the main theorems of Kohlberg and Mertens demonstrates the existence, for each finite game, of a hyperstable component of equilibria. These sets are offered as an appropriate refinement of Nash equilibrium . We begin by giving an alternative proof of the Kohlberg-Mertens existence theorem. Our proof shows the relation between hyperstability and the problem of studying how the fixed points of a continuous function vary when this function is perturbed. We show that hyperstability generalizes to games where the number of

players is still finite, say n, but where some player might have an infinite number of pure strategies. Existence of a fully stable component of equilibria is obtained directly from the proof of the latter result.

The method used is the following. In finite games, Nash equilibria are the fixed points of a continuous function which we call the Nash map. This function is a map of the cartesian product of the spaces of mixed strategies of each player into itself. We establish the existence of a hyperstable component of equilibria using a theorem of S. Kinoshita (1952) on essential sets of fixed points. We point out the idea of this approach belongs to A. Mas-Colell (1985), who gave another proof of Kinoshita's theorem.

The framework for discussing Nash equilibria in infinite games is due to I. Glicksberg (1952). Let P_i be the set of pure strategies of player i and $C(P_i)$ the space of continuous functions over P_i. In Glicksberg's formulation, the set of mixed strategies of a player is a subset of the set of continuous linear functionals over $C(P_i)$. The mixed strategies are the linear functionals induced by the probability distributions over P_i. To obtain the Nash equilibria one first defines a best reply correspondence R_i for each player; this correspondence determines which is his optimal mixed strategy against the other players actions. Then, we consider the correspondence R that takes the vector of mixed strategies of the players $(m_1,m_2,\ldots,m_n)$ into $R(m_1,\ldots,m_n):= (R_1(m_{-1}),R_2(m_{-2}),\ldots,R_n(m_{-n}))$, where $m_{-i}:= (m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_n)$. We call this correspondence the best reply correspondence of the game; each of its fixed points is a Nash equilibrium and vice versa. R depends on the pure strategies payoffs in a way to be made precise later. We use an extension of Kinoshita's theorem, presented in one of the appendices, to establish the existence of a hyperstable component of Nash equilibria for R.

We finish the paper with a remark based on a theorem of Fort (1949): for most

games all components of equilibria are simultaneously hyperstable and fully stable.

2. Hyperstability in Finite Games

For $n \geq 1$, let $S(n)$ be the standard (n-1)-dimensional simplex, that is $S(n):= \{(x_1,x_2,...,x_n) \mid x_j \geq 0 \text{ for all } j \text{ and } \Sigma_j x_j = 1\}$. $S(n)$ is a closed convex subset of $\Re^{n-1}$; it is a topological space with the restricted topology inherited from the euclidean topology of $\Re^{n-1}$. A basis for the topology of $S(n)$ is given by the open sets $V(x,\varepsilon):= B(x,\varepsilon)\cap S(n)$, where x is any point of $S(n)$ and $B(x,\varepsilon)$ is the open ball of center x and radius $\varepsilon > 0$ in $\Re^{n-1}$.

A normal form game is a triple $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$, where $I= \{1,...,n\}$ is the set of players, P_i is the set of pure strategies for player i, card $P_i = k_i$, and the r_i are maps r_i: $\Pi_{j\in I}P_j \rightarrow \Re$, the payoffs in pure strategies. For each player i the set of mixed strategies is $S(k_i)$. . We extend each r_i to E_i: $\Pi_{j\in I}S(k_j) \rightarrow \Re$, by putting $E_i(\lambda_1,\lambda_2, ...,\lambda_n)= \Sigma_{j_1,j_2,...,j_n}\lambda_{j_1,j_2,...,j_n}r_i(z_{j_1},z_{j_2},...,z_{j_n})$ for any n, where z_{j_i} belongs to P_i. A Nash equilibrium is a point $z^* \in \Pi_{j\in I}S(k_j)$, such that for every i we have $E_i(z^*) \geq E_i(z_{-i}^*|z_i)$ for all $z_i \in S(k_i)$, where $z= (z_1^*,...,z_n^*)$ and $(z_{-i}^*|z_i)= (z_1^*,...,z_{i-1}^*,z_i,z_{i+1}^*,...,z_n^*)$.

A Nash equilibrium of a normal form game $z^* \in \Pi_{j\in I}S(k_j)$ is hyperstable if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|r_i(z)-r_i^{\sim}(z)| < \delta$ for all $z \in \Pi_{j\in I}P_j$, then the

game with pure strategy payoffs given by r_i~ has an equilibrium in $V(z^*,\varepsilon)$.

A Nash equilibrium of an extensive form game is hyperstable if it is a hyperstable equilibrium of a reduced normal form of the game.

In the followind sequence of examples we suggest why hyperstability is an attractive concept.

Example 1: (Selten, 1965)In the extensive form game below there are two equilibria. The first is: player I goes to the right; while the second is: player I goes to the left and then player II goes to the left too. The first equilibrium should be rejected. Indeed, player I knows that if player II knows that he has to play, then he will play left and both get a higher outcome. The first equilibrium is not sequential, it is not supported by any credible behaviour of the first player. (For a description of game trees see Kuhn, 1952).

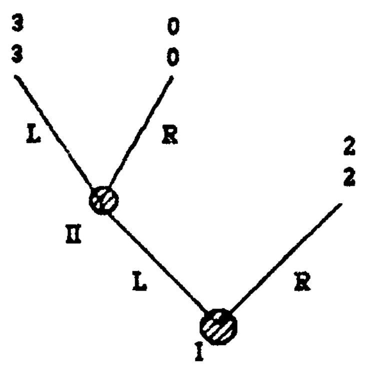

Fig. 1

A hyperstable equilibrium encompasses the usual refinements of Nash equilibrium like properness and sequentiality. In Examples 2 and 3 a component of equilibria which does not contain any proper equilibrium becomes a component where all equilibria are proper when we add a dominated strategy. Examples 2 and 3 show that hyperstability is robust to

droping a dominated strategy.

