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On the existence of stable population in life cycle models

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HIGHLIGHTS

- We study the demography of life cycle general equilibrium models.
- The existence of stable population is established.
- This result is crucial for the aggregation of individual decisions in this class of models.

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ABSTRACT

A common assumption adopted in life cycle general equilibrium models is that the population is stable at steady state, that is, its relative age distribution becomes constant over time. An open question is whether the demographic assumptions commonly adopted in these models in fact imply that the population becomes stable. In this article we prove the existence of a stable population in a demographic environment where both the age-specific mortality rates and the population growth rate are constant over time, the setup commonly adopted in life cycle general equilibrium models. Hence, the stability of the population do not need to be taken as assumption in these models.

1. Introduction

Quantitative life cycle general equilibrium models are one of the main tools used by economists and policy makers to conduct economic analysis and *ex ante* policy evaluations. The use of these models started with the seminal works of Imrohoroğlu et al. (1995) and Huggett (1996). A common assumption adopted in these models is that the population is stable at steady state, that is, its relative age distribution becomes constant over time. This assumption is important because it allows the calculation of a stationary distribution of individuals, which in turn permits the aggregation of individual decisions. An open question is whether we can prove the existence of a stable population from the demographic assumptions commonly adopted in these models.

A natural approach to prove this result would be to use the *Ergodic Theorems of Demography*, which is a set of results that provides conditions for the stability of a population.[1] These theorems make assumptions about the age-specific fertility and mortality rates of a population. However, these results cannot be applied to life cycle general equilibrium models, as these models are silent about fertility rates. These rates do not even appear in this type of model. In contrast, life cycle models make assumptions about the rate of population growth, assuming that it is constant over time, while the theorems are silent about this rate.

In this article we prove the existence of a stable population in a demographic environment where both the age-specific mortality rates and the population growth rate are constant over time, the setup commonly adopted in life cycle general equilibrium models. We make no assumptions about fertility rates. To our knowledge, no formal proof of this result exists in the economics

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[1] For references on stable population theory, see Sykes (1969), McFarland (1969), Parlett (1970), Cohen (1979), Arthur (1981), and Arthur (1982).

and demographics literature. To do this, we model a population in a generic transition path, that is, out of the steady state, and show that the population becomes stable when reaching the steady state. Important results of the theories of difference equations and polynomials are used in the formalization. We then simulate computationally the dynamics of a population to illustrate this theoretical result.

2. Demographic environment

Time is discrete and denoted by $t \in \{0, 1, 2, \ldots\}$. The population grows at a constant rate over time, where the growth factor is denoted by $g > 1$.[2] The age of individuals is denoted by $j \in \{0, \ldots, J\}$. The survival probability from age j to age $(j+1)$ is given by p_j, which is constant over time. Individuals younger than J have strictly positive survival probability, that is, $p_j \in (0, 1]$ for all $j < J$.[3] Individuals die with certainty at the end of age J, which means that $p_J = 0$. The unconditional probability of being alive at age $j > 0$ is given by $q_j = p_0 \ldots p_{j-1}$, where $q_0 = 1$. The number of individuals with age j at time t is denoted by $N_{j,t}$. The size of the population at time t is given by $N_t = N_{0,t} + \cdots + N_{J,t}$. The share of individuals with age j at time t is given by $M_{j,t} = N_{j,t}/N_t$. Because only a fraction p_j of the population with age j survives until age $(j+1)$, we have that $N_{j+1,t+1} = p_j N_{j,t}$ for all (j, t). Because the population grows at a constant factor g each period of time, we have that $N_{t+1} = gN_t$ for all t. The proposition below states two important results regarding the dynamics of the population.

Proposition 1. *Given the demographic environment, the following results are true:*

(i) $M_{j+1,t+1} = (p_j/g)M_{j,t}$ *for all* (j, t);
(ii) $M_{j,t} = (q_j/g^j)M_{0,t-j}$ *for all* (j, t) *such that* $j \le t$.

Proof. To prove the first result, note that because $N_{j+1,t+1} = p_j N_{j,t}$ for all (j, t) and $N_{t+1} = gN_t$ for all t, we have that

$$M_{j+1,t+1} = \frac{N_{j+1,t+1}}{N_{t+1}} = \frac{p_j N_{j,t}}{gN_t} = \frac{p_j}{g} M_{j,t}.$$

To prove the second result, we must apply the first result recursively and use the definition of q_j. By doing this, we conclude that for all (j, t) such that $j \le t$ it is true that

$$\begin{aligned} M_{j,t} &= \frac{p_{j-1}}{g} M_{j-1,t-1} = \frac{p_{j-2}p_{j-1}}{g^2} M_{j-2,t-2} \\ &= \cdots = \frac{p_0 \cdots p_{j-1}}{g^j} M_{0,t-j} = \frac{q_j}{g^j} M_{0,t-j}. \quad \square \end{aligned}$$

3. Stability of population

In this section we show that our demographic environment implies the existence of a stable population in the steady state. We start by defining the concept of stable population.

