

Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

Nicolae Gârleanu*

Stavros Panageas†

Current Draft: July 2012

Abstract

We study the implications of preference heterogeneity for asset pricing. We use recursive preferences in order to separate heterogeneity in risk aversion from heterogeneity in the intertemporal elasticity of substitution, and an overlapping-generations framework to obtain a non-degenerate stationary equilibrium. We solve the model explicitly up to the solutions of ordinary differential equations, and highlight the effects of overlapping generations and each dimension of preference heterogeneity on the market price of risk, interest rates, and the volatility of stock returns. We find that the model has the potential to address some well-documented asset-pricing puzzles, while also being consistent with certain salient facts about cross-sectional consumption and wealth inequality.

*UC Berkeley, Haas School of Business, Centre of Economic Policy Research (CEPR), and National Bureau of Economic Research (NBER), email: garleanu@haas.berkeley.edu.

†University of Chicago, Booth School of Business and National Bureau of Economic Research (NBER), email: stavros.panageas@chicagobooth.edu. We would like to thank Andy Abel, Hengjie Ai, Nicole Branger, John Cochrane, Mike Gallmeyer, John Heaton, Leonid Kogan, Debbie Lucas, Monika Piazzesi, Kathy Yuan, four anonymous referees and participants at seminars and presentations at Berkeley Haas, the 2008 CARESS-Cowles Conference, Chicago Booth, Drexel, the 2007 Duke UNC Asset Pricing Conference, the 2011 Finance UC Conference, HEC Paris, the 2012 ITAM Finance Conference, the 2007 NBER Summer Institute (AP), Princeton University, the 2010 Royal Economic Society, the 2007 SAET conference, the 2008 SED conference, Stanford GSB, Studienzentrum Gerzensee, the 2008 UBC Winter Finance Conference, University of Melbourne, University of Tokyo, and the 2008 Utah Winter Finance Conference for useful comments.

Macro-finance models routinely postulate economies populated by a “representative agent” with constant relative risk aversion and intertemporal elasticity of substitution (IES). A legitimate question is how preference heterogeneity along both these dimensions, a widely documented and intuitively plausible feature of reality, affects the conclusions of this parsimonious setup. We answer this question by identifying novel qualitative implications of such heterogeneity and showing analytically and numerically its ability to match asset-pricing as well as cross-sectional consumption and wealth data.

We study a continuous-time, overlapping-generations (OLG) economy (Blanchard (1985)) populated by agents with heterogeneous, recursive preferences (Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989)). The main differences between our setup and the existing literature on preference heterogeneity are *(i)* OLG and *(ii)* *recursive* preferences. The choice of these features is motivated by two considerations.

First, the OLG structure rules out the degenerate long-run wealth distributions obtaining in models with infinitely-lived heterogeneous agents. Specifically, in such models it is quite common that one type of agents dominates the economy in the long run, since preference heterogeneity translates into persistent cross-sectional differences in average growth rates of wealth. An immediate consequence is that all asset-pricing quantities (price-dividend ratios, equity premia, etc.) converge to the constant levels obtaining when the economy is populated by only one type of agent. As a result, it becomes difficult to compare these models to a large body of empirical literature predicated on a non-degenerate stationary distribution for these quantities.¹ By contrast, our framework generates non-degenerate cross-sectional distributions of wealth and consumption, through the combination of finite lifetimes and the constant arrival of newly born agents endowed with the same labor earnings. As a side benefit, the OLG structure enables an empirically disciplined link between the model’s cross-sectional wealth and consumption distributions and the paths of life-cycle earnings observed in the data.

¹To provide an example, the empirical literature on return predictability considers regressions of excess returns on a constant and various predictors, such as the price-to-dividend ratio. The identifying assumption of ordinary least squares (in large samples) requires that the regressors have non-degenerate asymptotic distributions, i.e., do not converge to constants (otherwise, the variance-covariance matrix is singular).

Second, by using recursive rather than time-additive preferences we can separate — both conceptually and quantitatively — the effects of risk-aversion and IES heterogeneity, and study their interaction.

We show that these features of the model — OLG and heterogeneity of recursive preferences — help address a number of stylized facts in asset pricing, which can be grouped in three broad categories, concerning *(i)* the market price of risk, *(ii)* the interest rate, and *(iii)* the return volatility and equity premium. We discuss each of them in turn.

The market price of risk, defined as the ratio of stock-market excess return to stock-market volatility, can increase above the levels that would obtain with heterogeneous CRRA utilities. To be specific, in the special case of heterogeneous CRRA utilities our OLG structure preserves a familiar result in the literature: The market price of risk is negatively correlated with the underlying aggregate-consumption shock and given by a weighted average of the market prices of risk in two representative-agent economies, populated exclusively by the more risk-averse, respectively the less risk-averse, agents of our model. Novel results obtain when agents have heterogeneous recursive preferences. The optimal consumption-allocation rule endogenously generates persistence in individual agents' consumption growth, although aggregate-consumption growth is i.i.d.. Except in the special CRRA case, agents' marginal rates of substitution are affected by such persistence, and so is the market price of risk. We provide conditions under which the resulting market price of risk exceeds the one under CRRA preferences (keeping risk-aversion coefficients fixed). Interestingly, in some illustrative examples the market price of risk exceeds even the value obtaining in the economy populated exclusively by the agent with the higher risk-aversion coefficient.

Our second set of results concerns interest rates. We show that even if all agents have homogeneous preferences, interest rates in an OLG economy are lower than in the respective infinitely-lived agent economy for the low values of IES that generate the risk-free puzzle in such an economy (Weil (1989)). The reason is that finitely-lived agents faced with realistic life-cycle earnings profiles need to save in order to provide for themselves in old age. These increased savings lower the interest rate.

Our third set of results concerns the volatility of stock-market returns and the equity premium. We first show that, if agents have different IES, but the same risk-aversion coefficient, then the volatility of the stock market and the equity premium (as well as the market price of risk) are constant and equal to their respective values in an economy populated by a single agent with the same risk aversion. In our model, therefore, risk-aversion heterogeneity is essential for addressing certain empirical asset-pricing properties.

However, for a given degree of risk-aversion heterogeneity, heterogeneity in the IES can affect the results significantly. Under certain conditions, we show that the primary effect of IES heterogeneity is to control variability in the real interest rate, while the primary effect of risk aversion heterogeneity is to control time variation in the market price of risk. Accordingly, the two dimensions of heterogeneity can be combined to produce discount-rate variability that is negatively related to the underlying aggregate-consumption shock and is predominantly driven by the variability of the equity premium, rather than that of the real interest rate.

Quantitatively, we find that — for realistically calibrated life-cycle earnings profiles and an appropriate choice of the joint distribution of IES and risk aversion across agents — we can obtain a small and non-volatile interest rate along with a substantially larger and more variable equity premium. A novel feature of our paper is that the same assumptions on preference specifications that are required to reproduce asset-pricing facts also imply cross-sectional consumption and wealth populations that share many salient features of the data. In particular, the model can simultaneously produce large variation in asset prices and relatively low variation in measures of cross-sectional consumption and wealth dispersions.²

Our paper relates primarily to the analytical asset-pricing literature on preference heterogeneity.³ As already mentioned, this literature does not separate IES heterogeneity from risk-aversion heterogeneity. Furthermore, with the notable exception⁴ of Chan and Kogan

²Constantinides and Duffie (1996) show that their heterogeneous-income model can generate an arbitrary stochastic discount factor. However, this may require a large degree of time variation in cross-sectional consumption dispersion.

³Contributions include Dumas (1989), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2009), Longstaff and Wang (2008), and Zapatero and Xiouros (2010).