Example 2: For the extensive form game of Figure 1a, let X,Y and Z be the probabilities that each of the players assigns to going to the left. Then Figures 1b, 1c and 1d give the reaction surfaces for players I, II and III respectively. The reaction surfaces are, by definition, the optimal responses to the others actions.

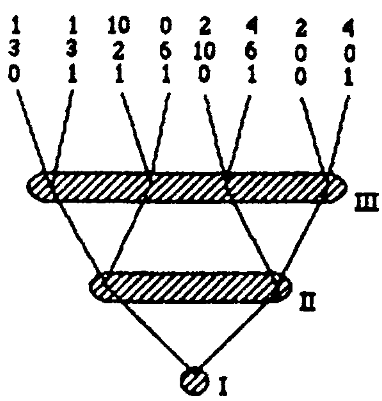

Fig. 2a

Reaction Surface for Player 1

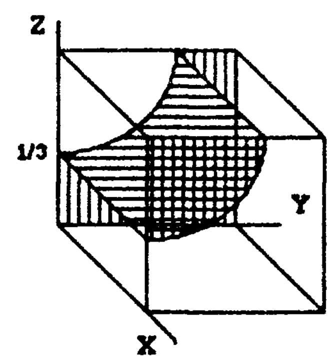

Fig. 2b

Reaction Surface for Player 2

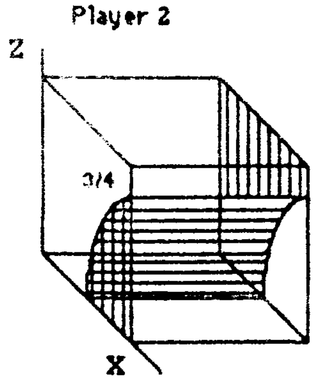

Fig. 2c

Reaction Surface for Player 3

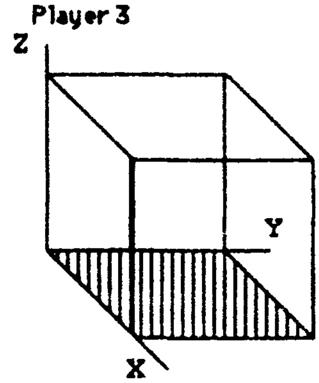

Fig. 2d

There are two distinct sets of equilibria: an isolated equilibrium at (0,1,0) and the line segment from (1,0,1/3) to (1,0,3/4). If one changes the pure strategy payoffs a little bit, one

is most likely to alter the shape of the reaction surface of the third player, while for the other players the difference is negligeable. The equilibria on the line segment are destroyed. The only hyperstable equilibrium is (0,1,0). This equilibrium is perfect. Indeed, if one restrict the mixed strategies to some closed parallelepiped contained in $[0,1]^3$, then the reaction surfaces of players I and II will be the intersection of the original reaction surfaces and this cube.For the third player, we will have a flat surface corresponding to the bottom of the parallelepiped. This implies that (0,1,0) is a sequential equilibrium.

Example 3: In the following modification of the previous game we have added a dominated strategy to the set of choices of the second player. The set of equilibria is not modified. However, as the reader might check, all the equilibria are now proper [1].

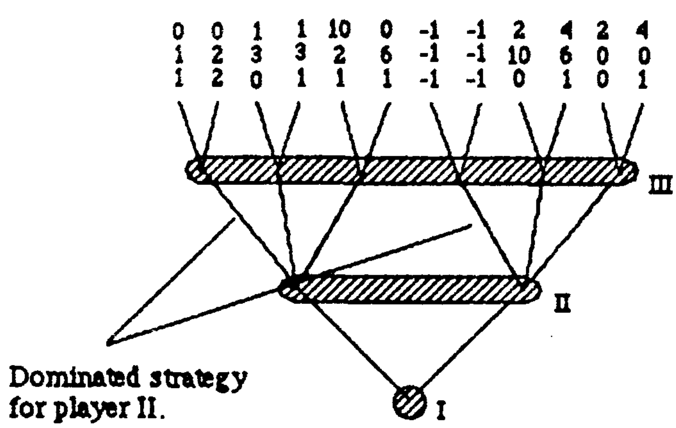

Fig. 3

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1 Let $S_\varepsilon(n)$ be the subset of S(n) which is the convex-hull of the vectors obtaining by considering all possible permutations of the coordinates of the vector $(1-\varepsilon)^{-n}(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{n-1})$. An ε-proper equilibrium is an equilibrium where the players are constrained to pick their strategies from $S_\varepsilon(k_j)$. A proper equilibrium is a point which is the limit of a sequence of ε-proper equilibria when $\varepsilon \to 0$.

There are games where no equilibrium is hyperstable. One needs to use certain sets of equilibria to find a refinement of Nash equilibrium that works for all games. These sets are called components and we define them formally in section 3 below.

Example 4: In a 2x2 bimatrix game where all the payoffs are zero, let X and Y be the probabilities with which the first and the second player pick the first row and the first column respectively. Then the set of equilibria of the game is the gray square of Figure 4a. For any r > 0 the games with pure strategies given in 4b and 4c have unique equilibria at A and B respectively. Therefore, no equilibrium in this game is hyperstable.

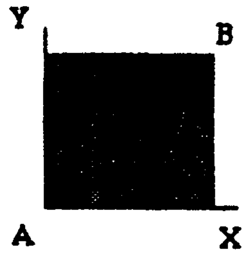

Fig. 4a

	r	l
R	(0,0)	(0,r)
L	(r,0)	(r,r)

Fig. 4b

	r	l
R	(r,r)	(r,0)
L	(0,r)	(0,0)

Fig. 4c

3. The Kohlberg- Mertens theorem

In this section we prove the Kohlberg- Mertens theorem for finite games.

