FUNDAÇÃO GETULIO VARGAS
ESCOLA DE ECONOMIA DE SÃO PAULO

KRISTOPHER KRISHNAMURTI HOMEM DE MELLO ROBINSON

PRICING THE CONVEXITY PREMIUM OF INTEREST RATE DERIVATIVES INDEXED TO CDI USING A HJM MULTI-FACTORIAL MODEL

SÃO PAULO
2021

KRISTOPHER KRISHNAMURTI HOMEM DE MELLO ROBINSON

PRICING THE CONVEXITY PREMIUM OF INTEREST RATE DERIVATIVES INDEXED TO CDI USING A HJM MULTI-FACTORIAL MODEL

Dissertação apresentada ao Programa de Mestrado da Escola de Economia de São Paulo da Fundação Getulio Vargas, como requisito para a obtenção do título de Mestre em Economia.

Área de concentração:
Engenharia Financeira.

Orientador: Prof. Dr. Afonso de Campos Pinto

SÃO PAULO
2021

Robinson, Kristopher Krishnamurti Homem de Mello.
Pricing the convexity premium of interest rate derivatives indexed to CDI using a HJM multi-factorial model / Kristopher Krishnamurti Homem de Mello Robinson. - 2021.
78 f
Orientador: Afonso de Campos Pinto
Dissertação (mestrado profissional MPFE) - Fundação Getulio Vargas, Escola de Economia de São Paulo.

1. Taxas de Juros. 2. Derivativos (Finanças). 3. Risco (Economia). 4. Mercado financeiro. 5. Modelos matemáticos. I. Pinto, Afonso de Campos. II. Dissertação (mestrado profissional MPFE) - Escola de Economia de São Paulo. III. Fundação Getulio Vargas. IV. Título.

CDU 336.781.5

Ficha catalográfica elaborada por: Isabele Oliveira dos Santos Garcia CRB SP-010191/O
Biblioteca Karl A. Boedecker da Fundação Getulio Vargas - SP

KRISTOPHER KRISHNAMURTI HOMEM DE MELLO ROBINSON

PRICING THE CONVEXITY PREMIUM OF INTEREST RATE DERIVATIVES INDEXED TO CDI USING A HJM MULTI-FACTORIAL MODEL

Dissertação apresentada ao Programa de Mestrado da Escola de Economia de São Paulo da Fundação Getulio Vargas, como requisito para a obtenção do título de Mestre em Economia.

Área de concentração:
Engenharia Financeira.

E aprovado em 20/07/2021
Pela comissão organizadora:

Prof. Dr. Afonso de Campos Pinto
EESP - FGV

Prof. Dr. André Cury Maialy
EESP - FGV

Prof. Dr. Marcos Eugênio da Silva
FEA - USP

I dedicate this and everything else to my family.

ACKNOWLEDGEMENTS

I would like to thank my thesis adviser and also coordinator of the Financial Engineering master course of the *Escola de Economia de São Paulo* at *Fundação Getulio Vargas*, Prof. Dr. Afonso de Campos Pinto, that from the very first interview of my enrollment on the master course showed me a great willingness to help me achieve the potential he saw I had and deliver a work that, hopefully, meets his high expectations.

I would also like to thank my class colleagues on the master course, that helped me during the classes and on the execution of this work. Ivan and Luísa, I definitely would not have completed the master course without your help and a special thanks to André, who assisted me in the Python implementation of the polynomial regressions in a priceless manner. I could not forget to thank Laura Gâmbera, that helped me a lot during the course on the mathematical front, even from another country.

Additionally, I would like to thank my mentors, Rafael Pistelli Gomes and Rafael de Godoy Oliveira Andrade, for having effectively formed me as a trader and passing to me the core stone of my knowledge of finance. Also I would like to thank Bruno Bernardes for all the useful and insightful talks.

And last but not at all least, I would like to thank the love and support of my family and my amazing wife Lívia, not only during the execution of this work, but throughout all my accomplishments.

Muito obrigado.

"O Brasil não é para principiantes."

Antonio Carlos Jobim

ABSTRACT

The stochastic element of the dynamics of interest rates combined with the compounded capitalization of the CDI index generates risks of first and second order on derivatives indexed to a percentage different from 100% of CDI. There is literature focused on this, recurrently brought as reference in this work, however in practice the financial market in a general manner prices correctly the first order risk (*delta*) without giving such diligent attention to the second order risk (*gamma*), related to the convexity. Such risk, as it is evidenced throughout this work, is positively related to the volatility of the yield curve, to the difference from 100% CDI to the percentage of CDI that the derivative is indexed, it is also related to the maturity of the derivative, being higher on longer tenor derivatives; altogether with the level of the interest rates yield curve.

The current context of low interest rates across the globe, may let agents to dismiss the impact of such second order risks, due to the low level of rates curves, but the lower level of interest rates also led to lengthen the tenors of issued debts and the derivatives associated with such emissions, making the amount of this exposure on longer tenor possibly higher now more than ever. With a potential future normalization of monetary policies taking action on different economies, it is expected that the second order risk related to the convexity of these derivatives appears rapidly then. Therefore, it could be said that it is convenient to address this particular issue now, once again.

The aim of this work is to contribute to the literature by analysing the convexity of these derivatives by the robust HJM framework and also by proposing as a tool a polynomial equation that could be used as an estimator of the convexity premium as a function of the percentage of CDI the swap in question is indexed and its maturity; to obtain in a quick manner an educated guess of the value of such convexity, that must be considered when pricing such derivatives. The work is based on historical closing prices of DI futures to estimate the volatility parameters of a multi-factorial HJM model to, with these modeled dynamics, then price through Monte Carlo simulations the premiums associated to the convexity of derivatives indexed to percentages different from 100% of CDI on different maturities.

Keywords: Interest Rate Derivatives, Convexity, HJM multi-factor model.

RESUMO

A estocasticidade das taxas de juros somada à capitalização composta do índice CDI geram riscos de primeira e de segunda derivada em derivativos indexados a um percentual diferente de 100% do CDI. Há literatura acerca do tema, muito utilizada como bibliografia referência para execução deste trabalho, entretanto na prática o mercado financeiro de forma generalizada costuma apreçar corretamente o risco de primeira derivada sem dar a devida atenção ao segundo fator de risco, relacionado à convexidade. Tal risco, conforme evidenciado neste trabalho, é relacionado ao quão maior for a volatilidade da curva de juros; ao percentual do CDI ao qual o derivativo está indexado; quão maior for o nível da curva de juros, e quão mais longo for o prazo até o vencimento do derivativo. O atual contexto de baixas taxas de juros em muitas geografias pode levar agentes a relevar o impacto de tais riscos de segunda ordem, devido ao baixo nível das curvas de juros, porém este mesmo ambiente de juros baixos também levou ao alongamento do prazos de dívidas, o que impactará o prazo dos derivativos associados a tais emissões, levando o montante desta exposição em vencimentos mais longos potencialmente ao maior nível observado no mercado. Com uma possível normalização das políticas monetárias expansionistas em diferentes países, é esperado que os riscos de segunda ordem atrelados à convexidade destes derivativos então se revele de forma abrupta. Portanto, poderia ser dito que é conveniente adereçar esse tema em particular ainda hoje, uma vez mais. Este trabalho busca contribuir para a literatura ao analisar a convexidade de tais derivativos sob o arcabouço teórico do modelo HJM e propor como ferramenta a obtenção de uma equação polinomial que possa ser utilizada como estimador do prêmio associado à convexidade presente em derivativos indexados à um percentual acima de 100% do CDI, em função deste percentual e em função do vencimento do derivativo, para obter de uma maneira rápida uma estimativa do valor de tal convexidade, a ser considerada quando apreçando derivativos desta natureza. São utilizados dados históricos de fechamento de futuros de DI para estimar os parâmetros de volatilidade de um modelo multifatorial HJM para então através de simulações de Monte Carlo apreçar os prêmios relacionados à convexidade presente em derivativos indexados à percentuais do CDI diferentes de 100%.

Palavras-chave: Derivativos de Taxa de Juros, Convexidade, Modelo HJM multi-fatorial.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1. INTRODUCTION

Through personal experience in the financial market, pricing derivatives indexed to a percentage of CDI different from 100% notably led to identify a mispricing by part of the market of a second order risk associated to the convexity of this derivative, that should be embedded in the price but that often it is not. Then, for trading purposes the study of the topic would be already justified. The literature about the theme already exists. Yam (2010) modeled the structure of the yield curve under a general equilibrium of interest rates framework, more specifically the Cox, Ingersoll, and Ross (1985) model and the Vasicek (1977) model. Gomes (2015) made use of a non-arbitrage framework, with a Black, Derman, and Toy (1990) model approach, adding an analysis of which strategy would be more efficient to hedge the *delta* risk that the convexity generates depending on the movements of prices of the yield curve. The motivation of this work is to contribute to the literature by providing estimates of such convexity premium under the robust HJM framework and by founding a polynomial equation in order to have a quick but fair estimate of the premium associated to this convexity, as a function of the percentage of CDI that is being considered in the swap and its maturity. This is done by using a Heath, Jarrow, and Morton (1992) framework to model the structure of the yield curve, extracting from historical data from closing prices of DI futures three principal components to be used as the volatility structure for the whole curve, and using this approach to then apply Monte Carlo simulations in order to price the fair value of the convexity of derivatives indexed to a percentage of CDI that differ from 100% . The resultant vectors of these simulations are then interpolated by a polynomial equation.

The main objective of this work is to find a polynomial equation that estimates the convexity premium that is required to be embedded in the price of derivatives indexed to a percentage of CDI different from 100%, more specifically bullet swaps. Not having a closed formula for the convexity premium, the HJM framework is validated by back-testing the parameters of the model with historical data of prices negotiated in the Brazilian DI futures market, via Monte Carlo approach simulating thousands of scenarios of the yield curve path, adding to it a dynamic hedging strategy of the first order risk that appears in a portfolio composed by a swap indexed to a percentage of CDI different from 100%; and a determined quantity of DI future contracts. The relevance of this topic, within a context of historically low interest rates is justified by the fact that, since the convexity risk is positively related to the nominal level of the interest rate curve, the agents might not consider this risk correctly, since at the actual levels, thinking it

might not be relevant as it used to be with higher rates. But the same environment led firms and governments to lengthen their debts and the derivatives associated with those emissions increased in maturity. And maturity is also positively related to *gamma* and now might be higher than at any other moment in the past, at least here in Brazil. The potential normalization of monetary policies across the globe, after the largest coordinated easing environment ever seen on rates globally, can make the second order risks of these derivatives surge abruptly, making the precautionary assessment of this topic relevant within the horizon ahead.

This work is divided into eight chapters, including this brief introduction. Chapter 2 presents a literature review regarding both the HJM model to be implemented in order to generate the scenarios simulated and the literature about pricing of derivatives indexed to a percentage of CDI. On Chapter 3, it is contextualized the Brazilian interest rate derivatives market and specifically derivatives indexed to a percentage of CDI. On Chapter 4, the HJM model is presented in detail. On Chapter 5, it is explained how the HJM framework is applied to the specific case of pricing the derivatives indexed to a percentage of CDI. On Chapters 6 and 7, the estimations and results are presented and based on the results of the Monte Carlo simulations, it is estimated a polynomial equation that could serve as a way for obtaining a fair estimate of the convexity premium to be embedded in these derivatives, as a function of the percentage of CDI it is indexed and of its maturity. On Chapter 8 the developments presented in this work and the conclusions from the results obtained are summarized and suggestions of possible extensions upon the work realized are presented.

2. LITERATURE REVIEW

Although it has already been largely studied and much has been developed related to models to price interest rate derivatives and the hedging of portfolios composed by them, there is not yet a consensus on an unique model that would be the best to be used. The main models are based on the construction of a term structure of the yield curve, which are primarily divided between general equilibrium models and non-arbitrage models.

The first models of the general equilibrium approach were presented by Vasicek (1977), and later by Cox, Ingersoll, and Ross (1985), generally making assumptions about economic variables (such as the economy's state variables and investor preferences), and then describing a process for the spot rate, r, that once obtained is used in the pricing of bonds and interest rate derivatives, such as options on bonds.

The non-arbitrage approach counts with the models presented by Ho and Lee (1986), Hull and White (1990), Black, Derman, and Toy (1990) and Heath, Jarrow, and Morton (1992). What they all have in common is the goal to obtain results that are consistent with market prices, modeling the evolution of the yield curve from market prices observed in a certain moment and calibrating parameters with historical data.

It is also common to categorize interest rates models by the number of factors they consider, from models that consider only one source of randomness (one-factor models) to models that consider more than one (multi-factor models).

The framework that Heath, Jarrow and Morton presented in 1992 was a non-arbitrage multi factorial model, that aimed to describe the dynamics of the whole yield curve forward rate structure, with a high degree of freedom with respect to the volatility structure considered, being then a more complex model than the previously presented.

The HJM model develops the dynamics of the instantaneous forward rate structure on a continuous capitalization environment, making it necessary to discretize it somehow in order to relate to discrete data obtained from the market. The key change presented by the HJM framework was the concept of stochasticity on the *drift* of the dynamics. Glasserman (2003) presents different ways to approximate a continuous *drift* to a discrete one. From those many, it is used in this work the one in which discounted bond prices (and in the particular case studied, DI future's prices) maintain their martingale properties, solution that is also applied by Nojima (2013) and Sena (2019), works applied to the Brazilian interest rate market.

Renò and Uboldi (2002) presented the implementation of a model under the HJM frame-

work, indicating the capacity of this framework to represent the main characteristics of the dynamics of the structure of the yield curve, by construction excluding arbitrage opportunities. On one hand, their implementation made explicit the attractiveness of the HJM model, by its consistency and its flexibility, but on the other hand showed the complexity of the model's implementation, in which the authors made use of a technique called Principal Component Analysis (PCA) to estimate the volatility parameters.

Litterman and Scheinkman (1991) developed a work in which they identify that up to 96% of the historical returns of the secondary market of bonds from the American Treasury could be explained by only 3 orthogonal vectors, or components, that were related to the level of the yield curve, its steepness and its curvature. The method they applied to identify these main components is known as Principal Component Analysis, in order to identify only 3 main factors that will contribute to explain the stochastic component of the dynamics of interest rates. It was used by Renò and Uboldi (2002) in their implementation of the HJM framework, also used by Nojima (2013), Sena (2019) and in this work.

Regarding contributions to the implementation of a HJM framework model, Brace and Musiela (1994) proposed a new way to consider the time intervals for the forward rates modeled, leaving the original HJM approach to an instantaneous forward rate ($f(t, T)$) that depended on both of the moment of observation (t) and on the fixed maturity (T), to instead consider a substitution of T to an interval $t + \tau$ so that the forwards remain much conveniently with the same period of time, τ, between its initial and final vertices, disregarding its moment of observation or maturity.

Another issue to be considered when dealing with the HJM framework is the fact that the model is non-Markovian, which means not only the present state of the variable determines the possible future outcomes, but also previous states have an impact on future states, although random. This characteristic appears due to the fact that the model assumes that the function defining the *drift* of a tenor is impacted by the volatility of previous tenors, removing the Markovian essence from this model. There are then few functions of volatility that allows an analytical solution for the stochastic differential equation of the HJM model, making a numerical method approach necessary in order to obtain its solution. On top of that, its flexibility impacts the complexity of the implementation, what could also be seen as a reason to the model not being so largely used by practitioners.

Björk, Szepessy, Tempones, and Zouraris (2012) however implement the HJM framework on their work focused on numerical approximation on the pricing of financial instruments, using Monte Carlo simulation as a numerical method to price bonds and other financial instruments that depend on the interest rates modeled by the HJM. Monte Carlo simulations are used to this end broadly on finance literature, as in Andrade (2014) and in Gomes (2015), to estimate convexity premiums, being also the numerical method implemented in this work, with the same purpose.

The HJM framework has already been used in studies about Brazilian volatility market, such

as in Barbedo, Vicente, and Lion (2009) in which the authors implemented three versions of a HJM model, with three, two and only one factor to price IDI options (options over an index related to the accumulated CDI) with its volatility structure estimated by the Principal Component Analysis technique. In this work, the authors concluded that the one factor model lead to better results than the two and three factor models, based on a metric of standard deviation from market prices of the IDI options. Although this result may corroborate the use of a single-factor HJM model, in this work it is implemented a three factor model due to the fact that the purpose is to generate a dynamic for the interest rate curve so that the Monte Carlo simulations of the strategy of *delta* hedging the swap indexed to a percentage of the CDI different from 100% resembles more accurately the one observed in the market. There is evidence, as in Litterman and Scheinkman (1991), even if not related to the Brazilian case, that as an estimate of the term structure dynamics more than one factor (three in this specific case) fits better its behavior than only one factor would.

Related to pricing derivatives indexed to CDI, Hübner (2003) introduces the analysis of risks using a HJM model to price derivatives indexed to a percentage of the CDI significantly superior to 100%, coming with evidence that these instruments have *delta* risks related to the fixed rate level in the moment they are priced, positively related to the difference from 100% of the CDI and also noticing the side of the risk related to the side in which each counterpart of the derivative is facing. So in a way, beyond just pricing the derivatives under a HJM framework, the author also made explicit the *delta* risk and how each counterpart should manage this first order risk in the moment the trade is dealt. According to the author, by the time the work was published such risks were not correctly managed by a significant part of the financial industry, which (unclear the influence of such paper on this fact) now is not the case anymore.

Later, Eid and Ohanian (2005) studied the sensitivity of these kind of derivatives to movements on the underlying rate curve. In their work, the authors positively relate this sensitivity to the level of the underlying curve and also to the maturity of the derivative and identify a negative convexity on derivatives in which the holder is receiving the percentage of the CDI under 100%. The maturities analyzed by Eid and Ohanian were between six months and one year, which in posterior literature has been proved to lack relevant convexity. Yam (2010) investigates in her work using general equilibrium models such as Cox-Ingersoll-Ross (CIR) and Vasicek the convexity of derivatives indexed to a percentage higher than 100% of the CDI, obtaining objective results for this convexity premium for derivatives of 10 years of maturity and indexed to 200% assuming a 15% annual volatility and a mean reversal rate of 50% per year, in which she estimates being of 343 basis points throughout the life of the derivative, using the CIR model.

Following Yam's investigation, Gomes (2015) elaborates his work on this convexity premium for derivatives indexed to a percentage of the CDI using the non-arbitrage model of Black-Derman-Toy to estimate the dynamic of the yield curve and reaches different results than Yam. By his analysis, in which different *delta* hedging strategies were considered, the estimate

for the convexity effective premium for a derivative maturing in 10 years, indexed to 200% of the CDI and assuming a 15% annualized volatility is around (depending on the hedging strategy and different assumptions over costs related to the *hedge*) 1.700 basis points throughout the life of the derivative. Beyond this difference in results on similar derivatives analyzed, Gomes also presents in his work the results on different maturities, hedging strategies and hypothesis related to transaction costs when hedging. As an extension suggestion, he proposes that this convexity premium be studied on an HJM framework, which is now the scope of this work.

The present work approaches the pricing of convexity premiums of interest rate derivatives indexed to CDI using a HJM multi factorial model, extracting from historical data volatility vectors to generate potential future paths for the yield curve and through Monte Carlo simulation, estimate a fair premium for the convexity embedded in derivatives with different maturities and indexed to different percentages of CDI. Additionally, the matrix of results of convexity premiums for each analyzed combination of tenors and percentages of CDI is used to obtain a polynomial equation that could serve as a tool to have an estimate of convexity premiums for maturities and percentages of CDI other than the ones estimated by the simulations. This tools shows its relevance by allowing practitioners to obtain in a quicker manner an estimate for convexity premiums, being useful for pricing these kind of derivatives without having to implement each time the whole process of Monte Carlo simulations to a new set of combinations of percentages of CDI and maturities, only periodically when considered appropriate to readdress the volatility inputs of the model.

3. AN OVERVIEW OF THE BRAZILIAN INTEREST RATE MARKET

This chapter presents an overview of the Brazilian interest rate market and introduces the characteristics of the portfolio that is analyzed on this work.

3.1 The CDI

Even though being considered an emerging economy, Brazil has a peculiar well developed derivatives market, concentrating regarding both currency and interest rate markets, far more liquidity on derivatives than on its underlying asset markets. Just to exemplify the importance of the derivatives market in Brazil: it is easier to trade nominal interest rate futures on the B3 exchange than federal government nominal rate bonds over the counter. As a matter of fact, these bonds are traded basically over the counter if traded together with a derivative with the same maturity.

During the late 1980's and early 1990's Brazil suffered from hyperinflation, which caused perceptible evolution regarding its banking and liquidation system, in order to overcome the highly frequent changes in prices. This same fact also led to a high concentration of real economy contracts based on floating indexes, such as the CDI.

The CDI (which stands for "*Certificado de Depósito Bancário*", or in free translation, Banking Deposit Receipt) is an index published daily by B3 Exchange[1] which, in theory, represents the mean of the negotiated rates between financial institutions to exchange cash volumes over night. This rate is correlated to SELIC, which is the economy's basic target rate defined at each COPOM[2] meeting. There are some particularities that differ the two, but for the past few years they have been traded at the same level due to fallback rules that took place which made the two rates, in a practical manner, the same. Therefore, the CDI has been considered through literature regarding Brazilian markets, the basic interest rate of the economy and it is used in many studies, such as in this present work, as the instantaneous short term rate, $r(t)$.

[1]the CDI index was previously published by CETIP, "*Central de Custódia e Liquidação Financeira de Títulos*", or in free translation, Central of Custody and Financial Liquidation of Bonds; but as of March 2017, CETIP and BMF&Bovespa, the former Brazil Stock and Futures Exchange, merged into the name of B3, that now publishes the CDI index.

[2]the Brazilian Monetary Policy Committee.

3.2 The DI future

The main tradable standard derivative related to this index is the DI future, a derivative contract traded on B3 exchange that has as its underlying the CDI index between the negotiation date and its expiry. The agents that trade this contract are betting that the accumulated CDI index between these dates is going to be higher (or lower, depending on the side of each counterpart on the trade) than a fixed rate in which the contract is being negotiated at the moment. As it is frequent when dealing with interest rate markets and its instruments, besides the fact that what is indeed being traded is an interest rate between two dates, the asset (in this case, the derivative) has a predetermined future value, and the interest rate traded is the one that will relate the present value to the predetermined future value. In the case of the DI future, this future value is set to R$ 100,000 and the day count between the negotiation date and its expiry is defined by business days in a 252 business days per year basis.