⁴See also Zapatero and Xiouros (2010) for a discrete-time version of Chan and Kogan (2002).

(2002), these models imply generically non-stationary dynamics. The key assumption that ensures stationarity in Chan and Kogan (2002) is that an agent's utility derives exclusively from her consumption relative to aggregate consumption. We use instead an OLG approach to obtain stationarity (see, e.g.,⁵ Spear and Srivastava (1986)), which allows us to study standard CRRA and recursive-utility specifications. In doing so, we can better isolate the effects of preference heterogeneity, while at a practical level we can avoid the large volatility of the real interest rate that is associated with (multiplicative) external habit formation.

A theoretical literature has considered preference aggregation for infinitely-lived agents with heterogeneous, recursive preferences. Early contributions studied deterministic environments.⁶ Subsequently, other authors investigated the existence of equilibrium and recursive algorithms for the construction of a representative agent in stochastic environments.⁷ From a methodological perspective, we develop a novel approach to determining equilibrium allocations, which is distinct from the central-planning approaches developed in Kan (1995) and Dumas et al. (2000). Our alternative is better suited for an OLG setup, since the OLG feature precludes usage of the existing central-planning approaches. A useful byproduct of our approach is that we can provide analytic expressions for the equilibrium prices and quantities up to the solution of a system of ordinary differential equations. Besides allowing an accurate and efficient computation of the equilibrium, these expressions make it possible to characterize the new equilibrium properties associated with recursive preferences (as compared with CRRA preferences).⁸

Unlike the case of infinitely-lived agents, in which survival of all agent types is parameter dependent,⁹ in our OLG framework survival obtains generically. Accordingly, we can analyze the effect of parameter constellations (e.g., the heterogeneous CRRA case) that are incompatible with stationarity when agents are infinitely lived. Additionally, by analyzing

⁵Spear and Srivastava (1986) discusses the existence of stationary Markov equilibria in OLG models. However, it focuses on heterogeneous goods (rather than preferences), and the Markov state space contains past prices rather than the consumption distribution.

⁶See, e.g., Lucas and Stokey (1984) and Epstein (1987).

⁷See, e.g., Ma (1993), Kan (1995), Duffie et al. (1994), and Dumas et al. (2000).

⁸Backus et al. (2008) give a closed-form solution for the special case where one type of agents is risk neutral and has zero IES, whereas the other type of agents is infinitely risk averse and has infinite IES.

⁹See the recent contribution of Branger et al. (2012).

equilibrium quantities around the steady state, we can obtain a more detailed characterization of asset-pricing quantities than the previous literature.

Some of our model's channels are reminiscent of leading representative-agent models. Similar to Campbell and Cochrane (1999), the market price of risk increases following negative shocks. In Campbell and Cochrane (1999) this property is due to the assumed sensitivity to shocks of the representative-agent risk aversion, while in our model it arises as a result of the endogenous redistribution of wealth from the less to the more risk averse agents in bad times.¹⁰ Also, similar to Bansal and Yaron (2004), persistent consumption growth coupled with recursive preferences generate a higher market price of risk. However, in contrast to Bansal and Yaron (2004), this persistence is the endogenous result of the consumption-allocation rule and it pertains to the consumption growth of *individual agents*. It obtains even if *aggregate* consumption growth is i.i.d.

Another relevant literature employs numerical methods to solve life-cycle-of-earnings models in general equilibrium. The paper most closely related to ours is Gomes and Michaelides (2008).¹¹ Gomes and Michaelides (2008) obtains joint implications for asset returns and stock-market participation decisions in a rich setup that includes costly participation, heterogeneity in both preferences and income, and realistic life cycles. This paper also assumes, however, that the volatility of output, the volatility of stock market returns, and the equity premium are driven by exogenous, random capital-depreciation shocks, rather than the commonly assumed total-factor-productivity shocks. This assumption implies that stock-market volatility is due almost entirely to exogenous variation in the quantity of capital, rather than to fluctuations in the price of capital (Tobin's q), in contrast to the data where most of the return volatility is due to fluctuations in the price of capital. This exogenous source of return

¹⁰This result was first shown for CRRA utilities by Dumas (1989), and emphasized by Chan and Kogan (2002). We also note that a large asset-pricing literature obtains time-varying market prices of risk as a result of the coexistence of multiple goods or production factors. Indicative examples include Piazzesi et al. (2007), Santos and Veronesi (2006), Tuzel (2010), Papanikolaou (2010), and Gomes et al. (2009). Our model differs in that the result is not due to assumptions on relative endowments of different goods and production factors, but to the interaction of agents with different preferences.

¹¹Other related work includes Storesletten et al. (2007) and Guvenen (2009). Storesletten et al. (2007) study an OLG economy with homogeneous preferences and uninsurable income shocks. Similar to the OLG model of Gomes and Michaelides (2008), Guvenen (2009) studies preference heterogeneity and limited market participation, but his model features infinitely lived agents.

volatility is essential for their model to produce a non-negligible equity premium.¹²

We consider exclusively endowment shocks in a Lucas (1978)-style economy. While this more conventional asset-pricing framework abstracts from modeling investment, it has the advantage that stock-market fluctuations are due to endogenous variations in the price of capital. It also allows us to readily compare our results to the large asset-pricing literature using such a framework. More importantly, endogenizing the price of capital generates new insights compared to Gomes and Michaelides (2008). For instance, even though our theoretical results on the market price of risk support the conclusions of Gomes and Michaelides (2008), we also find that some of the joint distributions of intertemporal elasticity of substitution and risk aversion discussed in Gomes and Michaelides (2008) may increase the market price of risk at the cost of attenuating the volatility of returns and the equity premium.

Another difference with the calibration-oriented literature is the tractability of our framework, which allows us to obtain analytical results. Therefore, our work is complementary to quantitative exercises that feature richer setups, but must sacrifice partly the transparency of the mechanisms involved.

We also relate to the literature that analyzes the role of OLG in asset pricing.¹³ Many of these models combine the OLG structure with other frictions or shocks to drive incomplete risk sharing across generations, so that consumption risk is disproportionately high for cohorts participating predominantly in asset markets. Even though we think these channels important for asset pricing, we do not include them in order to isolate the intuitions pertaining to preference heterogeneity. Another source of difference is that we model births and deaths in continuous time, similar to Blanchard (1985). As a result, our model produces implications for returns over any duration (month, year, etc.), and not just over the lifespan of a generation.¹⁴

The paper is structured as follows. Section 1 presents the model. Section 2 discusses

¹²As Gomes and Michaelides (2008) acknowledge, “Without those shocks, our economy would still have a high market price of risk, but a negligible equity premium.”

¹³Examples of such papers are Abel (2003), Storesletten et al. (2007), Constantinides et al. (2002), Farmer (2002), Piazzesi and Schneider (2009), Heaton and Lucas (2000), and Gârleanu et al. (2009).

¹⁴Farmer (2002) analyzes a stochastic version of Blanchard (1985) in discrete time, but all agents maximize the same logarithmic preferences.

the solution obtaining with preference homogeneity, which allows us to isolate the effects of overlapping generations. Section 3 introduces preference heterogeneity, and develops the analytical implications of the model for asset pricing. Section 4 discusses quantitative implications and Section 5 concludes. All proofs are in the appendix.

