

# AUCTIONS OF IDENTICAL OBJECTS WITH SINGLE-UNIT DEMANDS: A SURVEY\*

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## Abstract

The theory of auctions of a single object generalizes to a situation where identical objects are sold either sequentially or simultaneously but individuals can only buy one object. In this context, I will present a survey of the main results regarding the ranking of auctions based on revenue, bidding behaviour, effects of entry fees and reserve prices, and other strategic issues.

## Resumo

A teoria de leilões de um único objeto pode ser generalizada para o caso no qual diversos objetos idênticos são vendidos simultanea ou sequencialmente desde que os participantes demandem apenas um único objeto. Este artigo apresenta uma resenha dos principais resultados da literatura com relação ao ranking de leilões de acordo com receita esperada, ao comportamento individual na elaboração de lances em equilíbrio, aos efeitos de taxas de entrada e de preços mínimos, além de outras considerações de natureza estratégica.

*Palavras-Chave:* Auctions of identical objects; single-unit demands.

*Código JEL:* D44.

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## 1. Introduction.

Suppose you own a unique object such as a rare painting or a medieval fifty-room castle. As there may not be established markets for the sale of either of these commodities, you may decide to employ an auction. You will be able to obtain very clear advice about single-object auctions as there is a well-established theory<sup>1</sup> based on the pioneer work of Vickrey (1961) and on the seminal paper of Milgrom and Weber (1982).

Now suppose you own several unique objects such as plots of land facing undeveloped parkland or 1-800 phone numbers that you can assign to particular individuals. Unfortunately, clear advice on how to proceed in this circumstance is not available — the theory of multi-object auctions is still in incipient form. Nevertheless, the theory of auctions of a single object can be applied here, albeit in a rather special case. Thus, the special case analysed in this paper is the auction of identical objects when individuals demand only one object.

In this context it is worth explaining the differences that might arise when analysing multi-object auctions vis-a-vis single-object auctions. When competing for a single object — assuming that we limit ourselves to auctions where the winner is the individual with the highest bid — each buyer will try to outbid everyone else. To do so, it suffices for a bidder to outbid the opponent with the highest bid. Therefore, in some sense (i.e., in a symmetric equilibrium), each bidder can concentrate on estimating a single bid.

Now suppose  $n$  bidders are competing for  $k$  identical objects with  $n > k$ . Depending on the auction rules, it is possible that

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<sup>1</sup>There are several surveys and introductory papers available, including Engelbrecht-Wiggans (1980), Klemperer (1998), McAfee and McMillan (1987), Milgrom (1985, 1987, 1989), Riley (1989) and Wolfseter (1997).

each individual bidder has to forecast up to  $k$  bids. However, when individuals only want one object and in the case of auctions where the bidders with the  $k$  highest bids win one object each, then each individual needs only to forecast a single bid – the highest of the other players – in a symmetric equilibrium. Given this analogy, it is not surprising that both the intuition and the theoretical results from single-object auction theory apply in this case.

Among auctions of identical objects we can distinguish between simultaneous sales – where objects are sold at the same time – and sequential sales — where objects are sold one after another in a pre-arranged fashion. To use the terminology of Weber (1983), we can distinguish between simultaneous-dependent auctions, in which bidders make only one bid and the objects are awarded based upon those bids, and simultaneous-independent auctions, in which bidders must simultaneously submit bids in several different auctions of individual objects, and the award of an object to a bidder is independent of the awards of other objects. In a situation where individuals only want one object, simultaneous-independent auctions may expose a bidder to the risk of obtaining more than one object, which may cause inefficiencies. Thus, we will concentrate on simultaneous-dependent auctions and we will refer to such auctions as simultaneous auctions.

Sequential auctions are potentially more complex than simultaneous auctions because there may be reputation or signalling effects. For example, a buyer who has a very high valuation may not want to bid very aggressively in earlier rounds if she believes that by doing so she will convince other players that her valuation is low and, therefore, other players may take that into account when bidding for later units. Nevertheless, we will show that under some circumstances, both sequential and simultaneous auctions may generate the same expected revenue. Moreover, both types of auctions can be made optimal — in the sense that they maximize the seller's expected revenue —by introducing a reserve price.

We will concentrate on the analysis of the following types of auctions of  $k$  objects with  $n$  bidders ( $n > k$ ) with single-unit demand : simultaneous-dependent discriminatory and uniform-price auctions, and sequential first- and second-price auctions. In a discriminatory auction, each of the  $k$  highest bidders pays the amount she bids. In a uniform-price auction, each bidder pays the amount of the  $k + 1^{st}$  highest bid, that is, the highest rejected bid. A first-price sequential auction is simply a sequence of  $k$  first-price sealed-bid auctions, where the winner in each round is the bidder with the highest bid and she pays her bid. A second-price sequential auction is defined similarly as a sequence of  $k$  second-price sealed-bid auctions where the winner in each round is the bidder with the highest bid and she pays the equivalent of the second highest bid in that particular auction.

Auction models typically fall into three categories. In private values model, each potential buyer knows her own value for the object, which is not influenced by how other potential buyers value it. If individuals' valuations are independent from each other — for example, one may think of valuations being determined by independent draws from a fixed distribution — then we have the independent private value (IPV) model. If valuations are dependent of each other, then we have a dependent private value model. A private values model might be most appropriate for nondurable goods with no resale value.