Definition 1. A population is called stable if its relative age distribution is constant over time.

By the definition above, we must show that the sequences $\{M_{j,t}\}$ are convergent for all j. The proposition below provides a sufficient condition for the convergence of all these sequences.

Proposition 2. *If the sequence* $\{M_{0,t}\}$ *is convergent, then the sequences* $\{M_{j,t}\}$ *are convergent for all* $j > 0$.

Proof. The result follows from the item (ii) of Proposition 1. $\square$

The above result ensures that if the share of newborns converges, then the share of all other ages also converges, which implies that the relative age distribution of the whole population is convergent. Therefore, our objective now is to prove that the share of newborns converges, i.e., that the sequence $\{M_{0,t}\}$ is indeed convergent. The first step is to show that we can express $M_{0,t}$ as a difference equation for all t. Using the definition of N_t, we can write that

$$N_{0,t} + N_{1,t} + N_{2,t} + \cdots + N_{J,t} = N_t.$$

Dividing both sides of the above expression by N_t and using the item (ii) of Proposition 1, we conclude that

$$M_{0,t} + \frac{q_1}{g} M_{0,t-1} + \frac{q_2}{g^2} M_{0,t-2} + \cdots + \frac{q_J}{g^J} M_{0,t-J} = 1$$

for all t such that $J \le t$. For ease of notation, define $c_j = q_j/g^j$ for all j. Thus, we can write the above difference equation as

$$c_0 M_{0,t} + c_1 M_{0,t-1} + c_2 M_{0,t-2} + \cdots + c_J M_{0,t-J} = 1. \tag{1}$$

From the theory of difference equations, we know that the general solution of Eq. (1) is given by $M_{0,t} = P_t + H_t$, where P_t is called the particular solution and H_t is called the homogeneous solution. To find the particular solution P_t we use the *guess and verify* method. First, we assume that the particular solution is a constant P that does not depend on t. Then, to check that this is the case, we replace $M_{0,t-j}$ by P in equation (1) and solve for P. By doing this, we find that

$$P_t = P = \frac{1}{c_0 + c_1 + c_2 + \cdots + c_J}. \tag{2}$$

To find the homogeneous solution H_t, we need to work with the homogeneous equation associated with equation (1), which is given by

$$c_0 M_{0,t} + c_1 M_{0,t-1} + c_2 M_{0,t-2} + \cdots + c_J M_{0,t-J} = 0. \tag{3}$$

The characteristic polynomial associated with the homogeneous equation (3) is given by

$$c_0 \lambda^J + c_1 \lambda^{J-1} + c_2 \lambda^{J-2} + \cdots + c_J = 0, \tag{4}$$

where λ is a complex number. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be the distinct roots of Eq. (4), with multiplicities given by $\{m_1, m_2, \ldots, m_r\}$. From Corollary 2.24 of Elaydi (2005), we have that the general solution of the homogeneous equation (3) is given by

$$H_t = \sum_{i=1}^{r} \lambda_i^t \left(a_{i,0} + a_{i,1} t + a_{i,2} t^2 + \cdots + a_{i,m_i-1} t^{m_i-1} \right), \tag{5}$$

where $\{a_{i,0}, a_{i,1}, \ldots, a_{i,m_i-1}\}$ are complex numbers for all i.

Using the results (2) and (5), we can write the general solution of Eq. (1) as

$$\begin{aligned} M_{0,t} = &\frac{1}{c_0 + c_1 + c_2 + \cdots + c_J} \\ &+ \sum_{i=1}^{r} \lambda_i^t \left(a_{i,0} + a_{i,1} t + a_{i,2} t^2 + \cdots + a_{i,m_i-1} t^{m_i-1} \right). \end{aligned} \tag{6}$$

Note that the above general expression for the share of newborns depends on the population growth rate, survival probabilities, and initial population. It depends on the population growth rate and survival probabilities through the constants $\{c_0, \ldots, c_J\}$ and roots

[2] Our main result remains true if we consider a constant growth rate of newborns rather than a constant population growth rate.

[3] The assumption that $g > 1$ means that the population growth rate is strictly positive. We could relax this assumption and assume that $g \ge 1$ to consider zero population growth. In this case, to prove the result, we would have to assume that $p_0 < 1$ or $p_{J-1} < 1$.