1 Model

1.1 Demographics and preferences

Our specification of demographics follows Blanchard (1985) and Yaari (1965). Time is continuous. Each agent faces a constant hazard rate of death $\pi > 0$ throughout her life, so that a fraction π of the population perishes per unit of time. Simultaneously, a cohort of mass π is born per unit of time. Given these assumptions, the time- t size of a cohort of agents born at some time $s < t$ is given by $\pi e^{-\pi(t-s)}ds$, and the total population size is $\int_{-\infty}^t \pi e^{-\pi(t-s)}ds = 1$.¹⁵

To allow for the separation of the effects of the IES and the risk aversion, we assume that agents have the type of recursive preferences proposed by Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989) in discrete-time settings and extended by Duffie and Epstein (1992) to continuous-time settings. Specifically, an agent of type i maximizes

$$V_s^i = E_s \left[\int_s^\infty f(c_u^i, V_u^i) du \right], \quad (1)$$

where $f^i(c, V)$ is given by

$$f^i(c, V) \equiv \frac{1}{\alpha^i} \left(\frac{c^{\alpha^i}}{((1 - \gamma^i) V)^{\frac{\alpha^i}{1 - \gamma^i} - 1}} - (\rho + \pi)(1 - \gamma^i) V \right). \quad (2)$$

The function $f^i(c, V)$ aggregates the utility arising from current consumption c and the value function V . The parameter $\gamma^i > 0$ controls the risk aversion of agent i , while $(1 - \alpha^i)^{-1}$ gives

¹⁵We assume no population growth for simplicity. Introducing population growth is a straightforward extension.

the agent's IES. We assume that $\alpha^i < 1$, so that the IES ranges between zero and infinity. The parameter ρ is the agent's subjective discount factor. The online appendix¹⁶ gives a short derivation of the objective function (1) as the continuous-time limit of a discrete-time, recursive-preference specification with random times of death.

To study the effects of heterogeneity in the most parsimonious way, we assume that there are two groups of agents labeled A and B : agents in group A have risk aversion γ^A and IES $(1 - \alpha^A)^{-1}$, whereas agents in group B have risk aversion γ^B and IES $(1 - \alpha^B)^{-1}$. At every point in time a proportion $v^A \in (0, 1)$ of newly born agents are of type A , while the remainder $v^B = 1 - v^A$ are of type B . For the rest of the paper we maintain the convention $\gamma^A \leq \gamma^B$, and use superscripts to denote the type $i \in \{A, B\}$ of an agent.

1.2 Endowments, earnings, and dividends

Aggregate output in the economy is given by

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dB_t, \quad (3)$$

where μ_Y and σ_Y are constant parameters and B_t is a standard Brownian motion.¹⁷

At time t , an agent born at time s is endowed with earnings $y_{t,s}$, where

$$y_{t,s} = \omega Y_t [G(t - s)], \quad \omega \in (0, 1). \quad (4)$$

$G(t - s) \geq 0$ is a function of age that controls the life-cycle earnings profile. For some results in the paper we can consider an arbitrary function $G(t - s)$. For tractability, some results use the parametric form

$$G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}. \quad (5)$$

This parametric form is flexible enough to reproduce the hump-shaped pattern of earnings

¹⁶Available at <http://faculty.chicagobooth.edu/stavros.panageas/research/OLGextapp.pdf>.

¹⁷We fix throughout a probability space and a filtration generated by B satisfying the usual conditions.

observed in the data, as we illustrate in the next section.

Aggregate earnings are given by

$$\int_{-\infty}^t \pi e^{-\pi(t-s)} y_{t,s} ds = \omega Y_t \int_{-\infty}^t \pi e^{-\pi(t-s)} G(t-s) ds = \omega Y_t \int_0^{\infty} \pi e^{-\pi u} G(u) du. \quad (6)$$

We assume throughout that $\int_0^{\infty} \pi e^{-\pi u} G(u) du < \infty$, and normalize the value of this integral to 1 by scaling G . The aggregate earnings are therefore given by ωY_t , the remaining fraction $1 - \omega$ of output Y_t being paid out as dividends $D_t \equiv (1 - \omega) Y_t$ by the representative firm. Following Lucas (1978), we assume that the representative firm is a “Lucas tree”, i.e., it simply pays dividends without facing any economic decisions.

1.3 Markets and budget constraints

Agents can allocate their portfolios between shares of the representative firm and instantaneously maturing riskless bonds, which pay an interest rate r_t per dollar invested. The supply of shares of the firm is normalized to one, while bonds are in zero net supply. The price S_t of each share evolves according to

$$dS_t = (\mu_t S_t - D_t) dt + \sigma_t S_t dB_t, \quad (7)$$

where the coefficients μ_t and σ_t are determined in equilibrium, as is the interest rate r_t .

Finally, agents can access a market for annuities through competitive insurance companies as in Blanchard (1985). Specifically, an agent born at time s who owns financial wealth equal to $W_{t,s}$ at time t can enter a contract with an annuity company that entitles the agent to receive an income stream of $\pi W_{t,s}$ per unit of time. In exchange, the insurance company collects the agent’s financial wealth when she dies. Entering such an annuity contract is optimal for all living agents, given that they have no bequest motives. The law of large numbers implies that insurance companies collect πW_t per unit of time from perishing agents, where W_t denotes aggregate financial wealth. This allows them to pay an income flow of πW_t to survivors and break even.

Letting $\theta_{t,s}$ denote the dollar amount invested in shares of the representative firm,¹⁸ an agent's financial wealth $W_{t,s}$ evolves according to

$$dW_{t,s} = (r_t W_{t,s} + \theta_{t,s} (\mu_t - r_t) + y_{t,s} + \pi W_{t,s} - c_{t,s}) dt + \theta_{t,s} \sigma_t dB_t, \quad W_{s,s} = 0. \quad (8)$$

1.4 Equilibrium

The definition of equilibrium is standard. An equilibrium is given by a set of adapted processes $\{c_{t,s}, \theta_{t,s}, r_t, \mu_t, \sigma_t\}$ such that (i) the processes $c_{t,s}$ and $\theta_{t,s}$ maximize an agent's objective (1) subject to the dynamic budget constraint (8), and (ii) markets for goods clear, i.e., $\int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s} ds = Y_t$, and markets for stocks and bonds clear as well: $\int_{-\infty}^t \pi e^{-\pi(t-s)} \theta_{t,s} ds = S_t$ and $\int_{-\infty}^t \pi e^{-\pi(t-s)} (W_{t,s} - \theta_{t,s}) ds = 0$.

2 Homogeneous preferences

Our model setup has two main features: OLG and preference heterogeneity. In preparation for the main results of the paper, this section clarifies the asset-pricing implications of OLG (absent preference heterogeneity).

Proposition 1 *Suppose that all agents have the same preferences ($\alpha^A = \alpha^B = \alpha$, $\gamma^A = \gamma^B = \gamma$), and consider the following non-linear equation for r :*

$$r = \rho + (1 - \alpha) (\mu_Y + \pi (1 - \beta)) - \gamma (2 - \alpha) \frac{\sigma_Y^2}{2}, \quad (9)$$

where

$$\beta = \omega \left(\int_0^\infty G(u) e^{-(r + \pi + \gamma \sigma_Y^2 - \mu_Y)u} du \right) \left(\pi + \frac{\rho}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \left(r + \frac{\gamma}{2} \sigma_Y^2 \right) \right). \quad (10)$$

Suppose that \bar{r} is a root of (9) and $\bar{r} > \mu_Y - \gamma \sigma_Y^2$.¹⁹ Then there exists an equilibrium where

¹⁸We assume standard square integrability and transversality conditions. See, e.g., Karatzas and Shreve (1998) for details.

¹⁹Lemma 2 contains sufficient — but not necessary — conditions on the parameters for the existence of

the interest rate, the expected return, and the volatility of the stock market are all constant and given, respectively, by $r_t = \bar{r}$, $\mu_t = r + \gamma\sigma_Y^2$, and $\sigma_t = \sigma_Y$.