Thus, being t_b the number of business days between the date of negotiation and the start of the derivative; T_b the number of business days until the expiry (or, equivalently, the maturity) of the derivative, and $y_{(t_b,T_b)}$ the interest rate, annualized on 252 business days basis, observed in the market at $t_b = 0$ to T_b business days, we may define that the price at time $t_b = 0$ of a DI future contract maturing after T_b days related to a rate $y_{(t_b,T_b)}$ as $DI(y, t_b, T_b)$:

$$DI(y, t_b, T_b) = \frac{100{,}000}{(1 + y_{(t_b,T_b)})^{\frac{(T_b - t_b)}{252}}}. \tag{3.1}$$

Additionally, we may also define the payoff at maturity of this derivative. Being $CDI_{(t_b,T_b)}$ the daily annualized rate (on the 252 business days basis) of the CDI accumulated throughout the period of the contract (since negotiation to its expiry), it is possible to define the payoff of this derivative at its maturity as:

$$Payoff_{(T_b)}[DI(y, t_b, T_b)] = \left[100{,}000 - \frac{100{,}000}{(1 + y_{(t_b,T_b)})^{\frac{(T_b - t_b)}{252}}} \prod_{t_b=1}^{T_b} (1 + CDI_{(t_b-1,t_b)})^{\frac{1}{252}}\right]. \tag{3.2}$$

3.3 The portfolio

This work analyzes a portfolio composed by DI future contracts and another derivative, a non-standardized bullet[3] swap, commonly traded over-the-counter between two counterparts, that has one of its legs indexed to a percentage φ of CDI, and the other indexed to 100%CDI plus (or minus, depending on the percentage of CDI of the other leg) a fixed spread. The percentage of CDI is a well-known metric of profitability in Brazil, and the demand for this kind of swap

[3] A "bullet" financial instrument is an instrument that pays its interest and amortizes its outstanding only at the agreed maturity, with no intermediate payments of any nature.

is considerable due to this particular method of measuring profitability/debt-cost related to the CDI. In order to provide a simplified example of the future value of such derivative, consider a swap that has a notional of N; but in which one of the legs is fixed at the rate $fix_{(t_b,T_b)}$ to be capitalized in a day-count convention of 252 business days per year; that will expire after T_b business days from the negotiation date, with t_b being the number of business days between the date of negotiation and the start of the derivative (in the specific case $t_b = 0$); and the other leg is indexed to a percentage φ of CDI, different from 100%. This way, we may define the payoff of this swap at its maturity, $\text{Payoff}_{(T_b)}[\text{Swap}(fix, t_b, T_b, \varphi)]$, as below:

$$N \times \left[\prod_{t_b=1}^{T_b} \left(1 + \left(\left(1 + CDI_{(t_b-1,t_b)}\right)^{\frac{1}{252}} - 1\right)\varphi\right)\right] - N \times \left[\left(1 + fix_{(t_b,T_b)}\right)^{\frac{(T_b-t_b)}{252}}\right] \tag{3.3}$$

What is worth being pointed out here is the future value of the leg indexed to a percentage φ of CDI in the swap, so to develop the rationale about the convexity embedded in these derivatives. Selecting only the future value of this leg we have that:

$$N \times \left[\prod_{t_b=1}^{T_b} \left(1 + \left(\left(1 + CDI_{(t_b-1,t_b)}\right)^{\frac{1}{252}} - 1\right)\varphi\right)\right] \tag{3.4}$$

From which we can understand the importance of the path of the CDI index on a derivative indexed to a percentage φ of CDI, since the index is compounded daily on a exponential basis, multiplying each day by the factor of the CDI index leveraged by the φ percentage of the contract. But this is the formula that shows the future value of such derivative. Often throughout the period between its negotiation and its expiry, the counterparts of the contract need to assess the value of this contract subject to market conditions, or in other words, "mark to market"[4] its value.

3.4 The MTM

From the future value formula 3.4 of the floating leg of this derivative, it is clear that in order to mark to market properly one should have an estimate of what would be the expected cumulative daily CDI index between the date in question and the expiry of the derivative. And it is at this point that practitioners begin to diverge.

Most practitioners [5] consider that the expected cumulative daily CDI extracted from the DI future interest rate negotiated to the same maturity of the swap (or if the swap is not due on a date that does not match a tenor of a negotiated DI future, some interpolation of two tenors

[4]The term is often abbreviated in the financial industry by the acronym "MTM".

[5]As can be found on pricing manuals of the Brazilian Association of Financial and Capital Markets, as know as ANBIMA; and in pricing manuals of many Brazilian financial institutions, such as asset management firms and banks.

may be considered in order to approximate what would be the interest rate of a DI future to the maturity of the swap) would be a good approximation of the real accumulated daily CDI. For these practitioners, the mark to market of the floating leg of this swap would be given, considering $y_{(t_b,T_b)}$ the fixed rate observed in the market at $t_b = 0$ related to the DI future with the same maturity of the swap, after T_b business days, by $\theta^m(CDI, y, t_b, T_b, \varphi)$ as below:

$$N \times \left[\prod_{t_b=1}^{T_b} \left[1 + \left(\left(1 + CDI_{(t_b-1,t_b)}\right)^{(1/252)} - 1\right) \varphi\right] \frac{\left[\left(\left(1 + y_{(t_b,T_b)}\right)^{(1/252)} - 1\right) \varphi + 1\right]^{(T_b-t_b)}}{\left(1 + y_{(t_b,T_b)}\right)^{(T_b-t_b)/252}} \right] \tag{3.5}$$

What is worth to notice is that are two assumptions/approximations being made in this formula[6], one of them being reasonable to be made and has a negligible effect in practice [7] but the other one could be considered unrealistic and has an apropos impact. The reasonable approximation made is to consider as the effective factor of daily CDI index embedded in a DI future within all the business days throughout the period of the contract as the annualized daily factor of the rate of such contract due on T_b business days. To put in mathematical terms, let $y_{(t_b-1,T_b)}$ be the embedded forward between $t_b - 1$ and T_b business days from the negotiation date of such contract, and φ still being the percentage of CDI that such a derivative is indexed, therefore what is being considered in this first approximation is that:

$$\begin{aligned} &\left(1 + \left(\left(1 + y_{(t_b,T_b)}\right)^{\frac{1}{252}} - 1\right) \varphi\right)^{(T_b-t_b)} \approx \left(1 + \left(\left(1 + y_{(0,1)}\right)^{\frac{1}{252}} - 1\right) \varphi\right) \times \\ &\left(1 + \left(\left(1 + y_{(1,2)}\right)^{\frac{1}{252}} - 1\right) \varphi\right) \times \dots \left(1 + \left(\left(1 + y_{(T_b-1,T_b)}\right)^{\frac{1}{252}} - 1\right) \varphi\right) \end{aligned} \tag{3.6}$$

Considering still only this first approximation, most practitioners already correctly consider the embedded so called *delta* risk when the percentage φ of CDI of such derivative is different from 100% related to movements on the yield rate curve. Aiming to hedge such risk, one should neutralize it with a *delta* hedging strategy with a standard instrument that has its price related to the yield curve and the chosen instrument to do so in practice is the, already presented in this work, DI future. Being Δ the number of contracts of DI future that neutralizes the risk related to a movement of ε basis points on the yield curve; N the notional of the swap; φ the percentage of CDI that the swap is indexed; $y_{(t_b,T_b)}$ the interest rate from the negotiation date until the maturity of the derivative; T_b the number of business days between the negotiation date and the maturity of the derivative and $t_b = 0$ the number of days between the negotiation date and the start date of the derivative, one can estimate for that certain level of the yield curve the quantity Δ of contracts of such derivative as being:

[6] These approximations were mentioned on the work of both Yam (2010) and Gomes (2015). The approach developed in this section is broadly based upon their work.

[7] The practical effect was measured by Gomes (2015), in which he finds that the approximation differs from the effective CDI factor by the order of 10^{-7}.

$$\Delta(N, y, t_b, T_b, \varepsilon, \varphi) = N \times \left[\frac{ \frac{\left[\left((1+y_{(t_b,T_b)}-\varepsilon)^{\frac{1}{252}}-1\right)\varphi+1\right]^{(T_b-t_b)}}{(1+y_{(t_b,T_b)}-\varepsilon)^{\frac{((T_b-t_b))}{252}}} - \frac{\left[\left((1+y_{(t_b,T_b)})^{\frac{1}{252}}-1\right)\varphi+1\right]^{(T_b-t_b)}}{(1+y_{(t_b,T_b)})^{\frac{(T_b-t_b)}{252}}} }{ 100{,}000 \left[\frac{1}{(1+y_{(t_b,T_b)}-\varepsilon)^{\frac{(T_b-t_b)}{252}}} - \frac{1}{(1+y_{(t_b,T_b)})^{\frac{(T_b-t_b)}{252}}} \right] } \right] \tag{3.7}$$

Considering the second assumption/approximation made on (3.5), this static initial *delta* hedge would be enough. However, as it is better explored later on this work, one should not consider that this hedge is effective if not managed dynamically, due to the convexities embedded both in the DI future and in the swap indexed to a percentage different from 100% CDI. What this means is that for each new level of the yield curve the quantity of DI future contracts to hedge properly the *delta* risk is different, what is usually called as *gamma*. What is evidenced in the related literature and in experiments made in this work is the fact that the *gamma* that the counterpart receiving a percentage of CDI larger than 100% has is strictly positive (so strictly negative for the counterpart paying the percentage of CDI larger than 100%).

In this case, one should be willing to pay a premium to open a position receiving percentage of CDI on a level higher than 100%, as much as one should demand the same premium in order to open a position as the counterpart in such trade, in which one would have a negative *gamma* to manage until the derivative expires, with a negative expected value. In practice, for each movement of the yield curve, if one would estimate over again the *delta*, a new quantity of DI future contracts that would effectively hedge this risk at that level of the yield curve would be calculated. Adjusting this quantity of DI future contracts at this new level of the yield curve would be the dynamic hedging.

To put in other words, for an upward movement of the yield rate curve, the agent receiving the percentage of CDI greater than 100% has to sell [8] at this new higher rate DI future contracts in order to offset the *delta* risk of the portfolio, as its counterpart has to buy DI future contracts at this new higher rate. Another important insight is the positive relation between the magnitude of the movements with the results of such dynamic hedging strategy, meaning that a larger volatility would lead to larger results. Therefore, estimating the expected value of such results is the way to estimate what would be a fair premium to be exchanged by the counterparts at the start date of the trade, in order to each manage this positive/negative *gamma* until maturity, with an expected value of zero, now considering the exchanged fair premium. And in order to estimate that properly, one should model the interest rate curve future dynamics with a framework that can be a fair approximation of reality. This work uses a Heath, Jarrow, and Morton (1992) multi-factorial model and the chosen way to obtain the volatility parameters for it is to

[8] Selling contracts of DI future means receiving that rate, as buying contracts of DI future means paying the interest rate; this would be equivalent to buying bonds and selling bonds, in a regular fixed income trading approach, but B3 exchange adapted the view of the contract for the rate instead of its price, so that it became more intuitive for market participants that were actually trading the related interest rate.

use Principal Component Analysis on historical data to extract three main vectors of volatility.

The second assumption/approximation made on (3.5) is related to the stochastic aspect of interest rates and of the CDI index; which is often ignored and has an important impact on the management of the risks on this portfolio composed by the swap indexed to a percentage of CDI different from 100% and a certain quantity of contracts of DI future. In order to better understand the origin of such aspect it is worth presenting briefly the impact of randomness when modeling derivatives with payoffs being convex functions, such as options but also as DI future contracts and swaps indexed to a percentage of CDI different from 100%.

3.4.1 The randomness

One aspect that is considered crucial to understand is the non-linearity present on derivatives that have convexity, being options, DI future contracts or swaps indexed to a percentage of CDI different from 100%. The origin of such non-linearity is found on the randomness of the underlying of those derivatives. One way to better understand this connection[9] is to look at what has been called as Jensen's Inequality. Suppose there is an asset which today is priced at R$20. Consider that the only two possible outcomes for the price of this asset in the time range of a month are being priced at R$10 or R$30, being both scenarios equally likely. In this hypothetical case, two intuitive ways that one might consider to price an option over this asset (let's say a call option expiring in one month with strike at the money being R$20) would be, first, to consider the average of the possible outcomes in the future ($(R\$10+R\$30)/2 = R\$20$), and with that say that the payoff of such option is 0, due to the fact that the strike will coincide with what is being considered that the asset will be priced in the future, and then bring this payoff of 0 to present value which would still be 0. This first approach might seem intuitive but it is not the correct way to price an option.

A second and still intuitive way would be to calculate the payoff of the option in each scenario and then estimate the expectation of them. In the first scenario, in which the asset price fell from R$20 to R$10, the payoff of a call with strike R$20 would be 0. In the second scenario, in which the asset price rose from R$20 to R$30, the payoff of a call with strike R$20 would be R$10. The average of these payoffs would be then R$5 at the expiry of the option, which then brought to its present value could be considered an estimate of the option price today. This second way is more similar to what is done in practice when pricing options, but the calculations involving randomness shows where the non-linearity appears in these kind of derivatives. We can see where the Jensen's Inequality appears when we put this same example in mathematical terms:

$$\text{Expected Value [Payoff (Asset Price)]} = \text{R\$ } 5 \tag{3.8}$$

If the function of the payoff is convex, lets say $c(A)$, being c the convex payoff function of

[9]Such a way is presented in Wilmott (2006).

the call option, and A the asset price, which is a random variable, then we have that:

$$E[c(A)] \geq c(E[A]) \tag{3.9}$$

Which we can measure how larger $E[c(A)]$ is than $c(E[A])$ considering a Taylor series approximation around the mean of A:

$$A = \bar{A} + \varepsilon \tag{3.10}$$

In which $\bar{A} = E[A]$ and $E[\varepsilon] = 0$, so we have that:

$$\begin{aligned} E[c(A)] &= E[c(\bar{A}+\varepsilon)] = E\left[c(\bar{A}) + c'(\bar{A})\varepsilon + \frac{1}{2}c''(\bar{A})\varepsilon^2 + \cdots\right] \\ &\approx c(\bar{A}) + \frac{1}{2}c''(\bar{A})E\left[\varepsilon^2\right] = c(E[A]) + \frac{1}{2}c''(E[A])E\left[\varepsilon^2\right] \end{aligned} \tag{3.11}$$

Therefore, with (3.11) it becomes clear that what makes $E[c(A)] \geq c(E[A])$, is (approximately) the term $\frac{1}{2}c''(E[A])E\left[\varepsilon^2\right]$, from which two important points come to light: first, it shows that on any non-linear derivative[10] (such as options, but as is also the case of DI futures and swaps indexed to a percentage of CDI different from 100%) it is **essential** to consider the randomness of the underlying, what appears in the term $E\left[\varepsilon^2\right]$, and may be pointed in a variance term. To properly price these derivatives then one should have a model for the underlying that consider this randomness aspect, and that is what is done in this work, with the HJM framework obtaining the volatility vectors from historical data, through the technique of Principal Component Analysis. And secondly, now it is clear where the convexity of non-linear derivatives appears: in the term $c''(E[A])$. For the specific case of the swaps indexed to a percentage of CDI different from 100%, ignoring this expected value of the CDI is precisely what makes the most widespread formula of marking to market this kind of derivative, presented previously, incorrect.

That being understood, the remaining question then is what is after all the correct way to mark to market this floating leg of such derivative? It is required that one consider the stochastic nature of the interest rates underlying these derivatives. Making the negligible approximation explained previously regarding the CDI, one may call $r = \ln(1+y)$ the continuously compounded spot interest rate, with $\mathbb{E}_{\mathbb{Q}}(.)$ being the risk-neutral expected value for the stochastic dynamics of the short rate, $r(t_b)$; the formula for the MTM θ^m (3.5) that most practitioners consider may now be rewritten as θ, properly considering the stochasticity intrinsic to the underlying interest rate:

$$\theta(r, t_b, T_b, \varphi) = N \times \left[e^{\varphi \int_0^{t_b} r(t_b)dt_b} \cdot \mathbb{E}_{\mathbb{Q}}\left(e^{-(1-\varphi)\int_{t_b}^{T_b} r(t_b)dt_b}\right)\right] \tag{3.12}$$

[10]A linear derivative would be one such that for a given increment on the price of the underlying asset, the derivative's future value will have a linear increase or decrease, no matter the level of such increment on the underlying asset. Therefore, a non-linear derivative is one that does not have such linear property on its payoff.

Or, to put in evidence the discrete spot rate observed in the market, y, instead of its continuous formulation, r, the MTM, θ, may be written as:

$$\theta(y, t_b, T_b, \varphi) = N \times \left[e^{\varphi \int_0^{t_b} (\ln(1+y_{(t_b)}))dt_b} \cdot \mathbb{E}_{\mathbb{Q}} \left(e^{-(1-\varphi) \int_{t_b}^{T_b} (\ln(1+y_{(t_b)}))dt_b} \right) \right] \tag{3.13}$$

3.5 The convexity

In order to properly estimate and manage the risks of a portfolio composed by a swap indexed to a percentage of CDI and a certain quantity of contracts of DI future, that hedges its first order risk (as known as *delta*) one should consider then the convexity, or the second order risk (as known as *gamma*), of both derivatives composing the portfolio. Due to the stochastic nature of the underlying interest rate, the expected value of such convexities must be found for a given interval of time by a numerical method but it is possible to determine the impact of such risk at each point in time, by taking the second derivative with respect to the underlying interest rate, for each element that composes the portfolio.

To do so, lets recall the first order risk related to the underlying interest rate, y, present in each relevant element of the portfolio analyzed, the floating leg of a swap indexed to a percentage φ of CDI and a DI future contract, respectively. Below are calculated the first derivative with respect to $y_{(t_b,T_b)}$ of both the MTM, $\theta(y, t_b, T_b, \varphi)$, and of a DI future, $DI(y, t_b, T_b)$, respectively:

$$\frac{\partial\theta(y, t_b, T_b, \varphi)}{\partial y_{(t_b,T_b)}} = \frac{\theta(y, t_b, T_b, \varphi) \cdot T_b(\varphi - 1)}{(1 + y_{(t_b,T_b)}) \left[((1 + y_{(t_b,T_b)})^{\frac{1}{252}} - 1)\varphi + 1 \right]} \tag{3.14}$$

$$\frac{\partial DI(y, t_b, T_b)}{\partial y_{(t_b,T_b)}} = \frac{-DI(y, t_b, T_b) \cdot T_b}{(1 + y_{(t_b,T_b)})} \tag{3.15}$$

Then, if one takes the second partial derivative with respect to the underlying interest rate, $y(t_b, T_b)$, obtains the following second order risk for each element of the portfolio, swap and DI future, respectively:

$$\frac{\partial^2\theta(y, t_b, T_b, \varphi)}{\partial y^2_{(t_b,T_b)}} = \frac{\theta(y, t_b, T_b, \varphi) \cdot T_b(\varphi - 1)}{(1 + y_{(t_b,T_b)})^2} \times (T_b(\varphi - 1) - 1) \tag{3.16}$$

$$\frac{\partial^2 DI(y, t_b, T_b)}{\partial y^2_{(t_b,T_b)}} = \frac{DI(y, t_b, T_b) \cdot T_b(1 + T_b)}{(1 + y_{(t_b,T_b)})^2} \tag{3.17}$$

Based on these equations and assuming the spot rate curve observed at the first business day of January 2021 as shown below, it is possible to plot the impact of the equations (3.14), (3.15), (3.16) and (3.17) considering a 100mm BRL notional swap and the quantity of DI future contracts estimated by (3.7), for its static hedging, so that such calculated first and second

derivatives can be visually interpreted.

From which one may take a few insights, when leaning over the practical implications of those formulas. Figure 3.1 brings the estimated quantity of contracts of DI future that statically hedges the swap, estimated by Equation (3.7). Figure 3.2 shows visually the first derivative of the swap indexed to a percentage of CDI with respect to the underlying spot rate curve, $y(t_b, T_b)$, that is estimated upon Equation(3.14). Figure 3.3 is the equivalent for the DI future contract, which has its first derivative calculated in Equation (3.15). The second derivative with respect to $y(t_b, T_b)$ of the swap and of the DI future contract, shown in Equation (3.16) and (3.17), are presented in Figures 3.4 and 3.5. Summarizing the portfolio estimated convexity, calculated by the sum or subtraction of the convexities of each component of the swap (in the case of positive quantities of DI future contracts, one is paying rates, and therefore paying the convexity of the contracts; in the case of negative quantities of DI future contracts, one is receiving rates, and therefore receiving the convexity of such contracts.) is presented in Figure 3.6.

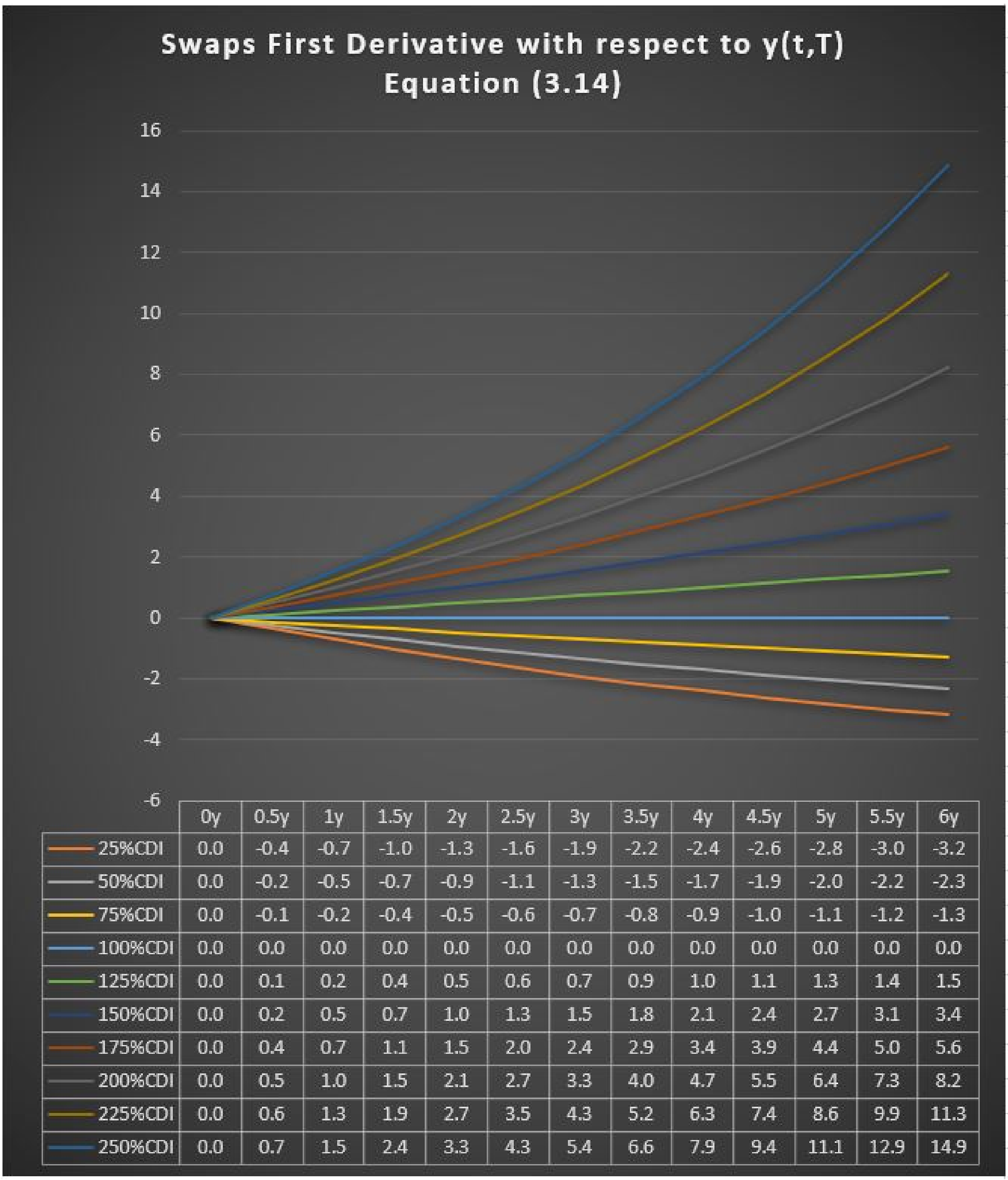
Spot Rate curve 1st Business Day January 2021						
Tenor	0y	0.5y	1y	1.5y	2y	2.5y
Yield (annual.)	1.93%	2.18%	2.94%	3.72%	4.33%	4.85%
3y	3.5y	4y	4.5y	5y	5.5y	6y
5.21%	5.54%	5.78%	5.98%	6.18%	6.35%	6.51%

Table 3.1: Spot Rate Curve first business day of January 2021.