$\{\lambda_1, \ldots, \lambda_r\}$, and depends on the initial population through the constants $\{a_{i,0}, \ldots, a_{i,m_i-1}\}$ for all $i \in \{1, \ldots, r\}$.[4]

Because the particular solution is a constant that does not depend on t, we have to concentrate only on the convergence of the homogeneous solution. The following proposition provides a sufficient condition for the convergence of the homogeneous solution.

Proposition 3. *If $|\lambda_i| < 1$ for all $i \in \{1, \ldots, r\}$, then $H_t \to 0$.*

Proof. Because the homogeneous solution is a summation, it is sufficient to prove that each element of this summation converges to zero. A typical element of this summation is given by $\lambda_i^t a_{i,k} t^k$ for all $i \in \{1, \ldots, r\}$ and for all $k \in \{0, \ldots, m_i - 1\}$. Because $a_{i,k}$ is a constant that does not depend on t, we can just focus on the convergence of $\lambda_i^t t^k$. We also know that $\lambda_i^t t^k \to 0$ is equivalent to $|\lambda_i|^t t^k \to 0$. For convenience, set $b_i = 1/|\lambda_i|$ for all i. Because by assumption $|\lambda_i| < 1$ for all i, then $b_i > 1$ for all i. From the item (d) of Theorem 3.20 of Rudin (1976), we have that $t^k/b_i^t \to 0$, which implies that $H_t \to 0$. □

The previous proposition guarantees that if all roots of the characteristic polynomial (4) are strictly inside the unit circle in the complex plane, then the homogeneous solution converges to zero. Now we are left to show that this condition is satisfied. The following proposition states an important result that will allow us to conclude that the sufficient condition of Proposition 3 is satisfied.

Proposition 4. *The coefficients of the characteristic polynomial (4) are strictly positive and strictly monotonic, that is,*

$$c_0 > c_1 > c_2 > \cdots > c_{J-2} > c_{J-1} > c_J > 0.$$

Proof. Because $g > 1$ and $q_j > 0$ for all j, then $c_j > 0$ for all j. Besides, for all $j > 0$ we have that

$$c_j = \frac{q_j}{g^j} = \frac{p_0 \cdots p_{j-1}}{g^j} = \frac{p_0 \cdots p_{j-2}}{g^{j-1}} \cdot \frac{p_{j-1}}{g} = \frac{p_{j-1}}{g} c_{j-1} < c_{j-1}. \quad \square$$

According to Theorem 1 of Nguyen et al. (2007), the result of Proposition 4 is a sufficient condition for the characteristic polynomial (4) to be *Schur stable*, which means that all its roots are strictly inside the unit circle in the complex plane.[5] Technically, this guarantees that $|\lambda_i| < 1$ for all $i \in \{1, \ldots, r\}$, which implies by Proposition 3 that $H_t \to 0$. Therefore, the sequence $\{M_{0,t}\}$ converges to the particular solution, and by Proposition 2, the sequences $\{M_{j,t}\}$ are also convergent for all $j > 0$. The limits of these sequences can be found using Proposition 1. Hence, we proved that the population is stable in the steady state.

A property of populations that converge to stability is that the growth rate of each age cohort converges to the population growth rate. The proposition below formalizes this argument.

Proposition 5. *If the sequences $\{M_{j,t}\}$ are convergent for all j, then the growth rate of each age cohort converges to the population growth rate.*

Proof. Let $g_{j,t}$ be the growth factor of individuals with age j from time t to $(t + 1)$. Using the definition of $M_{j,t}$, we conclude that

$$g_{j,t} = \frac{N_{j,t+1}}{N_{j,t}} = \frac{M_{j,t+1}N_{t+1}}{M_{j,t}N_t} = g\frac{M_{j,t+1}}{M_{j,t}}$$

for all (j, t). Because the sequences $\{M_{j,t}\}$ are convergent for all j, the item (ii) of Proposition 1 and the fact that the shares must sum to one ensure that the limits of these sequences are different from zero. Then, we have that $M_{j,t+1}/M_{j,t} \to 1$ for all j, and therefore $g_{j,t} \to g$ for all j. □

4. Discussion

Changes in population age distribution play an important role in economics, especially in understanding the role of intergenerational transfers. Our demographic model and result of stability provide insights into the relationship between mortality, population growth rate, and age distribution. As noted by Tuljapurkar (2008), a stable population's age pyramid is steeper for faster growing populations and shallower for low-mortality populations. To see this through our model, after reaching stability, Proposition 1 implies that $M_j/M_0 = q_j/g^j$ for all j, which means that the ratios between the share of individuals with any age and the share of newborns are proportional to the unconditional survival probabilities and the inverse of population growth rate. Therefore, an increase in the population growth rate causes the shares of older individuals to become increasingly smaller in relation to the share of newborns, making the age pyramid steeper. On the other hand, by the same reasoning, a decrease in mortality (or increase in survivorship) makes the pyramid shallower.

