

Extracting Default Probabilities from Sovereign Bonds*

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Abstract

Sovereign risk analysis is central in debt markets. Considering different bonds and countries, there are numerous measures aiming to identify the way risk is perceived by market participants. In such environment, probabilities of default play a central role in investors' decisions. This article contributes by providing a parametric arbitrage-free dynamic model to estimate defaultable term structures of sovereign bonds. The proposed model builds on Duffie and Singleton's (1999) general reduced-form model by proposing a piecewise constant structure for the conditional probabilities of defaults. Once an average recovery rate value is fixed for the whole market, the proposed model estimates implied probabilities of defaults from bond prices, working as a parsimonious tool to quantify investor's perception of credit risk. We apply this methodology to analyze the behavior of default probabilities within the Brazilian sovereign fixed income market at three different recent economic moments.

Keywords: Credit Risk, Term Structure of Interest Rates, Recovery Values, Jump Processes.

JEL Codes: C5, C51.

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1. Introduction

Implicit in any defaultable bond price is a premium related to the possibility that the issuer of the bond might not fulfill its payment obligations. For this reason, fixed income prices of risky bonds contain valuable information on the way risk is perceived by market participants. In this work, we propose and implement a simple dynamic term structure model, compatible with absence of arbitrages in the market, which helps with the extraction of probabilities of defaults for coupon-bearing bonds in emerging markets.

Credit risk modeling has been attracting the attention of researchers for at least a couple of decades. Merton (1974) proposed a jump-diffusion model to capture credit risk; Litterman and Iben (1991) studied a discrete-time model for defaultable bonds with zero recovery and obtained that risky interest rates could be represented by an adjusted short-term rate. Jarrow and Turnbull (1995) introduced discrete-time Markov chains to model rating migrations in corporate bond markets. Duffie and Singleton (1999) presented general results on reduced-form term structure models. Collin Dufresne et al. (2004) proposed general formulas for the pricing of defaultable claims.

Our model is a particular yet effective version of Duffie and Singleton's (1999) proposal. It builds on the discrete-time motivational model presented in the first part of their paper, where conditional probabilities of defaults are fixed as constant values. In contrast, we write the conditional probabilities of defaults as piecewise constant functions. The duration in time for each constant part is motivated by aiming at having a model with default factors capturing default probabilities whose perceptions of risk are related to different horizons. Despite being motivated by a discrete-time model, it is actually a continuous-time arbitrage-free model with a simple functional form for the conditional probabilities of default.¹ The most interesting features of the proposed model are its parsimonious structure, its simplicity in implementation, and its ability to easily interpret market perception of defaultable bond risks by simple quantities, such as annualized conditional probabilities of defaults and recovery rates for face values of bonds. These measures are quite intuitive and well established in use by market participants. In addition, it can be effectively applied to price new issues of bonds, as we shall observe in subsection 3.3.

A number of credit risk crises have occurred during recent times. In 1997, the Asian market suffered severe losses, and in August of 1998, the Russian market collapsed. After Russia defaulted its ruble-denominated debt obligations, there has been a thoughtful reevaluation of the concept of credit risk in global financial markets.² The number of complex credit derivative products, like collateralized

¹Note that the model is applied to a cross-section of data, and in this sense it works like a static model. However, we could also have applied a dynamic version of the model to a time-series dataset of defaultable bonds data.

²At this point, we could also cite Brazil's 1999 currency devaluation crisis, Argentina's 2000-

debt obligations (CDOs) and credit default swaps (CDSs), traded in international markets tremendously increased. People were much more concerned with the understanding and pricing of credit risk. From the empirical viewpoint, all these crises motivated research where the main focus was to extract implied probabilities of default and recovery rates from market data to give more support to trading desks on the design and evaluation of credit derivative products. For instance, Merrick (2001) introduced a joint implied parameter approach to extract the expected recovery rates and default probability term structure from Russian bonds. Andritzky (2002) adapted Jarrow and Turnbull's model (1995) to estimate default probabilities and recovery ratios from Argentinean bonds.³ Berardi et al. (2004) adopted a recursive logit type model to predict default probabilities of Global bonds⁴ in different emerging markets and make use of these predictions to propose trading strategies for defaultable bond portfolios. Pan and Singleton (2007) applied different term structure models to extract default probabilities and recovery values from term structures of sovereign CDS spreads.