Quantity of DI Future Contracts

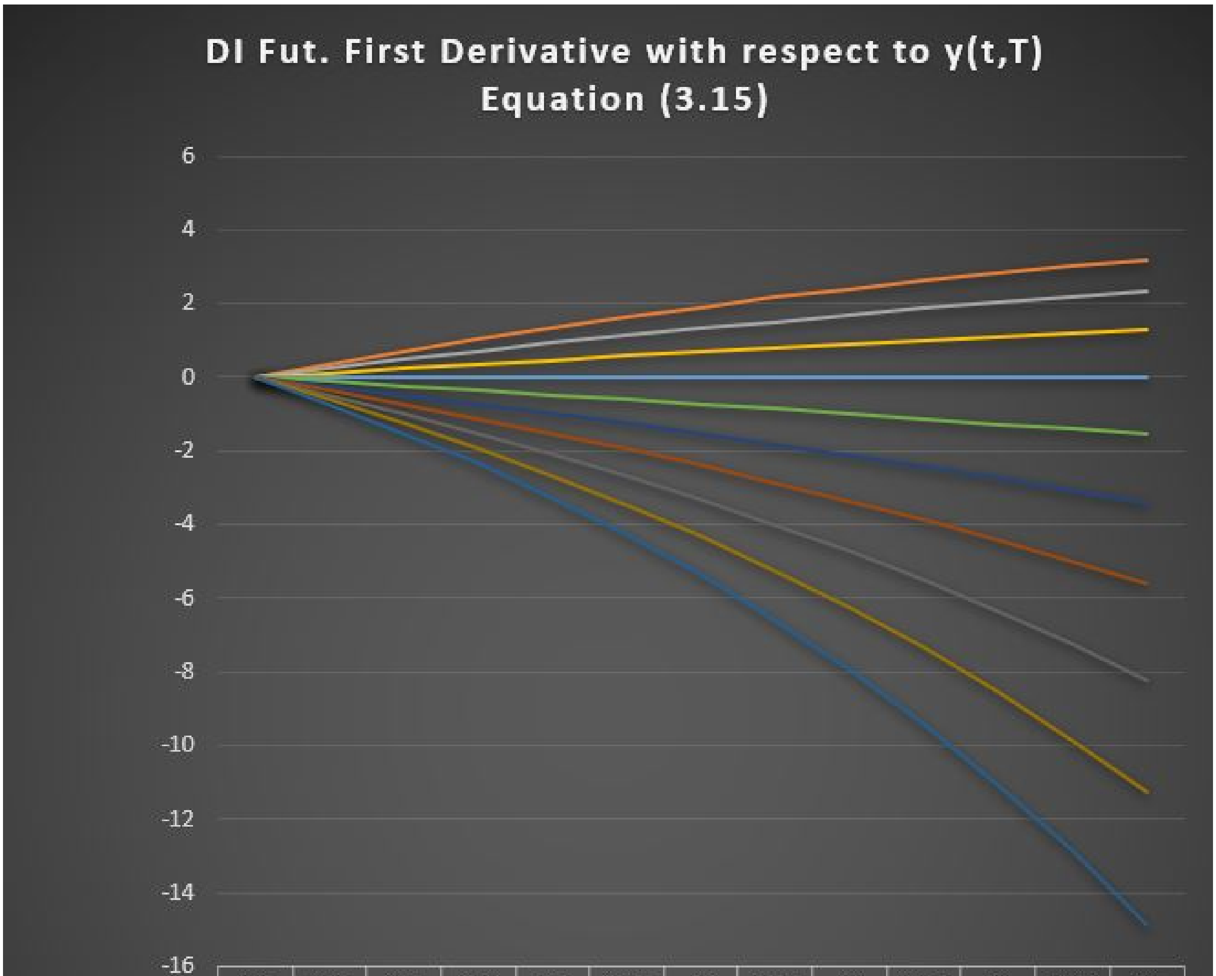
	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	750	752	755	760	766	772	779	786	793	801	808	816	824
50%CDI	500	503	507	514	522	530	540	549	559	570	581	592	604
75%CDI	250	252	255	260	266	273	280	288	296	304	313	322	332
100%CDI	0	0	0	0	0	0	0	0	0	0	0	0	0
125%CDI	-250	-253	-259	-268	-278	-290	-302	-316	-331	-346	-363	-382	-401
150%CDI	-500	-508	-522	-543	-568	-597	-628	-663	-700	-740	-783	-830	-881
175%CDI	-750	-764	-789	-825	-870	-922	-979	-1,043	-1,111	-1,184	-1,267	-1,355	-1,453
200%CDI	-1,000	-1,022	-1,059	-1,116	-1,184	-1,266	-1,356	-1,458	-1,567	-1,686	-1,820	-1,966	-2,130
225%CDI	-1,250	-1,280	-1,334	-1,414	-1,512	-1,631	-1,760	-1,910	-2,072	-2,249	-2,452	-2,675	-2,926
250%CDI	-1,500	-1,541	-1,612	-1,720	-1,853	-2,015	-2,194	-2,403	-2,630	-2,881	-3,171	-3,493	-3,858

Figure 3.1: Quantity of DI future contracts hedging statically the page portfolio.



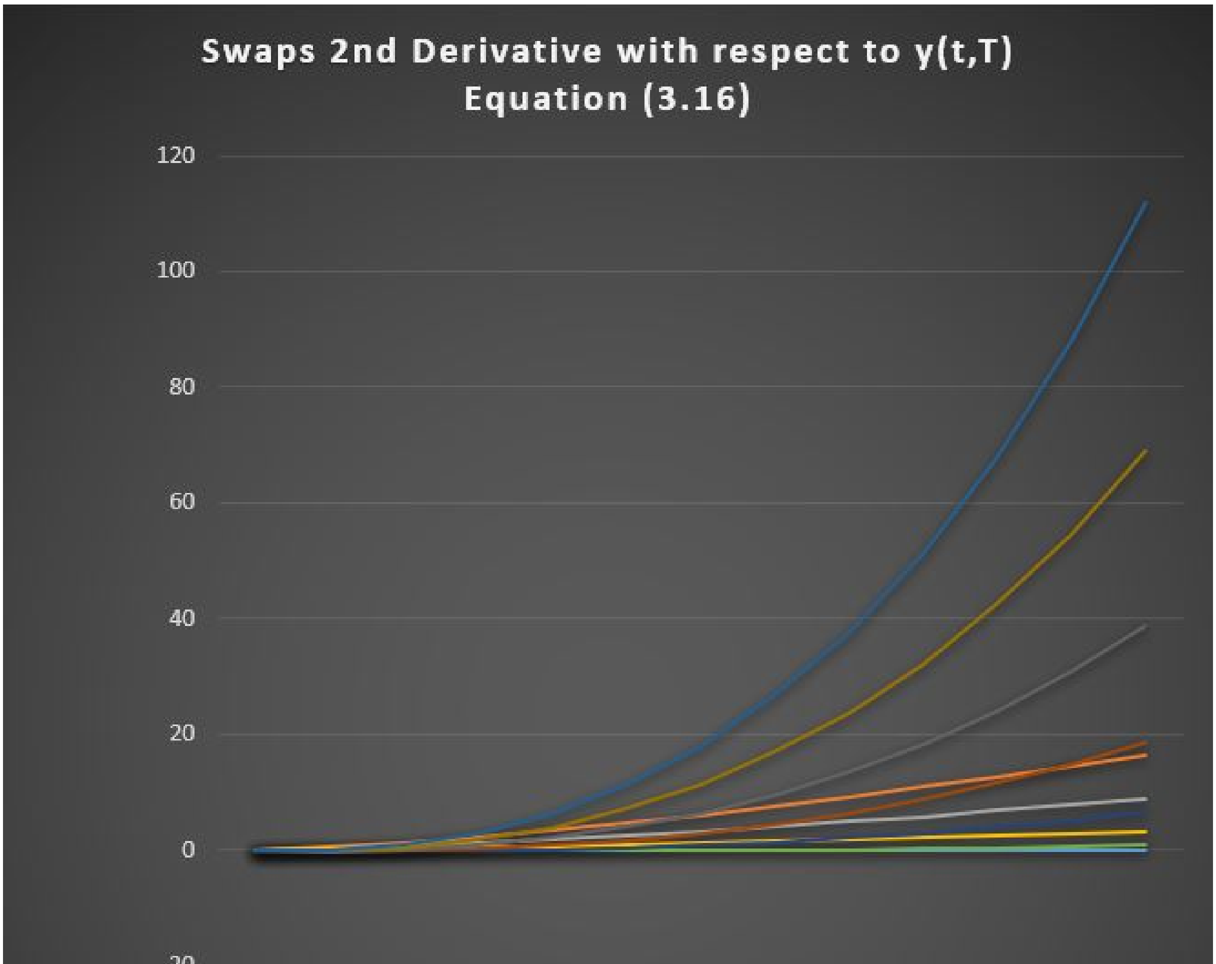
	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	0.0	-0.4	-0.7	-1.0	-1.3	-1.6	-1.9	-2.2	-2.4	-2.6	-2.8	-3.0	-3.2
50%CDI	0.0	-0.2	-0.5	-0.7	-0.9	-1.1	-1.3	-1.5	-1.7	-1.9	-2.0	-2.2	-2.3
75%CDI	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1.0	-1.1	-1.2	-1.3
100%CDI	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
125%CDI	0.0	0.1	0.2	0.4	0.5	0.6	0.7	0.9	1.0	1.1	1.3	1.4	1.5
150%CDI	0.0	0.2	0.5	0.7	1.0	1.3	1.5	1.8	2.1	2.4	2.7	3.1	3.4
175%CDI	0.0	0.4	0.7	1.1	1.5	2.0	2.4	2.9	3.4	3.9	4.4	5.0	5.6
200%CDI	0.0	0.5	1.0	1.5	2.1	2.7	3.3	4.0	4.7	5.5	6.4	7.3	8.2
225%CDI	0.0	0.6	1.3	1.9	2.7	3.5	4.3	5.2	6.3	7.4	8.6	9.9	11.3
250%CDI	0.0	0.7	1.5	2.4	3.3	4.3	5.4	6.6	7.9	9.4	11.1	12.9	14.9

Figure 3.2: Swaps first derivative with respect to $y(t_b, T_b)$.



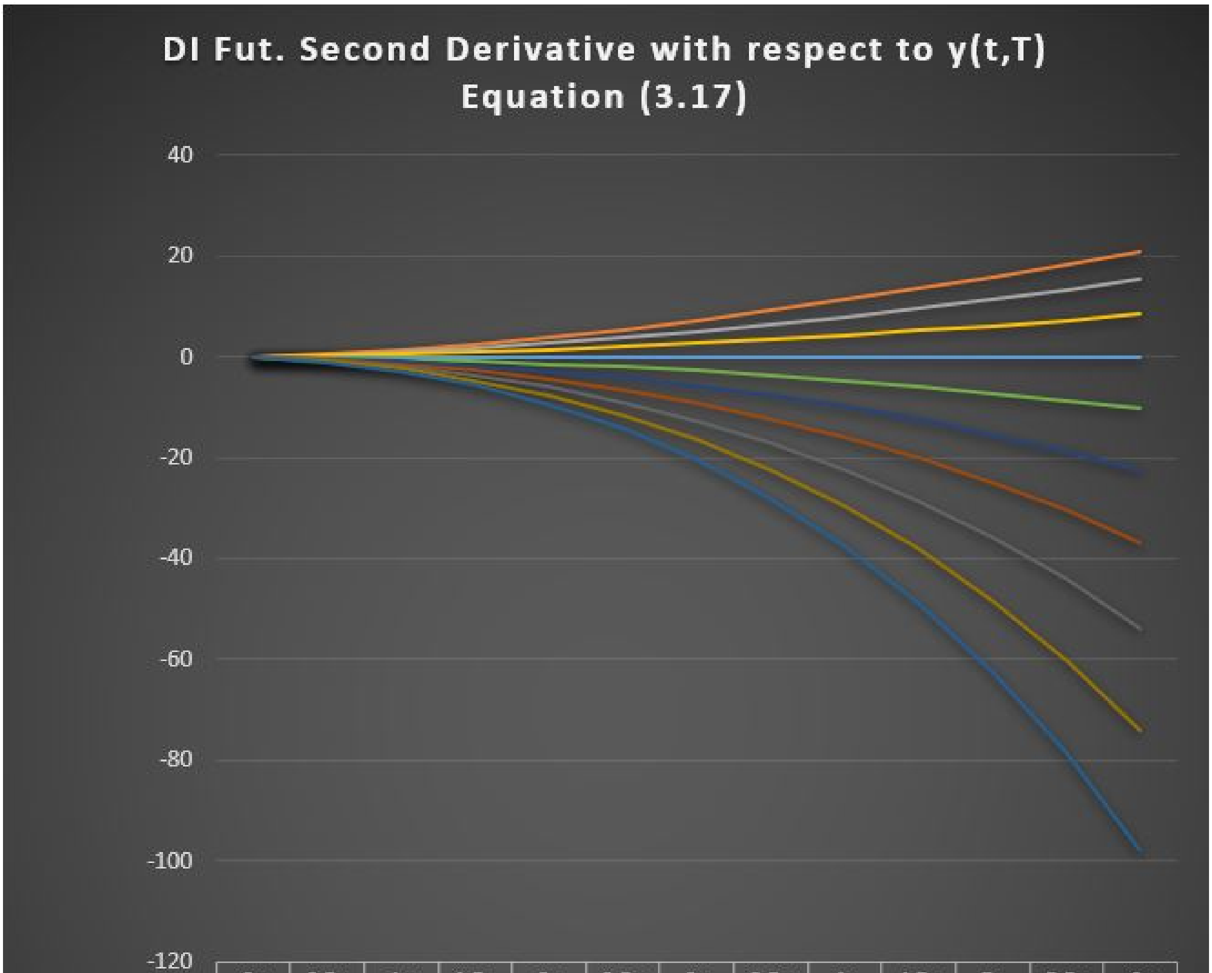
	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	0.0	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.4	2.6	2.8	3.0	3.2
50%CDI	0.0	0.2	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.0	2.2	2.3
75%CDI	0.0	0.1	0.2	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
100%CDI	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
125%CDI	0.0	-0.1	-0.2	-0.4	-0.5	-0.6	-0.7	-0.9	-1.0	-1.1	-1.3	-1.4	-1.5
150%CDI	0.0	-0.2	-0.5	-0.7	-1.0	-1.3	-1.5	-1.8	-2.1	-2.4	-2.7	-3.1	-3.4
175%CDI	0.0	-0.4	-0.7	-1.1	-1.5	-2.0	-2.4	-2.9	-3.4	-3.9	-4.4	-5.0	-5.6
200%CDI	0.0	-0.5	-1.0	-1.5	-2.1	-2.7	-3.3	-4.0	-4.7	-5.5	-6.4	-7.3	-8.2
225%CDI	0.0	-0.6	-1.3	-1.9	-2.7	-3.5	-4.3	-5.2	-6.3	-7.4	-8.6	-9.9	-11.3
250%CDI	0.0	-0.7	-1.5	-2.4	-3.3	-4.3	-5.4	-6.6	-7.9	-9.4	-11.1	-12.9	-14.9

Figure 3.3: DI Future first derivative with respect to $y(t_b, T_b)$.



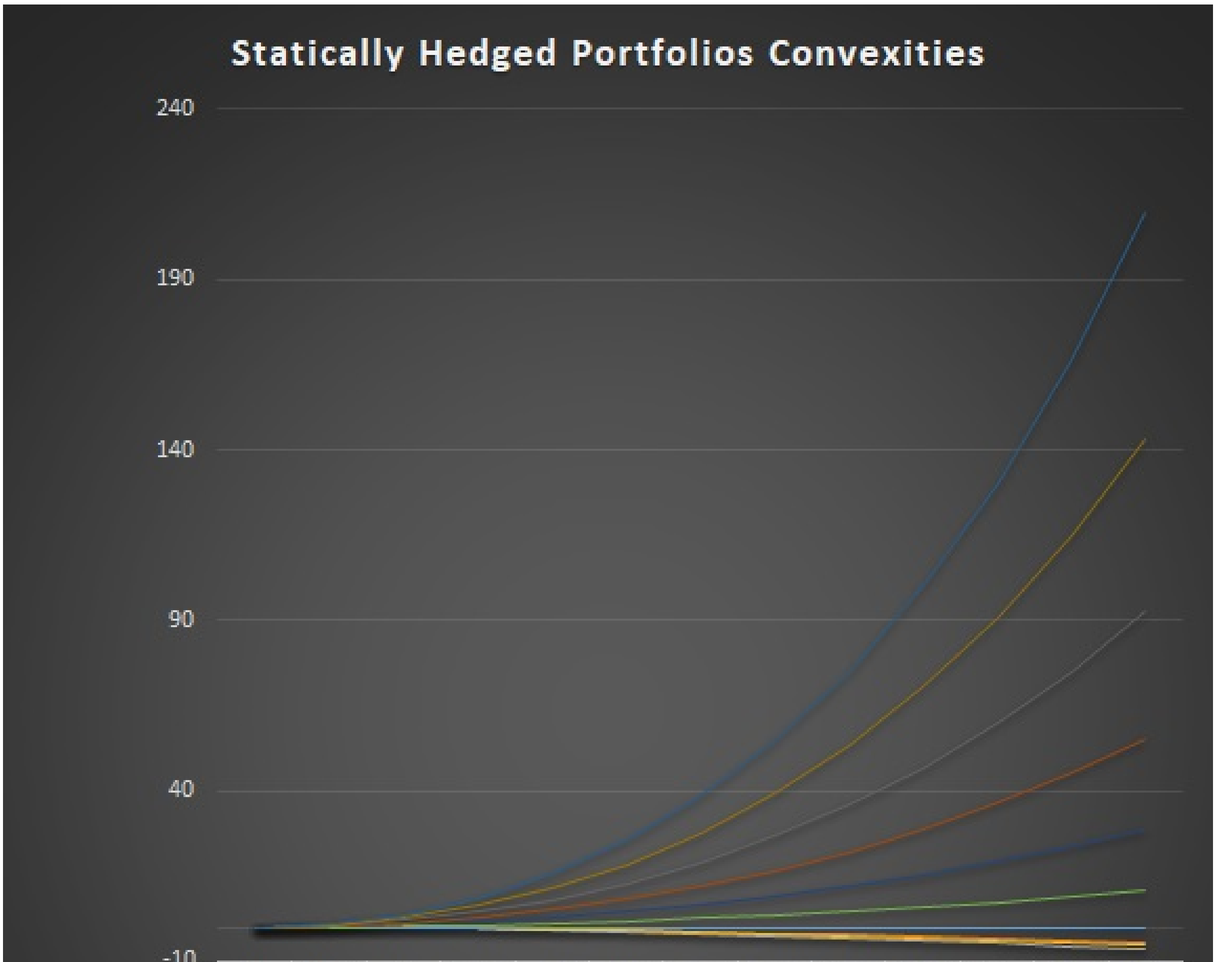
	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	0.0	0.5	1.2	2.1	3.2	4.5	5.9	7.4	9.1	10.8	12.6	14.5	16.4
50%CDI	0.0	0.3	0.7	1.2	1.8	2.4	3.1	3.9	4.8	5.7	6.7	7.7	8.8
75%CDI	0.0	0.1	0.3	0.5	0.7	0.9	1.1	1.4	1.7	2.0	2.3	2.7	3.0
100%CDI	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
125%CDI	0.0	-0.1	-0.2	-0.2	-0.2	-0.2	-0.2	-0.1	0.0	0.1	0.3	0.5	0.7
150%CDI	0.0	-0.2	-0.2	-0.2	0.0	0.3	0.7	1.3	2.0	2.9	3.9	5.0	6.4
175%CDI	0.0	-0.2	-0.2	0.1	0.7	1.6	2.8	4.4	6.3	8.7	11.5	14.7	18.4
200%CDI	0.0	-0.2	0.0	0.7	2.0	3.8	6.3	9.5	13.4	18.2	23.9	30.7	38.6
225%CDI	0.0	-0.2	0.3	1.6	3.8	7.0	11.3	16.8	23.7	32.1	42.3	54.5	68.9
250%CDI	0.0	-0.2	0.7	2.8	6.3	11.2	17.9	26.6	37.6	51.1	67.8	87.8	111.9

Figure 3.4: Swaps second derivative with respect to $y(t_b, T_b)$.



	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	0.0	0.5	1.4	2.5	3.9	5.5	7.3	9.2	11.3	13.6	15.9	18.4	20.9
50%CDI	0.0	0.4	0.9	1.7	2.6	3.8	5.0	6.4	8.0	9.7	11.5	13.3	15.3
75%CDI	0.0	0.2	0.5	0.9	1.3	1.9	2.6	3.4	4.2	5.2	6.2	7.3	8.4
100%CDI	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
125%CDI	0.0	-0.2	-0.5	-0.9	-1.4	-2.0	-2.8	-3.7	-4.7	-5.9	-7.2	-8.6	-10.2
150%CDI	0.0	-0.4	-1.0	-1.8	-2.9	-4.2	-5.8	-7.8	-10.0	-12.6	-15.5	-18.7	-22.4
175%CDI	0.0	-0.5	-1.4	-2.7	-4.4	-6.5	-9.1	-12.2	-15.9	-20.1	-25.0	-30.5	-36.9
200%CDI	0.0	-0.7	-1.9	-3.7	-6.0	-9.0	-12.6	-17.1	-22.4	-28.6	-35.9	-44.3	-54.0
225%CDI	0.0	-0.9	-2.4	-4.7	-7.7	-11.5	-16.4	-22.4	-29.6	-38.2	-48.3	-60.3	-74.2
250%CDI	0.0	-1.1	-3.0	-5.7	-9.4	-14.3	-20.4	-28.1	-37.5	-48.9	-62.5	-78.7	-97.9

Figure 3.5: DI Future second derivative with respect to $y(t_b, T_b)$.



	0y	0.5y	1y	1.5y	2y	2.5y	3y	3.5y	4y	4.5y	5y	5.5y	6y
25%CDI	0.0	0.0	-0.2	-0.4	-0.6	-1.0	-1.4	-1.8	-2.3	-2.8	-3.3	-3.9	-4.5
50%CDI	0.0	-0.1	-0.2	-0.5	-0.9	-1.3	-1.9	-2.5	-3.2	-4.0	-4.8	-5.6	-6.6
75%CDI	0.0	0.0	-0.2	-0.4	-0.7	-1.0	-1.5	-2.0	-2.5	-3.2	-3.9	-4.6	-5.4
100%CDI	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
125%CDI	0.0	0.1	0.3	0.7	1.2	1.8	2.6	3.6	4.7	6.0	7.5	9.1	10.9
150%CDI	0.0	0.2	0.7	1.6	2.9	4.5	6.6	9.1	12.0	15.4	19.3	23.8	28.7
175%CDI	0.0	0.3	1.3	2.9	5.1	8.2	12.0	16.6	22.2	28.8	36.4	45.2	55.3
200%CDI	0.0	0.5	1.9	4.4	8.0	12.8	18.9	26.6	35.8	46.8	59.8	75.0	92.6
225%CDI	0.0	0.7	2.8	6.3	11.5	18.5	27.7	39.1	53.2	70.3	90.7	114.8	143.1
250%CDI	0.0	0.9	3.7	8.5	15.6	25.5	38.3	54.7	75.1	100.0	130.3	166.6	209.8

Figure 3.6: Statically hedged portfolios convexities.