A compact set C can always be decomposed into the union of compact connected C_α, such that, for any given α, C_α and $C \backslash C_\alpha$ have non-intersecting neighborhoods. We call the C_α the components of C.

The set of Nash equilibria is a compact set. Indeed, because the only requirement for a point z^* to be a Nash equilibrium is that $r_i(z^*) \geq r_i(z_{-i}^*|z_i)$ for all i, it is easy to check that

this is a closed set and since it is contained in the compact set $\prod_{j\in I}S(k_j)$ it is also compact. We may therefore speak of a component of Nash equilibria.

A component C of Nash equilibria is hyperstable if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $|r_i(z)-r_i\sim(z)| < \delta$ for all $z \in \prod_{j\in I}P_j$, then the game with pure strategy payoffs given by $r_i\sim$ has an equilibrium in $\cup_{z^*\varepsilon C}V(z^*,\varepsilon)$. In Example 4, the gray square of Figure 4a is a hyperstable component of equilibria.

Brouwer's fixed point theorem says that the set of fixed points of a continuous function $f: \prod_{j\in I}S(k_j) \to \prod_{j\in I}S(k_j)$ is nonvoid. It is easy to check that this set is compact. A component C of fixed points of f is essential if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\sup\{ |f(z)-g(z)| : z \in \prod_{j\in I}S(k_j) \} < \delta$, then the continuous function $g: \prod_{j\in I}S(k_j) \to \prod_{j\in I}S(k_j)$ has a fixed point in $\cup_{z^*\varepsilon C}V(z^*,\varepsilon)$.

Theorem 4: (S. Kinoshita, 1952) Every function $f: \prod_{j\in I}S(k_j) \to \prod_{j\in I}S(k_j)$ has an essential component of fixed points.

Theorem 5: (Kohlberg and Mertens 1984) Every finite game has a hyperstable component of equilibria.

Proof: Following Nash (1951), there is a continuous map $f_r = (f_{11},\ldots,f_{1k1},\ldots,$

$f_{n1},\ldots,f_{nk_n})$: $\Pi_{j\in I}S(k_j) \rightarrow \Pi_{j\in I}S(k_j)$ for which the set of fixed points coincides with the set of Nash equilibria of the game with payoffs given by r. This map is defined in the following way: let $c_{ij}(z) := \max(0, r_i(z_{-i}|e_{ij}) - r_i(z))$, we put $f_{ij}(z) := (z_{ij}+c_{ij}(z))/(1+\Sigma_k c_{ik}(z))$.

It is easy to check that the map f_r varies continuously with the r_i: $\Pi_{j\in I}P_j$. Suppose no hyperstable component of equilibria exists, then given any neighborhood U of the set of fixed points of f_r we would always find an r~ as close as we want to r such that $f_{r\sim}$ has no fixed point in U. This contradicts Kinoshita's theorem and we conclude that a hyperstable component of equilibria always exists.

Corollary 6: (Wu and Jiang 1962) If a game has a finite number of equilibria, then one of them is hyperstable.

Corollary 7: Except for a set of Lebesgue measure zero in the space of pure strategy payoffs, every normal form game has a hyperstable equilibrium.

Proof: Because of Sard's theorem for piecewise linear maps we know that, except for a set of Lebesgue measure zero in the space of pure strategy payoffs, every normal form game has an odd number of equilibria. QED

4. Infinite Games

In this section we extend the Kohlberg- Mertens theorem to games where the players may have an infinite number of pure strategies. The proof of existence of a hyperstable component of equilibria given here encompasses the Kohlberg- Mertens theorem for finite

games.

An n- person normal form game is a triple $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$, where $I=\{1,...,n\}$ is the set of players, the P_i 's are the set of pure strategies and each $r_i: \Pi_{j\in I}P_j \rightarrow \Re$ is the payoff in pure strategies to player i. We assume that the P_i are compact metric spaces and that the r_i are continuous maps.

We consider the set M_i of regular Borel probability measures [2] over P_i. M_i is the set of mixed strategies for player i. Each player has an extended payoff function $E_i:\Pi_{j\in I}M_j \rightarrow \Re$ defined by $E_i(m_1, m_2, ..., m_n):= \int_{P_1}\int_{P_2}...\int_{P_n} r_i(p_1,p_2,...,p_n)\, dm_1 dm_2...dm_n$. A Nash equilibrium is a point $m^* \in \Pi_{j\in I}M_j$ such that for all i we have $E_i(m^*) \geq E_i(m_{-i}^*|m_i)$ for all $m_i \in M_i$, where $(m_{-i}^*|m_i)= (m_1^*, m_2^*, ..., m_{i-1}^*, m_i, m_{i+1}^*, ..., m_n^*)$.

Let $C(P_i)$ be the space of continuous real functions over P_i. This space is a Banach space when we endow it with the norm defined by $|f-g|:= \sup\{|f(p)-g(p)| : p \in P_i\}$. We consider its dual $C(P_i)^*$, that is, the space of continuous real linear functionals over $C(P_i)$.

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2 A measure is said to be regular if the measure of a set S can be calculated in either of the two following ways: by computing the measures of the compact sets contained in S and taking the sup of these numbers; or by considering the measures of the open sets containing S and taking the inf.

This space is a topological space when endowed with the weak*- topology. In this topology a net $\{L_\delta\}$ is by definition convergent if for every element f of $C(P_i)$ one has that $L_\delta(f)$ converges.

Lemma 7: The set M_i can be identified with a subset of $C(P_i)^*$ through an isomorphism $\mathfrak{I}$: $M_i \rightarrow C(P_i)^*$ by putting $\mathfrak{I}(m) := L_m$, where $L_m(f) := \int_{P_i} f dm$ for every $m \in M_i$.