In our empirical application, we study the behavior of investor's risk perception within an important fixed income emerging market: the Brazilian global bonds market. By using the common market practice of fixing a predetermined consensual value for the recovery rate, we extract implied probabilities of default during three periods under clearly distinct economic situations: the international crisis around September 11, 2001, the Brazilian local crisis during the presidential elections in 2002, and an example of an economically stable period at the end of 2004.

As a second part of the application, we test the model's ability to price an out-of-sample bond. The ability of a model to price an out-of-sample bond is closely related to its ability to price newly issued bonds. As it is usual to have countries issuing new bonds, especially when their credit receives better classifications, it is relevant to perform this out-of-sample test. Considering the 2004 dataset adopted in the first part of the empirical application, we here estimate the model by excluding the 2030 global bond from the estimation dataset. This bond presents the higher spread over treasury among all Brazilian global bonds on this date, and for this reason, it represents a challenge if priced out-of-sample by the model. We show that the model correctly prices the global 2030 out-of-sample bond, offering a statistical error of less than five basis points on the spread over treasury of this bond, an error of the order of one tenth of the statistical error obtained under a linear interpolation scheme.

2001 currency devaluation crisis, and the September 11, 2001 attack on the twin towers in New York.

³The hazard rate, defining instantaneous probabilities of default, was modeled by using a Gumbel probability distribution.

⁴A Global bond is a sovereign coupon-bearing bond issued in a foreign currency (usually in dollars), in a foreign market.

The rest of the paper is organized as follows. Section 2 presents the theoretical model and the estimation methodology. Section 3 presents the model estimation results and discusses the empirical application outlined above. Section 4 summarizes the paper and presents some concluding remarks.

2. The Model

Consider a defaultable contingent claim that promises to pay $X_{t+\tau}$ at its maturity date $t + \tau$. Assume the existence of a risk-neutral measure Q under which discounted prices of defaultable bonds are martingales. This insures the absence of approximate arbitrages in the market (see Duffie (2001)).

Now, let r_s represent the default-free short-term rate,⁵ h_s denotes the probability of a default happening between s and $s + 1$, conditional on information available up to time s , in the event of no default by s , and ϕ_s denotes the recovery values in dollars, in the event of a default. It is immediate to verify that the fair price of the defaultable contingent claim at time t , as a function of its price at time $t + 1$, should be written as a sum of two terms: one paying the recovery value ϕ_{t+1} in the event of default happening between times t and $t + 1$; and the other one paying the discounted fair price at time $t + 1$, in the event of no default:

$$V_t = h_t e^{-r_t} E_t^Q(\phi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}) \quad (1)$$

Duffie and Singleton (1999) applied a recursive argument to show that the price at time t is the following function of the payoff $X_{t+\tau}$:

$$\begin{aligned} V_t = & E_t^Q \left(\sum_{j=0}^{\tau-1} h_{t+j} e^{-\sum_{k=0}^j r_{t+k}} \phi_{t+j+1} \prod_{l=0}^j (1 - h_{t+l-1}) \right) \\ & + E_t^Q \left(e^{-\sum_{k=0}^{\tau-1} r_{t+k}} X_{t+\tau} \prod_{l=0}^{\tau-1} (1 - h_{t+l-1}) \right) \end{aligned} \quad (2)$$

They obtained the following important result: If, under the event of a default at time $s + 1$, the recovery value is a fraction $1 - L_s$ of the market value of the bond at time $s + 1$ (not a function of its face value), then equation (2) simplifies to:

$$\begin{aligned} V_t = & h_t e^{-r_t} (1 - L_t) E_t^Q(V_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}) \\ = & E_t^Q \left(e^{-\sum_{k=0}^{\tau-1} R_{t+k}} X_{t+\tau} \right) \end{aligned} \quad (3)$$

⁵Represented in this paper by the short-term rate implied by U.S. swaps.

with

$$e^{-R_t} = (1 - h_t)e^{-r_t} + h_te^{-r_t}(1 - L_t) \quad (4)$$

Duffie and Singleton (1999) noted in addition that for annualized rates and time periods of small length, equation (4) can be further simplified to yield that the defaultable short-term rate is approximately equal to the default-free short-term rate plus the product of the loss rate L_t and the conditional probability of default h_t at time t : $R_t = r_t + h_tL_t$. They show that this approximation for the defaultable short-term rate is exact in continuous time (Theorem 1 on page 697). No restriction is imposed on the hazard rate process h in order to render the theorem valid, and we use this fact to propose our model.