In the common value model, the object is worth the same to every potential bidder, but this value is unknown at the time of bidding. Typically, individuals have some information about the (unknown) true value of the object. If information is correlated across individuals, then we have a dependent common value model. If information is independent across individuals, then we have an independent common value model. The common value model is appropriate for analysing the sale of mineral rights and offshore oil drilling leases.

Finally, Milgrom and Weber (1982) introduce the notion of affiliated values, which includes both private and common values as special cases. Roughly speaking, affiliated values capture the idea that individuals' valuations for an object have a private component but are influenced by how other people value it. In most sales we can imagine, a bidder's valuation for the object being sold does have a private component, but that valuation it is also influenced by her participation of other individuals' valuations. For instance, when bidding for a house, one takes into account both the personal value of the house as well as how easily it would be to resell it in the future.<sup>2</sup>

In general, the single-unit demand and independent private values assumptions oversimplify the problem by ignoring interesting and relevant issues. However, there are many examples where such assumptions might be justifiable. One such example is the sale of randomly generated toll-free phone numbers where resale of the numbers is not allowed. Given the existing technology, firms will typically need only one such number. Given that resale is not allowed, bidding for a strategic reason will be limited. That is, the unit demand assumption seems reasonable in this case. Moreover, such phone numbers do not have any value per se. It is simply a sales/marketing tool and its value to a firm might not be influenced by how other firms value it. In addition, we may think of firms' values across different industries as being determined independently from each other. That is, the independent private values assumption might be justifiable.

Thus I present, in section 2, the main results for the IPV model. Namely we will characterize bidding strategies for a family of auctions and show that these auctions should generate, on average, the same revenue. We will then characterise the auction that maximizes

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<sup>2</sup>The discussion above relates to how individuals value objects. Of course, even in the IPV model, bidders' bidding behaviour will depend on how they think others will bid.

the seller's expected revenue, and examine the effects of entry fees, reserve prices and increased competition. In section 3 we will discuss a more general model and some of the results. Section 4 outlines some of the existing research on multi-unit auctions and some of the existing empirical evidence. Finally, section 5 concludes.

## 2. The Independent-Private Value Model.

Let  $v_i$ ,  $i = 1, \dots, n$ , denote player  $i$ 's value for any of the  $k$  ( $k < n$ ) identical objects. It is assumed that potential buyers demand only one object. Each buyer knows her own valuation but only knows the distribution of her opponents' valuations. It is assumed that bidder  $i$ 's valuation ( $i = 1, \dots, n$ ) is determined by an independent draw from a fixed distribution  $F(v)$  with positive density  $f(v)$  on  $[0, \bar{v}]$ . Buyers are assumed to be risk-neutral. For simplicity we consider the case when the seller assigns zero value for the objects.

For convenience, let  $v_1^n, v_2^n, \dots, v_n^n$  denote the order statistics of the  $n$  draws. For instance,  $v_1^n$  is the first-order statistics of  $n$  samples from this fixed distribution  $F(\cdot)$ ,  $v_2^n$  is the second-order statistics from  $n$  samples, and so on. As we will use the uniform distribution on  $[0, 1]$  in most of our examples, the next fact summarizes some important properties of this distribution.

**Fact 1:** The expected value of the  $m^{th}$  order-statistics of  $n$  draws from a uniform  $[0, 1]$  distribution is  $\frac{n+1-m}{n+1}$ ,  $1 \leq m \leq n$ .

**Fact 2:** Conditional on  $v$  being the highest value, the expected value of the  $(m-1)^{th}$  order-statistics from  $n-1$  draws from a uniform  $[0, v]$  distribution is  $\frac{n+1-m}{n}v$ ,  $1 \leq m-1 \leq n-1$ .

The equilibrium notion we use is that of a Bayesian Nash equilibrium. This notion, due to Harsanyi, is a natural extension of the concept of Nash equilibrium (each player's strategy is a best

response to the best response of other players) to static games of incomplete information. In this case, each player chooses a strategy that maximizes her expected payoff given that other players are also choosing strategies to maximize their expected payoffs. We will focus on symmetric equilibrium where every bidder chooses the same bidding function.<sup>3</sup>

## 2.1. Sequential auctions.

We now look at first and second-price sequential auctions. As in the remainder of this survey, we start with an example and then we state the corresponding theorems.

**Example 1:** *Suppose three potential buyers ( $n = 3$ ) are bidding for two identical objects ( $k = 2$ ) and each bidder only wants one object. Suppose the seller assigns zero value for the objects, and there is no reserve price or entry fee. We suppose further that each player  $i$ ,  $i = 1, 2, 3$ , knows her own value  $v_i$ , but only that her opponents' values are independent draws from the uniform  $[0, 1]$  distribution. Objects are sold sequentially and, at the end of each auction, the auctioneer announces the price at which the object was sold. Recall that at a first-price sequential auction, the winner in each round is the bidder with the highest bid and she pays her bid.*

### First-price sequential auctions.

We will solve the game backwards. At the end of the first round there are two potential buyers and one object left. This is simply a one-shot first-price auction where a bidder with value  $v$  bids in such a way to just outbid her opponent conditional on  $v$  being the highest

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<sup>3</sup> Although the sequential auctions we examine are dynamic games, the assumptions we impose allow us to think of them as static games. This should be clear later.

value. Given that her own value is  $v$ , and considering that bids are increasing in values (one has to check that this is indeed the case in equilibrium) so that the bidder with the highest value in each round wins, her bid is the expected value of the largest of the other players (there is only one opponent in this case) given that her value is  $v$  and she has the highest value, i.e., she bids

$$b_{FP}^2(v) = E[v_3^3 \mid v_2^3 = v] = \frac{v}{2}.$$

(Just apply Fact 2 with  $n = m = 2$ .)