These properties carry over qualitatively to real populations. According to Tuljapurkar (2008), many industrialized countries in the 21st century have small positive or even negative growth rates, as well as low mortality. In addition, the author also argues that todays populations have a larger fraction of older individuals and a smaller fraction of young than the populations of the 19th and early 20th centuries.

5. Simulation

In this section we simulate the dynamics of a population with the same characteristics as the one defined in Section 2 to illustrate its stability. We set the population growth factor to $g = 1.01$, the maximum age to $J = 3$, the survival probabilities to $\{p_0, p_1, p_2, p_3\} = \{95\%, 75\%, 50\%, 0\%\}$, and the number of individuals at time $t = 0$ to $\{N_{0,0}, N_{1,0}, N_{2,0}, N_{3,0}\} = \{100, 75, 50, 25\}$. We simulated the dynamics of the population over 50 periods. Fig. 1 shows that although the sizes of each age cohort grow over time, its shares become stable over time.

6. Conclusion

In this article we prove the existence of a stable population in a demographic environment where both the age-specific mortality rates and the population growth rate are constant over time, the setup commonly adopted in life cycle general equilibrium models. To our knowledge, no formal proof of this result exists in the economics and demographics literature. This result is important because it allows the calculation of a stationary distribution of individuals in life cycle general equilibrium models, which in turn permits the aggregation of individual decisions.

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[4] The coefficients of the characteristic polynomial in Eq. (4) are formed by the constants $\{c_0, \ldots, c_J\}$, which implies that the roots $\{\lambda_1, \ldots, \lambda_r\}$ depend on these constants, and therefore also depend on the population growth rate and survival probabilities. The constants $\{a_{i,0}, \ldots, a_{i,m_i-1}\}$ are obtained from the initial conditions of the model for all $i \in \{1, \ldots, r\}$, and therefore depend on the initial population. See Example 2.25 of Elaydi (2005) on how to find these constants from the initial conditions.

[5] The condition of Proposition 4 is known in the literature of polynomials as the *Monotonic Condition for Schur Stability of Real Polynomials*.

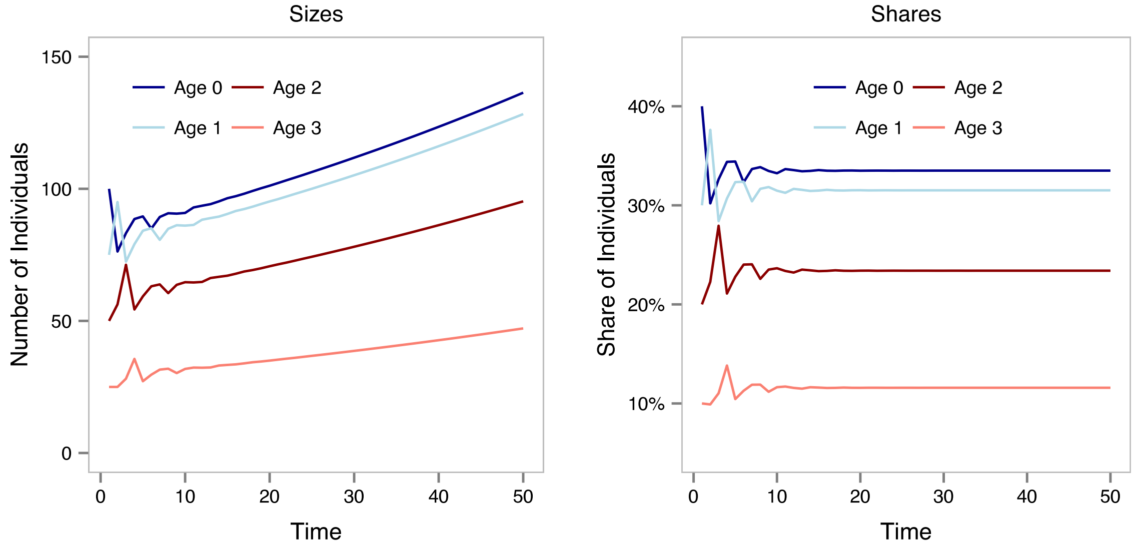

Fig. 1. Dynamics of the population. Left: Sizes of each age cohort over time. Right: Shares of each age cohort over time.

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