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A note on systems with ordinary and impulsive controls

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We investigate an everywhere defined notion of solution for control systems whose dynamics depend non-linearly on the control u and state x, and are affine in the time derivative $\dot{u}$. For this reason, the input u, which is allowed to be Lebesgue integrable, is called *impulsive*, while a second, bounded measurable control v is denominated *ordinary*. The proposed notion of solution is derived from a topological (non-metric) characterization of a former concept of solution which was given in the case when the drift is v-independent. Existence, uniqueness and representation of the solution are studied, and a close analysis of effects of (possibly infinitely many) discontinuities on a null set is performed as well.

Keywords: impulse controls; pointwise defined measurable solutions; input–output mapping; commutative control systems.

1. Introduction

Control systems of the form

$$\dot{x} = f(t,x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [a,b], \tag{E}$$

$$x(a) = \bar{x}, \tag{IC}$$

can be given a classical interpretation as soon as the control u is an absolutely continuous function and the control v is Lebesgue integrable. This paper is devoted to the investigation of a notion of solution for the Cauchy problem (E) and (IC), when one assumes the following hypotheses:

(i) the vector fields g_α *commute*, namely $[g_\alpha, g_\beta] \equiv 0$, for all $\alpha, \beta = 1, \ldots, m$, where $[\cdot,\cdot]$ denotes the Lie bracket;

(ii) the inputs u belong to the space $\mathcal{L}^1([a,b];U)$ of everywhere defined Lebesgue integrable functions.

Loosely speaking, the denomination 'impulsive' comes from the fact that, due to the affine dependence of the dynamics on the control's derivative $\dot{u}$, a discontinuity in u may cause a discontinuity in the

corresponding trajectory x. On the other hand, the bounded, measurable input v can be regarded as an 'ordinary' control.

Let us observe that the case where u is taken in the class of bounded variation functions (and the commutativity in (i) is not necessarily verified) has received most of the attention (see, e.g. Rishel, 1965; Bressan & Rampazzo, 1988; Dal Maso & Rampazzo, 1991; Silva & Vinter, 1996 and references therein). In these articles, the authors studied the technique that is nowadays known as *graph completion.* An extension of this concept, also dealing with trajectories with bounded variation, was investigated in Karamzin (2006) and Arutyunov *et al.* (2011) for systems of the form (E), while a more general framework allowing the dependence of the vector fields g_α on the ordinary control v, was analysed in Arutyunov *et al.* (2012).

Even in the case where u can have unbounded variation, a notion of solution valid for systems where f is independent of the ordinary control v, and both (i) and (ii) are met, has already been investigated (see, e.g. Bressan & Rampazzo, 1991; Sarychev, 1991; Dykhta, 1994). This solution can be defined pointwise and verifies nice properties of uniqueness and continuity on the data. The main goal of the present note consists in investigating a suitable generalization of this concept of solution to the case when f is actually v-dependent. Incidentally, let us observe that a system like

$$\dot{x} = f(t,x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x,u)\dot{u}_\alpha \tag{1.1}$$

reduces to (E) as soon as one adds m extra state variables $z_1,\ldots,z_m$ and the additional equations

$$\dot{z}_\alpha = \dot{u}_\alpha, \quad \text{for } \alpha = 1,\ldots,m.$$

In this case, the commutative hypothesis (i) reads: (i′) $\left[\sum_{j=1}^n g_\alpha^j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial z_\alpha}, \sum_{j=1}^n g_\beta^j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial z_\beta}\right] \equiv 0$, for all $\alpha,\beta = 1,\ldots,0$, where $[\cdot,\cdot]$ denotes the Lie bracket for vector fields on $\mathbb{R}^{n+m}$.

In fact, several applications justify the introduction in the dynamical equations of the ordinary, bounded, control v besides the impulsive control $(u,\dot{u})$. For instance, in Lagrangian mechanics, if the control u denotes the shape of a concatenation C of rigid bodies and the input v is, say, an external force or torque acting on C, then the whole motion of C in space is determined by equations of the form (1.1). More generally, in a $N+m$-dimensional Lagrangian system (where $N = n/2$) the input u might represent a portion of a local system of coordinates (q,u), while x would be identified with (q,p), p being the *momenta* corresponding to the free coordinates q (see Bressan, 1989; Rampazzo, 1999). Let us point out that the commutativity assumption is actually verified is some situations of practical interest (Bhat & Tiwari, 2009).

The main results of the paper, including existence, uniqueness, continuous dependence of solutions on data, state-response to measure-zero changes of u, are stated in Section 2. The latter is concluded by Theorem 2.4, where a representation of solutions is given in terms of a diffeomorphism constructed through an application of the Multiple Flow-Box Theorem to the vector fields $\{g_1,\ldots,g_m\}$. All proofs can be found in Section 3.

Notation and assumptions. Let h be a locally Lipschitz vector field on $\mathbb{R}^n$, and let $\bar{x} \in \mathbb{R}^n$. Whenever the solution to

$$\dot{x}(t) = h(x(t)), \quad h(0) = \bar{x}$$

is defined on an interval I containing 0, we use $\exp(th)(\bar{x})$ to denote the value of this solution at time t.

Let I be a closed interval and let E be a subset of an Euclidean space $\mathbb{R}^d$. We use $\mathcal{L}^1(I;E)$ to denote the set of pointwise defined Lebesgue integrable functions from I to $\mathbb{R}^d$ with values in E, while

$L^1(I;E)$ will denote the corresponding family of equivalence classes (with respect to the Lebesgue measure). We write $AC(I;E)$ for the set of absolutely continuous maps from I to E. For an open subset $\Omega \subseteq \mathbb{R}^n$, $\mathcal{C}^k(\Omega;\mathbb{R}^d)$ will denote the space of k-times continuously differentiable $\mathbb{R}^d$-valued functions defined on Ω.

Throughout the paper we shall assume the following hypotheses on the control system (E) and (IC).
Hypothesis H:

(i) U is a compact subset of $\mathbb{R}^m$ such that, for every bounded interval $I \subset \mathbb{R}$, for each $\tau \in I$, and for every function $u \in \mathcal{L}^1(I;U)$, there exists a sequence $(u_k^\tau) \subset AC(I;U)$ verifying

$$|u_k^\tau(\tau) - u(\tau)| + \|u_k^\tau - u\|_1 \to 0,$$

when $k \to \infty$.
(Convex sets verify this hypothesis.)

(ii) The set $V \subset \mathbb{R}^l$ is compact.

(iii) The map $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \times V \to \mathbb{R}^m$ is such that,

- for each $(x,u,v) \in \mathbb{R}^n \times \mathbb{R}^m \times V$, the map $t \mapsto f(t,x,u,v)$ is measurable on $[a,b]$;
- for each $t \in [a,b]$, the function $(x,u,v) \to f(t,x,u,v)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m \times V$ and, moreover,
- the map

$$(x,u) \mapsto f(t,x,u,v)$$

is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$, uniformly in $(t,v) \in [a,b] \times V$.