Now, prior to the first round there are three players and two objects left. Moreover, bidders know that if they miss out on the first object, then given the second-round strategies, the loser with the highest value wins the other object. Therefore, to win one of the objects it suffices to outbid the opponent with the second-highest value, i.e., to win an object it suffices to outbid the bidder with the lowest value. That is, a bidder with value  $v$  bids in the first round as follows:

$$b_{FP}^1(v) = E[v_3^3 \mid v_1^3 = v] = \frac{v}{3}.$$

(Just apply Fact 2 with  $n = 3$  and  $m = 3$ .)

Notice that the bidding functions  $b_{FP}^1(v)$  and  $b_{FP}^2(v)$  are increasing in  $v$ . Thus, a bidder is guaranteed to win one of the two objects if her value is among the two highest values. If her value is the highest, she will win the first auction. If her value is the second highest, she will win the second object. This suggests that for a player to be indifferent between winning one of the two identical objects, the price she pays has to be the same. That is, the expected revenue must be identical in both rounds. The expected revenue in the first round is simply the expected value of the highest bid. From fact 1,



the expected value of the highest of three samples from the uniform  $[0,1]$  is  $\frac{3}{4}$ . Therefore,

$$Ep_{FP}^1 = \frac{3}{4} \frac{1}{3} = \frac{1}{4}$$

Similarly, since the expected value of the second highest of three samples from the uniform  $[0,1]$  is  $\frac{2}{4}$ , the expected price in the second round is given by

$$Ep_{FP}^2 = \frac{2}{4} \frac{1}{2} = \frac{1}{4}.$$

### Second-price sequential auctions.

At the end of the first round, there are two bidders and one object left. Therefore, it is simply a single-object second-price auction. In this case, each of the two bidders left bids their own valuations. That is,

$$b_{SP}^2(v) = E[v_3^3 \mid v_3^3 = v] = v$$

In the first round, each bidder again bids so as to be indifferent between winning one of the two objects. If a bidder has one of the two highest values, she will bid so as to outbid the opponent with the lowest value conditional on this information. That is,

$$b_{SP}^1(v) = E[v_3^3 \mid v_2^3 = v] = \frac{v}{2} \text{ (just apply Fact 2 with } n = m = 2 \text{)}$$

We can also compute the expected revenue of the seller. From fact 1, the expected value of the second highest of three samples from the uniform  $[0,1]$  is  $\frac{2}{4}$ . As you recall, the winner of each round is the bidder with the highest bid, but she pays the second highest bid. Therefore,

$$Ep_{SP}^1 = \frac{2}{4} \frac{1}{2} = \frac{1}{4}$$

Similarly, the expected value of the third highest of three samples from the uniform  $[0,1]$  is  $\frac{1}{4}$ . Therefore, the expected price in the second round is given by

$$Ep_{SP}^2 = \frac{1}{4}.$$

We can note a few regularities in the above example. First, for each type of auction, the expected revenue across rounds is the same. This can be understood as follows. There are two economic forces working. On one hand, there are fewer opportunities to win objects as the auction proceeds (increased competition for later units); on the other hand, there are less competitors in later sales (the winner always drops out from the auction). The fact that these two effects cancel out in the above example is not a coincidence. Second, both auction types generate the same expected revenue. This is no coincidence either as the next theorem states.

**Theorem 1:** (Weber (1983), Milgrom (1985)) In the IPV model with  $n$  risk-neutral bidders,  $k$  objects ( $n > k$ ), single-unit demands, and no entry fees or reserve price, the following holds:

- i.  $b_i^l(v) = b_{FP}^l(v) = E[v_{k+1}^n \mid v_i^n = v]$ ,  $i = 1, \dots, n$ , and  $l = 1, \dots, k$  is the unique symmetric equilibrium of the first-price sequential auction.
- ii.  $b_i^l(v) = b_{SP}^l(v) = E[v_{k+1}^n \mid v_{i+1}^n = v]$ ,  $i = 1, \dots, n$ , and  $l = 1, \dots, k$  is the unique symmetric equilibrium of the second-price sequential auction.
- iii. Both auction formats generate the same expected revenue, namely,  $kE[v_{k+1}^n]$ . Moreover, the sequence of prices in each auction format is a martingale.

This expected revenue equivalence result holds more generally than in Theorem 1 above and it will be explained in more detail later. The martingale result can be understood as part of the equilibrium condition. Perhaps it is easier to understand why different expected prices cannot occur in equilibrium; given that the objects are identical, if an individual expects the price of the object to fall in later rounds, she will wait until then driving down the price in earlier rounds. By the same token, if bidders expect the price to rise in later rounds, then there will be additional competition in earlier rounds driving the price up in earlier rounds. This is simply a restatement of the *law of one price*.<sup>4</sup>

## 2.2. Simultaneous auctions.

We motivate our analysis with a simple example of simultaneous discriminatory and uniform-price auctions.

**Example 2:** *As in example 1, suppose three potential buyers ( $n = 3$ ) are bidding for two identical objects ( $k = 2$ ) and each bidder only wants one object. Suppose the seller assigns zero value for the object, and there is no reserve price or entry fee. We suppose further that each player  $i$ ,  $i = 1, 2, 3$ , knows her own value  $v_i$ , but only that her opponents' values are independent draws from the uniform  $[0, 1]$  distribution.*

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<sup>4</sup>In section 4 we mention an empirical literature reporting violations of the law of one price in sequential auctions and some of the theoretical explanations for the such violations.