2.1 The piecewise constant model

In this paper, we propose to have a constant loss rate L and conditional probabilities of default expressed by a piecewise constant function:

$$h_t = \sum_{j=1}^N H_j I(t_{j-1} < t < t_j) \quad (5)$$

where I represents the indicator function which assumes one or zero depending on t being within the indicated interval or not, and H_1, H_2, \dots, H_N are constants. Figure 1 depicts an example of the conditional probabilities of default function defined by equation (5).

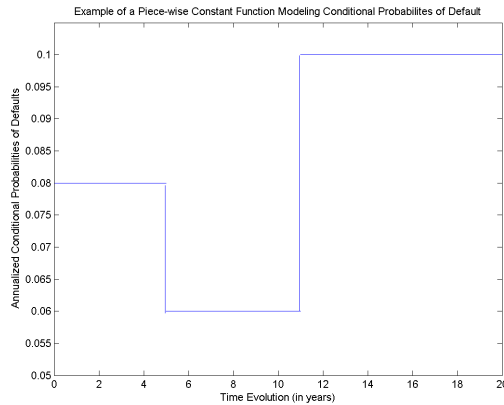


Figure 1
An example of conditional probabilities of default through time

The piecewise constant function is a particular hazard rate process and by Theorem 1 in Duffie and Singleton (1999) our version of their model is arbitrage-free if we consider that the defaultable short-term rate is given by $R_t = r_t + h_t L_t$ at time t . The implications of this particular choice of conditional probabilities for the shape of the yield curve can be obtained by analyzing the price of a bond with time to maturity T :

$$\begin{aligned} P(t, t+T) &= e^{-L \sum_{k=t+1}^{t+T} h_k} E_t^Q \left(e^{\sum_{k=t+1}^{t+T} r_k} \right) \\ &= e^{-L(\sum_{k=1}^M H_k(t_k - t_{k-1}) + H_{M+1}(T - t_M))} P_{defaultfree}(t, t+T) \end{aligned} \quad (6)$$

where M is such that $t_M < T < t_{M+1}$ and $P_{defaultfree}(t, t+T)$ represents the time t price of a default-free bond with time to maturity T .

Note that this model implies a parametric arbitrage-free model⁶ for the term structure of interest rates:

$$R(T) = R_{defaultfree}(T) + \frac{L}{T} \left(\sum_{k=1}^M H_k(t_k - t_{k-1}) + H_{M+1}(T - t_M) \right) \quad (7)$$

The term structure is expressed as a risk-free curve added to a constant and a hyperbolic function which changes its curvature according to the different levels of conditional probabilities of default. An interesting point of this model is the existence of a closed-form formula for the term structure of interest rates and its simultaneous compatibility with absences of arbitrage in the market, a characteristic which is not shared by most of the parametric term structure models.⁷ In this paper we estimate cross-sectional versions of the model and therefore only deal with the risk-neutral probability measure. On the other hand, if we were interested in analyzing a dynamic version of the model by considering transition densities of the bonds, a stochastic hazard rate would be more appropriate in order to produce a stochastic spread curve. Moreover, the piecewise constant structure implies a parametric arbitrage-free version of the flat-forward method, frequently adopted by market participants to interpolate interest rate curves. The flat-forward method assumes that forward rates between observed interest rate zero-coupon rates are taken to be piecewise constant with no economic justification. In some sense, our model brings economic justification to this widely adopted method.

⁶Parametric term structure models are those which present a closed-form formula for the term structure of interest rates. For instance, Nelson and Siegel (1987) represent the term structure as a linear combination of exponential functions.

⁷See Sharf and Filipovic (2004) for an exception: an example of a parametric term structure model which is also arbitrage-free.

Figure 2 presents an example of an implied term structure of spreads over the risk-free rate, for a fixed loss fraction $L = 0.5$, when we adopt respectively one, two or three default factors, represented here by the conditional probabilities of default H_1 , H_2 and H_3 from equation (5). The top plot considers the spread structure when only H_1 describes all conditional probabilities of defaults. In this case, the spread is constant and captures, as a particular case, the static spread method widely adopted by market participants. When a medium-term default factor is introduced (represented by the existence of H_2), the medium-term spreads take the form of a concave function as can be observed in the second plot of Figure 2. Finally, when a third default factor, representing long-term defaults, is included, long-term spreads are allowed to have a curvature distinct from medium-term spreads, offering more flexibility to the shape of the spread curve. This is shown in the third plot. The values adopted to generate this example were $H_1 = 0.08$, $H_2 = 0.1$ and $H_3 = 0.12$, $t_0 = 0$; $t_1 = 5$; $t_2 = 10$; $t_3 = 20$.