One may notice looking at Figure 3.6 that for a swap indexed to a percentage of CDI larger than 100%, the sum of the convexities of the components of the portfolio (swap and the quantity of contracts of DI future that statically hedges the first derivative risk of the swap) will result in a portfolio known as long *gamma*, which means that every movement that the underlying yield curve makes, no matter in which direction, the increase in the first order risk, *delta*, will be such that the result from the hedge of this new *delta*, will always be positive. On the other hand, for the counterpart of such a portfolio (the counterpart of the swap will be paying the percentage of CDI larger than 100%, and hedging its *delta* risk with DI futures), will have a portfolio known as short *gamma*, with *delta* risks appearing always on the negative side, always with negative results, no matter the direction that the underlying interest rate takes. Additionally, there is also the case in which the percentage that the swap is indexed is less than 100% of CDI. In that case the sum of the convexities of the swap and the quantity of contracts of DI future that hedges the first order risk results negative, leaving the portfolio in these scenarios short *gamma* for the receiver of the percentage of CDI. However, notably it is significantly less short *gamma* in these scenarios than it was long *gamma* in the prior cases, due to the fact that in the scenarios of a percentage greater than 100% of CDI, the first order risk of the swap is positive, therefore to hedge it one will need to buy DI contracts, paying rates and the convexity of it, that in these cases, when subtracted from the small positive convexity of the swap, results in negative convexity for the portfolio.

Besides the visualization of the equations for the first and second derivatives for the components of the swap, it may help to develop a better understanding of the magnitude of the convexity over the management of the risks to observe visually some results of these portfolios. Below are presented in detail in Figure 3.7 the results of each component of the portfolio and the result of the portfolio statically hedged:

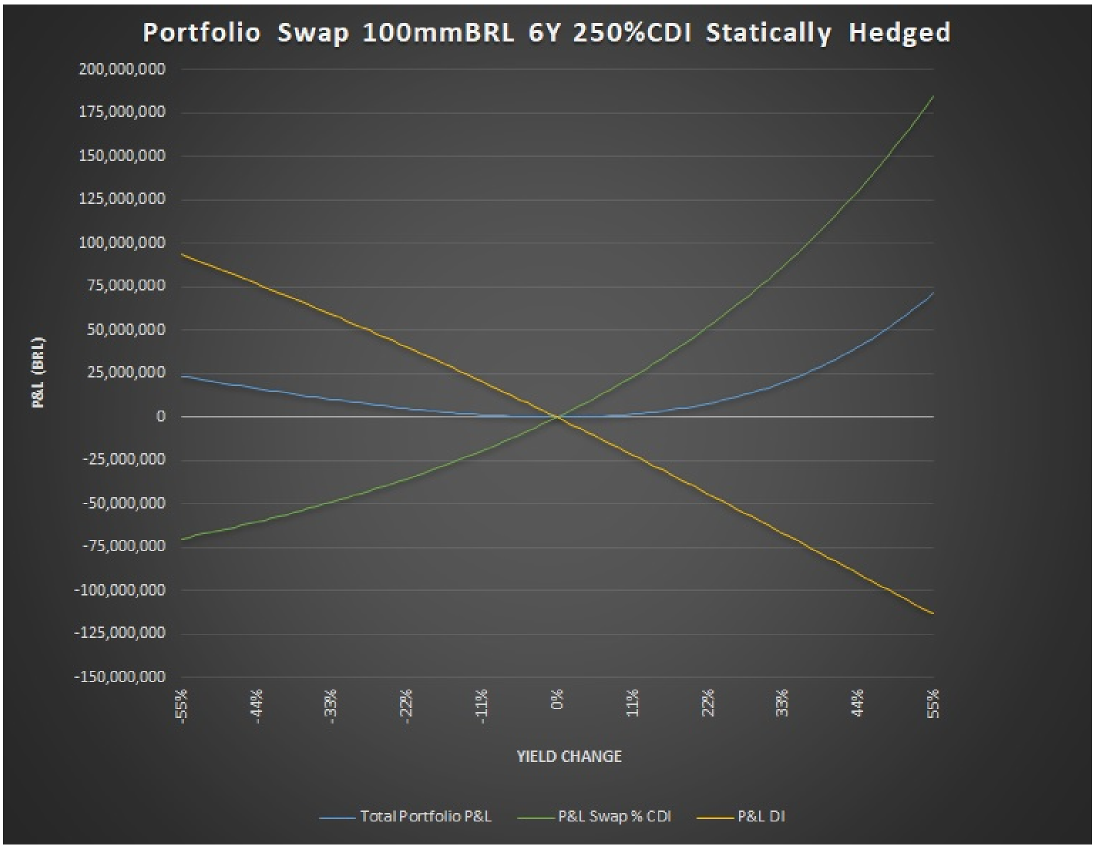


Figure 3.7: Results of a statically hedged swap indexed to 250%CDI maturing in 6 years.

And to have a more comprehensive view of the results of these convexities, still in portfolios with swaps maturing 6 years statically hedged, are shown below the results for different percentages of CDI in Figure 3.8 and more detailed in the negative cases in Figure 3.9:

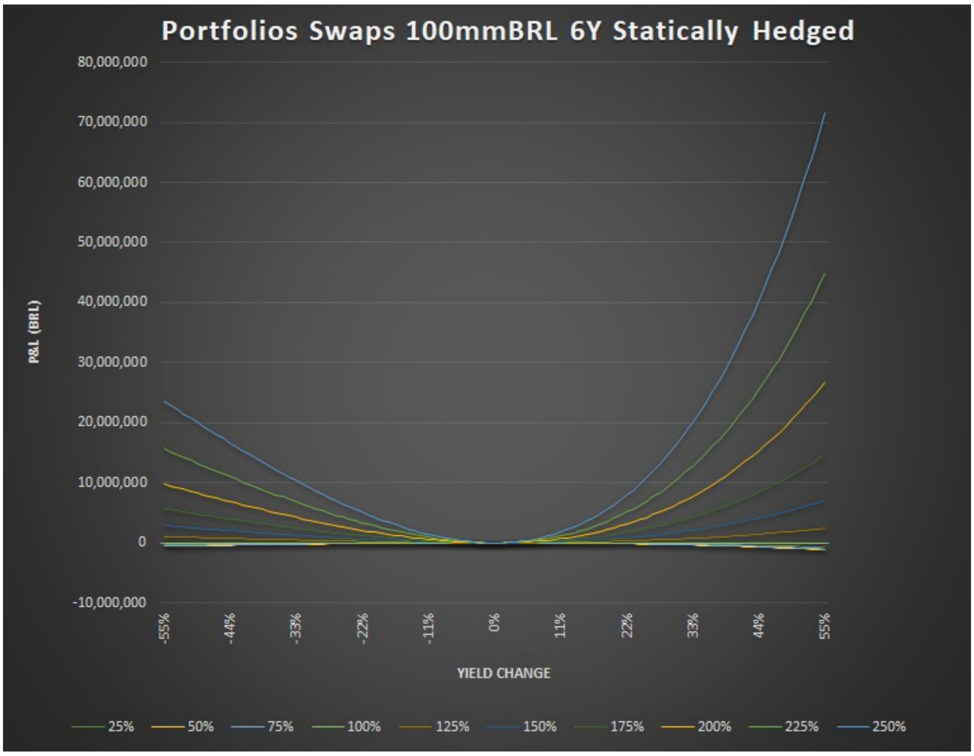


Figure 3.8: Results of statically hedged swaps indexed different percentages of CDI maturing in 6 years.

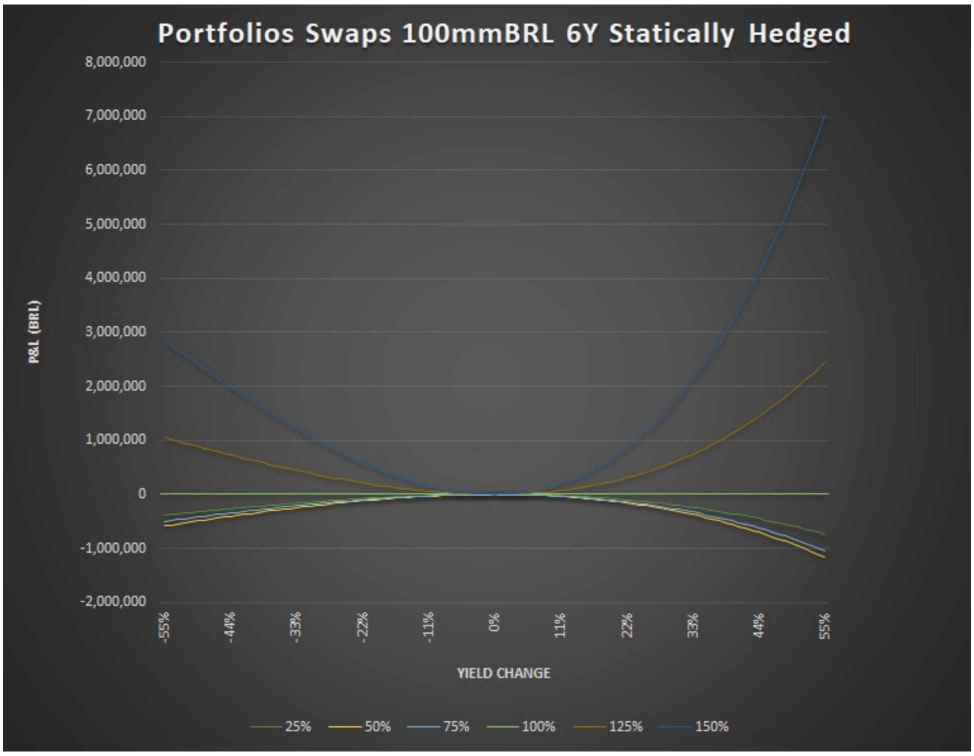


Figure 3.9: Results of selected statically hedged swaps indexed different percentages of CDI maturing in 6 years.

Additionally, Equations (3.16) and (3.17) also allows one to understand the relation between the convexity of such portfolio and the maturity of the derivatives in question, altogether with the percentage of CDI of the swap composing the portfolio. The longer the maturity of the derivative from the given point in time being analyzed, greater will be the convexity of the portfolio, in module[11]. The percentage of CDI will have an exponential impact when larger than 100%, increasing the convexity of the portfolio in module; and if it is less than 100%, then it will have a negative impact on the convexity, but only a fraction on the overall convexity of the portfolio, due to the fact that it will only subtract from the convexity present on the DI future hedging the *delta* risk.

The results obtained on this work, both on the simulated scenarios and on the historical analyzed results on such portfolios (used on the back-tests) corroborate these insights, serving as empirical evidence that the effects of such developments can be observed in practice.

[11]It is in module generically, because if not considering in module, then the magnitude of the convexity will depend on the side of the swap, if receiving or paying the percentage of CDI

4. THE HEATH-JARROW-MORTON MODEL

In this chapter are presented the Heath, Jarrow, and Morton (1992) model, the adaptations proposed by Brace and Musiela (1994) on the HJM framework and the approach made to discretize the model.

4.1 The model

The literature about modeling interest rates before the work of Heath, Jarrow, and Morton (1992) was broadly approaching the issue by modeling the so-called short term interest rate and from it then deducing the yield curve derived from the dynamics of the modeled short rate. What Heath, Jarrow, and Morton (1992) proposed could be considered a breakthrough in the field of modeling the dynamics of interest rates and therefore on the pricing of fixed income instruments. In their work, the starting point of the model would be the extraction of volatility parameters from real data and the model they presented would courageously (and successfully) attempt to reach the whole yield curve at once, by using as object the instantaneous forward rates already observed in market data, instead of modeling the short rate and from it deriving the forward rates that would compose the yield curve structure. The model they presented was then vastly implemented throughout the literature, becoming what is now called in a more general sense a framework.

In the HJM framework what is being modeled is the instantaneous forward rates that make up the yield curve structure. More specifically, the process of such rates through a predetermined time frame is what the model is attempting to describe. When it comes to interest rates modeling, although the interest rate is the main subject being studied, the actual tradable object is a fixed-income instrument, for instance, a zero-coupon bond that is traded at a given time t, that matures at a given time T and that pays R$ 1 then. So, to start the deduction of the HJM model, if one considers $f(t,T)$ the forward rate at time t for the maturity T, then one could say that the price of such zero-coupon bond would be:

$$B(t,T) = e^{-\int_t^T f(t,s)ds} \tag{4.1}$$

Additionally, one could describe the process of such zero-coupon bond prices through time

with a stochastic differential equation:

$$dB(t,T) = \alpha(t,T)B(t,T)dt + \sigma(t,T)B(t,T)dX(t) \tag{4.2}$$

But what the HJM model aim to describe is not the price of the bond $B(t,T)$, nor the process of the price of the bond $dB(t,T)$ through time, but instead the forward rate $f(t,T)$ related to this price; more specifically, the process of such forward rate $df(t,T)$ through time.

To do that, one might try to rearrange (4.1) in order to define $f(t,T)$ and from it, derive $df(t,T)$. This attempt could start by taking the natural logarithm[1] on both sides of the equation:

$$-\ln B(t,T) = \int_t^T f(t,s)ds \tag{4.3}$$

Then, to remove the stochastic integral of the right hand side, one may want to take the partial derivative with respect to T on both sides of the equation:

$$-\frac{\partial}{\partial T}[\ln B(t,T)] = \frac{\partial}{\partial T}\int_t^T f(t,s)ds \tag{4.4}$$

What would require to apply the Leibniz Rule such as generally described below:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)dt\right] = f(x,b(x))\cdot\frac{db(x)}{dx} - f(x,a(x))\cdot\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial f(x,t)dt}{\partial t} \tag{4.5}$$

So that:

$$\begin{aligned}\frac{\partial}{\partial T}\int_t^T f(t,s) &= f(t,T)\cdot\frac{\partial T}{\partial T} - f(t,t)\cdot\frac{\partial t}{\partial T} + \int_t^T \frac{\partial}{\partial T}f(t,s)ds \\ &= f(t,T)\cdot 1 - f(t,t)\cdot 0 + \int_t^T 0ds \\ &= f(t,T)\end{aligned} \tag{4.6}$$

Which, then if only one rearrange to isolate the forward rate $f(t,T)$ on the left hand-side of the equation, one would have:

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T) \tag{4.7}$$

Which then could be considered the relation between the zero-coupon bond traded at time t and maturing at time T, $B(t,T)$ and the forward rate $f(t,T)$ for the same time interval.

So, considering (4.2) and (4.7), one may try to find then $df(t,T)$. To manipulate these equations, one may follow the logic that from $dB(t,T)$ it could be found $d(\ln B)$ and, from that, then deduct what could describe the wanted process $df(t,T)$.

[1] Also known as Napierian logarithm, sometimes mentioned through the literature by $\log_e$.

In order to do so, lets define $G = \ln B$[2] and then try to calculate[3] dG as being:

$$dG = \frac{\partial G}{\partial t} \cdot dt + \frac{\partial G}{\partial B} \cdot dB + \frac{1}{2} \cdot \frac{\partial^2 G}{\partial B^2} \cdot (dB)^2 + \dots \tag{4.8}$$

In which:

$$(dB)^2 \cong \alpha^2 B^2 (dt)^2 + \sigma^2 B^2 (dX)^2 + 2\alpha\sigma dX dt \tag{4.9}$$

That only the second term is of order dt, since the first term is of order $(dt)^2$ and the third of order $(dt)^{\frac{3}{2}}$, so that:

$$(dB)^2 \cong \sigma^2 B^2 dt \tag{4.10}$$

So, rearranging (4.8):

$$dG = d(\ln B) = \left(\frac{\partial G}{\partial t} + \alpha B \cdot \frac{\partial G}{\partial B} + \frac{\sigma^2 B^2}{2} \cdot \frac{\partial^2 G}{\partial B2}\right) dt + \sigma B \cdot \frac{\partial G}{\partial B} dX \tag{4.11}$$

That one only need to substitute the partial derivatives such as calculated below:

$$\frac{\partial G}{\partial t} = 0$$

$$\frac{\partial G}{\partial B} = \frac{1}{B}$$

$$\frac{\partial^2 G}{\partial B^2} = -\frac{1}{B^2}$$

Reaching that:

$$dG = d(\ln B) = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dX \tag{4.12}$$

That if recalling (4.7):

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T)$$

One then may notice that it will only require to obtain from (4.12) the partial derivative with respect to T such that:

$$d\left(-\frac{\partial}{\partial T} \ln B(t,T)\right) = -\frac{\partial}{\partial T}[d \ln B(t,T)] \tag{4.13}$$

[2] In order to clean the notation, it is written only B and also only f, without its parameters (t,T), but the reader must consider they are implicit throughout this development.

[3] Here it is applied Itô's Lemma, which is considered known by the reader due to its vast application throughout the literature.

That, rearranging:

$$-\frac{\partial}{\partial T}[d\ln B] = -\frac{\partial}{\partial T}\left[\left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dX\right] = -\frac{\partial}{\partial T}\left(\alpha - \frac{\sigma^2}{2}\right)dt - \frac{\partial}{\partial T}\sigma dX \tag{4.14}$$

Or, just rearranging once more:

$$-\frac{\partial}{\partial T}[d\ln B] = \frac{\partial}{\partial T}\left(\frac{\sigma^2}{2} - \alpha\right)dt - \frac{\partial}{\partial T}\sigma dX \tag{4.15}$$

Which then, could be considered the process to describe the forward rate, $df(t,T)$:

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{\sigma^2(t,T)}{2} - \alpha(t,T)\right]dt - \frac{\partial}{\partial T}\sigma(t,T)dX \tag{4.16}$$

Until this point, the model is considering parameters under the real probability measure, $\mathbb{P}$, however in order to price derivatives (or any financial instrument) related to the interest rates being modeled is necessary to enter the risk-neutral probability measure, $\mathbb{Q}$. The function that is used to this end is commonly named through the literature as the Market Price of Risk, usually identified by λ, which could be interpreted as the return obtained in excess to a risk-free investment per unit of risk, dX, allocated. The market price of risk λ connects both real probability measure $\mathbb{P}$ to the risk-neutral probability measure $\mathbb{Q}$ by assuming the absence of arbitrage opportunities, a strong hypothesis present on all models developed under a more general framework known as No-Arbitrage Pricing Theory.

An arbitrage opportunity is by definition an opportunity to earn from trades a return without risk and without any initial invested capital. The no-arbitrage principle assumes that any arbitrage opportunity is instantly captured by the market, making them in practice non-existent. In consequence of that, any risk-free investments will only return the risk-free rate and nothing in excess to it.

This principle is based on the assumption that markets are rational and efficient and that any two portfolios with the same payoff (or equivalently, the same return) will have the same price, which is known in the literature as the Law of One Price. Also, it assumes that markets are complete, which means that any and all payoff can be obtained from the combination of existing instruments negotiated on the market, or in other words, any instrument can be replicated by a linear combination of available instruments on the market.

To give an example, already linking with the HJM framework, let's assume a single factor HJM model for a simplification purpose of understanding the Market Price of Risk. Under this hypothetical model, one may have a hedged portfolio composed of two zero-coupon bonds, maturing on different dates, T_0 and T_1:

$$\Pi = B(t,T_0) - \Delta B(t,T_1) \tag{4.17}$$

And from this, one might describe the change in such portfolio with the stochastic differential equation below:

$$\begin{aligned} d\Pi &= dB\left(t,T_0\right) - \Delta dB\left(t,T_1\right) \\ &= B\left(t,T_0\right)\left(\alpha\left(t,T_0\right)dt + \sigma\left(t,T_2\right)dX\right) - \Delta B\left(t,T_1\right)\left(\alpha\left(t,T_1\right)dt + \sigma\left(t,T_1\right)dX\right) \end{aligned} \tag{4.18}$$

So that if by choosing a Δ quantity of bonds maturing on T_1 respecting the relationship below:

$$\Delta = \frac{\sigma\left(t,T_0\right) \times B\left(t,T_0\right)}{\sigma\left(t,T_1\right) \times B\left(t,T_1\right)} \tag{4.19}$$

Then such portfolio is perfectly hedged or in other words, returns only the risk free rate, $r(t)$.

If one may then isolate on each side of the equation the different maturities, T_0 and T_1, will obtain:

$$\frac{\alpha\left(t,T_0\right) - r(t)}{\sigma\left(t,T_0\right)} = \frac{\alpha\left(t,T_1\right) - r(t)}{\sigma\left(t,T_1\right)} \tag{4.20}$$

Which could only be true if both sides are independent of the maturity date, so then the *drift* may be written as:

$$\alpha(t,T) = r(t) + \lambda(t)\sigma(t,T) \tag{4.21}$$

In which, λ is here the Market Price of Risk function that will relate both measures of probability $\mathbb{P}$ and $\mathbb{Q}$.

In the risk-neutral world, $\lambda = 0$, and therefore the *drift* of the price of the zero-coupon bond is equivalent to the risk-free rate: $\alpha(t,T) = r(t)$. And so, in the risk-neutral world then, one may rewrite (4.16) as:

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{\sigma^2(t,T)}{2} - r(t)\right]dt - \frac{\partial}{\partial T}\sigma(t,T)dX \tag{4.22}$$

So that, since the partial derivative of $r(t)$ with respect to T is zero, one may then consider that the stochastic differential equation that describes the process of the forward rate in the risk-neutral world as being:

$$df(t,T) = \frac{\partial}{\partial T}\frac{\sigma^2(t,T)}{2}dt - \frac{\partial}{\partial T}\sigma(t,T)dX \tag{4.23}$$

Then, one may define the volatility parameter of such dynamics by ν as:

$$\nu = -\frac{\partial}{\partial T}\sigma(t,T) \tag{4.24}$$

So that if taken the integral on both sides from t to T:

$$\int_t^T \nu(t,s)ds = -\frac{\partial}{\partial T}\int_t^T \sigma(t,s)ds \tag{4.25}$$

That if one again apply the Leibniz Rule, will have that:

$$\int_t^T \nu(t,s)ds = -\sigma(t,T) \tag{4.26}$$

So, just rearranging :

$$\sigma(t,T) = -\int_t^T \nu(t,s)ds \tag{4.27}$$

Which means that the volatility of the zero-coupon bond price at any given time will depend on the volatility of the forward rates from previous tenors, being $t \leq T$.