Proof: It is clear that $\mathfrak{I}$ is a homomorphism. If $\mathfrak{I}(m) = \mathfrak{I}(m')$, then for all $f \in C(P_i)$ we have $\int_{P_i} f dm = \int_{P_i} f dm'$ and since the weak* topology is a Hausdorff topology we have that m= m'. Therefore, $\mathfrak{I}$ is an isomorphism. QED

Lemma 8: The set $\mathfrak{I}(M_i)$ is a compact convex subset of $C(P_i)^*$.

Proof: That $\mathfrak{I}(M_i)$ is convex is obvious. We claim that $\mathfrak{I}(M_i)$ is closed. Indeed, take a net $\{L_\delta\}$ of elements of $\mathfrak{I}(M_i)$. Suppose that L_δ converges to L, then L is a continuous linear functional, that is $L \in C(P_i)$. By Riesz's representation theorem (see Rudin, 1973), there exists a regular Borel measure m such that $L(f) = \int_{P_i} f dm$ for every $f \in C(P_i)$. It is easy to

prove that m is a probability measure and therefore $\mathfrak{I}(M_i)$ is closed. The Banach-Alaoglu theorem (Rudin, 1973) implies that the set of elements L of $C(P_i)^*$ such that $|L(f)| \leq \max f$ is compact. We conclude that $\mathfrak{I}(M_i)$ is a closed subset of a compact set and thus is compact.

Lemma 9: $\mathfrak{I}(M_i)$ is a metric space.

Proof: This is a consequence of the theorem in Appendix 2. By that theorem there is a homeomorphism h between $\mathfrak{I}(M_i)$ and a subset of the normed space ℓ^2. Put $d(x,y) := |h(x)-h(y)|$ and this defines a distance on $\mathfrak{I}(M_i)$.

QED

From now on, we will write M_i for both M_i and $\mathfrak{I}(M_i)$. The situation will make clear which one we are using.

Lemma 11: The functions $E_i : \Pi_{j \in I} M_j \to \Re$ are continuous. They are also continuous in the functions r_i.

Proof: The proof of these two facts is trivial. We leave them to the reader.

The set of Nash equilibria is a closed subset of the set $\Pi_{j \in I} M_j$ and therefore, as in

the finite case, it is a compact set. Let $V_i(m_i,\varepsilon) := \{m' \in M_i : d(m',m_i) < \varepsilon\}$ and $V(m_1, m_2, \ldots, m_n, \varepsilon) := \prod_{j\in I} V_j(m_j,\varepsilon)$. A component of Nash equilibria is hyperstable if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $|r_i(z) - r_i{\sim}(z)| < \delta$ for all $z \in \prod_{j\in I} P_j$, then the game with pure strategy payoffs given by $r_i{\sim}$ has an equilibrium in $\cup_{z^*\in C} V(z^*,\varepsilon)$.

We proceed to prove the existence of a hyperstable component of equilibria, but first we discuss how can one transform the problem of finding a Nash equilibrium into that of finding the fixed point of a correspondence. Our framework is that of Glicksberg (1952). We start by giving a collection of definitions.

Definitions 11: Let X and Y be two compact metric spaces. Let P(Y) be the set which elements are the nonempty subsets of Y. A correspondence is a map $R: X \to P(Y)$. A correspondence R is upper-semicontinuous if for every converging sequence $(x_n,y_n) \in X \times Y$ such that $y_n \in R(x_n)$, the limit (x,y) satisfies $y \in R(x)$. A correspondence R is lower-semicontinuous if given a converging sequence $x_n \in X$ and $y \in R(\lim x_n)$, then there exists $y_n \in R(x_n)$ such that $y = \lim y_n$. A correspondence is continuous if it is both upper- and lower- semicontinuous.

When Y is a vector space, we say that a correspondence R is convex valued if R(x) is a convex set.

When Y= X, we say that R has a fixed point if there exists $x \in X$ such that $x \in$

$R(x)$.

The following theorem is the core of Glicksberg formulation. We provide the proof in order to bring out all the problems we will have to solve.

Theorem 12: The problem of finding Nash equilibria can be transformed into the problem of finding the fixed points of an upper-semicontinuous convex valued correspondence R: $\Pi_{j\in I}M_j \rightarrow \Pi_{j\in I}M_j$. This correspondence is called the best reply correspondence of the game.

Proof: For every player i we define a correspondence R_i: $\Pi_{j\in I\setminus\{i\}}M_j \rightarrow P(M_i)$ by putting $R_i(m_{-i}) := \{ m_i \in M_i: m_i$ maximizes $E_i(m_{-i} \mid m_i) \}$. First, we point out that R_i really goes into $P(M_i)$. This is true because each E_i is a continuous function and each M_i is compact. Second, if $E_i(m_{-i,n} \mid m_{i,n}^*) \geq E_i(m_{-i,n} \mid m_i)$ and $m_{-i,n} \rightarrow m_{-i}$ and $m_{i,n}^* \rightarrow m_i^*$, then by the continuity of the E_i we have $E_i(m_{-i} \mid m_i^*) \geq E_i(m_{-i} \mid m_i)$. This implies that R_i is upper-semicontinuous. Third, if $E_i(m_{-i} \mid m_i^*) \geq E_i(m_{-i} \mid m_i)$ and $E_i(m_{-i} \mid m_i^{**}) \geq E_i(m_{-i} \mid m_i)$, then the linearity of the integral on the measure says that $E_i(m_{-i} \mid \lambda m_i^* + (1-\lambda)m_i^{**}) \geq E_i(m_{-i} \mid m_i)$ for any $\lambda \in (0,1)$. This proves that R_i is convex valued.