(iv) For every $\alpha = 1,\dots,m$, $g_\alpha \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$.

(v) There exists $A > 0$ such that

$$|(f(t,x,u,v), g_1(x),\dots,g_m(x))| \leqslant A(1 + |(x,u)|),$$

for every $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ uniformly in $(t,v) \in [a,b] \times V$.

Note that hypotheses (ii)–(v) above imply that, for every initial value $\bar{x} \in \mathbb{R}^n$, and each pair $(u,v) \in AC([a,b];\mathbb{R}^m) \times L^1([a,b];V)$, the Cauchy problem (E) and (IC) has a unique (Carathéodory) solution, here denoted by $x[\bar{x},u,v]$.
Hypothesis CC:

(CC_1) the vector fields g_α are complete[1] and

(CC_2) $g_1,\dots g_m$ verify the *global commutativity hypothesis* on $\mathbb{R}^n$, namely for every Lipschitz continuous loop

$$u : [0,1] \to \mathbb{R}^m, \quad u(0) = u(1),$$

[1] We say that g_α is *complete* if the solution to the Cauchy problem $\dot{x} = g_\alpha(x),\ x(0) = \bar{x} \in \mathbb{R}^n$ is (uniquely) defined on $\mathbb{R}$.

and each $\bar{x}\in\mathbb{R}^n$ such that there exists a (unique) Carathéodory solution to the Cauchy problem

$$\dot{x}(t)=\sum_{\alpha=1}^{m} g_\alpha(x(t))\dot{u}_\alpha(t),\quad t\in[0,1],\qquad x(0)=\bar{x};$$

the solution x is a loop, that is, it verifies $x(0)=x(1)=\bar{x}$.

REMARK 1.1 Let us define the *Lie bracket* of g_α and g_β as

$$[g_\alpha,g_\beta]:=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g_{\beta,i}}{\partial x_j}g_{\alpha,j}-\frac{\partial g_{\alpha,i}}{\partial x_j}g_{\beta,j}\right)\frac{\partial}{\partial x_i}.$$

It is trivial to verify that for the domain $\mathbb{R}^n$ the *null bracket condition*

$$[g_\alpha,g_\beta]\equiv 0,\qquad \alpha,\beta=1,\dots,m, \tag{1.2}$$

is necessary and sufficient for $g_1,\dots,g_m$ to verify the global commutativity hypothesis. Actually, if instead of $\mathbb{R}^n$ one considered an open subset $\Omega\subset\mathbb{R}^n$ (or a differential manifold) as state space, the null bracket condition (1.2) would be no longer sufficient for global commutativity. As a trivial example, one can take the vector fields $g_1:=(1,0,-x_2/(x_1^2+x_2^2))^\top$, $g_2:=(0,1,x_1/(x_1^2+x_2^2))^\top$, which verify the null bracket condition (1.2) on $\Omega:=\mathbb{R}^2\backslash\{0\}$, but *do not* match the global commutativity hypothesis. Indeed, if $u(t):=(\cos(2\pi t),\sin(2\pi t))^\top$, for $t\in[0,1]$, and x is the corresponding solution to $\dot{x}=g_1(x)\dot{u}_1+g_2(x)\dot{u}_2$, $x(0)=(1,0,0)^\top$, one has $x(1)=(1,0,2\pi)^\top\neq x(0)$.

2. Limit solutions

In this section, we give the definition of limit solution and state the main results. The corresponding proofs have been placed in Section 3.

DEFINITION 2.1 (Limit Solution) Consider an initial data $\bar{x}\in\mathbb{R}^n$ and controls $(u,v)\in\mathcal{L}^1([a,b];U)\times L^1([a,b];V)$. We say that an $\mathcal{L}^1-$map $x:[a,b]\to\mathbb{R}^n$ is a *limit solution* of the Cauchy problem (E) and (IC) if, for every $\tau\in[a,b]$, there exists a sequence $(u_k^\tau)\subset AC([a,b];U)$ such that

$$|(x_k^\tau,u_k^\tau)(\tau)-(x,u)(\tau)|+\|(x_k^\tau,u_k^\tau)-(x,u)\|_1\to 0, \tag{2.1}$$

where $x_k^\tau:=x[\bar{x},u_k^\tau,v]$.

REMARK 2.1 Let us point out that x is a limit solution associated to u if, for every $\tau\in[a,b]$, (x,u) can be approximated, in the sense of (2.1), by sequences of absolutely continuous paths (x_k^τ,u_k^τ) that verify (E) in the classical, Carathéodory sense. We also observe that no direct distributional meaning can be given to the derivative of u or to (E) (see some general considerations on the subject in Hájec, 1985), essentially because of two facts: on one hand the g_α are not constant; on the other hand we look for everywhere defined solutions.

THEOREM 2.1 (Existence and uniqueness) For every $\bar{x}\in\mathbb{R}^n$, and every control pair $(u,v)\in\mathcal{L}^1([a,b];U)\times L^1([a,b];V)$, there exists a unique limit solution of the Cauchy problem (E) and (IC) defined on $[a,b]$.

Given $\bar{x} \in \mathbb{R}^n$, and a control pair $(u, v) \in \mathcal{L}^1([a, b]; U) \times L^1([a, b]; V)$, let $x[\bar{x}, u, v]$ denote the (unique) corresponding limit solution of (E) and (IC).

REMARK 2.2 Note that, for every input $u \in \mathcal{L}^1([a, b]; U)$, the map $t \mapsto \bar{x} + u(t) - u(a)$ is a limit solution of the trivial Cauchy problem

$$\dot{x} = \dot{u}, \quad x(a) = \bar{x}, \tag{2.2}$$

and, thanks to the above uniqueness result, it is in fact the only solution. Since in general $\mathcal{L}^1$ functions cannot be pointwise approximated by absolutely continuous functions (see, e.g. to Oxtoby, 1980), the fact that the choice of the approximating control sequence depends on the time τ is crucial for guaranteeing existence of everywhere defined solutions, even for the trivial equation (2.2).

EXAMPLE 2.1 Let $R \subset \mathbb{R}^2$ be the subset defined by

$$R \doteq \{(x, \hat{y}(x)) \quad x \in [0, 1]\} \cup \{(x, e^{-1/2}) \quad x \in [2, 3]\},$$

where

$$\hat{y}(x) := \begin{cases} e^x, & \text{for } x \in \left[0, \frac{1}{2}\right[, \\ e^{1/2}e^{-2} & \text{for } x \in \bigcup_{k=1}^{\infty} \left[1 - \frac{1}{2k}, 1 - \frac{1}{2k+1}\right[, \quad k \in \mathbb{N}, \\ e^{1/2} & \text{for } x \in \bigcup_{k=1}^{\infty} \left[1 - \frac{1}{2k+1}, 1 - \frac{1}{2k+2}\right[, \quad k \in \mathbb{N}, \\ e^{-1/2} & \text{for } x = 1, \end{cases}$$

and let us consider the optimal control problem

$$\min\{w(2) + (y(1) - e^{-1/2})^2 + (x(2) - 3)^2\}, \tag{2.3}$$

on the interval $[0, 2]$ subject to the dynamics

$$\begin{cases} \dot{x} = 1 + \dot{u}_2, \\ \dot{y} = yv + y\dot{u}_1, \\ \dot{w} = \mathrm{d}((x, y), R), \\ (x, y, w)(0) = (0, 1, 0), \end{cases} \tag{2.4}$$

where $v \in \{0, 1\}$ and $(u_1, u_2) \in [-1, 1] \times [0, 1]$. Note that (2.4) meets the general hypotheses, for the vector fields

$$g_1 \doteq \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix}, \quad g_2 \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f \doteq \begin{pmatrix} 1 \\ yv \\ \mathrm{d}((x, y), R) \end{pmatrix},$$

are Lipschitz continuous, and, moreover, g_1, g_2 are smooth and verify $[g_1, g_2] \equiv 0$.