Furthermore, looking again at (4.23) and now considering the *drift* of the process of the forward rate as:

$$\frac{\partial}{\partial T}\frac{\sigma^2(t,T)}{2} = \frac{1}{2}\cdot 2\sigma(t,T)\cdot\frac{\partial}{\partial T}\sigma(t,T) = \sigma(t,T)\cdot\frac{\partial}{\partial T}\sigma(t,T) \tag{4.28}$$

That if one substitutes on (4.23), will have:

$$df(t,T) = \sigma(t,T)\cdot\frac{\partial}{\partial T}\sigma(t,T)dt + \nu(t,T)dX \tag{4.29}$$

Which then can be rewritten considering (4.24) and (4.27) to obtain the desired stochastic differential equation for the forward rate process through time under the HJM model on the risk-neutral world:

$$df(t,T) = \left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX \tag{4.30}$$

This form of describing the process of the forward rate may give one a significant insight about the nature of the HJM model. What can be seen in equation (4.30) is that the *drift* of the forward rate is a function of the volatility of such forward rate but also of all the volatilities of forward rates of previous tenors, being being $t \leq T$. What this does in practice is to remove from the HJM framework a desirable feature when it comes to modeling financial variables known as Markovianity[4]. This is an issue that makes extremely harder to find closed solutions for the stochastic differential equation of the dynamics of the interest rate process under the HJM framework when comparing to models that describe its variables following Markov pro-

[4]A process that has Markovianity, or as also know as a Markov process, is such that only the present state of a random variable has relevance over potential future random states.

cesses and is also one of the reasons why many practitioners do not implement such model on their pricing of fixed-income instruments due to, in practice, the requirement of implementing numerical methods in detriment of closed formulas.

4.2 The Musiela parameterization

In their model, Heath, Jarrow, and Morton (1992) were considering fixed future dates T, while changing the date in which each forward rate was starting from, t. Although this way of developing led them successfully to a stochastic differential equation to describe the dynamics of the whole yield curve, in practical terms it is not convenient to deal with fixed future dates such as T, but instead it is more convenient to consider fixed terms τ, for instance, one, two, three, etc. years from t, so that at a certain moment $T = t + \tau$ but as time passes, the fixed terms are still being considered and not the initially considered maturity date T. This idea to modify the horizon of the forward rate to make it more convenient for practical implementation of the HJM framework was first presented in the literature by Brace and Musiela (1994). In their work[5], the authors considered the following modification:

$$\nu(t,T) = \bar{\nu}(t, T-t) \tag{4.31}$$

With $\tau = T - t$. In practice this means that the authors instead of considering fixed dates as it was being considered in the original work of HJM, they instead passed to consider fixed terms. Then, all it was required was to discover $\bar{f}(t,\tau) = f(t, t+\tau)$ that satisfy the stochastic differential equation:

$$d\bar{f}(t,\tau) = \bar{m}(t,\tau)dt + \bar{\nu}(t,\tau)dX \tag{4.32}$$

With $\bar{m}(t,\tau)$ being:

$$\bar{m}(t,\tau) = \bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial}{\partial\tau}\bar{f}(t,\tau)$$

Which for its turn is a significantly more convenient way to describe the evolution of the yield curve in the risk-neutral world and from it then price derivatives over those interest rates. In practice, instead of considering as maturities fixed dates from the start date considered, it is then considered a fixed time interval between the start date and the maturity. For example, if the start date t is considered 01/02/2021, then what would have been the maturities T, 07/01/2021, 01/02/2022, 07/01/2022, 01/03/2022, and so on, would then be considered τ intervals such as, 0.5 year, 1 year, 1.5 year, 2 years, and so on.

[5] In this work it will only be presented the implications of their parameterization. For further reading, the theorem the authors present has its proof on their work and also appears on the appendix section of Nojima (2013).

4.3 From continuous to discrete

So far, the HJM framework has been presented in a continuous environment, such as it is convenient for the development of the model. However, for its implementation, due to the fact that in only a very few specific cases it can be found an analytical solution for the stochastic differential equation proposed in the HJM framework (due to the fact that, in the majority of cases, the *drift* dependency of a given forward rate on the volatility from previous tenors removes the markovianity of the process described), the most common approach in the practical implementation of the model is to move the framework from the continuous time to a discrete time environment. There could be many ways to make this adaptation, but this work uses the approach presented by Glasserman (2003) combined with an adaptation presented by Nojima (2013). To do so, it considers time grids of the start date by the maturities and then the continuous integral for the *drift* is approximated, by Euler's method for discretization. This is done first under the original consideration of the HJM model of the maturity T, and then adapted to the already presented Musiela parameterization, in which the maturity date T is substituted for the tenor τ.

In order to approximate the dynamics of the forward rate described by the HJM model from the continuous time to the discrete time, as it is pointed by Glasserman (2003), both of the time parameters present in the stochastic differential equation must be approximated, or in other words, discretized. It is worth to recall the SDE[6] that it is being approximated:

$$df(t,T) = \left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX$$

For the purpose of simplification of the notation throughout this development, it is proposed the same notation used by Nojima (2013), in which for the first parameter of the function $f(t,T)$, the time from where the forward rate is beginning, t, is considered a time grid $0 = t_0 < t_1 < t_2 < ... < t_M$, in which t_0 is the initial time simulated, and t_M the final. And for the second parameter, the maturities of the forward rates, T, another time grid will then be considered, ranging from $T_0 < T_1 < ... < T_J$, with T_0 being the first maturity considered, and T_J the last. Then, considering that it can be made a relation between both grids in which it may be possible to write, to each simulated start time t_i, each corresponding maturity as a function of the first time grid, the maturities $T_0, T_1, ..., T_J$ can be referenced as $t_k, t_{k+1}, ..., t_{k+J}$, with t_k being the first maturity available in the simulation, after the first start time t_i. With such a notation one then might be able to represent a discrete version of the forward rate between a given start time t_i to a given maturity maturity t_j, $i \leq j$, with $\hat{f}(t_i, t_j)$, in which the circumflex accent will differ the discretized functions from the continuous ones. And with that one might be able then to represent the price of a zero-coupon bond that pays 1 R$ at maturity t_j in this discrete environment with:

[6] Stands for Stochastic Differential Equation.

$$\hat{B}(t_i,t_j) = \exp\left(-\sum_{l=i}^{j-1}\hat{f}(t_i,t_l)[t_{l+1}-t_l]\right) \tag{4.33}$$

Since the HJM framework requires that the model comply with the principle of no arbitrage for construction, the chosen approximation of the discrete *drift* from the continuous is such that the discounted price of bonds remain with their martingale properties.

So, being $\hat{B}_D(t_i,t_j)$ the discounted price in a date t_0 of a zero-coupon bond $\hat{B}(t_i,t_j)$ and being $\hat{f}(t_k,t_k)$ the spot (or short) rate in time t_k, it can be expressed in terms of:

$$\hat{B}_D\left(t_i,t_j\right) = B\left(t_i,t_j\right)\exp\left(-\sum_{k=0}^{i-1}\hat{f}\left(t_k,t_k\right)\left[t_{k+1}-t_k\right]\right) \tag{4.34}$$

From which, as it is presented in Glasserman (2003), it can be determined the discrete version of the *drift* of the forward rate on the HJM model, in which the discounted prices of bonds $\hat{B}_D(t_i,t_j)$ remain martingale, such as:

$$\hat{\alpha}(t_{i-1},t_j) = \frac{1}{2}\left[\left(\sum_{l=i}^{j}\hat{\sigma}(t_{i-1},t_l)[t_{l+1}-t_l]\right)^2 - \left(\sum_{l=i}^{j-1}\hat{\sigma}(t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right][t_{j+1}-t_j]^{-1} \tag{4.35}$$

This then may be generalized to a multi-factorial model, which is useful for the implementation made later on this work. Considering a generic number N of randomness factors $\hat{\sigma}_n$, compiled in a vector $\hat{\sigma} = (\sigma_1,\sigma_2,...,\sigma_N)$ then one may describe the *drift* of such multi-factor model, $\hat{\alpha}(t_{i-1},t_j)$, as:

$$\begin{aligned}\hat{\alpha}(t_{i-1},t_j) = \frac{1}{2}&\left[\sum_{n=1}^{N}\left(\sum_{l=i}^{j}\hat{\sigma}_n(t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right.\\ &\left.-\sum_{n=1}^{N}\left(\sum_{l=1}^{j-1}\hat{\sigma}_n(t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right][t_{j+1}-t_j]^{-1}\end{aligned} \tag{4.36}$$

From this then, this adaptation from continuous to discrete ise now merged with the parameterization proposed by Brace and Musiela (1994), presented in the previous subsection of this chapter, to find the now discrete *drift* to the model adapted by the proposed parameterization of the HJM framework, now on a multi-factorial model:

$$\begin{aligned}\hat{\hat{m}}(t_{i-1},\tau_j) = \frac{1}{2}&\left[\sum_{n=1}^{N}\left(\sum_{l=i}^{j}\hat{\hat{\nu}}_n(t_{i-1},\tau_l)[\tau_{l+1}-\tau_l]\right)^2\right.\\ &\left.-\sum_{n=1}^{N}\left(\sum_{l=1}^{j-1}\hat{\hat{\nu}}_n(t_{i-1},\tau_l)[\tau_{l+1}-\tau_l]\right)^2\right][\tau_{j+1}-\tau_j]^{-1}\end{aligned} \tag{4.37}$$

And, finally, then one may be able to describe, conveniently for the purpose of implementing the model, the process of the forward rate in this discrete environment for a multi-factorial and under Musiela parameterization with the following dynamics:

$$\hat{f}(t_i,\tau_j) = \hat{f}(t_{i-1},\tau_j) + [t_i - t_{i-1}]\left[\frac{\hat{f}(t_{i-1},\tau_{j+1}) - \hat{f}(t_{i-1},\tau_j)}{\tau_{j+1}-\tau_j} + \hat{\hat{m}}(t_{i-1},\tau_j)\right] + \sum_{n=1}^{N}\hat{\hat{\nu}}_n(t_{i-1},\tau_j)\sqrt{t_i - t_{i-1}}Z_{in} \tag{4.38}$$

In which, the start dates, t, indexation will go from $i = 1, 2, ..., M$ and the maturities, τ, indexation will go from $j = 1, 2, ..., J$. Also, the pseudo-random independent shocks with distribution $N(0, 1)$ are represented by the vectors $Z_i = (Z_{i1}, Z_{i2}, ..., Z_{iN})$; and the volatility parameters represented by $\hat{\hat{\nu}} = (\hat{\hat{\nu}}_1, \hat{\hat{\nu}}_2, ..., \hat{\hat{\nu}}_N)$, with N being the number of factors (i. e. sources of randomness) of the model. And the *drift* element of such forward rate dynamics, $\hat{\hat{m}}$, is defined in (4.37).

This is then the form of the HJM framework that is applied in this work, and the details of the implementation made specifically in the case of bullet swaps indexed to a percentage of CDI different from 100% are presented in the next chapter.

5. CDI-INDEXED DERIVATIVES PRICING UNDER HJM FRAMEWORK

In this chapter are presented the implementation's practical details of the multi-factorial HJM model in order to simulate by Monte Carlo and numerically obtain estimates of convexity premiums that should be embedded in a swap indexed to a percentage of CDI different from 100%.

5.1 The data

In this work, the raw material from which the model is developed upon and then made the back-testing of the obtained results are historical daily settlement prices of DI future contracts, obtained from Bloomberg platform. The data used comprehends the time interval between the first business day of January, 2011 and the last business day of December, 2020. Such data are treated in order to be used within the context desired. The first treatment is to change the fixed maturities that the DI future contracts were related (in example, fixed dates such as January 2012, January 2013, and so on) to fixed tenors[1] (such as one year, two years, and so on), which was done by interpolating[2] the yield curve of each day observed within the sample considered. From this interpolated yield curve with now fixed tenors then it is made a treatment in order to change the observed spot rates to forward rates between each of the designated tenors, by calculating from each pair of following spot rates y_t and y_T, a forward $\hat{f}_{(t,T)}$, with the following equation:

$$\hat{f}_{(t,T)} = \left(\frac{(1+y_T)^{\frac{T}{252}}}{(1+y_t)^{\frac{t}{252}}} \right)^{252/(T-t)} - 1 \tag{5.1}$$

And the last treatment then to the data is to convert this forward rates from discrete forwards ($\hat{f}$) to continuous (f) forwards, by converting them with the relation:

[1]For notation purposes, instead of writing in terms of t and $t+\tau$ to mean a fixed tenor, it is written from now on only t and T for the start and end date of each forward rate, where it must be considered that what is being mentioned with this are fixed tenors and not fixed maturity dates, as explained in the Musiela parameterization Section in Chapter 4, in which $T = t + \tau$.

[2]The method of interpolation in this work was the cubic *spline*, which is broadly used in the literature and so it is considered to be understood by the reader and will not be further explained here.

$$f_{(t,T)} = 252 \times \frac{\ln{(1 + \hat{f}_{(t,T)})}^{\frac{(T-t)}{252}}}{T-t} = \ln{(1 + \hat{f}_{(t,T)})} \tag{5.2}$$

5.2 The PCA

With such continuous forward rates between fixed tenors then, it is possible to analyze this database with the technique called Principal Component Analysis (also known in the industry for its acronym, PCA), in order to identify the volatility vectors corresponding to three different factors in such time frame and then apply it on the HJM model to generate future potential stochastic paths for the yield curve adherent to the volatility regime observed in the sample.

To do so, it is used the method developed on Shreve (2004) and also described and implemented by Nojima (2013). By this method, it is assumed that $\tau = T - t$ and $\tau \geq 0$, so that for a non-random function $\tilde{\sigma}(\tau)$ the volatility $\sigma(t, T)$ is generally defined by:

$$\sigma(t,T) = \tilde{\sigma}(\tau) \min\{L, f(t,T)\} \tag{5.3}$$

In which, L is a positive constant, to set a superior boundary in order to prevent an explosion of the estimated forward rate when τ is too small, which is not be the case in this work but it is considered important to notice that to generalize the concept it is necessary to define L if one is willing to consider significantly small time intervals to measure the volatility.

So, according to the model of the evolution of the forward rate then:

$$df(t,T) = \mu(t,T)dt + \tilde{\sigma}(\tau) \min\{L, f(t,T)\} dW(t) \tag{5.4}$$

Considering now that the forward rates observed in the sample are past observations $t_1 < t_2 < t_3 < \cdots < t_M < P$, with P referring to the present moment, and that such forward rates are related to the maturities $\tau_1 < \tau_2 < \tau_3 < \cdots < \tau_J$, so what this means is that the observed forward rates are in the form of $f(t_i, t_i + \tau_j)$ with $i = 1, 2, 3, \ldots, M$ and $j = 1, 2, 3, \ldots, J$. Furthermore, considering an increment δ small enough in order that $t_i + \delta < t_i + 1$ with $i = 1, 2, 3, \ldots, M - 1$ and $t_M + \delta \leq P$, then it may be written that:

$$f(t_i+\delta, t_i+\tau_j) - f(t_i, t_i+\tau_j) \approx \delta\mu(t_i, t_i+\tau_j) + \tilde{\sigma}(\tau_j) \min\{L, f(t_i, t_i+\tau_j)\} \left(W(t_i + \delta) - W(t_i)\right) \tag{5.5}$$

Additionally, one might define the relation:

$$D_{i,j} = \frac{f(t_i + \delta, t_i + \tau_j) - f(t_i, t_i + \tau_j)}{\sqrt{\delta} \min\{L, f(t_i, t_i + \tau_j)\}} \tag{5.6}$$

That being substituted in (5.5), makes:

$$D_{i,j} \approx \frac{\sqrt{\delta}\mu(t_i, t_i + \tau_j)}{\min\{L, f(t_i, t_i + \tau_j)\}} + \tilde{\sigma}(\tau_j)\frac{W(t_i + \delta) - W(t_i)}{\sqrt{\delta}} \tag{5.7}$$

Due to the fact that the first term on the right hand side of the equation (5.7) is multiplied by $\sqrt{\delta}$, this term becomes irrelevant, so if one writes:

$$X_i = \frac{W(t_i + \delta) - W(t_i)}{\sqrt{\delta}} \tag{5.8}$$

Then one might write $D_{i,j}$ instead as in (5.7), as approximately $\tilde{\sigma}(\tau_j)X_i$.

Then, the empirical and the theoretical covariance may be written, respectively, as:

$$C_{j_1,j_2} = M^{-1}\sum_{i=1}^{M} D_{i,j_1}D_{i,j_2} \tag{5.9}$$

$$\mathbb{E}\left[\tilde{\sigma}(\tau_{j_1})\tilde{\sigma}(\tau_{j_2})X_i^2\right] = \tilde{\sigma}(\tau_{j_1})\tilde{\sigma}(\tau_{j_2}) \tag{5.10}$$

Since $X_1, X_2, X_3, \ldots, X_M$ are normal and independent random variables, therefore this means that $D_{1,j}$,$D_{2,j}$, $D_{3,j}, \ldots, D_{M,j}$ can be considered independent observations of the forward rates measured at $t_1, t_2, t_3, \ldots, t_M$ for the tenor τ_j.

Then, if one define a matrix $(M \times J)$ related to the observations at times M to tenors J as:

$$\mathbf{D} = \begin{bmatrix} D_{1,1} & D_{1,2} & \ldots & D_{1,J} \\ D_{2,1} & D_{2,2} & \cdots & D_{2,J} \\ D_{3,1} & D_{3,2} & \cdots & D_{3,J} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M,1} & D_{M,2} & \cdots & D_{M,J} \end{bmatrix} \tag{5.11}$$

It is possible to define then a symmetrical and positive semi-defined empirical variance and covariance matrix over the observations **D** as:

$$\mathbf{C} = \begin{bmatrix} C_{1,1} & C_{1,2} & \ldots & C_{1,J} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,J} \\ C_{3,1} & C_{3,2} & \cdots & C_{3,J} \\ \vdots & \vdots & \ddots & \vdots \\ C_{J,1} & C_{J,2} & \cdots & C_{J,J} \end{bmatrix} = M^{-1}\mathbf{D}'\mathbf{D} \tag{5.12}$$

From that, if one then define $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$ as the eigenvalues of **C**; the column vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_J$ as its orthogonal eigenvectors and $\mathbf{e}'_j$ as the transposed of $\mathbf{e}_j$, one may obtain the Principal Components as:

$$\mathbf{C} = \lambda_1\mathbf{e}_1\mathbf{e}'_1 + \lambda_2\mathbf{e}_2\mathbf{e}'_2 + \lambda_3\mathbf{e}_3\mathbf{e}'_3 + \cdots + \lambda_J\mathbf{e}_J\mathbf{e}'_J \tag{5.13}$$

Approximating then $\tilde{\sigma}$ by:

$$\begin{bmatrix} \tilde{\sigma}(\tau_1) \\ \tilde{\sigma}(\tau_2) \\ \tilde{\sigma}(\tau_3) \\ \vdots \\ \tilde{\sigma}(\tau_J) \end{bmatrix} = \sqrt{\lambda_1}\mathbf{e}_1 \tag{5.14}$$

In order to better approximate **C**, one might consider more factors into the equation of the dynamics of the forward rate, for instance a total of three factors, as it is typically made when applying this technique to interest rate models. In this work it is considered three factors. Each of the volatilities are then defined by a vector: $\sqrt{\lambda_1}\mathbf{e}_1$, $\sqrt{\lambda_2}\mathbf{e}_2$ and $\sqrt{\lambda_3}\mathbf{e}_3$.

Additionally, in this work these non-parametrical volatility vectors are smoothed with Ordinary Least Square regressions: linear for the first volatility vector, quadratic for the second and cubic for the third. The goal with that smoothing process is to give a more comprehensive volatility as input for the HJM model, avoiding outlier observations to have an impact on the simulated future paths of the yield curve generated by the model.

5.3 The Monte Carlo Simulation

These smoothed vectors of volatility then are applied to the dynamics presented in the end of Chapter 4, in equation (4.38), generating one of the possible paths for the yield curve; more specifically generating the path of continuous forward rates. This specific path is then converted to discrete forwards, applying the inverse of what is presented in (5.2) to treat the original data. And on top of that then, again is applied the inverse of an equation already presented when treating the data, the equation (5.1), to convert the discrete forward rates to spot rates, resulting in the dynamics then of the spot rate curve for that specific simulated path.

This simulated spot rate curve path is then used to calculate the expected value for the convexity premium of the portfolio presented in Chapter 3.

It is calculated for each day, with the yield curve obtained from the model as described above, the result of the portfolio composed by a bullet swap indexed to a percentage of CDI different from 100% and a certain amount of contracts of DI future that hedges the *delta* risk related to the interest rate. At the end of each day, it is assumed that it is possible to hedge again the portfolio with the precise amount of contracts of DI future at the new level of interest rates. The amount of contracts required to hedge the portfolio at the new level of interest rates observed at the end of each day has already been presented, in Chapter 3, defined by the equation (3.7). The portfolio at each movement of the yield curve presents a result, due to the ineffectiveness of the hedge (calculated to the previous level of interest rates, instead of the new one). These daily results are brought to present value and then added, resulting in a expected present value of the dynamic hedging of such portfolio on this specific simulated path.

More specifically, for each portfolio analyzed (several different swaps are analyzed, to obtain a matrix of results to serve as input to estimate a polynomial equation, in order to have an approximate value for the convexity premium of swaps with different characteristics than the ones simulated) are defined a few key characteristics:

φ : percentage of CDI to which one leg of the bullet swap are indexed.
T : number of business days between the start date and the maturity of the bullet swap.
N : notional, in BRL, of the bullet swap.
S : spread over CDI, related to the other leg of the swap.

The characteristics above mentioned are arbitrarily chosen, with the exception of *S*, that are such that the present value of both legs of the swap are initially the same, in module. The financial result in itself is not useful for analysis, considering that it varies depending on the notional considered for such swap, so it is converted from a financial amount to an yearly compounded rate, with the following relation:

$$C_r = \left(\frac{C_f}{N} + 1\right)^{\left(\frac{252}{T}\right)} - 1 \tag{5.15}$$

Where C_r is the Convexity yearly compounded rate, C_f the financial expected value for the convexity premium, expressed in BRL, and T the initial number of business days until the maturity of the swap.

The method used to obtain the expected value for this convexity premium is Monte Carlo simulation, where it is be repeated the proceeding above described for each selected combination of key characteristics 10,000 times, simulating 10,000 different potential paths of the yield curve for each portfolio analyzed. Finally the mean of such results for each portfolio is then considered the convexity premium to be embedded into the rate of the swap with the determined characteristics.

Once obtaining the matrix of convexity premiums for each pair of characteristics of maturity and percentage of CDI, these data are transfered to be analyzed with a Python library called scikit-learn[3], with which are obtained polynomial equations to approximate on a three-dimension Cartesian plane, the surface composed by the points obtained on the Monte Carlo simulations, with the dimensions of such Cartesian plane being the maturities, the percentages of CDI and the convexity premiums.

To summarize the more general proceedings, it is presented below a table composed by each step realized:

[3]for further information about the documentation of such library access https://scikit-learn.org/stable/modules/linear_model.html

Step	Procedure
1	Gather historical data;
2	Treat the data;
3	Extract the volatility vectors with PCA;
4	Define the combination of key characteristics of the swap to be analyzed: percentage of CDI and tenor;
5	Simulate a scenario for the yield curve, using the HJM model with the volatility vectors extracted from historical data;
6	Calculate the present value of the dynamic *delta* hedge strategy of the portfolio with the simulated yield curve path;
7	Store the calculated present value (and/or convert it into annualized spread) of the dynamic *delta* hedge strategy of the portfolio;
8	Repeat steps 5, 6 and 7 again, until have stored 10,000 values. Take the mean of these values. Store this as the convexity premium for the selected combination;
9	Select another combination of key characteristics and proceed again from step 5, until all combinations wanted have been simulated.