Proposition 1 shows that in the absence of preference heterogeneity the equity premium and the volatility of the stock market in our OLG economy are identical to the respective quantities in a standard, infinitely-lived representative-agent model with the same preferences. (See, e.g., Weil (1989).) Accordingly, the OLG feature does not help address issues such as the equity premium, the excess volatility puzzle, the predictability of returns, etc.. The only interesting asset-pricing implication of the OLG assumption is that the interest rate differs from the respective infinitely-lived representative-agent economy: in the infinitely-lived representative-agent model the interest rate is given by $r = \rho + (1 - \alpha)\mu_Y - \gamma(2 - \alpha)\frac{\sigma_Y^2}{2}$, while equation (9) contains the additional term $(1 - \alpha)\pi(1 - \beta)$.

The source of the difference is that in an OLG economy only the Euler equation (and hence the per-capita consumption growth) of *existing* agents matters. The per-capita consumption growth of existing agents is in general different from the growth rate of *aggregate* consumption, because of deaths and births. Absent births, the death of a fraction π of the population per unit of time would imply a per-capita consumption growth rate of $\mu_Y + \pi$. In the presence of births, however, a portion of aggregate consumption accrues to arriving agents. Collectively, these agents consume $\pi c_{t,t}$, which is a fraction $\frac{\pi c_{t,t}}{C_t}$ of aggregate consumption. Therefore, the combined effect of births and deaths is that the per capita consumption growth of existing agents is $\mu_Y + \pi\left(1 - \frac{c_{t,t}}{C_t}\right)$.

As we show in the appendix, when all agents have homogeneous preferences, $\frac{c_{t,t}}{C_t}$ is constant and equal to the constant β given in Proposition 1. In the appendix (Lemma 2), we also extend some results of Blanchard (1985) and show that under certain conditions $\beta > 1$ and, consequently, the interest rate of our economy (equation (9)) is lower than the interest rate in the respective economy featuring an infinitely-lived agent.

Figure 1 depicts the quantitative magnitude of the difference in riskless rates between an economy featuring an infinitely lived agent and our OLG economy. To expedite the

such a root.

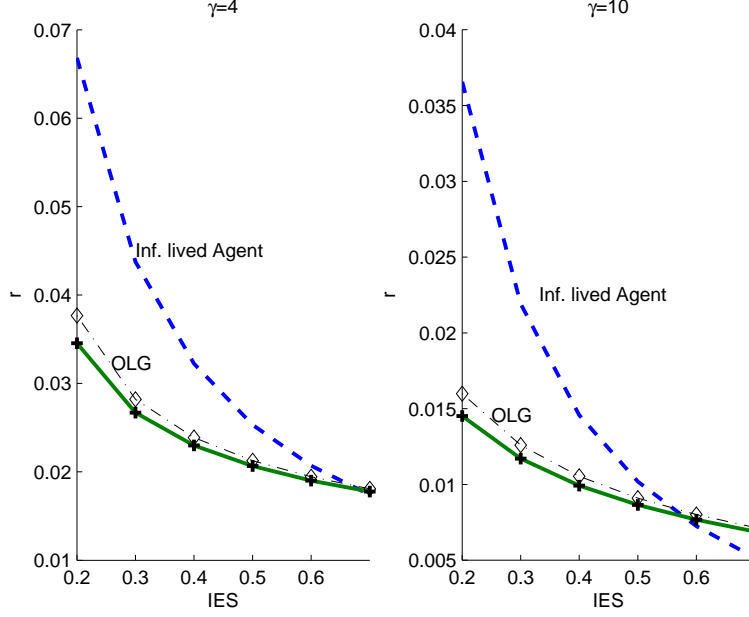


Figure 1: Interest rates when all agents have identical preferences. The dashed line pertains to an economy populated by an infinitely-lived agent, while the other two lines to our OLG model. The continuous line obtains when using the exact path for $G(u)$ reported in Hubbard et al. (1994) and performing the integration in equation (10) numerically. The dash-dot line obtains when using non-linear least squares to project $G(\cdot)$ on a sum of scaled exponential functions, and then calculating the integral in equation (10) exactly. For the computations, we set a positive, but small, value of $\rho = 0.001$ to illustrate that the high risk free rates (in the case where agents are infinitely lived) are not the result of a high discount rate. We set $\mu_Y = 0.02$ and $\sigma_Y = 0.041$ to match the mean and the volatility of the yearly, time-integrated consumption growth rate, $\pi = 0.02$ to match the birth rate, and $\omega = 0.92$ to match the share of labor income in national income. Details on the sources of the data and further discussion of these parameters are contained in the appendix (Section B).

presentation of the main results of our paper, we refer the reader to the appendix (Section B) for a detailed justification of the parameters that we use for calibration. In short, we choose μ_Y and σ_Y to reproduce average, annual, time-integrated consumption growth and its volatility, π to reproduce the average birth rate in the economy, and ω to capture the labor share of national income. For $G(t-s)$ we use the exact path of life-cycle earnings reported in Hubbard et al. (1994), which we depict in Figure 2. (For future reference, it is useful to note that interest rates remain basically unchanged when we use the parametric specification for $G(t-s)$ of equation (5) with parameters estimated by non-linear least squares.) Figure 1

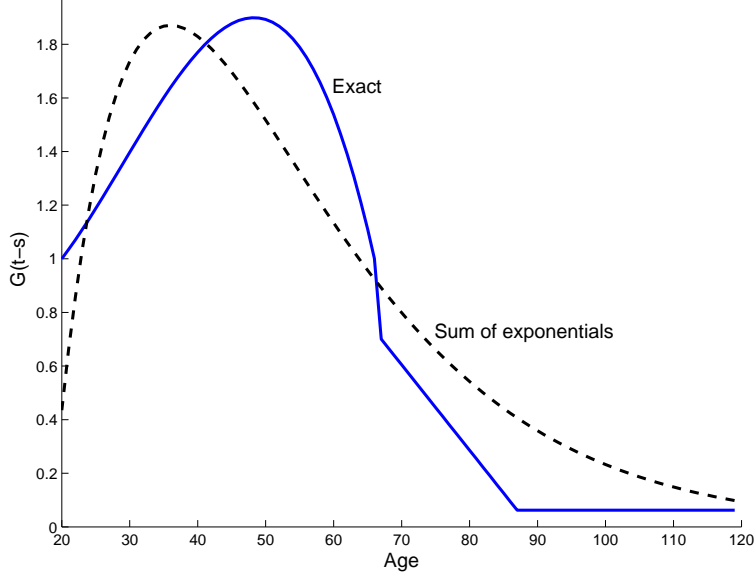


Figure 2: Hump-shaped profile of earnings over the life-cycle. The continuous line reports the profile estimated by Hubbard et al. (1994), while the broken line depicts the non-linear least-squares projection of the earnings profile in the data on a sum of scaled exponentials $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$. The estimated coefficients are $B_1 = 30.72$, $B_2 = -30.29$, $\delta_1 = 0.0525$, and $\delta_2 = 0.0611$.

shows that, for the low levels of IES that lead to implausibly high interest rates in the economy featuring an infinitely-lived agent, the OLG counterpart generates lower interest rates.

We summarize the role of the OLG feature of our model as follows: Besides helping lower the equilibrium interest rate, the OLG feature adds little in terms of resolving asset-pricing puzzles such as the equity premium, predictability, and excess volatility. However, as we show shortly, it plays a key role in terms of ensuring stationary consumption and wealth distributions with preference heterogeneity.

3 Heterogeneous preferences

We can now turn to the main results of the paper, by allowing agents to be heterogeneous ($\gamma^A \neq \gamma^B$, $\alpha^A \neq \alpha^B$).