We claim that the limit solution (x, y, w) corresponding to the input

$$v(t) := \begin{cases} 1 & \text{for } t \in \left[0, \dfrac{1}{2}\right[, \\ 0 & \text{for } t \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

$$u_1(t) := \begin{cases} (-1)^{k+1} & \text{for } t \in \left[1 - \dfrac{1}{k}, 1 - \dfrac{1}{k+1}\right[, \quad k \in \mathbb{N}, \\ 0 & \text{for } t \in [1, 2], \end{cases}$$

$$u_2(t) := \begin{cases} 0 & \text{for } t \in [0, 1], \\ 1 & \text{for } t \in]1, 2], \end{cases}$$

is a minimum for problem (2.3). Indeed, on any subinterval $[t_1, t_2] \subset [1/2, 1]$ where u_1 is absolutely continuous, one has

$$y(t) = y(t_1) e^{u_1(t) - u_1(t_1)}. \tag{2.5}$$

On the other hand, one can easily check that

$$\begin{aligned} y(1 - 1/k+) &= y(1 - 1/k-) e^2 \quad \text{if } k \text{ is odd}, \\ y(1 - 1/k+) &= y(1 - 1/k-) e^{-2} \quad \text{if } k \text{ is even}, \\ y(1) &= e^{-1/2}, \end{aligned}$$

where $y(1 - 1/k-)$ and $y(1 - 1/k+)$ denote the left and right limits of y at $t = 1 - 1/k$, respectively. Moreover,

$$x = t \ \text{ for all } t \in [0, 1], \quad x = t + 1 \ \text{ for all } t \in]1, 2].$$

Hence $\mathrm{d}((x(t), y(t)), R) = 0$ for all $t \in [0, 2]$ and $x(2) = 3$, so the corresponding payoff is equal to zero. Therefore (x, y, w) is an optimal trajectory, since the payoff of every control-trajectory pair is non-negative.

Note that both (u_1, u_2) and (x, y, w) have infinitely many discontinuities and unbounded variation. Observe also that it is crucial that the input u and the solution are defined everywhere. In fact, the control $(v, \tilde{u}_1, u_2)$, with $\tilde{u}_1(t) = u_1(t)$ for all $t \neq 1$ and $\tilde{u}_1(1) = 1$, is not optimal, for the corresponding solution $(\tilde{x}, \tilde{y}, \tilde{w})$ is equal to (x, y, w) on $[0, 2] \backslash \{1\}$, while, in view of Theorem 2.3, $\tilde{y}(1) = e^{1/2}$.

THEOREM 2.2 (Continuous dependence) The following assertions hold true:

(i) for each $\bar{x} \in \mathbb{R}^n$ and $u \in \mathcal{L}^1([a, b]; U)$, the function $v \mapsto x[\bar{x}, u, v]$ is continuous from $L^1([a, b]; V)$ to $L^\infty([a, b]; \mathbb{R}^n)$;

(ii) for any $r > 0$, there exists a compact subset $K' \subset \mathbb{R}^n$, such that the trajectories $x[\bar{x}, u, v]$ have values in K', whenever we consider $|\bar{x}| \leqslant r$, $u \in \mathcal{L}^1([a, b]; U)$ and $v \in L^1([a, b]; V)$;

(iii) for each $r > 0$, there exists a constant $M > 0$ such that, for every $\tau \in [a,b]$, for all $|\bar{x}_1|, |\bar{x}_2| \leqslant r$, $u_1, u_2 \in \mathcal{L}^1([a,b];U)$ and for every $v \in L^1([a,b];V)$, one has

$$|x_1(\tau) - x_2(\tau)| + \|x_1 - x_2\|_1 \leqslant M[|\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1], \tag{2.6}$$

where $x_1 := x[\bar{x}_1, u_1, v]$, $x_2 := x[\bar{x}_2, u_2, v]$.

Since the limit solution depends on the pointwise definition of u, it is interesting to investigate the effects of a change of the u's values on a measure-zero subset of $[a,b]$.

THEOREM 2.3 (Pointwise dependence) Let us consider an interval $[a,b]$, an initial state $\bar{x} \in \mathbb{R}^n$ and an ordinary control $v \in L^1([a,b];V)$. Let $u, \hat{u} \in \mathcal{L}^1([a,b];U)$ be impulse controls that coincide a.e. in $[a,b]$ and that verify $u(a) = \hat{u}(a)$. Then, setting $x := x[\bar{x}, u, v]$, $\hat{x} := x[\bar{x}, \hat{u}, v]$, one has

$$x(t) = \exp\left(\sum_{\alpha=1}^{m}(u_\alpha(t) - \hat{u}_\alpha(t))g_\alpha\right)(\hat{x}(t)) \quad \text{for all } t \in [a,b]. \tag{2.7}$$

In particular,

$$x(t) = \hat{x}(t),$$

for every $t \in [a,b]$ such that $u(t) = \hat{u}(t)$, that is, almost everywhere.

In order to state the representation theorem below we need to introduce a change of coordinates induced by the g_α's flows.

Let us extend f, g_α, for $\alpha = 1, \ldots, m$ to functions $\tilde{f}, \tilde{g}_\alpha$ with values in $\mathbb{R}^{n+m}$ by setting, for every $(t,x,z,v) \in [a,b] \times \mathbb{R}^{n+m} \times V$,

$$\tilde{f}(t,x,z,v) := \sum_{j=1}^{n} f_j(t,x,z,v)\frac{\partial}{\partial x_j}, \quad \tilde{g}_\alpha(x,z) := \sum_{j=1}^{n} g_{\alpha,j}(x,z)\frac{\partial}{\partial x_j} + \frac{\partial}{\partial z_\alpha},$$

where $(\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial z_1, \ldots, \partial/\partial z_m)$ is the canonical basis of $\mathbb{R}^{n+m}$. [2] Let $\mathrm{Pr} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denote the canonical projection on the first factor, i.e.

$$\mathrm{Pr}(x,z) := x,$$

and let the function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be defined by

$$\varphi(x,z) := \mathrm{Pr} \circ \exp(-z_m\tilde{g}_m) \circ \cdots \exp(-z_1\tilde{g}_1)(x,z).$$

Finally, let us consider the map $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ given by

$$\phi(x,z) := (\varphi(x,z), z).$$

[2] Note that at this stage there is no more need to distinguish between equations (E) and (1.1), for the $\tilde{g}_\alpha$ are vector fields on $\mathbb{R}^{n+m}$, so it is irrelevant that their first n components are or are not dependent on u.