Table 5.1: Procedures to price CDI-indexed derivatives under HJM framework

6. EXPERIMENT

This chapter presents the experiments made in the attempt to estimate the convexity premiums of portfolios composed by a bullet swap with a leg indexed to a percentage of CDI different from 100% and another leg indexed to the CDI plus a fixed rate exponentially and yearly compounded, altogether with an amount of contracts of DI future that hedges in a dynamical strategy of daily offsets the interest rate risk of first order of the swap.

6.1 The procedures

The experiments developed in this work consist of the following procedures:

- Extract of the volatility vectors from historic data through PCA.
- Perform Monte Carlo simulations of 10,000 potential scenarios for the yield curve over 6 years, with the HJM model implemented as described on previous chapters.
- Calculate, based upon the simulated scenarios, the present value of a daily dynamic *delta* hedge strategy of the portfolio initially composed by a bullet swap indexed to a percentage of CDI and the amount of contracts of future of DI hedging its first order risk. A set of combinations of percentages of the CDI and tenors are simulated, each, 10,000 times. The mean of the simulations for each combination of percentage of CDI and tenor is considered the fair convexity premium for that determined combination.
- Back-test the convexity premiums estimated with the model comparing to the results of the same *delta* hedge strategy, in the actual realized path of the yield curve over the next 6 years. Compare the magnitude of the errors and analyze the results obtained with the model.
- Additionally, read with Python the matrix of convexity premiums for each combination of percentage of CDI and tenor mentioned above and with scikit-learn library use this data to obtain a polynomial equation that attempts to fit the surface of the simulated specific combinations, as a function of the percentage of CDI and the tenor of the swap, in order to have coefficients of a polynomial equation that could be used to estimate a fair convexity premium for different combinations of percentages of the CDI and tenors.

With that, it is supposed that the reader understands the procedures that are realized. The conditions of each experiment are detailed on the next section of this chapter.

6.2 The conditions

In order to obtain an estimate for the convexity premium of the portfolio specified above, a certain set of conditions must be defined to practically implement the simulations and back-tests. In the specific case of this work, given the data obtained, are implemented simulations based upon the volatility vectors extracted from data within three different time-frames: **2011-2012**, **2012-2013** and **2013-2014**. With these three different sets of volatility parameters, are simulated 10,000 potential paths for all combinations of the following key characteristics for a bullet swap composing the portfolio:

Tenors	Percentages of the CDI									
1 year	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%
2 years	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%
3 years	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%
4 years	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%
5 years	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%
6 years	25%	50%	75%	100%	125%	150%	175%	200%	225%	250%

Table 6.1: Tenors and percentages of the CDI selected for the experiments.

The six years horizon as the longest tenor considered in this work is a reflection of the availability of data obtained, in order to implement a back-test against a realized historic path of prices for the yield curve: the scenarios simulated with the vectors of volatility extracted from **2011-2012** period were compared to the next six years (from 2013 to, including, 2018); the scenarios simulated with the vectors of volatility extracted from **2012-2013** period were compared to the next six years (from 2014 to, including, 2019) and finally the scenarios simulated with the vectors of volatility extracted from **2013-2014** period were compared to the next six years (from 2015 to, including, 2020). It was possible to add a longer tenor on the first two cases, but for the purpose of comparison between the three experiments, the maximum tenor considered in all of them was limited to six years, so that all three experiments observed the same tenors, both on its simulations as it on its back-tests.

Below are presented the realized paths of the interest rates for each tenor that would be then simulated, for each time frame mentioned previously, being the tenor zero the CDI in itself:

In Figure 6.1 it is possible to notice that in this specific time period, from 2013 to 2016 the yield curve increased its level, with Brazil facing inflationary pressures, amid high fiscal and political risks, but then from 2017 to 2021, with a calmer political environment in Brazil and a lower inflationary scenario, not only in Brazil but globally, the yield curve decreased in level year after year. Besides understanding what was the realized path on the periods in which the

Figure 6.1: Evolution of the spot rate curve during **2013.2018**, **2014.2019**. and **2015.2020**, respectively

simulations were compared with on the back-tests, one might find Figure 6.2 helpful to have a glimpse of what were the spot rate curves structure at each beginning of year analyzed, in order to grasp the changes on the spot rate curve, not only in its level but also in its slope and curvature.

Looking at Figure 6.2, with the shape of the spot rate curves in each of the selected moments of time, it is possible to notice that the structure of the spot rate curve in January 2013 and January 2019 have a resemblance, but that the periods in between were nothing alike. In January 2014, compared to January 2013, the spot rate curve had increased significantly in level and a bit in steepness, reflecting worse financial conditions of the country. January 2015 compared to 2014 showed a perceptible flattening of the structure of the yield curve, with its level increasing from tenors up to 4 years and decreasing for longer tenors, what could be attributed to higher inflationary risks, that the market priced that was going to be dealt by a tighter monetary policy. But in January 2016, the political situation of Brazil deteriorated rapidly, what would lead to a process of impeachment of the president at the time, and the structure of the yield curve presented aversion for risks, evident on the higher level of the whole yield curve when

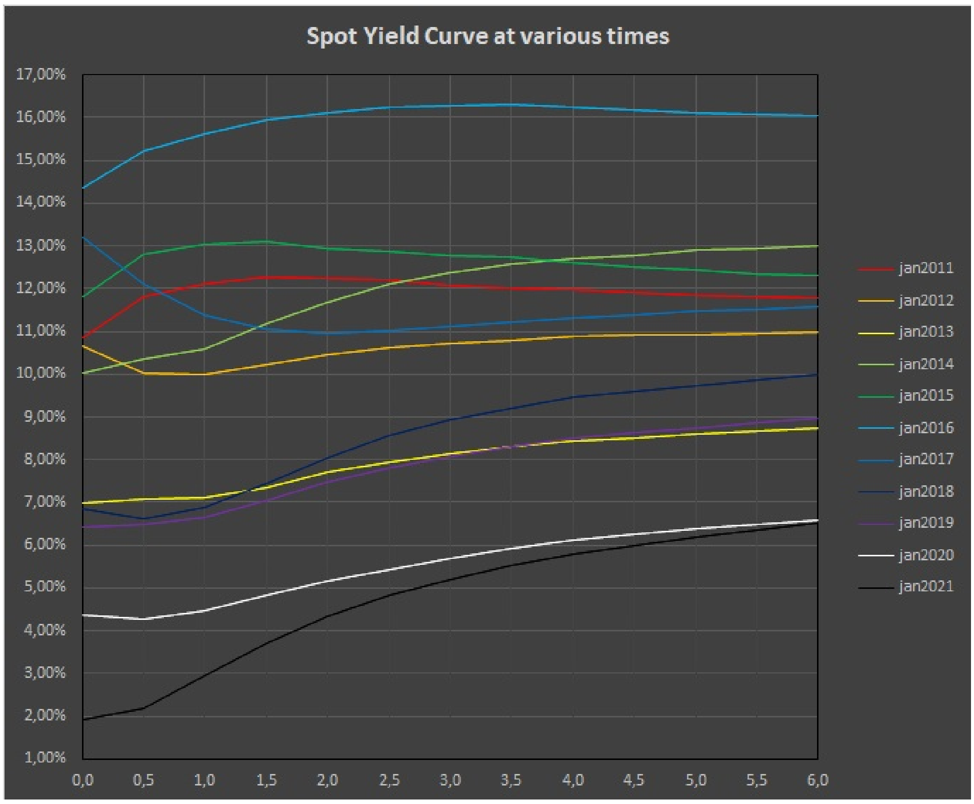


Figure 6.2: spot rate curves at various times, **2011.2021**.

compared to previous moments. After the impeachment, the political environment improved and so did the mood of the market, that began to price better financial conditions ahead, with a lower inflationary environment globally, lead to a structure of the yield curve lower and with negative slope, indicating that the next steps of the monetary policy were expected to be more loose. This process extended then all the way to January 2021, with lower levels of interest rates being observed one period after another. What is worth mentioning though is the slight increase on the steepness of the yield curve when comparing January 2020 to January 2021, after the fiscal responses to the COVID-19 pandemic, showing the beginning of market view of potential increases ahead on inflation levels, globally speaking. This is an ongoing process and the aftermath of it can not be presented in this work, but is consider a reason why the convexity premium of such derivatives must not be ignored although in a lower interest rates environment, due to risks of a reversion in this scenario ahead, which could lead to higher relevance of such convexity.

Additionally, it is presented below in Figure 6.3 the realized volatility of 3 months, 6 months and 12 months, from 2012 (2011 was not considered in this plot, due to the fact that the data required to obtain the volatility of 12 months could only begin 12 months after the sample of data used) to 2020 (the whole year of 2020). With this image, one may have a few insights about what was achieved as results using volatility vectors extracted from the beginning of this period, to forecast the convexity premium of the portfolio, highly correlated to the volatility, 6 years ahead. The results are analyzed in Chapter 7, but with this image one may already notice

that the volatility increased consecutively within the periods selected.

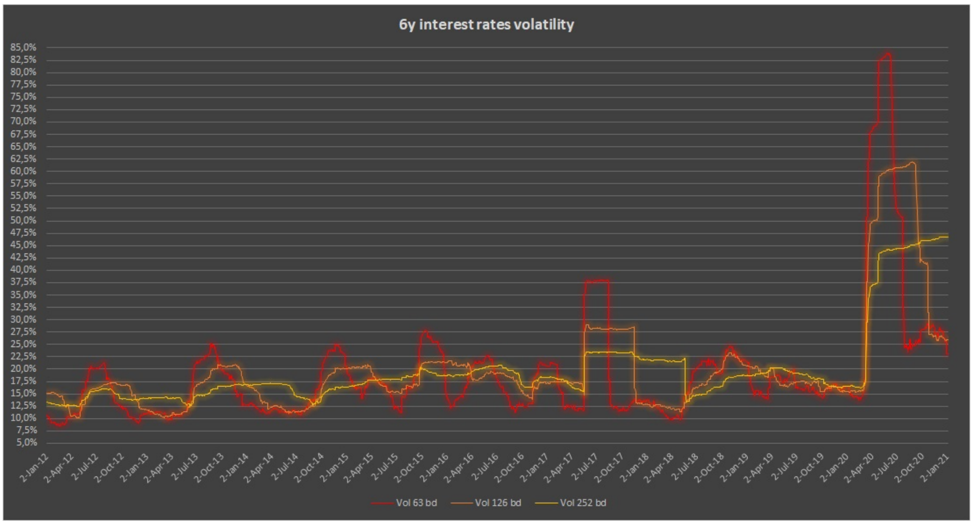


Figure 6.3: Realized volatility of 3, 6 and 12 months, during **2012.2020**.

It is considered valid to show the three first Principal Components obtained from each time period mentioned previously (**2011-2012**, **2012-2013**, **2013-2014**), in Figure 6.4, just to give an illustration for the reader of what was the product of such technique explained on the previous chapter. In this figure one may notice that the discrepancies between the Principal Components extracted from the periods of **2011-2012**, **2012-2013** and **2013-2014**, in which the second (blue line) and third (yellow line) component differs the most between the periods, with the first component (green line) maintaining a similar pattern in all the periods observed.

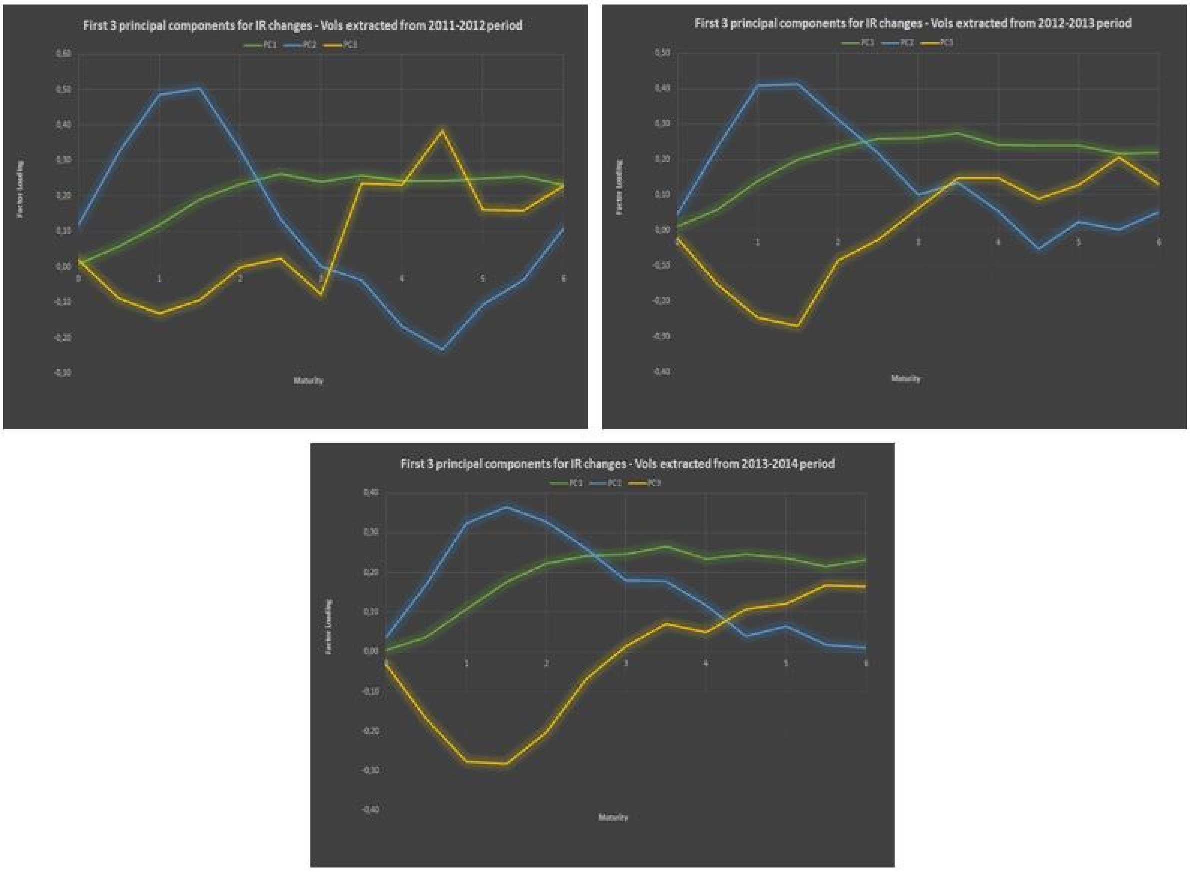


Figure 6.4: Principal Components obtained from **2011.2012**, **2012.2013**. and **2013.2014**., respectively.

Subsequently, in Figure 6.5 are then presented the regressions made upon each volatility vector extracted from the Principal Components, in order to smooth each vector before applying it on the HJM model. These are the vectors used in the simulations for each experiment. The goal with that smoothing process is to give a more comprehensive volatility as input for the HJM model, avoiding outlier observations to have an impact on the simulated future paths of the yield curve generated by the model.

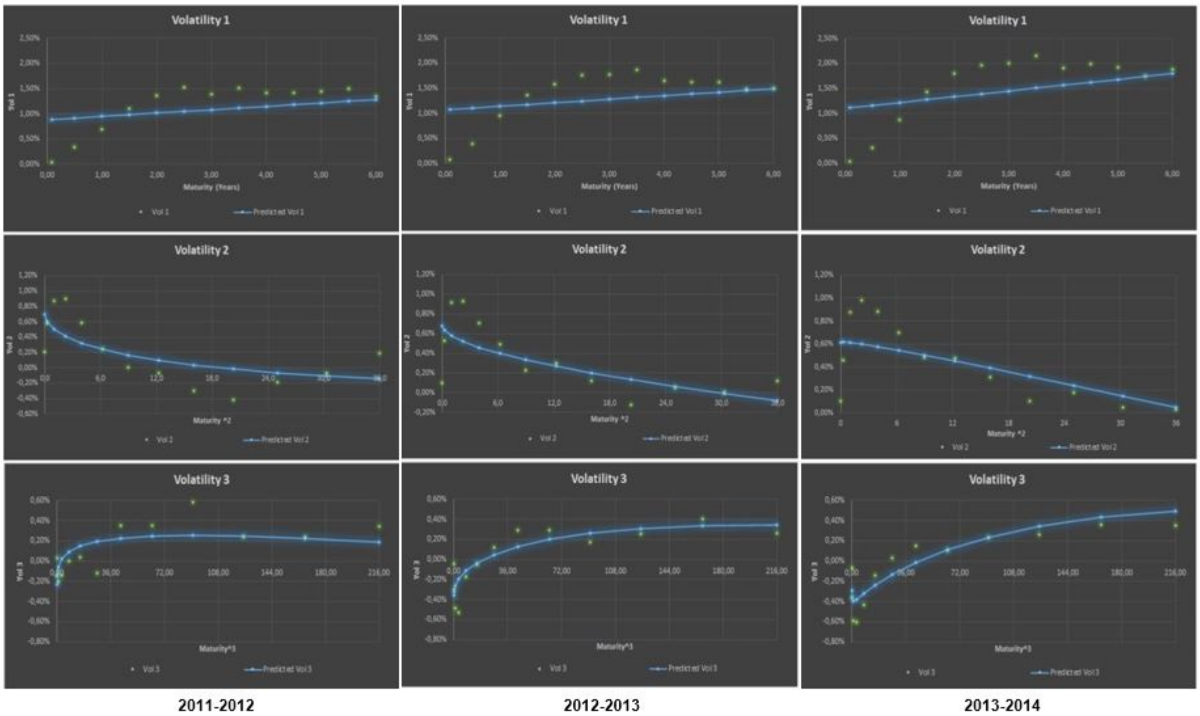


Figure 6.5: Smoothed volatility vectors obtained from **2011.2012**, **2012.2013**. and **2013.2014**., respectively.

What was then obtained from the model was potential paths of the yield curve to be used to calculate the present value of the dynamic *delta* hedging strategy of the portfolio. Each of the combinations of the key characteristics mentioned in the beginning of this chapter was simulated 10,000 times and the mean of these simulations was considered the fair estimate of the convexity premium for each combination of characteristics.

6.3 The polynomial equation

With such matrix of convexity premiums for each combination of tenor and percentage of CDI then, it was ran on a Python library named *scikit-learn*[1] a routine to obtain a polynomial equation that could be used to obtain the convexity premiums for tenors and percentages of the CDI that were not between the selected ones. The polynomial equation obtained is of the type:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_1 x_2 + \beta_5 x_2^2 + \beta_6 x_1^3 + \beta_7 x_1^2 x_2 + \beta_8 x_1 x_2^2 + \beta_9 x_2^3 \quad (6.1)$$

in which:

$y =$ convexity premium annualized, in basis points, to be added on the rate of the swap.
$x_1 =$ percentage of CDI.
$x_2 =$ maturity of the swap, in business days.
$\beta_n =$ coefficients of the polynomial equation, $n = 1, \ldots, 9$.

What is going to be obtained are the coefficients to be used in order to have a fair estimate for a convexity premium for different tenors and percentages of the CDI than those analyzed by the Monte Carlo simulations of the impacts of the yield curve paths on the present value of the dynamic *delta* hedging strategy of the portfolio. The polynomial equations for each matrix are presented graphically in a three dimensional surface, with the vertical axis being convexity premium annualized in basis points and the horizontal ones being the percentage of CDI and the maturity of the swap, in business days. The practical use of such polynomial equation is to provide the practitioner, after providing one round of Monte Carlo Simulations and acquiring a matrix of convexity premiums for each combination of tenor and percentage of CDI selected to be simulated, a tool to obtain an estimate for different tenors and percentages of the CDI. The practitioner would have to simulated, periodically, with new volatility inputs, if considers that the past parameters used in the previous simulations are no longer valid. But nevertheless, this periodic rounds of simulations would only be required when and if the practitioner consider that the last matrix of results is no longer a good input for the polynomial equation to estimate the convexity premium, in practice reducing the time cost of such pricing.

The results of such experiments will now be presented in the next chapter.

[1] for further information about the documentation of such library access https://scikit-learn.org/stable/modules/linear_model.html. Additionally, consider as reference Pedregosa et al. (2011).

7. RESULTS

This chapter presents the results obtained in the experiments described in Chapter 6. The back-test results are analyzed as well as the polynomial equations to approximate the matrix of results obtained from the Monte Carlo simulations.

7.1 The back-test results

What is done in this work to test the validity of the model is to compare the results obtained from 10,000 simulations of Monte Carlo with the realized path of the interest rates through the period analyzed, and then compare the estimate of the premium with the present value of the effective gain that would have been obtained on such strategy if the realized path was indeed what was realized.

Tables 7.1, 7.2 and 7.3 presented below show the results of simulations (column "Estimated Value (BRL)") compared (column "Real / Estimated") to the present value (at the date of pricing the swaps) of the results that would have been obtained if the realized paths of the interest rates were indeed what ended up happening (column "Real Value (BRL)"). The results are presented in financial terms (columns "Estimated Value (BRL)" and "Real Value (BRL)"), related to a swap with notional of 100 million BRL. Results are also presented as in annualized spread to be added on the rate of the swap (columns "Estimated Ann. Spread" and "Real Ann. Spread").