To motivate our construction of the equilibrium and obtain some basic implications of heterogeneity, we derive a few results for heterogeneous CRRA preferences ($1 - \alpha^i = \gamma^i$, $\gamma^A < \gamma^B$). We start with two definitions. First, we let X_t denote the fraction of aggregate output consumed by type- A agents:

$$X_t \equiv \frac{v^A \pi \int_{-\infty}^t e^{-\pi(t-s)} c_{t,s}^A ds}{Y_t}. \quad (11)$$

Second, we define the “market price of risk” (or Sharpe ratio) as

$$\kappa_t \equiv \frac{\mu_t - r_t}{\sigma_t}. \quad (12)$$

Since agents can dynamically trade in a stock and a bond, they face dynamically complete markets over their lifetimes. As a result, letting ξ_t denote the stochastic discount factor, their consumption processes satisfy the relation

$$e^{-(\rho+\pi)(t-s)} \left(\frac{c_{t,s}^i}{c_{s,s}^i} \right)^{-\gamma^i} = e^{-\pi(t-s)} \frac{\xi_t}{\xi_s}, \text{ for } i \in A, B. \quad (13)$$

Using (13) to solve for $c_{t,s}^i$ and substituting into the definition of X_t gives

$$X_t \equiv \frac{v^A \pi \xi_t^{-\frac{1}{\gamma^A}} \int_{-\infty}^t e^{\left(-\pi - \frac{\rho}{\gamma^A}\right)(t-s)} c_{s,s}^A \xi_s^{\frac{1}{\gamma^A}} ds}{Y_t}. \quad (14)$$

Letting σ_X denote the diffusion coefficient of X_t , applying Ito’s Lemma to (14) and isolating diffusion coefficients yields

$$\sigma_X = \left(\frac{\kappa_t}{\gamma^A} - \sigma_Y \right) X_t, \quad (15)$$

where we have used (3) and the fact that the stochastic discount factor evolves as $\frac{d\xi_t}{\xi_t} = -r_t dt - \kappa_t dB_t$. Performing a similar computation to determine the diffusion coefficient of the consumption share of agent B and noting that the consumption share of agent B is equal

to $(1 - X_t)$ yields

$$-\sigma_X = \left(\frac{\kappa_t}{\gamma^B} - \sigma_Y \right) (1 - X_t). \quad (16)$$

Adding equations (15) and (16) and solving for κ_t yields

$$\kappa_t = \Gamma(X_t) \sigma_Y, \quad (17)$$

where

$$\Gamma(X_t) \equiv \left(\frac{X_t}{\gamma^A} + \frac{1 - X_t}{\gamma^B} \right)^{-1}. \quad (18)$$

Equation (17) has the familiar form one encounters in single-agent setups: the market price of risk is the product of the “representative agent’s” risk aversion $\Gamma(X_t)$ and the volatility of aggregate consumption. Alternatively phrased, the market price of risk with heterogeneous CRRA agents is identical to the market price of risk in an otherwise identical economy populated by a single, fictitious, expected-utility-maximizing “representative” agent with risk aversion given by $\Gamma(X_t)$.²⁰

An important difference with single-agent setups, however, is that the risk aversion of the representative agent is not constant, but rather time varying and negatively correlated with aggregate-consumption growth. To see this fact, use (17) inside (15) to obtain

$$\sigma_X(X_t) = X_t \left(\frac{\Gamma(X_t)}{\gamma^A} - 1 \right) \sigma_Y. \quad (19)$$

Since $\Gamma(X_t) > \gamma^A$ for $X_t \in (0, 1)$, equation (19) implies $\sigma_X > 0$. Hence, positive innovations to aggregate consumption increase the consumption share of type- A agents. At the same time, $\Gamma(X_t)$ is a declining function of X_t , so that whenever aggregate consumption experiences a positive innovation, $\Gamma(X_t)$ declines.

²⁰Similar results to equation (17) have been established in the literature. See, e.g., Dumas (1989), Chan and Kogan (2002), and Zapatero and Xiouros (2010).

The economic intuition behind this result is straightforward. Less risk-averse agents (type- A agents) invest more heavily in stocks than more risk-averse agents (type- B agents). As a result, a positive aggregate shock raises the wealth and consumption shares of less risk-averse agents. When less risk-averse agents own a larger fraction of aggregate wealth, the market price of risk is low, since these agents require a relatively smaller compensation for holding risk. Conversely, a negative shock increases the consumption and wealth shares of more risk averse agents, and consequently the market price of risk.

A mathematical implication of equations (17) and (19) is that both the Sharpe ratio κ_t and the diffusion coefficient σ_X only depend on the level of X_t . This observation motivates us to search for an equilibrium with the properties that the Sharpe ratio and the interest rate depend exclusively on X_t , and moreover X_t is Markov (i.e., its drift and volatility depend exclusively on X_t). The next proposition describes such an equilibrium, while also allowing for general, recursive preferences.

Proposition 2 *Let X_t and $G(t-s)$ be defined as in equations (11) and (5), respectively, and let $\beta_t^i \equiv \frac{c_{t,t}^i}{Y_t}$, $i \in \{A, B\}$, denote the consumption of a newly-born agent of type i as a fraction of aggregate output. Finally, let $\Gamma(X_t)$ be defined as in (18), let $\Theta(X_t)$, $\omega^A(X_t)$, $\omega^B(X_t)$, and $\Delta(X_t)$ be defined as*

$$\Theta(X_t) \equiv \frac{X_t}{1 - \alpha^A} + \frac{1 - X_t}{1 - \alpha^B}, \quad (20)$$

$$\omega^A(X_t) \equiv \frac{X_t}{\gamma^A} \Gamma(X_t), \quad \omega^B(X_t) \equiv 1 - \omega^A(X_t), \quad (21)$$

$$\Delta(X_t) \equiv \omega(X_t) \left(\frac{\gamma^A + 1}{\gamma^A} \right) + (1 - \omega(X_t)) \left(\frac{\gamma^B + 1}{\gamma^B} \right), \quad (22)$$

and assume functions $g^i(X_t)$ and $\phi^j(X_t)$, for $i \in \{A, B\}$, $j \in \{1, 2\}$, that solve the system of ordinary differential equations (A.18) and (A.21) in the appendix. Then there exists an equilibrium in which X_t is a Markov diffusion with dynamics $dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dB_t$

given by

$$\sigma_X(X_t) = \frac{X_t (\Gamma(X_t) - \gamma^A)}{\frac{\Gamma(X_t)}{\gamma^B} X_t (1 - X_t) \left[\frac{1 - \gamma^A - \alpha^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1 - \gamma^B - \alpha^B}{\alpha^B} \frac{g^{B'}}{g^B} \right] + \gamma^A} \sigma_Y, \quad (23)$$

$$\mu_X(X_t) = X_t \left[\frac{r(X_t) - \rho}{1 - \alpha^A} + n^A(X_t) - \pi - \mu_Y \right] + v^A \pi \beta^A(X_t) - \sigma_Y \sigma_X(X_t), \quad (24)$$

with

$$\kappa(X_t) = \Gamma(X_t) \sigma_Y + \sum_{i \in \{A, B\}} \omega^i(X_t) \left(\frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \sigma_X(X_t), \quad (25)$$

$$\begin{aligned} r(X_t) = \rho + \frac{1}{\Theta(X_t)} & \left\{ \mu_Y - \pi \left(\sum_{i \in \{A, B\}} v^i \beta^i(X_t) - 1 \right) \right\} \\ & - \frac{1}{\Theta(X_t)} [X_t n^A(X_t) + (1 - X_t) n^B(X_t)], \end{aligned} \quad (26)$$

$n^i(X_t)$ given by

$$\begin{aligned} n^i(X_t) = \frac{2 - \alpha^i}{2\gamma^i(1 - \alpha^i)} \kappa^2(X_t) + \frac{\alpha^i + \gamma^i - 1}{2\gamma^i\alpha^i} & \left(\frac{g^{i'}}{g^i} \sigma_X(X_t) \right)^2 \\ - \frac{\gamma^i - \alpha^i(1 - \gamma^i)}{\gamma^i(1 - \gamma^i)} \frac{\alpha^i + \gamma^i - 1}{(1 - \alpha^i)\alpha^i} & \left(\frac{g^{i'}}{g^i} \sigma_X(X_t) \right) \kappa(X_t), \end{aligned} \quad (27)$$

and $\beta^i(X_t) = g^i(X_t)(\phi^1(X_t) + \phi^2(X_t))$ for $i \in \{A, B\}$.