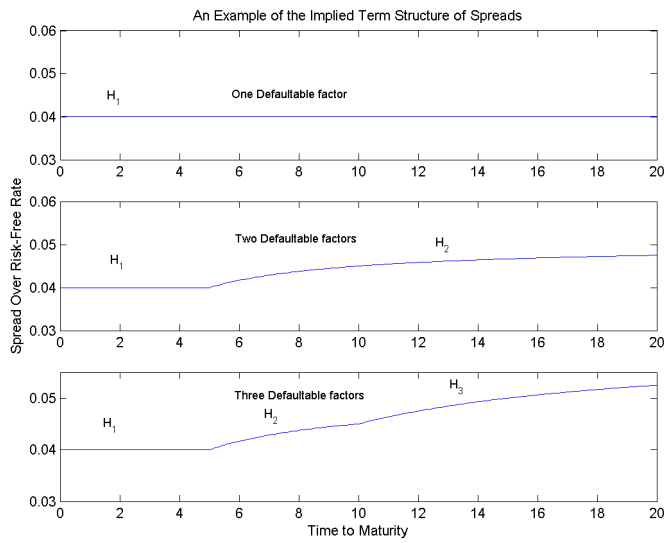


Figure 2
An example of implied term structure of spreads

2.2 Estimation process

We assume that the prices $P_j, j = 1, \dots, m$ and cash flows $C_{jk}, j = 1, \dots, m; k = 1, \dots, \gamma_j$ of m defaultable bonds are observed, with error. Cash flow k of bond j is paid at T_{jk} . Given a value for the loss rate L ,⁸ we apply a nonlinear least squares to minimize pricing errors so as to obtain the values of the constants $H_i, i = 1, \dots, M$ that determine the implied probabilities of default:

$$P_j = \sum_{k=1}^{\gamma_j} C_{jk} e^{-T_{jk} R(T_{jk})} + \epsilon_j \quad (8)$$

where $R(\cdot)$ is parameterized by equation (7).

An important issue on the definition and estimation of the model is related to the number of factors M that will drive the conditional probabilities of default. Since the residuals from the estimation process come from a nonlinear pricing equation (8), usual asymptotic t significance tests would not be a good choice. Instead, a formal statistical significance test could be performed with the use of a nonparametric bootstrap method (Davison and Hinkley, 1997), or alternatively with the use of information criteria methods (see, for instance Sakamoto et al. (1986)). However, in the empirical section, we choose to fix the number of factors at three, consistently with seminal results in Litterman and Scheinkman (1991), later confirmed by a large number of papers⁹ indicating that a large number of term structures around the world have their movements primarily driven by three factors. In particular, Almeida et al. (2003) applied principal components to a time series of data in the Brazilian global market (the market analyzed in this paper), finding that three movements are important to drive this term structure.

Another important point for discussion is the assumption of a constant predetermined loss rate, which is usual in the literature of reduced-form models. Duffie and Singleton (1999) clearly show that based on only bond data, the processes L_t and h_t are not separately econometrically identifiable.¹⁰ In this paper, we adopt a common market strategy that is to fix the loss rate at a certain value that is the market expected loss rate within each sovereign bond market (see Beinstein and Scott (2006) or Doctor and Goulden (2008)). This strategy has also been adopted in a number of recent academic papers including Houweling and Vorst (2005), Pan and Singleton (2007). What is important to bear in mind is that compared to implied probabilities of default, the loss rate is a secondary variable since usually market participants consider only a few possible values as possible

⁸Recovery rate is directly related to loss rate via $\phi = (1 - L)$.

⁹See, for instance Almeida et al. (2003) for an analysis of the Brazilian global market, Dai and Singleton (2000) for U.S. swaps, and Tang and Xia (2007), for Japan, U.S. and U.K., among many others.

¹⁰If data on nonlinear instruments, such as credit default options, is available, then these two processes can be separately identified. More recently, Pan and Singleton (2007) showed a way to separately identify these two processes from credit default swap data.

for a market expectation consensus like 0.15, 0.2, 0.25, 0.4, 0.5 or 0.75. So, even if a certain investor does not believe that the market expected loss rate is correct, or in other words, that bond prices are correct, he/she can back out implied probabilities of default with his/her model using the market expected loss rate and then recalculate bond prices with his/her own expected loss rates.