Back Testing				Out of Sample: 2013.2018		
%CDI	Tenors (Years)	Estimated Value (BRL)	Estimated Ann. Spread	Real Value (BRL)	Real Ann. Spread	Real / Estimated
250%	6	3,533,818	0.580%	12,255,781	1.952%	3.5
250%	5	1,728,533	0.343%	5,533,217	1.083%	3.2
250%	4	758,580	0.189%	2,068,252	0.513%	2.7
250%	3	278,183	0.093%	588,397	0.196%	2.1
250%	2	72,850	0.036%	132,139	0.066%	1.8
250%	1	8,296	0.008%	9,224	0.009%	1.1
225%	6	2,322,210	0.383%	7,753,851	1.257%	3.3
225%	5	1,164,274	0.232%	3,613,633	0.713%	3.1
225%	4	522,351	0.130%	1,395,888	0.347%	2.7
225%	3	196,180	0.065%	410,007	0.136%	2.1
225%	2	52,624	0.026%	94,825	0.047%	1.8
225%	1	6,122	0.006%	6,788	0.007%	1.1
200%	6	1,449,852	0.240%	4,652,374	0.763%	3.2
200%	5	743,753	0.148%	2,237,952	0.444%	3.0
200%	4	342,124	0.085%	893,315	0.223%	2.6
200%	3	131,290	0.044%	270,895	0.090%	2.1
200%	2	36,002	0.018%	64,520	0.032%	1.8
200%	1	4,275	0.004%	4,736	0.005%	1.1
175%	6	833,829	0.138%	2,576,727	0.426%	3.1
175%	5	438,631	0.088%	1,279,244	0.255%	2.9
175%	4	206,656	0.052%	527,620	0.132%	2.6
175%	3	81,125	0.027%	165,178	0.055%	2.0
175%	2	22,751	0.011%	40,513	0.020%	1.8
175%	1	2,756	0.003%	3,050	0.003%	1.1
150%	6	417,877	0.070%	1,242,966	0.207%	3.0
150%	5	225,451	0.045%	636,815	0.127%	2.8
150%	4	108,558	0.027%	271,370	0.068%	2.5
150%	3	43,611	0.015%	87,702	0.029%	2.0
150%	2	12,529	0.006%	22,151	0.011%	1.8
150%	1	1,548	0.002%	1,710	0.002%	1.1
125%	6	152,689	0.025%	437,303	0.073%	2.9
125%	5	84,413	0.017%	231,188	0.046%	2.7
125%	4	41,636	0.010%	101,780	0.025%	2.4
125%	3	17,125	0.006%	33,955	0.011%	2.0
125%	2	5,022	0.003%	8,831	0.004%	1.8
125%	1	633	0.001%	699	0.001%	1.1
100%	6	0	0.000%	0	0.000%	-
100%	5	0	0.000%	0	0.000%	-
100%	4	0	0.000%	0	0.000%	-
100%	3	0	0.000%	0	0.000%	-
100%	2	0	0.000%	0	0.000%	-
100%	1	0	0.000%	0	0.000%	-
75%	6	-70,766	-0.012%	-187,204	-0.031%	2.6
75%	5	-40,984	-0.008%	-105,351	-0.021%	2.6
75%	4	-21,153	-0.005%	-49,491	-0.012%	2.3
75%	3	-9,096	-0.003%	-17,591	-0.006%	1.9
75%	2	-2,798	-0.001%	-4,851	-0.002%	1.7
75%	1	-368	0.000%	-404	0.000%	1.1
50%	6	-82,973	-0.014%	-210,913	-0.035%	2.5
50%	5	-49,148	-0.010%	-122,444	-0.025%	2.5
50%	4	-25,988	-0.006%	-59,412	-0.015%	2.3
50%	3	-11,441	-0.004%	-21,796	-0.007%	1.9
50%	2	-3,586	-0.002%	-6,189	-0.003%	1.7
50%	1	-481	0.000%	-528	-0.001%	1.1
25%	6	-54,630	-0.009%	-133,696	-0.022%	2.4
25%	5	-33,135	-0.007%	-80,061	-0.016%	2.4
25%	4	-17,945	-0.004%	-40,121	-0.010%	2.2
25%	3	-8,073	-0.003%	-15,191	-0.005%	1.9
25%	2	-2,597	-0.001%	-4,442	-0.002%	1.7
25%	1	-354	0.000%	-389	0.000%	1.1

Table 7.1: Back-Testing results of the convexity premiums estimated with volatility vectors from **2011.2012** compared to realized convexity premiums during **2013.2018**. (100mm BRL Notional Swap).

In the first Table, 7.1, on the last column to the right, "Real / Estimated", it is possible to notice that the convexity premiums obtained by the simulations using the HJM model that had as input the volatility vectors extracted from the period of **2011-2012** (and therefore, priced the evolution of the portfolio through the next six years, from January 2013 to the end of 2018) underestimated the convexity premium that should have been considered, since the expected values of such convexity premiums of longer tenors in the scenario that the realized path was indeed what would happen, were approximately three times larger than what was estimated using the HJM model, with the volatility vectors extracted from the period of **2011-2012**. The fact that for shorter tenors, such as one or two years the premiums estimated with the model were closer to the expected value that would be obtained if the realized path indeed happened, is attributed here to the lower impact of the convexity for this shorter tenors. Once the maturity of the swap increases, the underestimation of the volatility has a larger impact on the convexity premium, which is intuitive, due to the positive relation between the volatility and convexity premiums. Besides, the estimation error increases with larger percentages of the CDI, also leading to an perception that the level of the percentage of CDI also plays a major impact on the convexity premium. With only Table 7.1, it is possible to empirically evidence the theoretical relations presented in Chapter 3, demonstrating in practice the positive impact of the maturity and the percentage of CDI of the derivative in the convexity premium of the portfolio.

On the second Table, 7.2, the error of the premiums obtained by the simulations decreases in comparison to the premiums on the scenario that the realized path actually happens. From an error of more than three and a half times when comparing a swap indexed to 250% of the CDI for 6 years, on Table 7.1, to a close to two times underestimate on the second case (observed on column "Real / Estimated", on the right side of the table). This could be related to the fact that the increase in the volatility during 2013 compared to 2011, helped the model to be more accurate when predicting the next six years, that had in fact an increase in its volatility regime. Howsoever, the insight that is possible to obtain from this results is the notion that the volatility inputs have a significant impact on the accuracy of the estimation.

The third Table, 7.3, remains close to the second one, when it comes to predicting the fair estimate for the convexity premiums, with an underestimation of such premiums in the area of 50% (the premiums obtained by the simulated paths were approximately half the premiums obtained in the scenario in which the realized path was indeed what happened in the yield curve).

What could additionally be noticed is the fact that the premiums are zero in the scenario of a swap indexed to 100%, what serves as proof that the simulations were indeed making correct calculations for the expected present value, so that in the case of a swap indexed to 100% no premiums were associated, since there was no convexity mismatch between the swap and the future of DI initially hedging its *delta* risk.

Back Testing				Out of Sample: 2014.2019		
%CDI	Tenors (Years)	Estimated Value (BRL)	Estimated Ann. Spread	Real Value (BRL)	Real Ann. Spread	Real / Estimated
250%	6	7,148,784	1.157%	15,342,476	2.409%	2.1
250%	5	3,346,087	0.660%	7,085,375	1.379%	2.1
250%	4	1,399,768	0.348%	2,647,402	0.655%	1.9
250%	3	484,214	0.161%	697,784	0.232%	1.4
250%	2	118,487	0.059%	91,375	0.046%	0.8
250%	1	12,413	0.012%	5,284	0.005%	0.4
225%	6	4,446,238	0.728%	9,582,124	1.538%	2.2
225%	5	2,150,140	0.426%	4,559,598	0.896%	2.1
225%	4	927,949	0.231%	1,758,158	0.437%	1.9
225%	3	331,972	0.111%	478,874	0.159%	1.4
225%	2	83,931	0.042%	64,789	0.032%	0.8
225%	1	9,067	0.009%	3,860	0.004%	0.4
200%	6	2,617,163	0.431%	5,675,818	0.925%	2.2
200%	5	1,309,325	0.260%	2,782,475	0.550%	2.1
200%	4	584,542	0.146%	1,107,131	0.276%	1.9
200%	3	216,206	0.072%	311,605	0.104%	1.4
200%	2	56,596	0.028%	43,556	0.022%	0.8
200%	1	6,289	0.006%	2,674	0.003%	0.4
175%	6	1,424,976	0.236%	3,103,490	0.511%	2.2
175%	5	737,262	0.147%	1,567,229	0.311%	2.1
175%	4	340,018	0.085%	643,426	0.160%	1.9
175%	3	129,809	0.043%	187,123	0.062%	1.4
175%	2	35,048	0.018%	27,022	0.014%	0.8
175%	1	4,026	0.004%	1,709	0.002%	0.4
150%	6	675,469	0.112%	1,478,044	0.245%	2.2
150%	5	360,761	0.072%	768,759	0.153%	2.1
150%	4	172,031	0.043%	325,625	0.081%	1.9
150%	3	67,846	0.023%	97,847	0.033%	1.4
150%	2	18,945	0.009%	14,598	0.007%	0.8
150%	1	2,243	0.002%	951	0.001%	0.4
125%	6	233,221	0.039%	513,420	0.085%	2.2
125%	5	128,873	0.026%	275,004	0.055%	2.1
125%	4	63,455	0.016%	120,168	0.030%	1.9
125%	3	25,909	0.009%	37,308	0.012%	1.4
125%	2	7,467	0.004%	5,750	0.003%	0.8
125%	1	909	0.001%	386	0.000%	0.4
100%	6	0	0.000%	0	0.000%	-
100%	5	0	0.000%	0	0.000%	-
100%	4	0	0.000%	0	0.000%	-
100%	3	0	0.000%	0	0.000%	-
100%	2	0	0.000%	0	0.000%	-
100%	1	0	0.000%	0	0.000%	-
75%	6	-96,354	-0.016%	-214,279	-0.036%	2.2
75%	5	-56,893	-0.011%	-121,675	-0.024%	2.1
75%	4	-29,886	-0.007%	-56,570	-0.014%	1.9
75%	3	-13,019	-0.004%	-18,746	-0.006%	1.4
75%	2	-4,006	-0.002%	-3,083	-0.002%	0.8
75%	1	-519	-0.001%	-220	0.000%	0.4
50%	6	-106,716	-0.018%	-238,383	-0.040%	2.2
50%	5	-65,034	-0.013%	-139,344	-0.028%	2.1
50%	4	-35,327	-0.009%	-66,817	-0.017%	1.9
50%	3	-15,892	-0.005%	-22,874	-0.008%	1.4
50%	2	-5,044	-0.003%	-3,887	-0.002%	0.8
50%	1	-674	-0.001%	-285	0.000%	0.4
25%	6	-66,462	-0.011%	-149,215	-0.025%	2.2
25%	5	-41,381	-0.008%	-89,776	-0.018%	2.1
25%	4	-23,458	-0.006%	-44,395	-0.011%	1.9
25%	3	-10,902	-0.004%	-15,700	-0.005%	1.4
25%	2	-3,580	-0.002%	-2,756	-0.001%	0.8
25%	1	-495	0.000%	-208	0.000%	0.4

Table 7.2: Back-Testing results of the convexity premiums estimated with volatility vectors from **2012.2013** compared to realized convexity premiums during **2014.2019**. (100mm BRL Notional Swap).

Back Testing				Out of Sample: 2015.2020		
%CDI	Tenors (Years)	Estimated Value (BRL)	Estimated Ann. Spread	Real Value (BRL)	Real Ann. Spread	Real / Estimated
250%	6	8,512,361	1.370%	19,030,306	2.409%	2.2
250%	5	4,062,887	0.800%	9,194,748	1.379%	2.3
250%	4	1,718,035	0.427%	3,979,546	0.655%	2.3
250%	3	593,487	0.197%	1,325,257	0.232%	2.2
250%	2	142,355	0.071%	266,797	0.046%	1.9
250%	1	14,098	0.014%	10,525	0.005%	0.7
225%	6	5,381,409	0.877%	11,815,154	1.538%	2.2
225%	5	2,637,865	0.522%	5,874,959	0.896%	2.2
225%	4	1,145,243	0.285%	2,620,255	0.437%	2.3
225%	3	407,942	0.136%	900,570	0.159%	2.2
225%	2	100,700	0.050%	187,457	0.032%	1.9
225%	1	10,259	0.010%	7,651	0.004%	0.7
200%	6	3,212,096	0.528%	6,958,813	0.925%	2.2
200%	5	1,618,713	0.322%	3,560,265	0.550%	2.2
200%	4	724,328	0.181%	1,636,034	0.276%	2.3
200%	3	264,951	0.088%	580,269	0.104%	2.2
200%	2	67,479	0.034%	124,882	0.022%	1.9
200%	1	7,077	0.007%	5,274	0.003%	0.7
175%	6	1,770,820	0.293%	3,784,335	0.511%	2.1
175%	5	918,844	0.183%	1,991,721	0.311%	2.2
175%	4	421,955	0.105%	942,835	0.160%	2.2
175%	3	159,082	0.053%	345,057	0.062%	2.2
175%	2	41,641	0.021%	76,777	0.014%	1.8
175%	1	4,509	0.005%	3,355	0.002%	0.7
150%	6	851,433	0.141%	1,792,933	0.245%	2.1
150%	5	454,188	0.091%	970,520	0.153%	2.1
150%	4	214,377	0.054%	473,189	0.081%	2.2
150%	3	83,179	0.028%	178,675	0.033%	2.1
150%	2	22,391	0.011%	41,102	0.007%	1.8
150%	1	2,499	0.002%	1,858	0.001%	0.7
125%	6	298,579	0.050%	619,717	0.085%	2.1
125%	5	163,602	0.033%	344,941	0.055%	2.1
125%	4	79,412	0.020%	173,190	0.030%	2.2
125%	3	31,670	0.011%	67,465	0.012%	2.1
125%	2	8,780	0.004%	16,044	0.003%	1.8
125%	1	1,010	0.001%	750	0.000%	0.7
100%	6	0	0.000%	0	0.000%	-
100%	5	0	0.000%	0	0.000%	-
100%	4	0	0.000%	0	0.000%	-
100%	3	0	0.000%	0	0.000%	-
100%	2	0	0.000%	0	0.000%	-
100%	1	0	0.000%	0	0.000%	-
75%	6	-127,020	-0.021%	-256,278	-0.036%	2.0
75%	5	-73,445	-0.015%	-150,736	-0.024%	2.1
75%	4	-37,670	-0.009%	-80,217	-0.014%	2.1
75%	3	-15,900	-0.005%	-33,246	-0.006%	2.1
75%	2	-4,676	-0.002%	-8,449	-0.002%	1.8
75%	1	-570	-0.001%	-423	0.000%	0.7
50%	6	-142,803	-0.024%	-283,907	-0.040%	2.0
50%	5	-84,838	-0.017%	-171,602	-0.028%	2.0
50%	4	-44,722	-0.011%	-93,992	-0.017%	2.1
50%	3	-19,398	-0.006%	-40,176	-0.008%	2.1
50%	2	-5,875	-0.003%	-10,554	-0.002%	1.8
50%	1	-737	-0.001%	-547	0.000%	0.7
25%	6	-90,214	-0.015%	-177,008	-0.025%	2.0
25%	5	-55,057	-0.011%	-109,924	-0.018%	2.0
25%	4	-29,812	-0.007%	-61,959	-0.011%	2.1
25%	3	-13,295	-0.004%	-27,310	-0.005%	2.1
25%	2	-4,145	-0.002%	-7,416	-0.001%	1.8
25%	1	-537	-0.001%	-397	0.000%	0.7

Table 7.3: Back-Testing results of the convexity premiums estimated with volatility vectors from **2013.2014** compared to realized convexity premiums during **2015.2020**. (100mm BRL Notional Swap).

Another interesting point to notice in Tables 7.1, 7.2 and 7.3 is the fact that the convexity premium associated with swaps indexed to a percentage of CDI lower than 100% are negative, but differently than the positive premiums associated with larger than 100% percentages of CDI swaps, they are not exponentially decreasing for lower percentages. The negative premiums decreases when comparing swaps indexed to 75% of the CDI to 50% of the CDI, but when comparing swaps indexed to 50% of the CDI to swaps indexed to 25% of the CDI, the premiums are still negative but less negative than they were when analyzing swaps indexed to 50% of the CDI. This fact was already noticed in the work of Gomes (2015), and is corroborated in this work with the simulations realized using the HJM multi factorial model. The understanding of such effect is that the convexity of the swap indexed to less than 50% of the CDI begins to have a smaller effect offsetting the convexity of the future of DI, which than begins to be less negative. This effect can be noticed Figures 7.1, 7.2 and 7.3 shown below, pointing the matrix of results obtained with the simulations for each combination of key characteristics, convexity premiums, maturities and percentages of the CDI. It is highlighted in the figures on the left side the zero-level of the premiums (with a light-yellow plain surface and pink dots), so that it is visualized where the estimated premiums crosses the zero level, and then begin to approximate again, when the percentages of the CDI goes below 50%. This fact can be observed when comparing the distance between the black dots (related to the premiums obtained by the simulated paths) to the pink dots (related to the level of zero on the three dimensional plot).

7.2 The matrices and the polynomial equations results

In addition to analyze the matrices of convexity premiums for each combination of percentage of CDI and tenor, it was considered useful to visualize in a three dimensional plot the estimated premiums for each combination of maturity and percentage of CDI, shown in the left side of Figures 7.1, 7.2 and 7.3. Besides the fact that what was obtained with the simulations were in fact only the black dots of each surface; it was considered more visual to present a (light blue) surface that fits the black dots. However, to determine a polynomial equation that could in a way approximately represent the surface that would connect the premiums obtained by the simulations, it was implemented, with the Python library called scikit-learn, a routine that would find the coefficients of a polynomial equation of a surface that attempts to fit the estimated points. The (red) surfaces represented by such Polynomial Equations are then shown in the right side of Figures 7.1, 7.2 and 7.3 to visually demonstrate the approximation of the polynomial equations to the surfaces that would hypothetically perfectly fit the dots obtained with the simulations.

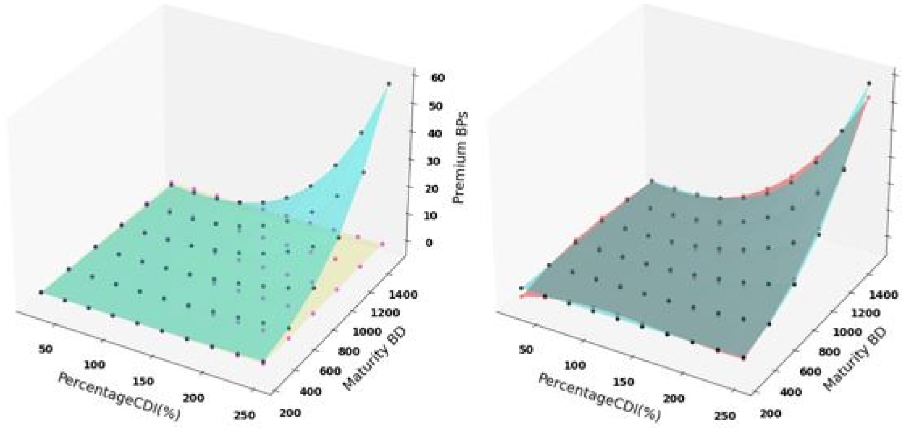


Figure 7.1: Convexity premiums obtained using volatility vectors from **2011.2012**.

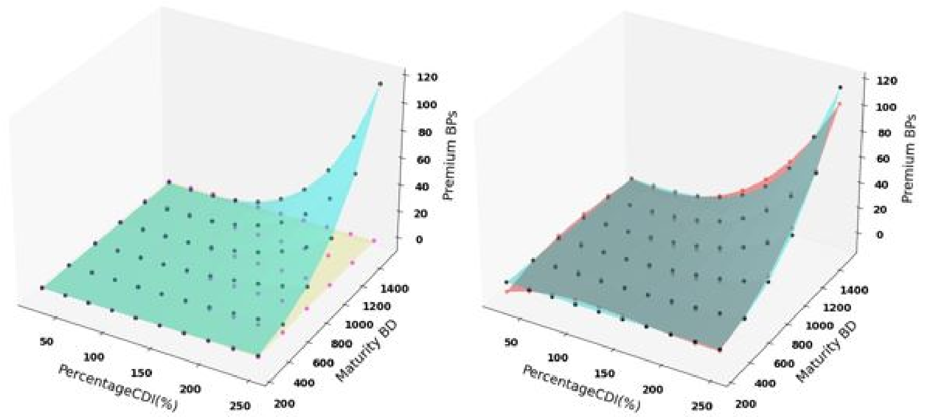


Figure 7.2: Convexity premiums obtained using volatility vectors from **2012.2013**.

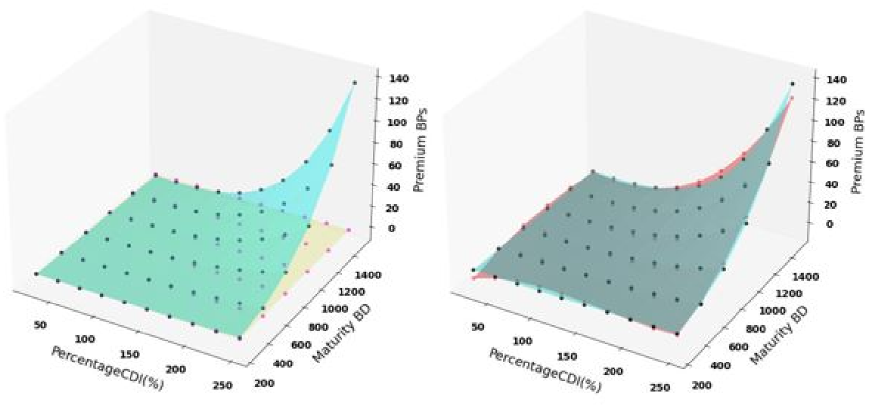


Figure 7.3: Convexity premiums obtained using volatility vectors from **2013.2014**.

Although the coefficients of each polynomial equation do not carry any significance by themselves, it is considered useful to display the coefficients for each period analyzed, for further research to have it available to comparison in any case of extending the developments made in this work. Additionally, the polynomial equations composed by them are the tools that were aimed to be obtained in this work, as quick and effective estimators for convexity premiums for different maturities and percentages of the CDI.

Just to contextualize, the polynomial equations will be of the format of Equation (6.1), shown again below:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_1 x_2 + \beta_5 x_2^2 + \beta_6 x_1^3 + +\beta_7 x_1^2 x_2 + \beta_8 x_1 x_2^2 + \beta_9 x_2^3$$

The coefficients of the polynomial equations for estimating the convexity premiums for derivatives considering the volatilities extracted from **2011-2012**, **2012-2013** and **2013-2014** periods, and maturities up to 6 years, are presented in Table 7.4 below.

β_n	**2011.2012 Coefficients**	**2012.2013 Coefficients**	**2013.2014 Coefficients**
β_0	-14.4573996900000000	-32.9925024000000000	-37.3441183500000000
β_1	$+0.28797316400000000$	$+0.67255417500000000$	$+0.76980004900000000$
β_2	$+0.03565032140000000$	$+0.07767400750000000$	$+0.08765766500000000$
β_3	-0.00147199706000000	-0.00358536655000000	-0.00408944749000000
β_4	-0.00049523951000000	-0.00105779687000000	-0.00123284393000000
β_5	-0.00002297226970000	-0.00004929189760000	-0.00005430486400000
β_6	$+0.00000210567400000$	$+0.00000552527484000$	$+0.00000621947597000$
β_7	$+0.00000143985060000$	$+0.00000300148695000$	$+0.00000353764117000$
β_8	$+0.00000015854608900$	$+0.00000032697262500$	$+0.00000038298918900$
β_9	$+0.00000000396241828$	$+0.00000000879875905$	$+0.00000000905610986$

Table 7.4: Polynomial Equation Coefficients: volatility vectors extracted from **2011.2012**, **2012.2013** and **2013.2014**, respectively.

In addition to the realized experiments above, it is also presented here the estimate for the convexity premiums for the same combinations of key characteristics, but with the volatility vectors extracted from more recent data obtained from 2019 to, including, 2020. In this case, there is not possible to back test, since the next six years are still unknown, but the premiums obtained with the multi factorial HJM model using this volatility vectors are considered important to be evidenced, so that further research also have comparison, if one want to, in addition to provide useful insights for the analysis made in this present work. Follows below Figure 7.4 of the three first Principal Components obtained from the period of **2019-2020**, in which is possible to notice an abrupt change in the second (blue line) and third (yellow line) components when comparing to the components extracted from the periods analyzed in the experiment. Additionally, it is presented Figure 7.5 with the smoothed volatility vectors extracted from them, evidencing the increase in the volatility when comparing to 6.5, besides the Table 7.6 with the convexity premiums estimated with the same process of 10,000 simulations with this inputs,

although not being possible to back-test them yet.