Lemma 2.1 Assume that the vector fields $g_1, \dots, g_m$ belong to $\mathcal{C}^r(\mathbb{R}^n;\mathbb{R}^n)$, with $r \geqslant 1$. Then, the mapping ϕ is a $\mathcal{C}^r$-diffeomorphism of $\mathbb{R}^{n+m}$ onto itself and, for every $(\xi,\zeta) \in \mathbb{R}^{n+m}$, one has

$$\phi^{-1}(\xi,\zeta) = (\varphi(\xi,-\zeta),\zeta).$$

The $\mathcal{C}^r$-diffeomorphism ϕ induces the $\mathcal{C}^{r-1}$-diffeomorphism $D\phi$ on the tangent bundle, where D denotes differentiation. For each $\alpha = 1,\dots,m$, $t \in [a,b]$, $(\xi,\zeta,v) \in \mathbb{R}^{n+m} \times V$, let us set

$$\tilde{F}(t,\xi,\zeta,v) := D\phi(x,z)\tilde{f}(t,x,z,v),$$
$$\tilde{G}_\alpha(\xi,\zeta) := D\phi(x,z)\tilde{g}_\alpha(x,z),$$

where $(x,z) := \phi^{-1}(\xi,\zeta)$. As a direct consequence of the *Simultaneous Flow-Box Theorem* (see, e.g. Lang, 1995), one obtains the following result (see Bressan & Rampazzo, 1991, Lemma 2.1 for a proof).

Lemma 2.2 For every $i = 1,\dots,n$ and $\alpha = 1,\dots,m$ one has

$$\tilde{F} = \sum_{i,j=1}^{n} \left(\frac{\partial \phi_i}{\partial x_j} f_j\right)\frac{\partial}{\partial x_i}, \quad \tilde{G}_\alpha = \frac{\partial}{\partial z_\alpha},$$

where we have set $\phi = (\phi_1,\dots,\phi_{n+m})$.

Note that the last m components of $\tilde{F}$ are zero. More precisely, $\tilde{F}$ can be written in components as $\tilde{F} = \binom{F}{0}$ with $F : [a,b] \times \mathbb{R}^{n+m} \times V \to \mathbb{R}^n$. Consider the Cauchy problem

$$\dot{\xi} = F(t,\xi,u,v) \quad \text{on } [a,b], \tag{2.8}$$
$$\xi(a) = \bar{\xi}. \tag{2.9}$$

For each $\bar{\xi} \in \mathbb{R}^n$, $(u,v) \in \mathcal{L}^1([a,b];\mathbb{R}^n) \times L^1([a,b];V)$, there exists a (unique) Carathéodory solution of (2.8) and (2.9), which will be here denoted by $\xi[\bar{\xi},u,v]$.

Theorem 2.4, which is trivial in the case $u \in AC$, provides a representation of the solutions of (E) (and of (1.1)) in terms of images of solutions of the simpler equation (2.8) through the map φ previously introduced.

Theorem 2.4 (Representation of limit solutions) For any $\bar{x} \in \mathbb{R}^n$, $(u,v) \in \mathcal{L}^1([a,b];U) \times L^1([a,b];V)$, one has

$$\xi[\bar{\xi},u,v](t) = \varphi(x[\bar{x},u,v](t),u(t)) \quad \text{for all } t \in [a,b],$$

where we have set $\bar{\xi} := \varphi(\bar{x},u(a))$.

The proof of this theorem is given in the next section.

3. Proofs of the results of Section 2

Since we are going to exploit the diffeomorphism $\phi:\mathbb{R}^{n+m}\to\mathbb{R}^{n+m}$ it is convenient to embed (E) and (IC) in the $n+m$-dimensional Cauchy problem

$$\begin{pmatrix}\dot{x}\\ \dot{z}\end{pmatrix}=\tilde{f}(t,x,z,v)+\sum_{\alpha=1}^{m}\tilde{g}_\alpha(x,z)\dot{u}_\alpha, \tag{3.1}$$

$$\begin{pmatrix}x\\ z\end{pmatrix}(a)=\begin{pmatrix}\bar{x}\\ \bar{z}\end{pmatrix}. \tag{3.2}$$

In view of the considered hypotheses, when $u\in AC([a,b];U)$, for every $(\bar{x},\bar{z})\in\mathbb{R}^{n+m}$ and $v\in L^1([a,b];V)$, there exists a unique solution to (3.1) in the interval $[a,b]$. We let $(x,z)[\bar{x},\bar{z},u,v]$ denote this solution.

We shall also consider the Cauchy problem

$$\begin{pmatrix}\dot{\xi}\\ \dot{\zeta}\end{pmatrix}=\tilde{F}(t,\xi,\zeta,v)+\sum_{\alpha=1}^{m}\tilde{G}_\alpha\dot{u}^\alpha, \tag{3.3}$$

$$\begin{pmatrix}\xi\\ \zeta\end{pmatrix}(a)=\begin{pmatrix}\bar{\xi}\\ \bar{\zeta}\end{pmatrix}. \tag{3.4}$$

Also for this problem, when $u\in AC([a,b];U)$, for every $(\bar{\xi},\bar{\zeta})\in\mathbb{R}^{n+m}$ there exists a unique solution to (3.3) and (3.4) in the interval $[a,b]$. We let $(\xi,\zeta)[\bar{\xi},\bar{\zeta},u,v]$ denote this solution.

Remark 3.1 When $u\in AC([a,b];U)$, the relation between the two systems is given by

$$(\xi,\zeta)[\bar{\xi},\bar{\zeta},u,v](t)=\phi((x,z)[\bar{x},\bar{z},u,v](t)),\quad \text{for all } t\in[a,b],$$

where $(\bar{\xi},\bar{\zeta}):=\phi(\bar{x},\bar{z})$.

Remark 3.2 The crucial difference between the two latter systems relies on the fact that the vector fields $\tilde{G}_\alpha=\partial/\partial z_\alpha$ are constant.

Theorem 3.1 will be utilized to prove Theorem 2.2, of which it is in fact a particular case.

Theorem 3.1 The following assertions hold:

(i) For each $(\bar{\xi},\bar{\zeta})\in\mathbb{R}^{n+m}$ and $u\in AC([a,b];U)$, the function $v\mapsto\xi[\bar{\xi},\bar{\zeta},u,v]$ is continuous from $L^1([a,b];V)$ to $L^\infty([a,b];\mathbb{R}^n)$.