We let R: $\Pi_{j\in I}M_j \rightarrow \Pi_{j\in I}M_j$ be defined by $R(m_1, m_2, ..., m_n) := (R_1(m_{-1}), ..., R_n(m_{-n}))$. R is clearly convex valued and upper-semicontinuous. It is easy to see that a fixed

point of R is a Nash equilibrium and vice versa. This proves the theorem.

QED

Theorem 13: The correspondence R is upper-semicontinuous in the pure strategy payoffs r_i.

Proof: Let $m_{i,n}$ be an element of $R_i(m_{-i})$ when the pure strategy payoff to player i is given by $r_{i,n}$, that is $\int_{P_1}\int_{P_2}\cdots\int_{P_n} r_{i,n}(p_1,p_2,\ldots,p_n)\, dm_1 dm_2 \ldots dm_{i-1} dm_{i,n} dm_{i+1} \ldots dm_n \geq$ $\int_{P_1}\int_{P_2}\cdots\int_{P_n} r_{i,n}(p_1,p_2,\ldots,p_n)\, dm_1 dm_2 \ldots dm_n$. Suppose that $m_{i,n} \rightarrow m_{i,\infty}$ and that $r_{i,n} \rightarrow r_i$ when $n \rightarrow \infty$, then the continuity of the integral with respect to m_i and r_i implies that $\int_{P_1}\int_{P_2}\cdots\int_{P_n} r_{i,\infty}(p_1,p_2,\ldots,p_n) dm_1 dm_2 \ldots dm_{i-1} dm_{i,\infty} dm_{i+1} \ldots dm_n \geq$ $\int_{P_1}\int_{P_2}\cdots\int_{P_n} r_{i,\infty}(p_1,p_2,\ldots,p_n)\, dm_1 dm_2 \ldots dm_n$. This says that each R_i is upper-semicontinuous on r_i. Therefore R is upper-semicontinuous on the r_i.

QED

Remark 14: One can prove that R is jointly upper- semicontinuous in both the m_i's and the r_i's by either putting together the proofs of the two theorems above or by directly using Berge's Maximum Theorem (Hildebrand, 1974, page 30). We shall use this fact in the proof of the lemma below. This lemma says that graphR is a continuous correspondence on the r_i's.

Lemma 15: Let R be the best reply correspondence. Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if for all i $|r_i - r_i\sim| < \delta$, then $\mathrm{graph}R_{r\sim} \subset B(\mathrm{graph}R_r, \varepsilon) := \{ (z,z') \in (\prod_{j\in I} M_j) x (\prod_{j\in I} M_j) : d((z,z'), \mathrm{graph}R_r) < \varepsilon\}$. ($R_r$ indicates the dependence of R on the r_i). Moreover, since r and r~ can be interchanged $\mathrm{graph}R_r$ is a continuous correspondence of the r_i's.

Proof: Suppose that we could find $r_n \to r$ and (x_n,y_n) such that $y_n \in R_{rn}(x_n)$ and such that $(x_n,y_n) \notin B(\mathrm{graph}R_{rn},\varepsilon)$. By passing to a sequence if necessary, we may assume that (x_n,y_n) converges to (x,y). Upper- semicontinuity (in both variables together) implies that (x,y) belongs to $\mathrm{graph}R_r$ and then for n big enough we have $(x_n,y_n) \in \{ (x',y') : d((x',y'), (x,y)) < \varepsilon \} \subset B(\mathrm{graph}R_r,\varepsilon)$. Finally, the last assertion of the lemma follows directly from the definitions.

QED

We state the main theorem of this section.

Theorem 16: Every normal form game has a hyperstable component of equilibria.

We make some observations before we prove this theorem. In Appendix I, we extend Kinoshita's theorem to upper- semicontinuous convex valued correspondences defined from a compact convex subset X of a normed space Y into itself. As in the finite case, one would like to use this theorem to obtain the existence of a hyperstable component

of equilibria. Unfortunately, one cannot directly apply the theorem in that appendix. There are two reasons.

First, M_i is not contained in a normed space unless P_i is finite. Indeed, the Hahn decomposition theorem says that any regular measure m can be written as $m = \lambda_1 m_1 - \lambda_2 m_2$ where $0 \le \lambda_1, \lambda_2 < \infty$ and m_1 and m_2 are regular probability measures. By the Riesz's representation theorem any element of $C(P_i)^*$ can be represented by the integral with respect to a regular measure m. Therefore, the smallest vector space containing M_i is $C(P_i)^*$. This space is not normable unless P_i is finite (Schaefer, 1980).

Second, R is not continuous in the pure strategy payoffs, R is only upper-semicontinuous on these variables.

Proof of the theorem 16: Let R be the best reply correspondence of the game. We consider the correspondence J_R: $(\Pi_{j\in I}M_j)x(\Pi_{j\in I}M_j) \rightarrow (\Pi_{j\in I}M_j)xP(\Pi_{j\in I}M_j)$ defined by $J_R(z,z') := ((z+z')/2, R((z+z')/2))$. J_R has three properties. First, $J_R((\Pi_{j\in I}M_j)x(\Pi_{j\in I}M_j))$ = graphR. Second, J_R is continuous in the r_i's. Third, the fixed points of J_R are of the type (z^*,z^*) where z^* is a fixed point of R and vice versa any fixed point of R induces a fixed point of J_R. The first property is easy to check. The second is a consequence of lemma 15. To see that the third is true, suppose that (z^*,z'^*) is a fixed point of J_R, then we have $z^* = (z^*+z'^*)/2$, which implies $z^* = z'^*$ and so $z^* \in R(z^*)$. That the fixed points of R induce fixed points of J_R is trivial. To prove that hyperstability holds for R it is enough to prove that

it holds for J_R.