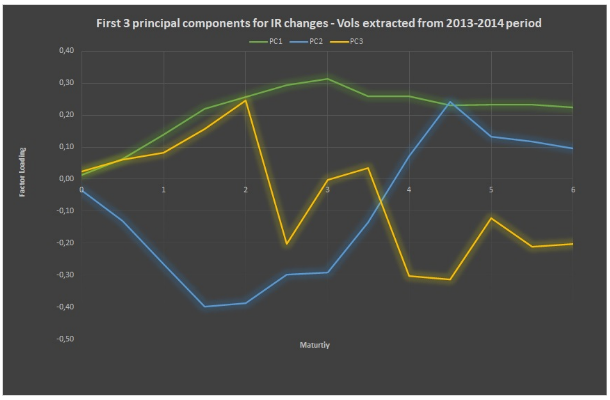
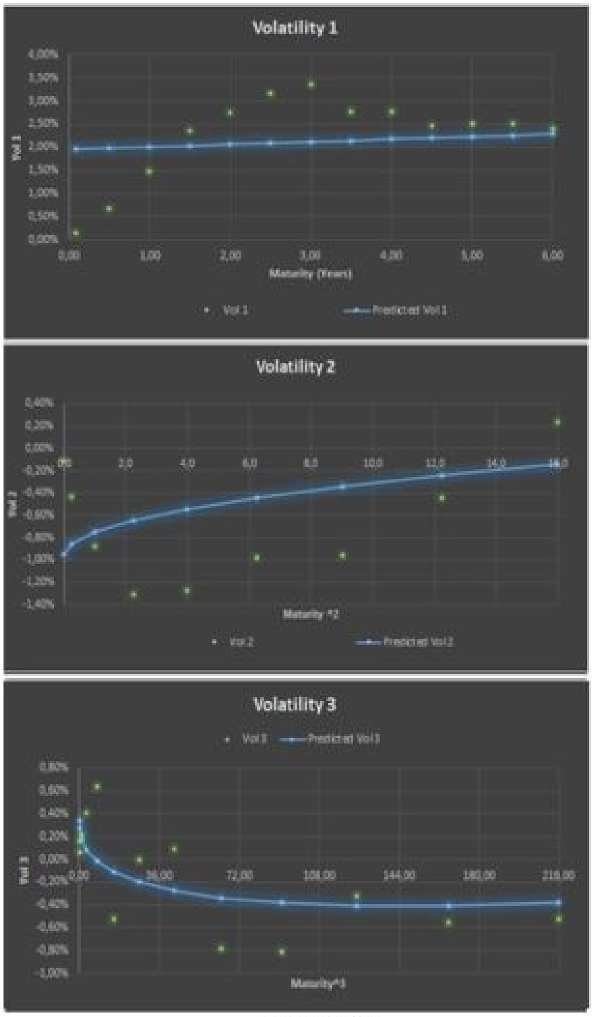


Figure 7.4: Principal Components obtained from **2019.2020**.



2019-2020

Figure 7.5: Smoothed volatility vectors obtained from **2019.2020**.

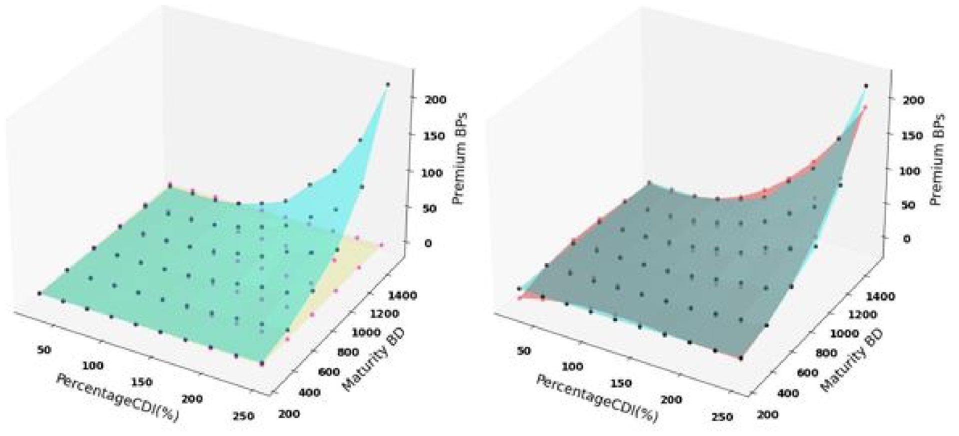


Figure 7.6: Convexity premiums obtained using volatility vectors from **2019.2020**

β_n	**2019.2020 Coefficients**
β_0	-63.653333360000000
β_1	$+1.0759822800000000$
β_2	$+0.1780896740000000$
β_3	-0.0051091841100000
β_4	-0.0019266749100000
β_5	-0.0001388301500000
β_6	$+0.0000068997668300$
β_7	$+0.0000051288394100$
β_8	$+0.0000006591942540$
β_9	$+0.0000000325170351$

Table 7.5: Polynomial Equation Coefficients: volatility vectors extracted from **2019.2020**.

Estimated Convexity Premiums							
%CDI	**Tenors (Years)**	**Estimated Value (BRL)**	**Estimated Ann. Spread**	**%CDI**	**Tenors (Years)**	**Estimated Value (BRL)**	**Estimated Ann. Spread**
250%	6	13,999,359	2,208%	125%	6	599,633	0.100%
250%	5	5,708,951	1.117%	125%	5	330,577	0.066%
250%	4	2,283,652	0.566%	125%	4	158,130	0.040%
250%	3	980,460	0.326%	125%	3	68,220	0.023%
250%	2	267,599	0.134%	125%	2	21,277	0.011%
250%	1	26,195	0.026%	125%	1	2,723	0.003%
225%	6	8,514,715	1.371%	100%	6	0	0.000%
225%	5	3,592,480	0.708%	100%	5	0	0.000%
225%	4	1,749,193	0.434%	100%	4	0	0.000%
225%	3	612,941	0.204%	100%	3	0	0.000%
225%	2	199,129	0.100%	100%	2	0	0.000%
225%	1	24,720	0.025%	100%	1	0	0.000%
200%	6	5,202,461	0.849%	75%	6	-285,252	-0.048%
200%	5	2,565,122	0.508%	75%	5	-163,284	-0.033%
200%	4	1,360,657	0.338%	75%	4	-83,634	-0.021%
200%	3	502,874	0.167%	75%	3	-33,517	-0.011%
200%	2	136,210	0.068%	75%	2	-10,902	-0.005%
200%	1	17,010	0.017%	75%	1	-1,599	-0.002%
175%	6	3,408,418	0.560%	50%	6	-316,472	-0.053%
175%	5	1,477,286	0.294%	50%	5	-196,593	-0.039%
175%	4	663,282	0.165%	50%	4	-104,839	-0.026%
175%	3	323,693	0.108%	50%	3	-43,905	-0.015%
175%	2	84,862	0.042%	50%	2	-15,469	-0.008%
175%	1	11,967	0.012%	50%	1	-1,477	-0.001%
150%	6	1,390,251	0.230%	25%	6	-226,005	-0.038%
150%	5	859,709	0.171%	25%	5	-126,424	-0.025%
150%	4	421,867	0.105%	25%	4	-79,827	-0.020%
150%	3	154,082	0.051%	25%	3	-40,531	-0.014%
150%	2	54,104	0.027%	25%	2	-9,613	-0.005%
150%	1	6,421	0.006%	25%	1	-1,215	-0.001%

Table 7.6: Convexity premiums estimated with volatility vectors from **2019.2020** to be compared to realized convexity premiums during **2021.2026**. (100mm BRL Notional Swap).

It is easy to notice comparing Tables 7.1, 7.2 and 7.3 to Table 7.6 an abrupt increase in the premiums estimated, what intuitively would be expected, considering the realized volatility observed in the period of **2019-2020**. But on the other hand, the level of the yield curve decreased significantly, what could have offset part of the increase in volatility, estimating somewhat larger (but not so larger as the difference in volatility between the periods) convexity premiums. This is summarized in the snippet table 7.7 below, that shows the top left part of each table previously presented, regarding the estimated convexity premiums associated to a swap indexed to 250% of CDI.

Comparison of estimated convexity premiums					
		2011-2012		**2012-2013**	
% CDI	**Tenors (Years)**	**Estimated Value (BRL)**	**Estimated Ann. Spread**	**Estimated Value (BRL)**	**Estimated Ann. Spread**
250%	6	3,533,818	0.580%	7,148,784	1.157%
250%	5	1,728,533	0.343%	3,346,087	0.66%
250%	4	758,580	0.189%	1,399,768	0.348%
250%	3	278,183	0.093%	484,214	0.161%
250%	2	72,850	0.036%	118,487	0.059%
250%	1	8,296	0.008%	12,413	0.012%
		2013-2014		**2019-2020**	
% CDI	**Tenors (Years)**	**Estimated Value (BRL)**	**Estimated Ann. Spread**	**Estimated Value (BRL)**	**Estimated Ann. Spread**
250%	6	8,512,361	1.370%	13,999,359	2.208%
250%	5	4,062,887	0.800%	5,708,951	1.117%
250%	4	1,718,035	0.427%	2,283,652	0.566%
250%	3	593,487	0.197%	980,460	0.326%
250%	2	142,355	0.071%	267,599	0.134%
250%	1	14,098	0.014%	26,195	0.026%

Table 7.7: Comparison between convexity premiums estimated with the different volatility vectors, for a selected percentage of CDI. (100mm BRL Notional Swap).

Additionally, it is considered pertinent to show in Figure 7.6 also what would be the surface of premiums per maturities and percentages of the CDI in the same three dimensional plot that the back-tested were shown, to let the reader compare visually the impact of the increase of volatility (observed in Figure 6.3) in the convexity premium. And to let registered the coefficients obtained to the Polynomial Equation to fit the simulated premiums, it is also mentioned below the coefficients of the polynomial equation for the simulations made using the volatility vectors extracted from the **2019-2020** period.

What can be seen comparing the surfaces of premiums between the selected periods in which the volatility vectors were extracted is the fact that the volatility vectors extracted from the period of **2011-2012** (Fig. 7.1) estimated lower convexity premiums, compared to both next periods and to the back-tested hedging strategy implemented in the 6 years horizon ahead. In both the periods of **2012-2013** (Fig. 7.2) and **2013-2014** (Fig. 7.3), the estimated premiums remained at a closer level from one another, and underestimated not much less than half the convexity premium observed in the back-tests, which is a perceptible smaller error if compared to the almost 3 times underestimation of the convexity premiums observed in the extraction of volatility vectors from the **2011-2012** sample. A fact that intuitively could explain this diver-

gence is the fact that the level of the yield curve rose from 2013 to 2016 (the majority of the period compared to the **2011-2012** volatility vectors simulations), but fell in level from 2016 ahead. Gomes (2015) had already noticed in his work the effect of the level of the yield curve in the convexity premiums and in this experiments the results corroborate this perception, especially if we take into consideration the simulated convexity premiums using the volatility vectors extracted from the **2019-2020** period, which had an incomparable increase in volatility, but at the lower levels of the yield curve yet observed. If we were to consider only the increase in the volatility regime affecting the convexity premiums, it would be expected significantly higher convexity premiums for the simulations using these vectors of volatility, however, considering the low level of the yield curve, the increase in the volatility of the curve was somehow offset by the decrease in the yield curve level. This fact, as mentioned before in this work, is one of the main reasons why the convexity premiums are relevant again, with the scenario ahead tilting towards an upward repricing of rates globally, in face of the fiscal stimulus observed during the COVID-19 throughout the world. Considering the increase of duration in corporate and governments debts, this effect may overshoot if the pace of this repricing is faster than expected by the markets, which could combine the increase in level of the yield curve with an increase in the volatility of rates, in an environment of risks more concentrated in longer tenors than previously seen.

The estimates for the convexity premiums obtained with the HJM model are considered useful in practice, even though underestimated the actual convexity premiums that would have happened in the realized path of the yield curve, so that when pricing such derivative, a bullet swap indexed to a percentage of CDI different from 100%, the practitioner takes into consideration at least the estimated premiums obtained with the HJM model in the rate of the swap. Depending on the side, if receiving or paying the percentage of CDI, one may need to consider a margin over the estimated premium, so that the results of the dynamic hedging strategy of the risk do not remain below zero, at the time the derivative matures. Indeed, it would be considered a better result if the model were able to provide a more accurate estimation of the convexity premium, but considering the relatively long period of time and uncertainty related to it, an estimation in the order of those obtained by the model in these experiments can be considered useful when pricing such derivative, taking into consideration the possibility that the volatility vectors used in the model may be underestimating the volatility ahead.

8. CONCLUSION

This final chapter has the objective of resuming the most important points made, discussing about the results observed in comparison to related literature and suggesting possible further extensions upon the work realized.

In this work it was presented an issue already seen in the literature, the existence of a convexity premium in a portfolio composed by a bullet swap indexed to a percentage of CDI different from 100% and an amount of DI future contracts that hedges its interest rate first order risk. It was brought to the reader the contributions made by previous authors on the theme, to put in context where the literature and the market practitioners have their understanding about the topic. The proposed contribution of this work was to perform an analysis under a HJM framework, to corroborate results presented in previous literature under other frameworks, such as the ones presented in Yam (2010) work (which used both Cox, Ingersoll, and Ross (1985) and Vasicek (1977) general equilibrium frameworks) and in Gomes (2015) (which developed his analysis under the Black, Derman, and Toy (1990) non-arbitrage model). In addition to the analysis under the robust HJM model, another contribution of this work was to present the idea to find potential polynomial equations that could serve as faster estimators for this convexity premiums, once obtained enough data through Monte Carlo simulations.

The main characteristics of the Brazilian interest rate derivatives market were presented briefly and the portfolio that would be later analyzed was explained in detail, so that became clear to the reader why the convexity premium must be considered when dealing with such derivatives indexed to a percentage of CDI different from 100%.

It was considered appropriate to develop in this work the rationale behind the HJM model to make the reader more familiar to its logic and have a better understanding of the contribution of such framework to the analysis developed in this work on convexity premiums embedded in derivatives. This development was broadly based upon the works of Heath, Jarrow, and Morton (1992), Brace and Musiela (1994) and Glasserman (2003). With the framework explained, it was then presented to the reader how it would be used to price the CDI-indexed derivatives and in this part of the work it was detailed the specific adaptations and techniques that would take place to implement the simulations. The work of Nojima (2013) was broadly referenced throughout this chapter, serving specially as reference for the most part of the practical details of the implementation of the Principal Component Analysis (first applied in finance by Litterman and Scheinkman (1991)) technique to extract the volatility vectors from the historical data.

Then the conditions under which the analysis would be performed were presented: the tenors and percentages of the CDI that would be considered in the experiments (annual tenors from 1 to 6 years; from 25% to 250% of the CDI, with steps of 25% in between each) and the periods in which the volatility vectors would be extracted from using the Principal Components Analysis technique (the periods of **2011-2012**, **2012-2013** and **2013-2014**, for the back-tested experiments, and **2019-2020** only for comparison purposes). The limitation over the maximum tenor (6 years) analyzed in the back-tests is an issue that for further extensions is suggested that be dealt with, since the impact of the tenor over the convexity premium is significant and longer tenors would give a more comprehensive analysis than the analysis made with tenors to up to 6 years only. But, as it was previously explained, this limitation emerges from the fact that in this work what was wanted was to compare different vectors of volatility for the same tenors, which was only possible at, maximum, the 6 years tenor, considering the historical data obtained from 2011 onwards.

The selected process of validation of the results obtained by the estimations made using the HJM model and Monte Carlo simulations was the back-test of the same daily dynamic *delta* hedging strategy, in the subsequent periods to the intervals in which the volatility vectors were extracted from, considering the actual realized path of the yield curve in each of these time intervals.

The results obtained from the experiments showed consistency and although underestimating the convexity premiums recurrently, the HJM model arrived at estimated convexity premiums, depending on the tenor analyzed, at roughly 50% of the realized convexity premium. The underestimation of the premiums was not the ideal expected result, but it brought to attention a few interesting points and helped form a more comprehensive understanding about the topic. The impact of the volatility input on the model is of most significance, but also the level of the yield curve makes an impact on the convexity premium of such portfolio. These understandings were already presented in previous literature (e.g. Gomes (2015)) but the results of the experiments made in this work, under a different framework, corroborates them.

To compare the results obtained with the HJM framework to results presented in previous literature, it was simulated using the **2013-2014** volatility vectors, the convexity premiums for a derivative indexed to 200% of the CDI, for three different tenors: 3 years, 5 years and 10 years. In Yam (2010), considering the CIR model with a 15% annual volatility and a convergence to the mean of 50% annually, were obtained the effective convexity premiums[1] of 0.41%, 1.12% and 3.43%, for the tenors of 3, 5 and 10 years, respectively. Gomes (2015) for its turn, found under the BDT model, with the same 15% annual volatility input, effective premiums of 0.39%, 1.84% and 17.86%. It is mentioned, although, in his work that with a 18% annual volatility input (which resembles more approximately the 2014 period, as shown in this work in Figure

[1] In the tables and figures presented in this work, the convexity premiums were shown in annualized terms. To convert it to effective premiums, it is only a matter of elevating it, in factor, to the number of years until the expiry of the derivative. For instance, a 0.322% annualized premium for a 5 year period would result in a 1.62% effective premium, such that $(1 + 0.322\%)^5 - 1 \approx 1.62\%$.

6.3), the effective premium for a 10 year derivative indexed to 200% of the CDI was estimated to be 29.77%. In this work, it was estimated using the HJM framework with the volatility vectors extracted from **2013-2014** period, effective premiums of 0.26%, 1.62% and 27.38%, for tenors of 3, 5 and 10 years respectively, considering the percentage of 200% of the CDI in the swap composing the portfolio. Comparing the tenors that could be back-tested, 3 and 5 years, the realized effective premiums were, respectively, 0.58% e 3.56%. The 10 year tenor is not yet possible to back-test, from 2015 onwards. It is worth noticing the proximity of the results obtained under both the BDT and the HJM frameworks, when the first used the annual volatility input of 18%.

Additionally, it was simulated in this work and it is considered worth mentioning for further works to compare, the effective convexity premiums for the same analyzed portfolios (composed by a bullet swap indexed to 200% of the CDI and its hedge, for the tenors of 3, 5 and 10 years), using the volatility vectors extracted from data of the period **2019-2020**, being them: 0.50%, 2.57% e 35.39%.

Beyond obtaining results to compare to previous literature, it was a goal of this work to introduce the idea of obtaining a polynomial equation to approximately estimate convexity premiums, for tenors and percentages not between the selected combinations simulated via Monte Carlo. The figures shown in Chapter 7, comparing the surfaces estimated by such polynomial equations to the points obtained in the Monte Carlo simulations, makes clear that this is a valid approximation for convexity premiums for different combinations than the specific combinations analyzed. However, it must be evidenced here that the main source of error from the realized premiums would not derive from the difference between the surfaces estimated by the polynomial equations to the points simulated via Monte Carlo, but would be inherently related to the volatility inputs to the model as well as the future potential level of the yield curve ahead. The impact of an increase in volatility and in the level of the yield curve, as it was perceptible when comparing the estimated convexity premiums to the realized ones, is of major significance. As a suggestion for further works on this topic, we propose that somehow be incorporated in the analysis an estimator for the future level of the yield curve, as to have a more broad set of estimators in different scenarios of the yield curve level, so that the practitioner takes into consideration the scenario they attribute higher probability and ponder the impact of such scenario in the estimation of the convexity premiums. Another suggestion for further developments is to consider the volatility in some way into the polynomial equation, due to the fact that this input have great relevance to the convexity premium, in addition to the percentage of CDI of the swap and its tenor. One other suggestion is to obtain the volatility parameters to input in the HJM model from the actual volatility structure, embedded in IDI options and swaptions of DI, as did Sena (2019) in her work, looking forward instead of backwards. Although today the market does not trade these instruments with enough extension to analyze a portfolio composed by a swap maturing in, at least, 5 years from its start date, once the market develops and begins to trade these instruments to longer tenors, this extraction would be achievable and, more, one

could further analyze the possibility of an arbitrage between these instruments and this portfolio with positive convexity, if the convexity premium is not well priced by the counterpart.

Also, what is considered a possible extension upon the work realized here is to develop an analysis considering *intraday* re-balances of the portfolio, which in practice can (and probably will) be done by the manager of the risks embedded in such portfolio and could have an impact on the convexity premiums that would be estimated considering *intraday* changes into the model.

8. Bibliography

Andrade, R. G. O. (2014). Relevância das diferenças entre contratos futuros e a termo: O caso do trio. *FGV, São Paulo*, 55 f.

Barbedo, C., J. V. M. Vicente, and O. Lion (2009). Pricing asian interest rate options with a three-factor hjm model. *Banco Central do Brasil Working Paper Series 188*, 01–33.

Björk, T., A. Szepessy, R. Tempones, and G. E. Zouraris (2012). Monte carlo euler approximations of hjm term structure financial models. *The Swedish National Network in Applied Mathematics*.

Black, F., E. Derman, and D. Toy (1990). A one factor model of interest rates and its application to treasury bond options. *Financial Analysts Journal 46*, 33–39.

Brace, A. and M. Musiela (1994). A multifactor gauss markov implementation of heath, jarrow, and morton. *Mathematical Finance 4*, 259–283.

Cox, J., J. Ingersoll, and S. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*, 385–407.

Eid, J. and G. Ohanian (2005). Operações indexadas ao percentual do cdi: Precificação e hedge dinâmico usando o contrato di futuro da bm&f. *Anais do Quinto Encontro Brasileiro de Finanças, Rio de Janeiro V*.

Glasserman, P. (2003). Monte carlo methods in financial engineering. *New York: Springer*.

Gomes, R. P. (2015). Apreçamento da convexidade de derivativos indexados ao percentual do cdi no modelo de black-derman-toy. *Insper, São Paulo*, 104 f.

Heath, D., R. Jarrow, and A. Morton (1992). Bond pricing and term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica 60*, 70–105.

Ho, T. and S. Lee (1986). Term structure movements and pricing interest rate contingent claims. *Journal of Finance 41*, 1011–1029.

Hull, J. and A. White (1990). Pricing interest rate derivative securities. *Review of Financial Studies 3*, 537–592.

Hübner, C. (2003). Risco e precificação de operações com taxa percentual do cdi. *FGV, Rio de Janeiro*, 35 f.

Litterman, R. and J. Scheinkman (1991). Common factors affecting bond returns. *Journal of Fixed Income*.

Nojima, N. K. (2013). Precificação de derivativos de taxas de juros utilizando o modelo hjm multifatorial com estrutura de volatilidade não parametrica. *FGV, São Paulo*, 72 f.

Pedregosa, F., G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay (2011). Scikit-learn: Machine learning in python. *Journal of Machine Learning Research 12*, 2825–2830.

Renò, R. and A. Uboldi (2002). Pca based calibration of an hjm model. *Quaderno IAC Q28-002*.

Sena, C. A. P. F. (2019). Aplicação do modelo hjm ao mercado brasileiro utilizando estruturas de volatilidades implícitas. *FGV, São Paulo*, 51 f.

Shreve, K. P. (2004). Calculus for finance ii: Continuous-time models. *New York: Springer*.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics 5*, 177–188.

Wilmott, P. (2006). Paul wilmott on quantitative finance. *ohn Wiley and Sons 2*, 507–625.

Yam, E. P. L. (2010). Apreçamento da convexidade de ativos indexados ao percentual do cdi nos modelos de vasicek e cox-ingersoll-ross. *Insper, São Paulo*, 82 f.