(ii) Furthermore, for any $r>0$, there exists a compact subset $K'\subset\mathbb{R}^{n+m}$ such that the trajectories $(\xi,\zeta)[\bar{\xi},\bar{\zeta},u,v]$ have values in K', whenever we consider $|(\bar{\xi},\bar{\zeta})|\leqslant r$, $u\in AC([a,b];U)$ and $v\in L^1([a,b];V)$.

(iii) Finally, for each $r > 0$, there exists a constant $M > 1$ such that, for every $\tau \in [a,b]$, for all $|(\bar{\xi}_1, \bar{\zeta}_1)|, |(\bar{\xi}_2, \bar{\zeta}_2)| \leqslant r$, for all $u_1, u_2 \in AC([a,b];U)$ and for every $v \in L^1([a,b];V)$, one has

$$\begin{aligned} &|(\xi_1,\zeta_1)(\tau) - (\xi_2,\zeta_2)(\tau)| + \|(\xi_1,\zeta_1) - (\xi_2,\zeta_2)\|_1 \\ &\quad \leqslant M[|(\bar{\xi}_1,\bar{\zeta}_1) - (\bar{\xi}_2,\bar{\zeta}_2)| + |u_1(a) - u_2(a)| + |u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1], \end{aligned} \tag{3.5}$$

where $(\xi_1,\zeta_1) := (\xi,\zeta)[\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1]$ and $(\xi_2,\zeta_2) := (\xi,\zeta)[\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2]$.

To prove Theorem 3.1, we shall exploit the following fixed-point result on parameterized contraction mappings (see e.g. Bressan & Piccoli, 2007, Theorem A.1).

LEMMA 3.1 Let X be a Banach space, Λ a metric space and $\chi : \Lambda \times X \to X$ be a continuous function such that

$$\|\chi(\lambda,x) - \chi(\lambda,y)\| \leqslant L\|x-y\|, \quad \text{for all } \lambda \in \Lambda,\ x,y \in X, \tag{3.6}$$

with $L < 1$. Then the following assertions hold:

(a) For every $\lambda \in \Lambda$, there exists a unique $x(\lambda)$ such that

$$x(\lambda) = \chi(\lambda, x(\lambda)).$$

(b) The map $\lambda \mapsto x(\lambda)$ is continuous, and one has

$$\|x(\lambda) - x(\tilde{\lambda})\| \leqslant \frac{1}{1-L}\|\chi(\lambda,x(\tilde{\lambda})) - \chi(\tilde{\lambda},x(\tilde{\lambda}))\|.$$

Proof of Theorem 3.1. Item (i) follows from classical results of continuity of the input–output map of a control system.

To prove the remaining assertions, assume momentarily that F is globally Lipschitz continuous with respect to the variable (ξ,ζ) with Lipschitz constant L. Later, we shall remove this extra assumption.

For $(\bar{\xi},\bar{\zeta},u,v) \in \Lambda := \mathbb{R}^{n+m} \times AC([a,b];U) \times L^1([a,b];V)$ and $(\xi,\zeta) \in X := AC([a,b];\mathbb{R}^{n+m})$, let us consider the mapping $\chi : \Lambda \times X \to X$ such that, for all $t \in [a,b]$,

$$\chi(\bar{\xi},\bar{\zeta},u,v,\xi,\zeta)(t) := \begin{pmatrix}\bar{\xi}\\ \bar{\zeta}\end{pmatrix} + \int_a^t \tilde{F}(s,\xi(s),\zeta(s),v(s))\,\mathrm{d}s + \sum_{\alpha=1}^m [u_\alpha(t) - u_\alpha(a)]\mathbf{e}_{n+\alpha},$$

where $\mathbf{e}_{n+\alpha}$ denotes the $(n+\alpha)$th vector of the canonical basis of $\mathbb{R}^{n+m}$. Observe that

$$\begin{pmatrix}\xi\\ \zeta\end{pmatrix} = \chi(\bar{\xi},\bar{\zeta},u,v,\xi,\zeta) \quad \text{if and only if} \quad \begin{pmatrix}\xi\\ \zeta\end{pmatrix} \text{ is solution of (3.3).}$$

We are therefore interested in applying the fixed-point result in Lemma 3.1 to the function χ.

Fix $\tau \in [a,b]$, and in the space Λ consider the norm

$$\|(\bar{\xi},\bar{\zeta},u,v)\|_Y := |(\bar{\xi},\bar{\zeta})| + |u(a)| + |u(\tau)| + \|u\|_1 + \|v\|_1,$$

and in X, define the norm

$$\|(\xi,\zeta)\|_X := \frac{e^{-4(b-a)L}}{4L}|(\xi,\zeta)(\tau)| + \int_a^b e^{-4sL}|(\xi,\zeta)(s)|\,\mathrm{d}s.$$

We shall prove that χ is continuous from $(\Lambda \times X, \|\cdot\|_Y + \|\cdot\|_X)$ to $(X, \|\cdot\|_X)$. By the Lipschitz continuity of the maps $(\xi,\zeta) \to \tilde{F}(t,\xi,\zeta,v)$, for any $(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_1,\zeta_1)$, $(\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2,\xi_2,\zeta_2)$ in $\Lambda \times X$, one has

$$\begin{aligned}
&\|\chi(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_1,\zeta_1) - \chi(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_2,\zeta_2)\|_X \\
&\quad = \frac{e^{-4(b-a)L}}{4L}\left|\int_a^\tau [\tilde{F}(s,\xi_1(s),\zeta_1(s),v_1(s)) - \tilde{F}(s,\xi_2(s),\zeta_2(s),v_1(s))]\,\mathrm{d}s\right| \\
&\qquad + \int_a^b e^{-4sL}\left|\int_a^t [\tilde{F}(s,\xi_1(s),\zeta_1(s),v_1(s)) - \tilde{F}(s,\xi_2(s),\zeta_2(s),v_1(s))]\,\mathrm{d}s\right|\mathrm{d}t \\
&\quad \leqslant \frac{1}{4}\|(\xi_1,\zeta_1) - (\xi_2,\zeta_2)\|_X, \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
&\|\chi(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_2,\zeta_2) - \chi(\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2,\xi_2,\zeta_2)\|_X \\
&\quad = \frac{e^{-4(b-a)L}}{4L}\left|\begin{pmatrix}\bar{\xi}_1 - \bar{\xi}_2\\ \bar{\zeta}_1 - \bar{\zeta}_2\end{pmatrix}\right| \\
&\qquad + \frac{e^{-4(b-a)L}}{4L}\left|\int_a^\tau [\tilde{F}(s,\xi_2(s),\zeta_2(s),v_1(s)) - \tilde{F}(s,\xi_2(s),\zeta_2(s),v_2(s))]\,\mathrm{d}s\right| \\
&\qquad + \int_a^b e^{-4sL}\left|\int_a^t [\tilde{F}(s,\xi_2(s),\zeta_2(s),v_1(s)) - \tilde{F}(s,\xi_2(s),\zeta_2(s),v_2(s))]\,\mathrm{d}s\right|\mathrm{d}t \\
&\qquad + \frac{e^{-4(b-a)L}}{4L}\left|\sum_{\alpha=1}^m [u_{1,\alpha}(\tau) - u_{1,\alpha}(a) - u_{2,\alpha}(\tau) + u_{2,\alpha}(a)]\mathbf{e}_{n+\alpha}\right| \\
&\qquad + \int_a^b e^{-4sL}\left|\int_a^t \sum_{\alpha=1}^m [u_{1,\alpha}(s) - u_{1,\alpha}(a) - u_{2,\alpha}(s) + u_{2,\alpha}(a)]\mathbf{e}_{n+\alpha}\,\mathrm{d}s\right|\mathrm{d}t. \qquad (3.8)
\end{aligned}$$