3. Empirical Results

3.1 Data

The data consist of prices and cash flows of Brazilian global bonds on three different days with very distinct market behavior: 10/08/2001, the day of worst results for global bonds in 2001, after the September 11 external crisis; 09/27/2002, the day of smallest global bond prices considering their whole history, reflecting a strong internal Brazilian crisis; and 11/19/2004, a day in the middle of a period reflecting economic stability. Table 1 presents global bonds characteristics, such as date of settlement, date of maturity, and coupon values.¹¹ Bootstrapped¹² U.S. swap rates represent our risk-free term structure of interest rates. These risk-free rates are obtained from a set of Libor short-term rates (3m, 6m, and 1 year) and U.S. swaps for maturities 2, 3, 5, 7, 10, 20, and 30 years. The data were collected at a Bloomberg terminal.¹³

Table 1
Cash flow characteristics of global bonds

Instrument	Date of settlement	Date of maturity	Coupon
Brasil 2004	04/30/1999	04/15/2004	11.625%
Brasil 2005	05/17/2001	07/15/2005	9.625%
Brasil 2007	07/26/2001	07/26/2007	11.25%
Brasil 2008	03/12/2002	03/12/2008	11.50%
Brasil 2009	10/25/1999	10/15/2009	14.50%
Brasil 2010	04/16/2002	04/15/2010	12.00%
Brasil 2011	08/07/2003	08/07/2011	10.00%
Brasil 2012	01/11/2002	01/11/2012	11.00%
Brasil 2013	06/17/2003	06/17/2013	10.25%
Brasil 2014	07/14/2004	07/14/2014	10.50%
Brasil 2019	10/14/2004	10/07/2019	8.88%
Brasil 2020	01/26/2000	01/15/2020	12.75%
Brasil 2024	03/22/2001	04/15/2024	8.88%
Brasil 2027	06/09/1997	05/15/2027	10.13%
Brasil 2030	03/06/2000	03/06/2030	12.25%
Brasil 2034	01/20/2004	01/20/2034	8.25%
Brasil 2040	08/17/2000	08/17/2040	11.00%

¹¹All globals pay semi-annual coupons.

¹²Bootstrapping is a recursive procedure, which extracts zero-coupon yields from a set of coupon-paying instruments, that might be bonds or swaps. See Brigo and Mercurio (2001) for a detailed explanation.

¹³For more information on this data feeder, see www.bloomberg.com.

3.2 Comparison of default probabilities in 2001, 2002 and 2004

Tables 2, 3, and 4 present observed and model-implied prices and spreads over treasury¹⁴ for globals on the three previous dates selected. Note how different the observed prices (and consequently the spreads) were during these three moments. For instance, the average price of the bonds presented in Table 2 was \$77.07. Compare this value to the average prices of this same subset of bonds extracted from Tables 3, and 4, which were, respectively, \$46.41 and \$120.24, to identify that, on average, these bonds had a loss of 40.2% of their value when investor's risk perception deteriorated from 10/08/2001 to 09/27/2002, and after that they had a 159% increase in value when the risk perception turned from very pessimistic in 2002 into considerably optimistic in 2004.

Table 2
Observed and estimated prices and spreads on 10/08/2001

Instrument	Estim. price	Observ. price	Estim. spread	Observ. spread
Brazil 2004	93.74	94.10	1129	1110
Brazil 2005	84.32	83.40	1122	1159
Brazil 2007	83.27	83.90	1127	1108
Brazil 2008	74.71	77.35	1112	1033
Brazil 2009	92.59	90.50	1147	1198
Brazil 2020	72.15	73.00	1288	1265
Brazil 2027	59.44	60.00	1205	1188
Brazil 2030	69.51	70.35	1256	1234
Brazil 2040	63.55	61.00	1218	1294

Table 3
Observed and estimated prices and spreads on 09/27/2002

Instrument	Estim. price	Observ. price	Estim. spread	Observ. spread
Brazil 2007	51.13	51.00	2854	2863
Brazil 2008	49.92	50.00	2794	2789
Brazil 2009	53.33	54.00	2785	2746
Brazil 2010	47.51	46.50	2636	2698
Brazil 2012	43.92	44.25	2469	2450
Brazil 2020	46.20	46.25	2416	2413
Brazil 2024	37.74	38.50	1989	1941
Brazil 2027	40.45	39.75	2129	2175
Brazil 2030	45.18	45.50	2340	2320
Brazil 2040	42.38	42.00	2216	2240

¹⁴The spread over treasury of a bond is defined as the difference between its internal rate of return and the internal rate of return of a U.S. bond with the same duration of the risky bond. For a definition of duration, see Fabozzi (2001).