## Uniform-price auction.

Each bidder submits a sealed-bid.<sup>5</sup> The individuals with the two highest bids win one object each and pay the price equivalent to the highest losing bid.<sup>6</sup> We claim that an individual with value  $v$  bids, in the symmetric equilibrium, according to the function

$$b_U(v) = v = E[v_3^3 \mid v_3^3 = v].$$

To see this, we can use an argument that is analogous to the one used in a single-object second-price auction. Let us examine the game from Player 1's point of view. Suppose players 2 and 3 are bidding their values. If 1 bids anything less than her value, then she might not win an object even though she might have a value that is greater than at least one of her competitors. She would increase her expected profits by increasing her bid. Now suppose Player 1 considers bidding more than her value. If her bid is still below Player 2 and 3 bids, then the change in her expected profits is zero. However, she might win an object and run the risk of paying more than her value.

One can immediately verify that the expected revenue from this auction is twice the expected value of the third-order statistics (the expected value of the highest losing bid), that is,  $2 \times \frac{1}{4} = \frac{1}{2}$ . (Just apply Fact 1 with  $n = m = 3$ .)

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<sup>5</sup>For some of the complications that might arise when multiple bids are allowed see, for example, Engelbrecht-Wiggans and Weber (1979).

<sup>6</sup>In the IPV model, this auction format is strategically equivalent to an oral auction formalized as follows. Bidders bid by pressing a button. Bidding starts at a low price which is increased continuously. The auction ends when one of the bidders releases the button — stopping the increase in the price — with the remaining two bidders winning one object each. Each winner pays the same price, namely the price at which the other bidder has released the button.

## Discriminatory auction.

Now each bidder submits a sealed-bid and the individuals with the two highest bids win one object each. Each winner pays her bid. Recall that in the uniform-price auction, a player's bid does not affect how much she pays if she wins an object; it only affects her probability of winning. In contrast, in a discriminatory auction a player's bid determines how much she pays if she wins an object. Therefore, a player will now want to evaluate more precisely the minimum bid she can place and still be expected to win one of the two objects. Thus, a bidder with value  $v$  bids, in the symmetric equilibrium, according to the function

$$b_D(v) = E[v_3^3 \mid v_3^3 < v] = \frac{2v}{5}.$$

The expected revenue can be computed as follows

$$\frac{2}{5} \frac{3}{4} + \frac{2}{5} \frac{2}{4} = \frac{1}{2}$$

(From Fact 1, the expected value of the first-order statistics is  $\frac{3}{4}$  and the expected value of the second-order statistics is  $\frac{2}{4}$ ).

Note that both uniform-price and discriminatory auctions generate the same expected revenue, even though bidders' expected payments differ across these two auction formats. The next result states the solution for the case of general distributions of values.

**Theorem 2:** (Vickrey (1962); Ortega-Reichert (1968); Weber (1983), Milgrom (1985)) In the IPV model with  $n$  risk-neutral bidders,  $k$  objects ( $n > k$ ), single-unit demands, and no entry fees or reserve price, the following holds:

- i.  $b_i^U(v) = E[v_{k+1}^n \mid v_{k+1}^n = v] = v$ , for  $i = 1, \dots, n$ , is the unique symmetric equilibrium of uniform-price auction.
- ii.  $b_i^D(v) = E[v_{k+1}^n \mid v_{k+1}^n < v]$ , for  $i = 1, \dots, n$ , discriminatory auction.
- iii. Both auction formats generate the same expected revenue, namely,  $kE[v_{k+1}^n]$ .

### 2.3. Revenue equivalence.

From the above theorems we can conclude that all the four auction formats considered — first and second-price sequential auctions and uniform-price and discriminatory auctions — generate the same expected revenue for the seller. This is indeed a special case of a more general result. Consider any auction format where the individuals with the  $k$  highest bids win one object each. Let us examine this generic auction format from the point of view of one of the players, say Player 1. Suppose the bidding strategies of Players 2, ...,  $n$  are fixed. Player 1's choice of strategy determines a probability of winning one of the objects,  $p$ , and an expected payment,  $e$ . Given independence of values, Player 1's problem is to choose a bidding strategy  $b_1$ , given that her value is  $v_1$ , to maximize  $v_1 p(b_1) - e(b_1)$ .

Our definition of equilibrium implies that each bidder  $i$  solves a differential equation relating the particular  $p_i$  and  $e_i$  that Player  $i$  faces. Thus, in equilibrium, all the  $n$  functions  $e_i$  are determined by the  $n$  functions  $p_i$ s and by  $n$  boundary conditions. As the seller's expected revenue is simply the sum of the expected payments from the  $n$  bidders, it is also determined by the  $p_i$ s. The boundary condition is such that a bidder with the lowest possible value must make an expected payment of zero. This is summarized in the next theorem.

**Theorem 3:** (Engelbrecht-Wiggans (1988), Maskin and Riley (1989), Myerson (1981); Milgrom and Weber (1982)) In the IPV

model with  $n$  risk-neutral bidders,  $k$  objects ( $n > k$ ), single-unit demands, and no entry fees or reserve price, any auction format in which, in a symmetric equilibrium, the bidders with the  $k$  highest values are guaranteed to win one object each and in which a bidder with the lowest possible value has an expected payment of zero, generates the same revenue in this equilibrium, namely  $kE[v_{k+1}^n]$ .