By the Dominated Convergence Theorem, for any fixed trajectory (ξ,ζ), the mapping

$$v \mapsto \tilde{F}(\cdot,\xi,\zeta,v)$$

is continuous from $L^1([a,\tau];V)$ to $L^1([a,\tau];\mathbb{R}^{n+m})$. Thus, by (3.7) and (3.8), for each $(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_1,\zeta_1) \in \Lambda \times X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if

$$\begin{aligned}
&|(\bar{\xi}_2,\bar{\zeta}_2) - (\bar{\xi}_1,\bar{\zeta}_1)| + |u_2(a) - u_1(a)| + |u_2(\tau) - u_1(\tau)| + \|u_2 - u_1\|_1 \\
&\quad + \|v_2 - v_1\|_1 + \|(\xi_2,\zeta_2) - (\xi_1,\zeta_1)\|_X < \delta
\end{aligned}$$

then

$$\|\chi(\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2,\xi_2,\zeta_2) - \chi(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_1,\zeta_1)\|_X < \varepsilon.$$

Hence, χ is continuous, and in view of (3.7) the inequality (3.6) holds true. To apply Lemma 3.1, let us identify λ with $(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1)$ and $\tilde{\lambda}$ with $(\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2)$. Then one has that there exist $(\xi_1,\zeta_1),(\xi_2,\zeta_2) \in X$ such that

$$\chi(\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1,\xi_1,\zeta_1) = (\xi_1,\zeta_1), \qquad \chi(\bar{\xi}_2,\bar{\zeta}_2,u_2,v_1,\xi_2,\zeta_2) = (\xi_2,\zeta_2),$$

that is $(\xi_1,\zeta_1) = (\xi,\zeta)[\bar{\xi}_1,\bar{\zeta}_1,u_1,v_1]$ and $(\xi_2,\zeta_2) = (\xi,\zeta)[\bar{\xi}_2,\bar{\zeta}_2,u_2,v_2]$.

In view of item (b) in Lemma 3.1, we get

$$\left\| \begin{pmatrix} \tilde{\xi} - \xi \\ \tilde{\zeta} - \zeta \end{pmatrix} \right\|_X \leqslant \frac{1}{2L}[|(\tilde{\bar{\xi}}, \tilde{\bar{\zeta}}) - (\bar{\xi}, \bar{\zeta})| + |\tilde{u}(a) - u(a)| + |\tilde{u}(\tau) - u(\tau)|] + \int_a^b e^{-4tL}|\tilde{u}(t) - u(t)|\, \mathrm{d}t. \tag{3.9}$$

Therefore, by the inequalities

$$\left\| \begin{pmatrix} \xi_2 - \xi_1 \\ \zeta_2 - \zeta_1 \end{pmatrix} \right\|_X \geqslant \frac{e^{-4(b-a)L}}{4L} \left| \begin{pmatrix} \xi_2(\tau) - \xi_1(\tau) \\ \zeta_2(\tau) - \zeta_1(\tau) \end{pmatrix} \right| + e^{-4bL} \left\| \begin{pmatrix} \xi_2 - \xi_1 \\ \zeta_2 - \zeta_1 \end{pmatrix} \right\|_1, \tag{3.10}$$

and (3.9) one obtains (3.5), with a constant M depending only on L, and hence item (iii) is proved. Note also that item (ii) is a consequence of the following standard result on ODEs.

LEMMA 3.2 (Bounds on solutions) Under the general hypothesis H, for each $r > 0$, there exists a compact set $K' \subset \mathbb{R}^{n+m}$, such that any solution (ξ, ζ) of (3.3) and (3.4) remains in K' whenever $|(\bar{\xi}, \bar{\zeta})| \leqslant r$, $u \in AC([a,b]; U)$ and $v \in L^1([a,b]; V)$.

This completes the proof of the theorem under the additional assumption that F is globally Lipschitz. To prove the general case we use a standard cut-off function argument.

Take $r > 0$, and let $K' \subset \mathbb{R}^{n+m}$ be the compact set provided by Lemma 3.2. Let $\rho \in \mathcal{C}^1(\mathbb{R}^{n+m})$ be a smooth real function such that $\rho = 1$ on K' and $\rho = 0$ outside a neighbourhood of K'. Define $\hat{F}(t, \xi, \eta, v) := \rho(\xi, \eta)\tilde{F}(t, \xi, \eta, v)$, and set

$$\Lambda := \{(\bar{\xi}, \bar{\zeta}) \in \mathbb{R}^{n+m} : |(\bar{\xi}, \bar{\zeta})| \leqslant r\} \times AC([a,b]; U) \times L^1([a,b]; V).$$

Then, for $(\bar{\xi}, \bar{\eta}, u, v) \in \Lambda$, the corresponding solution (ξ, η) of the Cauchy problem

$$\begin{pmatrix} \dot{\xi} \\ \dot{\zeta} \end{pmatrix} = \hat{F}(t, \xi, \zeta, v) + \sum_{\alpha=1}^{m} \tilde{G}_\alpha \dot{u}^\alpha, \quad \begin{pmatrix} \xi \\ \zeta \end{pmatrix}(a) = \begin{pmatrix} \bar{\xi} \\ \bar{\zeta} \end{pmatrix},$$

coincides with $(\xi, \eta)[\bar{\xi}, \bar{\eta}, u, v]$ (and remains inside K'). Now the function $\hat{F}$ is globally Lipschitz, and the procedure done before can be repeated for this new metric space Λ and for the function $\hat{F}$ in the place of $\tilde{F}$. Therefore, one can obtain the estimate (3.5) with a constant M depending only on the Lipschitz constant of the mapping $(\xi, \eta) \mapsto \tilde{F}(t, \xi, \eta, v)$ in the set K'. This completes the proof of Theorem 3.1. □

Proofs of Theorems 2.1 *and* 2.4. Let $\bar{x} \in \mathbb{R}^n$, $(u, v) \in \mathcal{L}^1([a,b]; U) \times L^1([a,b]; V)$, and let $\xi := \xi[\bar{\xi}, u, v]$ be the unique Carathéodory solution of (2.8) with the initial condition $\xi(a) = \bar{\xi} := \varphi(\bar{x}, u(a))$. Define, on $[a,b]$, the function

$$x := \varphi \circ (\xi, -u), \tag{3.11}$$

and let us show that x is the unique limit solution of (E) and (IC) associated with $\bar{x}$ and (u, v).