Table 4
Observed and estimated prices and spreads on 11/19/2004

Instrument	Estim. price	Observ. price	Estim. spread	Observ. spread
Brazil 2007	113.89	114.00	212	208
Brazil 2008	116.23	116.15	240	243
Brazil 2009	129.77	129.50	329	335
Brazil 2010	120.28	120.60	340	333
Brazil 2011	111.32	110.25	364	383
Brazil 2012	116.37	116.25	374	376
Brazil 2013	112.07	113.25	389	371
Brazil 2014	113.27	113.75	399	393
Brazil 2019	100.29	100.50	402	399
Brazil 2020	128.64	128.25	443	447
Brazil 2024	98.88	99.00	404	403
Brazil 2027	108.54	109.00	426	421
Brazil 2030	125.71	126.00	458	455
Brazil 2034	92.77	92.40	390	394
Brazil 2040	115.50	114.70	446	453

The model was estimated using the methodology described in Section 2. We decided to adopt a total of three constants to describe the conditional default probabilities defined by Equation (5). These three constants would stand for short-term, medium-term and long-term expected default rates, similarly to the interpretation of Nelson and Siegel's (1987) forward rate functions. More factors could be easily introduced, but for each introduced factor we would be specializing the ability of the default factors to smaller intervals of time, and the model would become less parsimonious. Based on the last observation, it is interesting to note that a second important decision concerns the interval sizes attached to each factor. For a formal econometric selection, we would incur serious computational time due to the combinatorial nature of the problem.¹⁵ Despite that, we decided to arbitrarily fix the short-term factor to be active from time to maturity zero up to the maturity of the shortest maturity global bond.¹⁶ The medium-term factor is active for five years, beginning at the point of inactivity of the short-term factor.¹⁷ Finally, the long-term factor is active from the point of inactivity of the medium-term factor up to the maturity of the longest maturity global bond, the 2040 global. For instance, on 09/27/2002, there were 5 years up to the maturity of the 2007 global. Then, the short-term factor was active from 0 to 5 years, the medium-term factor was active from 5 to 10 years, and the long-term factor from 10 to 38 years.

¹⁵For instance, we would have 40 years of maturity to split between three factors. The total number of possible ways to do that, considering only whole numbers on the split, is $\frac{42!}{40!2!} = 861$.

¹⁶The 2004 global when the 2001 dataset is adopted, and the 2007 global when the remaining datasets are adopted.

¹⁷With an exception in 2004, where it is active for 7 years, to compensate for low probabilities of default in the medium-term range.

Note in Tables 2, 3, and 4 how observed and estimated prices were close. The root mean square errors¹⁸ (RMSE) were respectively \$1.52, \$0.54 and \$0.51, which are considerably low values when compared to average prices. Table 5 presents the values of predetermined fixed recovery rates, conditional default probabilities, accumulated probabilities of default, and forward default probabilities¹⁹ obtained for the three datasets of global bond prices. Figure 3 depicts a plot of the conditional probabilities of default functions extracted from bond prices. Note how probabilities present clearly different orders. For instance, short-term conditional probabilities of default in 2002 were higher than all conditional default probabilities extracted from 2001 and 2004 global bond prices. The long-term probabilities, which reflect market fundamental expectations regarding the Brazilian financial credibility, are in 2002, around eight times the value obtained in 2004 (66.88% in 2002 against 8.79% in 2004). In 2002, close to the moment when President Lula assumed his position, the market had a catastrophic view for the future of the Brazilian financial market, practically expecting a default on a long-term basis, with certainty. This is directly quantified by the model that for a fixed recovery rate of 20% obtained a long-term conditional probability of default of 66.88%, with an accumulated probability of default of 97.25% for the period between 2002 and 2012. That is, the market expected in 2002, a 97.25% chance that default would occur in the next 10 years. This value dropped to 49.13% in 2004 as can be observed in Table 5. Figure 4 shows accumulated probabilities of defaults. The higher the slope, the faster the market would be expecting a default. For instance, in 2002, market participants expected a default with probability very close to one, in the next 15 years, in 2001, in the next 22 years, and in 2004, after the next 50 years. Other comparisons could be performed if we wanted to explore the term structure of default probabilities. For instance, we could use the forward probabilities of default, which are the conditional accumulated probabilities of default between two intervals of time, to compare one fixed interval of time (say the period between 2007 and 2010) among the three models estimated based on the three different Brazilian economic moments (2001, 2002 and 2004).