An affine homeomorphism is a homeomorphism that preserves convex combinations, that is $h(\lambda z+(1-\lambda)z')= \lambda h(z) + (1-\lambda)h(z')$. According to the lemma in Appendix II, for each M_i there exists an affine homeomorphism h_i: $M_i \rightarrow h_i(M_i) \subset \mathbb{R}^2$. Therefore, we may immerse $(\Pi_{j\in I}M_j)x(\Pi_{j\in I}M_j)$ into a compact convex subset of the normed space $(\mathbb{R}^2)^n x\ (\mathbb{R}^2)^n$ through an affine homeomorphism h defined by h:= $(h_1,...,h_n,h_1,...,h_n)$.

We consider the correspondence $(J_R)^*$: $h((\Pi_{j\in I}M_j)x(\Pi_{j\in I}M_j)) \rightarrow h((\Pi_{j\in I}M_j)x (\Pi_{j\in I}M_j))$ defined by $(J_R)^*:= h(J_R(h^{-1}))$. This correspondence is upper- semicontinuous in the m's and continuous in the r_i's. It is also convex valued. Using the theorem on essential components in Appendix I, we conclude that $(J_R)^*$ has an essential component of fixed points. Therefore, there exists a component of Nash equilibria that is hyperstable.

QED

Example 17: We use an example of Kreps (1985) to show how hyperstability can be used to cut down the number of equilibria. Kreps' example is an extensive form version of the screening problem first proposed by Spence (1974). The description of the game is as follows. There are four players: Nature, one employee and two employers. Nature chooses the type t= 1 or 2 of the employee. Nature has only one strategy, it chooses type 1 with probability 1/3 and type 2 with probability 2/3. The employee is informed of Nature's

choice. He then picks an effort level $e \in [0,K]$. The two employers observe this effort level and make a wage offer $w \in [0,2K]$. They bid to maximize the value $t \cdot e - w$. The one who bids the higher wage gets the employee, if both bid the same wage then the employee is randomly assigned with probability 1/2. The employee's objective is to maximize $w - k_t e^2$ where $k_1 > k_2$. Nature has payoff 0 whatever action is taken, we shall not consider perturbations in Nature's payoffs.

Kreps shows that this game has an infinite number of equilibria. The first type of equilibria is called screening equilibrium. For a pair (e_1,e_2) with $e_1 \neq e_2$ to give a screening equilibrium it is necessary and sufficient that

Type 1 prefers e_1 to 0: $\quad e_1 - k_1 e_1^2 \geq 0,$

Type 1 prefers e_1 to e_2: $\quad e_1 - k_1 e_1^2 \geq 2e_2 - k_1 e_2^2$

Type 2 prefers e_2 to e_1: $\quad 2e_2 - k_2 e_2^2 \geq e_1 - k_2 e_1^2$

It is clear that any point in the interior of the set of screening equilibria is hyperstable. Indeed, these points are characterized by strict inequalities in place of the $\geq$ above, and these are not destroyed by small variations in the parameters k_1, k_2.

A second sort of equilibrium is called pooling equilibrium. It has both types choosing the same effort level e^* and the wage will be $w=5e^*/3$. The condition for equilibrium is

Type 1 prefers e^* to 0: $\quad w - k_1(e^*)^2 \geq 0$

(the other condition is implied by this one). Again, the interior of the set of pooling equilibria is hyperstable.

A third type of equilibrium exists. It consists on having some randomization between some screening and some pooling. If, for example, one takes one of such equilibria for which the pooling part is described by an equality, then perturbing k_1 a little bit can make the equilibrium jump to pure screening. Such an equilibrium is not hyperstable.

5. Full Stability

In this section we give a formal definition of full stability and prove the existence of a fully stable component of Nash equilibria. Each fully stable component of equilibria contains a proper equilibrium. Moreover, deleting dominated strategies does not destroy full stability. These two facts were proven by Kohlberg and Mertens.

A finite game in restricted strategies is the same thing as a finite game, except that each player is constrained to choose his actions in a closed convex subset N_i of his simplex of mixed strategies. A restricted game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I}, \{N_i\}_{i\in I})$ is δ-close to the original game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$ if for every $i \in I$ we have $M_i \subset N_i + B(0,\delta)$. Restricted games are a particular case of convex games and one can prove the existence of Nash equilibria for these games (see, for example, Nikaido 1954).

A Nash equilibrium z^* of a game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$ is fully stable if given $\varepsilon > 0$,

there exists $\delta = \delta(\varepsilon) > 0$ such that any restricted game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I}, \{N_i\}_{i\in I})$ which is δ-close to $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$ has an equilibrium in $V(z^*,\varepsilon)$.

Every hyperstable equilibrium is fully stable (Van Damme, 1984) and one can prove that a hyperstable component of equilibria is fully stable as defined below. We give a new, direct, proof of this fact.

A component of Nash equilibria C of a game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$ is fully stable if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that any restricted game $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I}, \{N_i\}_{i\in I})$ which is δ-close to $(I, \{P_i\}_{i\in I}, \{r_i\}_{i\in I})$ has an equilibrium in $\cup_{z^*\in C} V(z^*,\varepsilon)$.

We shall need the following lemma for proving the existence of fully stable components of equilibria.

Lemma 18: Let M be a compact metric space and let FC(M) be the space of compact connected nonvoid subsets of M, then FC(M) is a compact metric space. Moreover, let FCC(M) be the subset of FC(M) that only contains convex subsets, then FCC(M) is a compact subset of FC(M).