Proposition 2 asserts that the consumption share of type- A agents (X_t) follows a Markov diffusion and that the market price of risk and the interest rate are exclusively functions of X_t . In preparation for the discussion of the asset-pricing implications of the model, it is useful to note some additional properties of X_t .

3.1 Consumption share X_t

A property of models featuring preference heterogeneity and infinitely lived agents is that in many circumstances the consumption distribution eventually becomes degenerate:²¹ Different agents have persistently different mean growth rates of wealth and eventually one group

²¹See, e.g., Dumas (1989) and Cvitanic and Malamud (2010).

dominates.

The OLG feature of our model, however, implies generically that no group of agents becomes extinct in the long run. To establish this result, we start by noting that equation (23) implies $\sigma_X(0) = 0$, while equation (24) gives $\mu_X(0) = v^A \pi \beta^A(0)$. Since the value of life-time earnings of any newly born agent is strictly positive, so is any newly-born agent's consumption. Thus, $\beta^A(X_t) = \frac{c_{t,t}^A}{Y_t} > 0$ and, as a consequence, $\mu_X(0) > 0$. Proceeding in a similar fashion, equations (18) and (23) imply $\sigma_X(1) = 0$, and from (24) we deduce $\mu_X(1) = -v^B \pi \beta^B(1) < 0$.

An implication of the above facts is that the process for X_t does not get absorbed at the points 0 or 1.²² Intuitively, although the mean growth rates of wealth differ across agents, eventually all agents perish, regardless of their accumulated financial wealth. All newly-born agents enter the economy with no financial wealth and with human capital that is equal across preference groups and proportional to the level of output. As a result, each group of agents receives a minimum inflow of new members whose consumption is a non-zero fraction of aggregate output, ensuring that no type of agent dominates the economy.

Equations (23) and (24) also help illustrate the importance of risk-aversion heterogeneity (as opposed to IES heterogeneity) in terms of obtaining non-trivial variation of the consumption distribution. Indeed, suppose that $\alpha^A \neq \alpha^B$ and $\gamma^A = \gamma^B$, so that agents differ in their IES, but not their risk aversion. Then equation (18) implies $\Gamma(X_t) = \gamma^A = \gamma^B$ for any value of X_t , and therefore equation (23) implies $\sigma_X = 0$ for all X_t . The implications of this fact are summarized in the following corollary to Proposition 2.

Corollary 1 *Consider the setup of Proposition 2 and impose the restriction $\gamma^A = \gamma^B = \gamma$. Then $\sigma_X = 0$ at all times, the market price of risk is given by the constant $\kappa = \gamma\sigma$ for all X_t , and there exists a steady state featuring a constant interest rate and a constant consumption share of type-A agents given by some value $\bar{X} \in (0, 1)$.*

Corollary 1 states that when agents differ only in their IES, X_t converges deterministically

²²Technical details on boundary behavior and an analytic approach for the computation of the stationary density (which we use when we derive the stationary means of the quantities in our quantitative analysis) are given in Karlin and Taylor (1981).

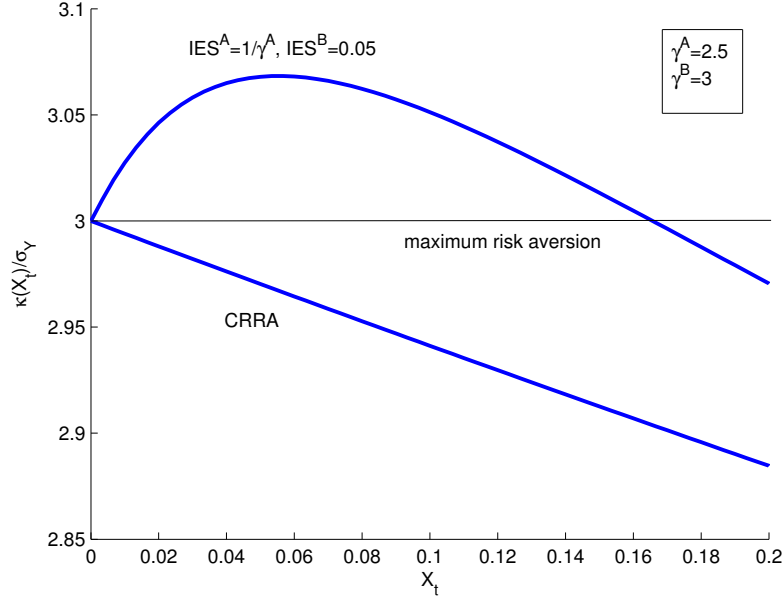


Figure 3: An illustration of Propositions 2 and 3. The ratio $\frac{\kappa(X_t)}{\sigma_Y}$ may be higher than the risk aversion of even the most risk-averse agent.

to a constant, and so does the interest rate. The market price of risk remains constant throughout at the level $\gamma\sigma$. In short, Corollary 1 asserts that risk-aversion heterogeneity is essential for the model to have any interesting asset-pricing implications. Therefore, for the rest of the paper we assume $\gamma^A < \gamma^B$.

3.2 Market price of risk (κ_t)

In the case where agents have heterogeneous CRRA preferences ($1 - \alpha^i = \gamma^i$ for $i \in \{A, B\}$) equations (25) and (23) become identical to equations (17) and (19), respectively. As we discussed earlier, this implies that the Sharpe ratio in our economy is identical to the Sharpe ratio of an economy populated by a representative agent featuring time-varying risk aversion. Moreover, the risk aversion of that representative agent is a weighted average of the risk aversions of the individual agents, with weights that are functions of each agents' consumption shares.

A novel implication of equation (25) is that when agents have recursive preferences ($1 -$

$\alpha^i \neq \gamma^i$ for some $i \in \{A, B\}$), the ratio $\frac{\kappa(X_t)}{\sigma_Y}$ is no longer equal to $\Gamma(X_t)$. Instead, it contains the additional term $\sum_{i \in \{A, B\}} \omega^i(X_t) \left(\frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$. This term depends on (i) the weights $\omega^i(X_t)$ of each agent, (ii) each agent's preferences for late or early resolution of uncertainty ($1 - \gamma^i - \alpha^i$), (iii) the preference parameters α^i , which control agents' IES, and (iv) the term $\frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$, which — as we show in the appendix — captures the volatility of each agent's consumption-to-total-wealth ratio normalized by the volatility of consumption.

It follows that $\frac{\kappa(X_t)}{\sigma_Y}$ no longer equals a simple weighted average of individual risk aversions. To illustrate this point, Figure 3 provides a numerical example where $\frac{\kappa(X_t)}{\sigma_Y}$ exceeds the risk aversion of any agent in the economy. In this example, a researcher using a standard, expected-utility-maximizing framework to infer the risk aversion of the representative agent (for such an exercise see, e.g., Ait-Sahalia and Lo (2000)) would obtain an estimate exceeding the maximum risk aversion in the economy.