One can readily check that the four auction formats considered above satisfy the conditions of the theorem. Given that bids are increasing in values, the individuals with the  $k$  highest values win one object each. Moreover, an individual with the lowest possible value, zero, bids zero in all four auction formats.

The above theorem, referred to as Revenue Equivalence Theorem, is one of the most celebrated results in the auction theory literature. The assumptions driving this result are risk-neutrality, independence, symmetry and single-unit demands. Indeed, revenue equivalence falls apart when any of these assumptions is violated. For example, by relaxing the risk-neutrality assumption, we obtain that the discriminatory auction yields more revenue than the uniform-price auction. This ranking is reversed when we assume that values are correlated instead of independent. (More on these issues later). There are several problems that arise when we relax either symmetry<sup>7</sup> or single-unit demand.<sup>8</sup>

## 2.4. The optimal auction and the effects of reserve prices and entry fees.

We now investigate the following question. Can we design an auction that maximises the expected revenue of the seller? For the

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<sup>7</sup>See, for example, Maskin and Riley (1986) and Lebrun(1995).

<sup>8</sup>See, for example, Ausubel and Cramton (1996), Menezes (1996), and Engelbrecht-Wiggans and Kahn (1998-a,b).

IPV model, the answer is yes. To motivate our analysis, let us revisit example 2 and concentrate on the uniform-price auction.

**Example 3:** *Suppose three potential bidders ( $n = 3$ ) are disputing two identical objects ( $k = 2$ ) and each bidder only wants one object. Suppose the seller assigns zero value for the object, and there is no reserve price or entry fee. We suppose further that each player  $i$ ,  $i = 1, 2, 3$ , knows her own value  $v_i$ , but only that her opponents' values are independent draws from the uniform  $[0, 1]$  distribution. For the uniform-price auction it was determined that in the unique symmetric equilibrium bidders submit a bid equal to their private values. The expected revenue was computed to be equal to  $\frac{1}{2}$ .*

Now let us suppose that the seller has established a reserve price equal to  $\frac{1}{3}$ . Players with values below  $\frac{1}{3}$  will either not bid or submit a zero bid. Thus, in the unique symmetric equilibrium a bid with value  $v$  bids according to the following function

$$b(v) = \begin{cases} v, & \text{if } v \geq \frac{1}{3} \\ 0, & \text{if } v < \frac{1}{3} \end{cases}$$

To compute the seller's expected revenue, notice that there are four relevant cases to consider. When all values are less than  $\frac{1}{3}$ , then the seller's expected revenue is equal to zero; when one of the values is less than  $\frac{1}{3}$  and two of the values are greater than  $\frac{1}{3}$ , then the seller's expected revenue is equal to  $2 \times \frac{1}{3}$  (recall that both winners expect to pay the equivalent to the maximum between the highest losing bid and the reserve price); when two of the values are less than  $\frac{1}{3}$  and one of the values is greater than  $\frac{1}{3}$ , then the expected revenue is  $\frac{1}{3}$  (the only winner is expected to pay the maximum between the reserve price and the highest losing bid — zero); and, finally, when all three players have values greater than  $\frac{1}{3}$ , the expected revenue is equal to  $2 \times \frac{1}{2}$  (both winners are expected to pay the maximum of the reserve price



and the highest losing bid  $-\frac{1}{2}$ ). Given the respective probabilities, the expected revenue when the reserve price is equal to  $\frac{1}{3}$  is

$$\frac{2}{3} \frac{12}{27} + \frac{1}{3} \frac{6}{27} + \frac{8}{27} = \frac{2}{3}$$

Notice that the expected revenue in this auction is bigger than in the case of a zero reserve price. By setting a positive reserve price the seller is effectively eliminating bidders with low valuations but extracting more surplus from bidders with high valuations, who now must pay not the highest losing bid but instead the maximum of such bid and the reserve price. The question is how should the seller set the reserve price. For example, we know that by setting a reserve price equal to zero, the expected revenue is  $\frac{1}{2}$ ; when the reserve price is equal to  $\frac{1}{3}$ , the expected revenue is equal to  $\frac{2}{3}$ ; and when the reserve price is equal to 1, the expected revenue is equal to zero. It turns out that we can write the expected revenue of the seller as a function of the number of players and the reserve price. Therefore, a simple maximization exercise implies that setting a reserve price equal to  $\frac{1}{2}$  is optimal. Let us compute the expected revenue in this case. There are three cases to consider. When all values are less than  $\frac{1}{2}$ , then the expected revenue is obviously zero; when one of the players has value below  $\frac{1}{2}$  and the two other players have value greater than  $\frac{1}{2}$ , then the expected revenue is  $2 \times \frac{1}{2}$ ; when two of the players have value below  $\frac{1}{2}$  and one player has value above  $\frac{1}{2}$ , then the expected revenue is equal to  $\frac{1}{2}$ ; Finally, when all three players have values above  $\frac{1}{2}$ , then the expected revenue is equal to  $2 \times \frac{5}{8}$ . Thus, taking into account the appropriate probabilities, the maximised value of the expected revenue is

$$\frac{3}{8} + \frac{3}{8} \frac{1}{2} + \frac{1}{8} \frac{10}{8} = \frac{23}{32}.$$

We now state the general theorem that characterises the optimal auction.