Proof: A theorem of Hausdorff (see Hildebrand, 1974) states that the space F(M) of closed subsets of a compact metric space M is a compact metric space with the metric $d(F,G) := \sup_{x\in M} \inf_{\varepsilon>0}\{F \subset B(G,\varepsilon) \text{ and } G \subset B(F,\varepsilon)\ \}$ is a compact metric space. FC(M) is a subset of this space. We prove that FC(M) is closed. Indeed, let $C_n \in$ FC(M) and $C \in$ F(M) be

such that $d(C_n,C)$ goes to zero when n goes to infinity. Suppose C is not connected, then we can write $C = (C\cap A) \cup (C\cap B)$ where A and B are open disjoint sets and both $C\cap A$ and $C\cap B$ are nonvoid. By the definition of the metric d, we have that for n big enough C_n is contained in the neighborhood $A\cup B$ of C. Since each C_n is connected, each C_n is contained in either A or B but not in both. Without loss of generality take a subsequence $C_{n'}$ which is contained in A, $C_{n'}$ converges to C and therefore C is contained in A and we have that $C\cap B$ is void. This contradiction yields that C is connected and that FC(M) is closed.

FCC(M) is a subset of FC(M). Let $C_n \in$ FCC(M) and $C \in$ FC(M) be such that when $\lim_{n\to\infty} d(C_n,C)= 0$. Let $x,y \in C$ and $\lambda \in (0,1)$, we claim that $z=\lambda x + (1-\lambda)y \in C$. Indeed, given $\varepsilon > 0$, for n big enough we can find $x_n, y_n \in C_n$ such that the distance from x_n to x and from y_n to y is smaller than ε, therefore the distance from $z_n := \lambda x_n + (1-\lambda)y_n \in C_n$ is smaller than 2ε. The z_n's converge to z, therefore z belongs to the closure of C. Since C is actually closed, we have that C is convex. FCC(M) is a closed subset of FC(M).

QED

Theorem 19: Every finite game has a fully stable component of equilibria.

Proof: From the proof of theorem 16 one can conclude that if X is a compact convex subset of a normed space, T is a metric space and $R: X\times T \to P(X)$ is an upper- semicontinuous

convex valued correspondence, then R has a component of fixed points which is robust to small perturbations in the $t \in T$. For each player i, one can define a best reply correspondence R_i in the restricted game by putting $R_i(m_{-i},N_i) := \{m_i \in N_i : m_i$ maximizes $E_i(m_{-i}|m_i)\}$. It is trivial that if $N_{i,n} \to N_{i,\infty}$, $m_{i,n} \to N_{i,n}$ and $m_{i,n} \to m_{i,\infty}$, then $m_{i,\infty} \in N_{i,\infty}$ and that each R_i is upper- semicontinuous. We define a best reply correspondence R for the game in the same way as we did before. This correspondence is upper-semicontinuous and the robustness of at least one of the components of fixed points of R to perturbations in the N_i's yields the existence of a fully stable component of equilibria.

QED

6. Genericity of Hyperstability and Full Stability

In this section we show that for a broad class of games all equilibria are hyperstable. This is based upon an idea of K. Fort (1949), who proved that an upper- semicontinuous correspondence from a complete metric space into a compact metric space is continuous in a dense set.

Let $C(\Pi_{j\in I}P_j, \Re)$ be the space of continuous real functions with domain $\Pi_{j\in I}P_j$ and let $T := \Pi_{i\in I}\, C(\Pi_{j\in I}P_j, \Re)$. An element f of T is a collection $(f_1, ..., f_n)$ of pure strategies payoff functions. T is a Banach space with the norm $|f-g| :=$

$\max_{j\in I} \sup\{|f_j(p)-g_j(p)|: p \in \Pi_{j\in I}P_j\}$.

Let $\Delta: T \to \prod_{j\in I}M_j$ be defined by $\Delta(f) := \{m \in \prod_{j\in I}M_j$: m is a Nash eq. of the game with pure strategy payoff f$\} \equiv \{ m \in \prod_{j\in I}M_j$: m is a fixed point of $R_f \}$.

Lemma 20: Δ is upper- semicontinuous.

Proof: Let $f^k \in T$ and $m^k \in \prod_{j\in I}M_i$ be such that $m^k \in \Delta(f^k)$ for all k. We have $m^k \in R_{f^k}(m^k)$ for all n. If $f^k \to f$ and $m^k \to m$, then the upper- semicontinuity of R in f and m simultaneously (remark 14) yields that $m \in R_f(m)$ or that $m \in \Delta(f)$. This proves that Δ is upper-semicontinuous. QED

Using Fort's theorem one immediately has that Δ is continuous on a dense subset H of T.

Theorem 21: For a dense set $H \subset T$, all games with pure strategy payoffs in H have all equilibria hyperstable.

Proof: The lower- semicontinuity of D on H says that given an equilibrium $z^*(f)$ of the game with payoff f, every game of any sequence of games with payoffs $f^k \to f$ has an equilibrium near z^*. In fact, we can find a sequence $z^*(f^k) \to z(f)$.

QED

Appendix I: Extension of Kinoshita's Theorem

Let X be a metric space and R: $X \rightarrow P(X)$ be an upper-semicontinuous convex valued correspondence. A component C of fixed points of R is essential if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that any other upper-semicontinuous convex valued correspondence G: $X \rightarrow P(X)$ obeying $d(G,R) := \sup_{x \in X} \inf_{\varepsilon > 0}\{R(x) \subset B(G(x),\varepsilon)$ and $G(x) \subset B(R(x),\varepsilon)\} < \delta$ has a fixed point in $\cup_{z \in C} V(x,\varepsilon)$.

The following theorem is Kinoshita's theorem for upper-semicontinuous convex valued correspondences. We follow his ideas closely.

<u>Theorem</u>: If X is a compact convex subset of a normed space Y and R: $X \rightarrow P(X)$ is an upper-semicontinuous convex valued correspondence, then R has an essential component of fixed points.