Since many standard asset-pricing models tend to produce a low market price of risk for plausible assumptions on agents' risk aversions, we are particularly interested in determining conditions that ensure that the term $\sum_{i \in \{A, B\}} \omega^i(X_t) \left(\frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y}$ in equation (25) is positive, so that the market price of risk $\kappa(X_t)$ exceeds the Sharpe ratio $\Gamma(X_t) \sigma_Y$ that one would obtain with heterogeneous, but expected-utility maximizing, agents. The following proposition addresses this question.

Proposition 3 *Consider the same setup as in Proposition 2 with the parameters restricted to lie in a (arbitrarily large) compact set, and let \bar{X} denote the stationary mean of X_t . Then — subject to technical parameter restrictions given in the appendix — for sufficiently small $|\alpha^B - \alpha^A|$ and sufficiently small $|\gamma^B - \gamma^A|$ (possibly depending on $|\alpha^B - \alpha^A|$) the Sharpe ratio satisfies $\kappa(\bar{X}) \geq \Gamma(\bar{X}) \sigma_Y$ if either*

(i) $\gamma^i + \alpha^i - 1 > 0$ for $i \in \{A, B\}$ and $\alpha^A < \alpha^B$, or

(ii) $\gamma^i + \alpha^i - 1 < 0$ for $i \in \{A, B\}$ and $\alpha^A > \alpha^B$.

We present the proof of Proposition 3 in the appendix. Here we only give a brief intuition behind this proposition using a graphical argument. We consider case (i), illustrated in

Figure 4; case (ii) can be treated symmetrically. The solid lines depict the paths of expected (log) consumption for an agent of type A and an agent of type B . According to case (i) of Proposition 3, agent A has a lower IES than agent B ($\alpha^A < \alpha^B$). This is reflected in that agent A 's expected log consumption path in Figure 4 is flatter than that of agent B . Now suppose that a positive aggregate-consumption shock arrives. The dotted lines describe the reaction of agents' log consumption in response to that shock. Since agent A is less risk averse, her consumption is more strongly exposed to aggregate risk. This is reflected in the larger vertical distance between the solid and the dotted line for agent A as compared to agent B .

An additional observation is that the immediate response in either agent's log consumption is *smaller* than the response in the long run. Differently put, there is a temporary increase in each agent's consumption growth rate, an outcome explained as follows. A positive aggregate shock increases the economic importance of less risk-averse agents (type- A agents), who, according to the assumptions of case (i), are also the agents with the lower IES. Since agents of type A become more important in the consumption distribution, equilibrium asset prices have to adjust to incentivize agents to accept a steeper consumption path.²³ This increase in agents' consumption growth is temporary; it dissipates in the long run as X_t returns to its long-run mean.

The fact that agents' consumption reacts more strongly in the long run than in the short run implies that individual agents' consumption growth exhibits "long-run risk" in the sense of Bansal and Yaron (2004). With standard expected utility, agents require compensation only for the immediate impact of a shock to their consumption. However, with recursive utility exhibiting a preference for early resolution of uncertainty (as in case (i) of the proposition) they command additional compensation for bearing long-run risk. We stress that in our setup predictable consumption components arise endogenously and *only* at the level of individual agents' consumption.²⁴ In contrast to Bansal and Yaron (2004), aggregate

²³If the now more economically important type- A agents kept on enjoying the same relatively flat consumption growth path going forward, feasibility would require an excessively steep path for type- B agents, which they are reluctant to accept at the old prices.

²⁴See Malloy et al. (2009) for empirical evidence of such predictability.

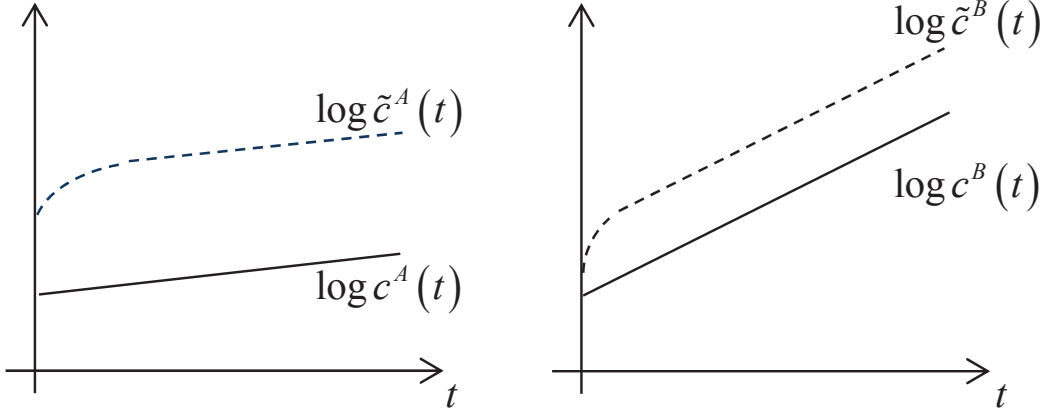


Figure 4: Illustration of case (i) of Proposition 3.

consumption growth is i.i.d. More importantly, our framework does not require stochastic volatility of aggregate consumption in order to produce variation in the market price of risk.

Finally, we remark that although Proposition 3 applies when preference heterogeneity is not “too large,” Figure 5 illustrates that the conclusion of the proposition holds even for non-trivial heterogeneity in preferences.

3.3 The interest rate (r_t)

In Section 2 we discussed why the OLG structure of our economy (absent preference heterogeneity) can lead to a lower level of the interest rate than in the respective infinite-horizon economy. With preference heterogeneity the interest rate becomes time varying. Motivated by the stylized facts listed in Campbell and Cochrane (1999), we are particularly interested whether there exist joint distributions of preference parameters for which the volatility of discount rates is due almost exclusively to equity-premium, rather than interest-rate, variability.

We can provide an affirmative answer to this question using an approximation (accurate when σ_Y is not too large) around the steady state.

Proposition 4 *Fix $D_1 > 0$ and $D_2 > 0$. Then there exist parameters γ^A , γ^B , and v^A , as*

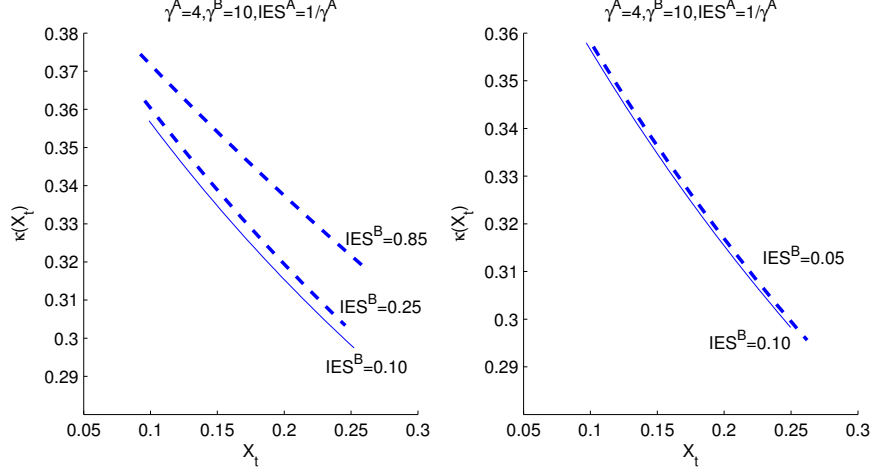


Figure 5: A numerical illustration of Proposition 3. In all cases, we fix agent A 's IES to be equal to the reciprocal of her risk aversion. The solid lines (in both the left and right panels) depict the case where agent B is also a CRRA agent. The dashed lines in the left panel depict parameter combinations corresponding to case (i) of Proposition 3, whereby agent B has preferences for early resolution of uncertainty and an IES at least as high as agent A . Similarly, the dashed line on the right hand side panel depicts a case where the more risk averse agent B has preferences for late resolution of uncertainty and (by implication) a lower IES than agent A .