**Theorem 4:** (Myerson (1981), Bulow and Roberts (1989); Maskin and Riley (1989)) Consider the IPV model with  $n$  risk-neutral bidders,  $k$  objects ( $n > k$ ), single-unit demands, and the seller's value for the objects is equal to zero. Let the distribution function satisfy the following property

$$v - \frac{1 - F(v)}{f(v)} \text{ is increasing in } v.$$

Then under the hypothesis of the Revenue Equivalence Theorem, any auction mechanism that allocates up to  $k$  units to buyers – one unit per buyer – with the highest values in excess of  $v_0$ , where

$$v_0 = \max \left\{ v : v_0 - \frac{1 - F(v_0)}{f(v_0)} = 0 \right\}$$

maximizes the seller's expected revenue. In particular, all four of the above auctions are optimal as long as the reserve price is set at  $v_0$ .

It follows from the above theorem that the auctions considered in examples 1 and 2 are optimal<sup>9</sup> when the reserve price is set at

$$v_0 - \frac{1 - F(v_0)}{f(v_0)} = 0$$

or

$$v_0 - 1 + v_0 = 0$$

that is

$$v_0 = \frac{1}{2}$$

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<sup>9</sup>For the uniform  $[0,1]$  distribution,  $v - \frac{1 - F(v)}{f(v)} = v - (1 - v) = 2v - 1$ , which is increasing in  $v$ .

The optimal reserve price from example 3 can be understood as follows. Suppose that the reserve price is equal to  $v_0$ . The experiment is to evaluate the effect on the seller's expected revenue of an increase in the reserve price to  $v_0 + \varepsilon$ . The seller obtains an extra  $\varepsilon$  if she is able to extract extra surplus from one of the three buyers with a value greater or equal than  $v_0 + \varepsilon$ . This happens with probability  $3v_0^2(1 - v_0 - \varepsilon)$ . On the other hand, the seller's expected revenue is reduced by the whole amount  $v_0$  if the highest bid (i.e., value) is between  $v_0$  and  $v_0 + \varepsilon$ . The probability of this event is  $3v_0^2(v_0 + \varepsilon - v_0)$ . Therefore, the change in the seller's expected revenue,  $d(\varepsilon)$ , from increasing the reserve price from  $v_0$  to  $v_0 + \varepsilon$  is given by

$$d(\varepsilon) = \varepsilon 3v_0^2(1 - v_0 - \varepsilon) - 3v_0^2(v_0 + \varepsilon - v_0)v_0$$

Dividing both sides by  $\varepsilon$  and taking the limit as  $\varepsilon \rightarrow 0$ , we obtain that  $v_0 = \frac{1}{2}$ . This is just another way to see that this value of the reserve price maximizes the seller's expected revenue;  $v_0$  is simply the average between the seller's value and highest possible value of any of the bidders.

There are a few issues regarding the above theorem that are worth mentioning. First, note that the optimal reserve price does not depend on the number of potential bidders,  $n$ . Second, since the reserve price is above the seller's value for the object (which was assumed to be zero in the analysis above), the optimal auction mechanism may not be ex-post efficient. In example 3, with  $v_0 = \frac{1}{2}$  there is a  $\frac{1}{8}$  chance that none of the two objects are sold and a  $\frac{3}{8}$  chance that one of the objects is not sold. (Efficiency requires all two objects to be sold as the seller's value is equal to zero.)

At the same time, the auction formats considered in examples 1 and 2 and all auction formats described in Theorem 3 are efficient. This leads us to the final issue regarding Theorem 4, namely that there is an implicit assumption that the seller has to commit herself

not to sell any unsold objects in the future. There is no definite answer yet for the characterization of the optimal auction when the seller cannot make such commitment. The answer will depend, however, on the nature of the secondary market.<sup>10</sup>

Moreover, it is not difficult to see that a seller could achieve the maximised value of the expected revenue by instead resorting to entry fees or a combination of reserve price and entry fees. The distinction, of course, is that with a reserve price the seller extracts more surplus from the winner whereas with an entry fee the extra surplus is extracted from every bidder who participates in the auction.<sup>11</sup>

It should not be surprising either that the seller's expected revenue increases with the number of potential buyers in the IPV model. This follows from the fact that, in the absence of a reserve price, the expected revenue is simply  $k$  times the expected value of the  $k^{th}$ -order statistics. This expected value increases with  $n$  as the expected value of the any order statistics increases with  $n$ .<sup>12</sup> In the optimal auction, the value of  $v_0$  does not change with  $n$ , but the

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<sup>10</sup> See, for example, McAfee (1993) who shows that, for a special dynamic model, the optimal auction is such that the reserve price is set at the seller's valuation.

<sup>11</sup> Engelbrecht-Wiggans (1993) and Levin and Smith (1994) consider a single-object auction model where bidders decide whether or not to enter the auction without knowing their values for the object. In this context again the seller should either set a reserve price or charge an entry fee or to reserve price above the seller's value. The reason is that bidders are ex-ante identical. Therefore, the private gain from further entry equals the gain to society. Menezes and Monteiro (1997-a) consider a single-object model where bidders decide whether or not to enter after seeing their values. They show that, as in the standard model, the seller should charge a reserve price or an entry fee (or a combination of both).