<u>Proof</u>: By Glicksberg fixed point theorem (Glicksberg, 1952) R has a fixed point. Let F be the set of fixed points of R. F is a compact set, we write F as the union of its components $F = \cup_\alpha C_\alpha$. Suppose that none of the C_α is essential. Then for every α and every $\delta > 0$, there exists an open set U_α such that $X \supset U_\alpha \supset C_\alpha$ and an upper-semicontinuous convex valued correspondence $R_\alpha : X \rightarrow P(X)$ such that $d(R_\alpha,R) < \delta$ and such that R_α has no fixed point

in U_α. We shall construct a correspondence G: $X \to P(X)$ that has no fixed point, contradicting Glicksberg's theorem.

The sets U_α are an open covering for the compact set F. Therefore, we may extract a finite subcovering $U_{\alpha^1}, U_{\alpha^2}, \ldots, U_{\alpha^n}$ of F. Take open sets W_i such that $clW_i \subset U_{\alpha^i}$. By the upper-semicontinuity of R there exists an $a > 0$ such that $\min\{|x-R(x)|: x \in X\backslash(\cup_\alpha W_i)\} > a$. Because of non-essentiallity, we may pick, for i=1 to n, upper-semicontinuous convex valued correspondences $R_{\alpha^i}: X \to P(X)$ having no fixed point in U_{α^i} and such that $d(R_{\alpha^i},R) < a$.

The correspondence G is constructed in the following way:

(a) $G(x)= R(x)$ for $x \in X\backslash(\cup_i U_{\alpha^i})$

(b) $G(x)= R_{\alpha^i}(x)$ for $x \in W_i$

(c) $G(x)= \lambda_i(x)G(x) + (1-\lambda_i(x))G_{\alpha^i}(x)$ for $x \in U_{\alpha^i}\backslash W_i$ where $\lambda_i : X \to \Re$ is the continuous function defined by

$$\lambda_i(x) := \frac{d(x,clW_i)}{d(x,clW_i) + d(x,X\backslash\cup_i U_{\alpha^i})}$$

It is easy to check that G is upper-semicontinuous. G is clearly convex valued for x

in $X\setminus\cup_i U_{\alpha^i}$ and x in W_i. For x in $U_{\alpha^i}\setminus W_i$, G(x) is the convex combination of two convex sets and is a convex set. Therefore, G is also convex valued.

G does not have fixed points. Indeed, G has no fixed point in either $X\setminus\cup_i U_{\alpha^i}$ or any of the W_i. On the other hand, for $x \in U_{\alpha^i}\setminus W_i$ we have $G(x)= \lambda_i(x)R(x) + (1-\lambda_i(x))R_{\alpha^i}(x)$ $\subset R(x) + B(0,a)$. Suppose we had a fixed point $x \in U_{\alpha^i}\setminus W_i$, then there exists $z \in R(x)$ and $v \in B(0,a)$ such that $x= z + v$, but $|x-z| > a$, a contradiction.

G does not have fixed points and satisfies all the conditions of Glicksberg theorem. This is a contradiction, therefore there must exist an essential component of fixed points.

QED

<u>Remark</u>: The fact that Y is a normed space was used to ensure that we could write that $\lambda_i(x)R(x) + (1-\lambda_i(x))R_{\alpha^i}(x)$ is contained in the set of points that are at a distance to R(x) smaller than a and that this latter set is equal to $R(x) + B(0,a)$.

Appendix II: Characterization of the Sets of Mixed Strategies

By definition the set M_i of mixed strategies of player i is the set of regular Borel probability measures over his set P_i of pure strategies. We considered M_i with the weak* topology, that is, a basis for the topology of M_i is given by the sets $V(m,\varepsilon) := \{m' \in M_i : |\int f dm' - \int f dm| < \varepsilon$ for every $f \in C(P_i)\}$, where m varies over all M_i and $\varepsilon > 0$. M_i is a metric space with this topology.

Theorem: Let the P_i be compact and metric. Then, either P_i is finite or M_i is affinely homeomorphic to a compact convex subset of ℓ^2.

Proof: Because P_i is compact and metric, we may use the Stone-Weirstrass theorem and ensure the existence of a dense countable family of functions $\{f_n\}_{n\in N} \subset C(P_i)$. This sequence induces a sequence $\{L_n\}$ of continuous linear functionals over M_i by putting $L_n(m) := \int f_n dm$. Since M_i is compact, there exist $C_n > 0$ such that $|L_n(m)| < C_n$ for all n. We define a sequence of maps $h_k: M_i \rightarrow \ell^2$ by putting $h_k(m) := (L_1(m)/C_1, L_2(m)/2C_2, ..., L_k(m)/kC_k, 0, ..., 0)$. Each h_k is a continuous map because $|h_k(m)-h_k(m')| \leq \max_{j=1,...k} |L_j(m)-L_j(m')|^2/C_j^2$. The sequence h_k converges uniformly since $|h_{k+p}-h_k|^2 \leq (1/k^2) +$

$(1/(k+1)^2) + ... + (1/(k+p)^2)$. Therefore, the map $h: M_i \to h(M_i) \subset \ell^2$ defined by $h(m) :=$ $(L_1(m)/C_1, L_2(m)/2C_2, ...)$ is continuous. If we prove that this map is injective, then since it clearly preserves convex combinations we will have an affine homeomorphism between M_i and $h(M_i)$. Suppose $h(m)= h(m')$, then $L_n(m)= L_n(m')$ for all n, since the $\{f_n\}$ are dense we have that for any continuous linear functional L over M_i, that $L(m)= L(m')$. Since the weak* topology is separating this implies that $m= m'$ and h is therefore injective. h clearly is affine.

If $h(M_i)$ is finite dimensional, then by Brouwer's Invariance of the Domain theorem (Hurewicz and Wallman, 1941) so is M_i. This implies that P_i is finite. This reasoning can be reversed to state that if P_i is infinite, then $h(M_i)$ is infinite dimensional.

QED

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