¹⁸Defined as $\frac{1}{m} \sum_{j=1}^m (P_j - \hat{P}_j)^2$.

¹⁹Letting p_t denote survival probability up to time t , fixed $s > t$, the forward default probability for the period between t and s is defined by $1 - \frac{p_s}{p_t}$.

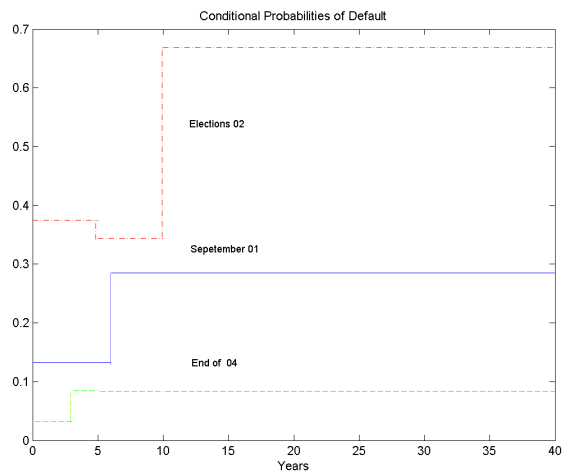


Figure 3
Conditional probabilities of default at different economic moments

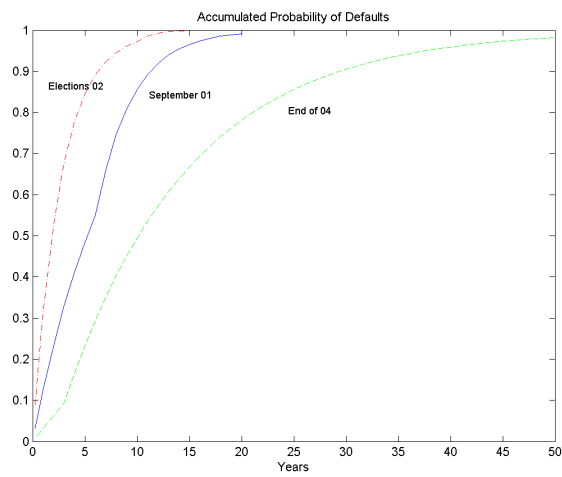


Figure 4
Accumulated probability of defaults at different economic moments

Table 5
Model-implied measures of default

Date	Default factor	Average recovery rate	Cond. default prob.	Accum. prob. default	Forw. default prob.
10/08/2001	short-term (0 to 3 years)	15.0%	12.63%	31.53% (up to 3 y)	31.53%
10/08/2001	medium-term (3 to 8 years)	15.0%	13.04%	64.43% (up to 8 y)	47.99%
10/08/2001	long-term (8 to 39 years)	15.0%	30.75%	100% (up to 39 y)	99.99%
09/27/2002	short-term (0 to 5 years)	20.0%	37.15%	83.84% (up to 5 y)	83.84%
09/27/2002	medium-term (5 to 10 years)	20.0%	34.00%	96.27% (up to 10 y)	81.22%
09/27/2002	long-term (10 to 38 years)	20.0%	66.01%	99.8% (up to 38 y)	99%
11/19/2004	short-term (0 to 3 years)	40.0%	3.40%	9.68% (up to 3 y)	9.68%
11/19/2004	medium-term (3 to 10 years)	40.0%	8.27%	49.50% (up to 10 y)	44.04%
11/19/2004	long-term (10 to 36 years)	40.0%	8.85%	95.53% (up to 36 y)	90.48%

Average recovery values were taken from the Bloomberg website as a consensus from the market around each of the three dates analyzed. Forward default probabilities are conditional for periods in between factors. For instance, in the second line last column, 47.99% indicates a conditional probability of default between 3 and 8 years.

One point which is worth mentioning is that the 2040 global bond presents an embedded call option within its cash flow. During the three dates analyzed here the option was deep out-of-the-money and we could price the bond without taking the option into account. However, if an analysis were to be performed with more recent data, the option would be in-the-money and should be priced. In this case, we would have to adopt an option pricing model such as Black et al. (1990), or Hull and White (1990), among others.