<sup>12</sup> Menezes and Monteiro (1997-a) show that for a single-object IPV model with endogenous participation the seller's expected revenue may or may not increase with the number of potential bidders. The reason is that entry has two effects. On one hand, an additional potential bidder tends to increase competition and, therefore, increase seller's expected revenue. On the other hand, an additional potential bidder reduces the ex-ante profits to the winner and, therefore, causes the cut-off value that determines which bidders will enter to increase.

probability that there will be at least  $k$  bidders with values above  $v_0$  does increase with  $n$  and so does the seller's expected revenue.

## 2.5. The effect of risk aversion.

The revenue equivalence result of section 2.3 is quite remarkable. This result, however, breaks down when agents are risk averse. Let us revisit example 2 to illustrate the effects of risk aversion on bidding behaviour.

**Example 4:** *Suppose three potential bidders ( $n = 3$ ) are disputing two identical objects ( $k = 2$ ) and each bidder only wants one object, and there is no reserve price or entry fee. We suppose further that each player  $i$ ,  $i = 1, 2, 3$ , has some private information  $x_i$ . Each player knows her own signal but only that her opponents' signals are independent draws from the uniform  $[0, 1]$  distribution. Now assume buyer  $i$ 's expected utility is given by  $v_i(\cdot) = \sqrt{(\cdot)}$ .*

### Uniform-price auction.

It should be clear that a bidder who has signal  $x$  bids, in the symmetric equilibrium, according to

$$b_U(x) = x$$

The argument is exactly the same as in example 2.

### Discriminatory auction.

The derivation of equilibrium strategies when bidders are risk averse is quite evolved. Instead, to show that the revenue equivalence result breaks down it suffices to explain why risk-averse buyers may bid more than is required to maximize expected profits. As Milgrom

(1985) explains, a small increase  $\Delta b$  in the bid from the maximizing level reduces expected profits on the order of  $(\Delta b)^2$  (this follows from the first-order condition), but such increase in the bid reduces the risk of the lottery a bidder faces by raising his chance of winning on the order of  $\Delta b$ .

Therefore, while risk-averse bidders still bid their true valuations in the uniform-price auction, they bid more aggressively than before in a discriminatory auction in order to buy partial insurance against losing. It follows that the latter generates more expected revenue than the former when bidders are risk averse.

The ranking of auction formats obtained above holds in a more general setting as stated next.

**Theorem 5:** (Matthews (1979), Holt (1980), Maskin and Riley (1980), Harris and Raviv (1982), Milgrom and Weber (1982), Weber (1983), Milgrom (1985)) In the IPV model with  $n$  bidders who are equally risk averse,  $k$  objects ( $n > k$ ), and single-unit demands, the discriminatory auction generates more expected revenue than the uniform-price auction.

### 3. The Correlated Values Model.

In many auctions we do expect bidders's valuations for the identical objects to be influenced by how other bidders value them. Milgrom and Weber (1982) provide a theory of auctions for the case when bidder's valuations are correlated in some particular way. It turns out that the revenue equivalence result also breaks down when values are correlated. The next example helps to motivate the analysis.

**Example 5:** *Suppose there are three bidders competing for two identical objects and that bidders only want one object. Now suppose that*

each bidder  $i$  observes a signal  $x_i$ ,  $i = 1, 2, 3$ , where  $f(x_1, x_2, x_3)$  denote the joint density of the three signals where all  $x_i$ 's are defined on  $[0, 1]$ . Suppose that  $f$  is symmetric and affiliated.<sup>13</sup> Let us suppose that bidder  $i$ 's valuation is given by  $v_i = u(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$ .<sup>14</sup> Using the same argument as above, it is easy to check that in the symmetric equilibrium of the uniform-price auction, each bidder bids her expected valuation. For example, bidder 1 bids according to

$$b_U(x_1) = E\left[\frac{x_1 + x_2 + x_3}{3} \mid X_1 = x_1, \min\{X_2, X_3\} = x_1\right]$$

In the discriminatory auction, however, the price paid by a bidder is influenced by the estimates of other bidders. Therefore, bidders tend to shade their bids by more than what their signals might suggest. As a consequence, the expected revenue from the uniform-price is higher than from the discriminatory price auction.

The next theorem states the general case.

**Theorem 6:** (Milgrom and Weber (1982), Weber (1983), Milgrom (1985)) Consider the sale of  $k$  identical objects to  $n$  potential buyers, where each buyer demands only one unit. Suppose that each buyer  $i$  receives a signal  $x_i$ ,  $i = 1, \dots, n$ , about the value of the object. Suppose  $f(x_1, \dots, x_n)$  denote the joint distribution of buyers' signals

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<sup>13</sup> $f$  is affiliated if for all points  $x = (x_1, x_2, x_3)$  and  $x' = (x'_1, x'_2, x'_3)$ , we have  $f(x \vee x')f(x \wedge x') \geq f(x)f(x')$ , where  $x \vee x'$  and  $x \wedge x'$  denote, respectively, the coordinate-wise maximum and minimum. Roughly speaking,  $f$  is affiliated if higher values of some variables make higher values of the other variables more likely.

<sup>14</sup>If we assume instead that the random variables  $X_1, X_2, X_3$  are independent, then this example collapses to the classical case of common value auctions, where the identical objects are worth the same to all bidders by that worth is not known at time of bidding. In this context, the winners in the discriminatory-price auction are those with the highest signals, and therefore, naive bidding may lead to a phenomenon known as the winners' curse; the winners pay more than the true worth of the object. See, for example, Hausch (1986).

and that  $f$  is symmetric and affiliated. Assume further that the value of buyer  $i$  for any of the  $k$  identical objects is given by  $v_i = u(x_1, \dots, x_n)$ , where  $u$  is nondecreasing in all arguments and  $E[v_i | X_i = x_i]$  is increasing in  $x_i, i = 1, \dots, n$ . Then the uniform-price auction generates more expected revenue than the discriminatory auction.