Choose $\tau \in [a,b]$, and consider a sequence of absolutely continuous controls $u_k^\tau : [a,b] \to U$ verifying

$$|u_k^\tau(a) - u(a)| + |u_k^\tau(\tau) - u(\tau)| + \|u_k^\tau - u\|_1 \to 0. \tag{3.12}$$

Consider equation (2.8) with the initial condition

$$\xi(a) = \bar{\xi}_k^\tau := \varphi(\bar{x}, u_k^\tau(a)).$$

Let $\xi_k^\tau := \xi[\bar{\xi}_k^\tau, u_k^\tau, v]$ be the (unique) corresponding Carathéodory solution. Then, by standard results of continuity with respect to the data, one has that

$$\xi_k^\tau \to \xi, \quad \text{uniformly on } [a,b]. \tag{3.13}$$

For the augmented system (3.3), (ξ_k^τ, u_k^τ) is the unique solution with the initial conditions $\xi(a) = \bar{\xi}_k^\tau$, $\eta(a) = u_k^\tau(a)$. In view of item (iii) in Theorem 3.1, the functions (ξ_k^τ, u_k^τ) have values in a compact set $\tilde{K}' \subset \mathbb{R}^{n+m}$. In view of Remark 3.1, the map

$$(x_k^\tau, z_k^\tau) := \phi^{-1} \circ (\xi_k^\tau, u_k^\tau),$$

is, for each $k \in \mathbb{N}$, the unique Carathéodory solution of (3.1) with the initial conditions $x(a) = \bar{x}$, $z(a) = u_k^\tau(a)$. Note that the functions (x_k^τ, z_k^τ) have values inside the compact set $K' := \phi^{-1}(\tilde{K}')$. In particular, the x_k^τ's are uniformly bounded. Observe as well that

$$|x(\tau) - x_k^\tau(\tau)| = |\varphi(\xi(\tau), -u(\tau)) - \varphi(\xi_k^\tau(\tau), -u_k^\tau(\tau))| \to 0,$$

due to (3.13) and (3.14) and the continuity of φ. Furthermore, since $u_k^\tau \to u$ almost everywhere and all these functions are uniformly bounded, one gets

$$\|x - x_k^\tau\|_1 = \int_a^b |\varphi(\xi(t), -u(t)) - \varphi(\xi_k^\tau(t), -u_k^\tau(t))| \, \mathrm{d}t \to 0,$$

thanks to the Dominated Convergence Theorem. Thus, x is a limit solution of (E) and (IC) associated with $\bar{x}$ and (u, v).

We shall now prove the uniqueness. Suppose on the contrary that there are two different limit solutions x and $\tilde{x}$ of (E) and (IC) associated with $(\bar{x}, u, v)$. Let $\tau \in]a, b]$ be such that $x(\tau) \neq x^*(\tau)$, and let (u_k^τ), $(u_k^{\tau,*})$ be sequences in $AC([a,b];U)$ verifying

$$|u_k^\tau(a) - u(a)| + |(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_1 \to 0, \tag{3.14}$$

and

$$|u_k^{\tau,*}(a) - u(a)| + |(x_k^{\tau,*}, u_k^{\tau,*})(\tau) - (x^*, u)(\tau)| + \|(x_k^{\tau,*}, u_k^{\tau,*}) - (x^*, u)\|_1 \to 0, \tag{3.15}$$

where $x_k^\tau := x[\bar{x}, u_k^\tau, v]$, $x_k^{\tau,*} := x[\bar{x}, u_k^{\tau,*}, v]$. Let $\xi_k^\tau := \xi[\bar{\xi}_k^\tau, u_k^\tau, v]$, $\xi_k^{\tau,*} := \xi[\bar{\xi}_k^{\tau,*}, u_k^{\tau,*}, v]$ be the solutions of (2.8) with initial conditions

$$\xi(a) = \bar{\xi}_k^\tau := \varphi(\bar{x}, u_k^\tau(a)), \quad \xi(a) = \bar{\xi}_k^{\tau,*} := \varphi(\bar{x}, u_k^{\tau,*}(a)),$$

respectively. Then, since both (u_k^τ), $(u_k^{\tau,*})$ converge to u in $L^1([a,b];U)$, one has that

$$(\xi_k^\tau),\ (\xi_k^{\tau,*}) \text{ converge uniformly to } \xi := \xi[\bar{\xi}, u, v],$$

with $\bar{\xi} := \varphi(\bar{x}, u(a))$. Observe that, from previous equation, (3.14) and (3.15), one obtains

$$|x(\tau) - x^*(\tau)| = \lim |x_k^\tau(\tau) - x_k^{\tau,*}(\tau)| = \lim |\varphi \circ (\xi_k^\tau, -u_k^\tau)(\tau) - \varphi \circ (\xi_k^{\tau,*}, -u_k^{\tau,*})(\tau)| = 0.$$

This contradicts our assumption, so the uniqueness of the limit solution of (E) and (IC) is proved. This ends the proof of Theorem 2.1.

Finally, since x was defined in (3.11) via the coordinates' transformation, by the uniqueness of the limit solution we also obtain the proof of Theorem 2.4. □

Proof of Theorem 2.2. We shall start by proving item (ii). For this, let $r > 0$ be arbitrary and consider the compact set $\tilde{K}' \subset \mathbb{R}^{n+m}$ provided by the item (ii) of Theorem 3.1. Then, for each $|\bar{x}| \leqslant r$ and $(u, v) \in \mathcal{L}^1([a,b]; U) \times L^1([a,b]; V)$, the corresponding limit solution $x[\bar{x}, u, v]$ has values inside the compact set K' defined as the projection of $\phi^{-1}(\tilde{K}')$ in the first n components. This proves item (ii).

To show part (i), fix $\bar{x} \in \mathbb{R}^n$ and $u \in \mathcal{L}^1([a,b]; U)$. Set $\bar{\xi} := \varphi(\bar{x}, u(a))$, and note that $v \mapsto \xi[\bar{\xi}, u, v]$ is continuous from $L^1([a,b]; V)$ to $AC([a,b]; \mathbb{R}^n)$, where the latter space is endowed with the $\mathcal{C}^0$-topology. Furthermore, for any $v \in L^1([a,b]; V)$, the trajectories $\xi[\bar{\xi}, u, v]$ remain inside a compact set $\hat{K}$. Let v_k converge to v in L^1, then $\xi[\bar{\xi}, u, v_k] \to \xi[\bar{\xi}, u, v]$ uniformly and, since φ is uniformly continuous on $\hat{K} \times (-U)$, one gets that

$$v \mapsto x[\bar{x}, u, v] = \varphi \circ (\xi[\bar{\xi}, u, v], -u),$$

is continuous from $L^1([a,b]; V)$ to $L^\infty([a,b]; \mathbb{R}^n)$, from which item (i) follows.