3.3 Pricing an out-of-sample bond: The 2030 global

Once a reduced-form model is used to extract the term structure of interest rates from sovereign bonds, it is a simple task to provide model-implied fair prices for all those bonds. However, one may argue that even when the model has a good in-sample fit, it might be only for statistical reasons (we are minimizing the RMSE), and not because the model explains important aspects of this assets valuation. In order to test the validity of the model, from an asset pricing perspective, we will perform a simple out-of-sample exercise: we will estimate the Brazilian term structure of sovereign bonds, excluding one bond, and then we will compare the model-implied fair price to the observed price of this bond. Considering practical purposes, this is useful whenever a new bond is issued in the market and needs to be priced.

Figure 5 presents a plot frequently adopted by financial institutions: a spread \times duration²⁰ plot for the Brazilian global bonds on 11/19/2004. Market values are represented by stars, model-implied values are represented by dots, and the whole line represents a quadratic polynomial fit of the observed market data usually adopted by practitioners to make decisions about investment opportunities in

²⁰Roughly speaking, it represents the cash flow of a coupon-bearing bond by a simple zero-coupon cash flow with time to maturity equal to a properly weighted average of all the cash flows of the original bond, and with yield to maturity equal to the internal rate of return of the original bond.

emerging markets. It is important to say that this picture was obtained after an estimation of the model including all global bonds presented in Table 4. At this point, we have an important choice to make: which bond will be left out of the sample estimation dataset? Observing Figure 5 we have chosen the 2030 global because it presents the highest spread among all Brazilian global bonds. By taking the bond with the highest spread, we want to show that our model performs clearly better than an interpolation scheme. On 11/19/2004, the market closing price for the 2030 global bond was \$ 126.00, which implied a value for the spread over treasury equal to 455.46 basis points (bps).²¹ While its spread market value corresponded to 455.46 bps, the model estimated a spread of 459.98 bps²² for this bond when it was included in the estimation sample. After re-estimating the model without considering the 2030 global, the out-of-sample implied spread was 463.4 bps, an error of less than 8 bps. If we were applying a linear interpolation scheme of the spreads, the error would be much higher, as can be observed by noting the line connecting the spreads of 2019 global and 2027 global, the two globals with duration values closest to the duration of the 2030 global. This line would imply a spread of 414.5 bps for the 2030 global, an error of 41 bps, five times greater than the model-implied error.

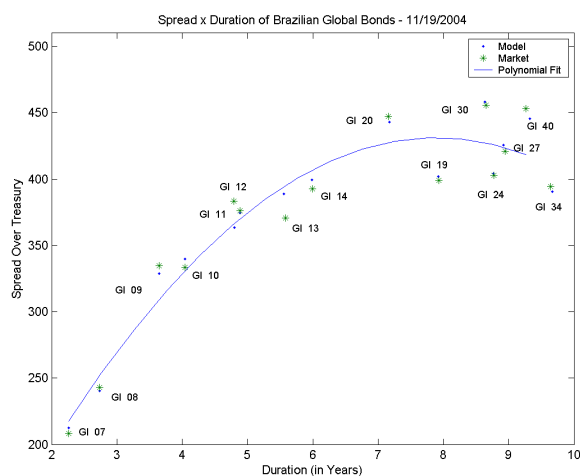


Figure 5

Spread over treasury \times duration plot for Brazilian global bonds on 11/19/2004

²¹Pointed out by a star in Figure 5 under the label “GL30”.

²²The model in fact estimated a fair price of \$ 125.5, which is readily converted to the spread of 459.98 bps using the formulas from Fabozzi (2001).

4. Conclusion

We propose and implement a dynamic term structure model useful to price defaultable bonds. The goal was to extract default probabilities from Brazilian global bond data and analyze the risk perception of investors during different economic periods. The model builds on Duffie and Singleton's (1999) work, writing the conditional probabilities of default driving the dynamics of bond prices as a piecewise constant function. In the empirical implementation, the dynamic factors driving defaults are specialized to three: a short-term, a medium-term and a long-term factor, resembling Nelson and Siegel's (1987) interest rate factors. It is shown that the model is able to offer intuitive results relating levels of default probabilities to observed bond prices, as suggested by a joint analysis of Tables 2, 3, 4, and Figure 4. An empirical out-of-sample pricing exercise of the 2030 global bond provides a consistent result, with an error of less than 8 bps, indicating that this model might be useful to price newly issued bonds. In summary, the model is consistent with no-arbitrage, parsimonious, and presents easy interpretation of its results. One possible extension for a future work would be to consider more general functions to represent the conditional probabilities of default, such as linear, or quadratic functions of maturity, or even stochastic functions like the one considered by Duffie et al. (2003).

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