#### 4. Some Empirical Evidence and Extensions.

The IPV model of single-demand auctions yields very precise predictions for bidding behaviour, seller's expected revenue, the design of optimal auctions and the effects of reserve prices, entry fees and increased competition. The correlated values model yields less precise predictions — for example, we do not know what the optimal auction should look like — but still provides us with very solid principles such as the ranking of auctions according to expected revenue.

There are several empirical papers that examine both simultaneous and sequential multi-unit auctions. Although these papers are not tests of the theory per se as some of the assumptions are not satisfied, they indicate the direction to which the standard single-demand auction theory might be generalized.

Among the papers that examine sequential auctions, Ashenfelter (1989) made pairwise comparisons of the prices for identical wine sold in the same lot size in three auction houses from 1985 to 1987. Although the most common pattern was for prices to remain constant, prices were at least twice as likely to decline as to increase. McAfee and Vincent (1993) adopted a similar approach to Ashenfelter and examine data from Christie's wine auctions at Chicago in 1987. In addition to pairwise comparisons they examined triples of identical wine sold in the same auction sale. Their results are very similar to those of Ashenfelter. These empirical findings show



a violation of the prediction that identical objects should fetch, on average, the same price.

Similar price anomalies have been identified in a number of other markets where sequential auctions are used; cable television licenses (Gandal (1997)); condominiums (Ashenfelter and Genesove (1992), and Vanderporten (1992-a,b)); commercial real estate (Lusht (1994)), dairy cattle (Engelbrecht-Wiggans and Kahn (1992)); stamps (Taylor (1991) and Thiel and Petry (1990)); and wool (Jones, Menezes and Vella (1998)).<sup>15</sup>

Most of the empirical literature on simultaneous auctions look at different aspects of Treasury bill auctions. For example, Cammack (1991) uses data from the U.S. market to show that the auction price of a bill is lower than the comparable secondary market price. Bayazitoglu and Kiefer (1997) examine Treasury bill auctions in Turkey and conclude that bidders' use a simple decision rule: bidders form minimum and maximum bid prices based on the result of the previous auctions. Gordy (1996) uses data from treasury bill auctions in Portugal to examine how bidders might submit multiple bids to hedge against the winner's curse. Finally, Tenorio (1993) examines data from Zambian foreign exchange auctions and concludes that uniform-price auctions generate more revenue and attract more bidders than discriminatory auctions.

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<sup>15</sup> Following these empirical findings, there were several attempts at providing a theoretical explanation for the price anomalies. For example, in a two-object model where individuals value the two objects differently, Black and de Meza (1992) show that the price declines whenever the winner of the first auction has an option of purchasing the second object at the same price. McAfee and Vincent (1993) show that introducing risk-aversion in the single-unit demand model may cause prices to fall throughout the auction. In the context of a single-demand auction where objects are stochastically identical (i.e., where bidders' valuations are independent draws from a fixed distribution), Menezes (1993), Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994) provide similar results.

The markets featured in this empirical literature differ from the above model in at least two dimensions. First, in these markets bidders typically demand more than one object. Second, in many cases similar but non-identical objects are sold. Thus, the empirical literature does suggest that the predictions of the above model would not go through in more general settings. Indeed, now there is a growing literature that stresses the complications that arise when bidders' strategies spaces involve both prices and quantities.<sup>16</sup> There is also a theoretical literature that relaxes the identical objects hypothesis in the context of sequential auctions with and without the single-demand assumption.<sup>17</sup> Unfortunately, however, these papers are still short of providing a general theory of multi-unit auctions. Nevertheless, the recent interest generated by the spectrum auctions in the U.S., Australia and elsewhere has generated some additional research<sup>18</sup>

## 5. Conclusion.

Consider the sale of identical objects in which potential buyers can only buy one such number and resale is not possible. If each buyer knows her value for the objects and this value is not influenced by how others value it, then the IPV model is appropriate. In this case, any of the four formats considered — the simultaneous discriminatory and uniform-price auctions and the sequential first and second-price auctions — yield the same revenue when agents are risk neutral. In addition, under the assumptions of the Revenue

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<sup>16</sup> See, for example, Ausubel and Cramton (1996), Menezes (1996), Tenorio (1997), Engelbrecht-Wiggans and Kahn (1998-a,b).

<sup>17</sup> Papers in the first category include Menezes (1993), Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994). Menezes and Monteiro (1997-b) fall into the second category.

<sup>18</sup> See, for example, McMillan (1994) and McAfee and McMillan (1996) and the special issue of the *Journal of Economic Strategy and Management* (September, 1997).

Equivalence Theorem, any of these four auctions will maximize the seller's expected revenue if an appropriate reserve price is chosen.

If bidders are risk-averse, then the revenue equivalence breaks down; for example, the discriminatory auction generates more revenue than the uniform-price auction. The revenue equivalence might also break down if buyers' values are influenced by how others value the identical objects. In particular, if values are affiliated then the uniform-price auction generates more revenue than the discriminatory auction. In this case, however, it is not known what format the optimal auction should take.

In summary, auction theory yields a series of predictions when identical objects are sold in a market where buyers can only buy one object and resale is not allowed. Auction theory, however, is yet to be generalized to the case where potential buyers demand more than one object.

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