We now prove item (iii). For an arbitrary $r > 0$, let $\tilde{r} > 0$ be such that the points $(\bar{\xi}_1, u_1(a)) := \phi(\bar{x}_1, u_1(a))$ and $(\bar{\xi}_2, u_2(a)) := \phi(\bar{x}_2, u_2(a))$ remain inside $\{|(\bar{\xi}, \bar{\zeta})| \leqslant \tilde{r}\}$ whenever $|\bar{x}_1|, |\bar{x}_2| \leqslant r$ and $u_1, u_2 \in \mathcal{L}^1([a,b]; U)$. Take $\tau \in [a,b]$, $|\bar{x}_1|, |\bar{x}_2| \leqslant r$, $u_1, u_2 \in \mathcal{L}^1([a,b]; U)$ and $v \in L^1([a,b]; V)$. Let $x_1 := x[\bar{x}_1, u_1, v]$ and $x_2 := x[\bar{x}_2, u_2, v]$ be the corresponding limit solutions of (E).

Let $(u_{1,k}^\tau)$, $(u_{2,k}^\tau)$ be sequences in $AC([a,b]; U)$ verifying

$$|u_{1,k}^\tau(a) - u_1(a)| + |u_{1,k}^\tau(\tau) - u_1(\tau)| + \|u_{1,k}^\tau - u_1\|_1 \to 0,$$

and

$$|u_{2,k}^\tau(a) - u_2(a)| + |u_{2,k}^\tau(\tau) - u_2(\tau)| + \|u_{2,k}^\tau - u_2\|_1 \to 0,$$

Consider the functions $\xi_{1,k}^\tau := \xi[\bar{\xi}_{1,k}^\tau, u_{1,k}^\tau, v]$, $\xi_{2,k}^\tau := \xi[\bar{\xi}_{2,k}^\tau, u_{2,k}^\tau, v]$, where $\bar{\xi}_{1,k}^\tau := \varphi(\bar{x}_1, u_{1,k}^\tau(a))$, $\bar{\xi}_{2,k}^\tau := \varphi(\bar{x}_2, u_{2,k}^\tau(a))$. Then $\xi_{1,k}^\tau$ converge uniformly to $\xi_1 := \xi[\bar{\xi}_1, u_1, v]$, where $\bar{\xi}_1 := \varphi(\bar{x}_1, u_1(a))$, and $\bar{\xi}_{2,k}^\tau$ converge uniformly to $\xi_2 := \xi[\bar{\xi}_2, u_2, v]$, where $\bar{\xi}_2 := \varphi(\bar{x}_2, u_2(a))$. Let $\tilde{M} > 1$ and $\tilde{K}'$ be the constant and the compact set provided by Theorem 3.1 for $\tilde{r}$, respectively. Then one has

$$\begin{aligned}
&|\xi_{1,k}^\tau(\tau) - \xi_{2,k}^\tau(\tau)| + \|\xi_{1,k}^\tau - \xi_{2,k}^\tau\|_1 \\
&\quad \leqslant (\tilde{M} - 1)[|\bar{\xi}_1 - \bar{\xi}_2| + |u_{1,k}^\tau(a) - u_{2,k}^\tau(a)| + |u_{1,k}^\tau(\tau) - u_{2,k}^\tau(\tau)| + \|u_{1,k}^\tau - u_{2,k}^\tau\|_1].
\end{aligned}$$

Letting k go to infinity in the previous inequality, one obtains

$$|\xi_1(\tau) - \xi_2(\tau)| + \|\xi_1 - \xi_2\|_1 \leqslant (\tilde{M} - 1)[|\bar{\xi}_1 - \bar{\xi}_2| + |u_1(a) - u_2(a)| + |u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1]. \tag{3.16}$$

Thus, for L a Lipschitz constant of φ on $\tilde{K}'$, one gets

$$\begin{aligned}
&|x_1(\tau) - x_2(\tau)| + \|x_1 - x_2\|_1 \\
&\quad = |\varphi \circ (\xi_1, -u_1)(\tau) - \varphi \circ (\xi_2, u_2)(\tau)| + \|\varphi \circ (\xi_1, -u_1) - \varphi \circ (\xi_2, u_2)\|_1 \\
&\quad \leqslant L[|(\xi_1, u_1)(\tau) - (\xi_2, u_2)(\tau)| + \|(\xi_1, u_1) - (\xi_2, u_2)\|_1] \\
&\quad \leqslant L(\tilde{M} - 1)[|\bar{\xi}_1 - \bar{\xi}_2| + |u_1(a) - u_2(a)| + |u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1] \\
&\qquad + L[|u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1] \\
&\quad \leqslant M[|\bar{x}_1 - \bar{x}_2| + |u_1(a) - u_2(a)| + |u_1(\tau) - u_2(\tau)| + \|u_1 - u_2\|_1],
\end{aligned}$$

for $M > 0$ depending on L and $\tilde{M}$, where the second inequality follows from (3.16). Thus, the desired estimate holds true, and this completes the proof. □

Proof of Theorem 2.3. Set $\bar{\xi} = \varphi(\bar{x}, u(a)) = \varphi(\bar{x}, \hat{u}(a))$ and observe that $\xi[\bar{\xi}, u, v](t) = \xi[\bar{\xi}, \hat{u}, v](t)$, for all $t \in [a, b]$. Let ξ denote the latter function. In view of Theorem 2.4, one has

$$\begin{aligned}
\xi(t) &= \varphi(x[\bar{x}, u, v](t), u(t)) = x[\bar{x}, u, v](t)e^{-u_\alpha(t)g_\alpha}, \\
\xi(t) &= \varphi(x[\bar{x}, \hat{u}, v](t), u(t)) = x[\bar{x}, \hat{u}, v](t)e^{-\hat{u}_\alpha(t)g_\alpha}.
\end{aligned}$$

Therefore, combining the last two equations and due to the commutativity of g_α, the relation (2.7) follows. □

4. Concluding remarks

A notion of everywhere defined solution for the control Cauchy problem on $[a, b]$

$$\dot{x} = f(t, x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x, u)\dot{u}_\alpha, \quad t \in [a, b], \quad x(a) = \bar{x},$$

has been provided, under a commutativity hypothesis on the fields g_α. In particular, we have proved results of existence, uniqueness and continuous dependence on the data, besides investigating the effects of u's changes on null sets. This concept of solution, which relies on an *extension by density* of the classical notion, turns out to verify consistency requirements. We point out that, by defining the output at every $t \in [a, b]$, we have departed from a topological picture based on normed spaces, instead framing the limiting processes in spaces endowed with family of semi-norms (this choice is concretely represented by the fact that approximating sequences in the solution's definition depend on τ, for every $\tau \in [a, b]$).

The paper is motivated by both applications (see Section 1 and Example 2.1) and the concern of constructing a suitable framework for further theoretical issues, like the study of the corresponding *adjoint equations*, a likely crucial object in the investigation of necessary conditions for minima.

We think that a generalization of the notion of limit solution to the non-commutative case (in particular, an extension that will agree with former concepts of solutions) might represent a natural direction for further investigations.

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