well as α^A and α^B possibly depending on σ_Y , so that at the steady-state mean \bar{X} ,

$$\frac{\kappa(\bar{X})}{\sigma_Y} = D_1 + \epsilon_1(\sigma_Y^2) \quad (28)$$

$$-\frac{\kappa'(\bar{X})}{\sigma_Y} = D_2 + \epsilon_2(\sigma_Y^2) \quad (29)$$

$$r'(\bar{X}) = 0 + \epsilon_3(\sigma_Y^2), \quad (30)$$

where the terms $\epsilon_i(\sigma_Y^2)$ satisfy $\lim_{x \rightarrow 0} \epsilon_i(x) = 0$ for $i \in \{1, 2\}$ and $\lim_{x \rightarrow 0} \frac{\epsilon_3(x)}{x} = 0$. Accordingly, the relative magnitude of the volatility of the equity premium to the volatility of the interest rate, $\frac{|(\kappa(\bar{X})\sigma(\bar{X}))'|_{\sigma_X(\bar{X})}}{|r'(\bar{X})|_{\sigma_X(\bar{X})}}$, can be made arbitrarily large.

The main thrust of Proposition 4 is that one can “reverse engineer” a joint distribution of IES and risk aversion so as to ensure (around the stationary mean of X_t) that the interest rate is substantially less volatile than the equity premium. The intuition for this result is quite simple. As discussed earlier, a positive aggregate shock leads to a decline in the Sharpe

ratio. To the first order in σ_Y the magnitude of this decline is determined by the magnitude of the differences in the risk-aversion coefficients γ of the two agents. However, this decline in the Sharpe ratio, which reflects the increased importance of relatively more risk-tolerant agents, also leads to lower aggregate precautionary savings and a higher interest rate. By choosing the differences in the IES of the two agents judiciously, it is possible to ensure that the reduction in aggregate precautionary savings is offset by an increase in savings due to intertemporal-smoothing reasons, leaving the interest rate effectively unchanged. Accordingly, the variation in the interest rate can be made arbitrarily smaller than the variation of the equity premium.²⁵

From a practical perspective one would also like to know how large these differences in risk aversion and IES have to be in order to match asset-pricing data. We turn to this quantitative question in the next section.

4 Quantitative Implications

4.1 Unconditional moments

In this section we illustrate the quantitative implications of the channels highlighted in the above propositions by determining numerically a population of agents associated with a high equity premium, volatile returns, and a low and non-volatile interest rate. Specifically, we choose the same parameters μ_Y , σ_Y , π , ω , and ρ as in Section 2 and the same estimates for the parametric specification of life-cycle earnings profile (B_1 , B_2 , δ_1 , and δ_2) as in Figure 2. (We justify these choices and give detailed references to the data we used in the appendix — section B). We treat the remaining parameters of the model, namely the risk aversions and IES's of agents A and B , as well as the share of type- A agents in the population (v^A) as free parameters. In the spirit of Proposition 4, we choose these parameters to approximately match asset-pricing moments and, in particular, the level of the equity premium, the volatility of returns, the level of the interest rate, and the volatility of the real rate. At a later stage we

²⁵In addition, it is a simple matter to ensure, through an appropriate parameter choice, that the interest-rate level $r(\bar{X})$ attains any value consistent with finite valuations.

Parameter	Data	Baseline	CRRA
Risk aversion of type- <i>A</i> agents		1.5	1.5
Risk aversion of type- <i>B</i> agents		10	10
IES of type- <i>A</i> agents		0.7	$\frac{1}{\gamma^A}$
IES of type- <i>B</i> agents		0.05	$\frac{1}{\gamma^B}$
Population share of type- <i>A</i> agents		0.01	0.01
Equity premium	5.2 %	5.1 %	2.1 %
Volatility of returns	18.2%	17.0 %	8.5 %
Average (real) interest rate	2.8%	1.4%	2.1 %
Volatility of the real interest rate	0.92%	0.93%	0.13 %

Table 1: Unconditional moments for the data, the baseline parametrization and a CRRA parametrization (restricting the IES to be the inverse of the relative risk aversion). The data for the average equity premium, the volatility of returns, and the level of the interest rate are from the long historical sample (1871-2011) available from the website of R. Shiller (<http://www.econ.yale.edu/~shiller/data/chapt26.xls>). The volatility of the “real rate” is based on the yields of 5-year constant maturity TIPS, as we explain in detail in the text.

also examine whether these parameters, which are chosen to match asset-pricing moments, also have the correct implications for cross-sectional consumption and wealth data. We place an additional requirement, namely that the risk aversion of agent *B* do not exceed 10, which is commonly considered the upper bound on plausible values of risk aversion. (See, e.g., Mehra and Prescott (1985)). This constraint limits the ability of the model to match all asset-pricing moments exactly, but seems nevertheless important in order to keep the results comparable with the literature.

Before proceeding, we make two remarks on relating the model’s results to the data. First, the volatility of the real interest rate is not directly observable in the data. Inferring this volatility from the difference between the nominal interest rate and realized inflation can be misleading, since unexpected inflation shocks raise the volatility of the realized real interest rate above the volatility of the “ex-ante” real interest rate. To circumvent this problem, we use the yield on the shortest constant-maturity (five years) treasury inflation-protected securities (TIPS) available from the Federal Reserve. The relatively small volatility (92 basis points) that we compute this way is a lower bound on the short-rate volatility, since yields on short-term bonds are more volatile than the respective long-term yields both inside our

model and in the data. By aiming to match this number, we make sure to avoid relying on implausibly volatile real rates.

Second, in order to compare the volatility of our all-equity financed firm to the levered-equity returns observed in the real world, we follow the straightforward approach advocated by Barro (2006). Barro (2006) proposes to treat levered equity as a zero-net-supply “derivative” security of the positive-supply unlevered equity. The introduction of a zero-net-supply claim leaves all allocations and prices unchanged, while the Modigliani-Miller theorem implies that levered equity has the same return as a (constantly rebalanced) replicating portfolio that is long unlevered equity and short debt. Specifically, given the debt value B_t and the value of levered equity $S_t - B_t$, the expected excess rate of return of levered equity is $\left(1 + \frac{B_t}{S_t - B_t}\right)$ times higher than that of unlevered equity. Following Barro (2006), we set $\frac{B_t}{S_t - B_t} = 0.5$ to reflect the average historical leverage ratio in NIPA data and report results for levered equity.

The first column of Table 1 gives the empirical values of the moments we are targeting. The second column gives preference parameters and the resulting model-implied moments. For comparison purposes, the last column of the table illustrates the CRRA case defined by the same risk aversions. In the appendix we investigate the model’s robustness to the choice of parameters; in particular, we show that the results are not affected much when v^A is constrained to take higher values (Table B.1).

We make two remarks about Table 1. First, the flexibility to choose agents’ IES independent of their risk aversion is important for the model’s quantitative implications. Without this flexibility the model doesn’t perform as well, as illustrated in the third column of the table. Interestingly, the main reason why the baseline model performs better is *not* its higher Sharpe ratio, which only increases from 0.25 to 0.3 when we compare the heterogeneous CRRA case to the recursive-preference case. Rather, the main reason for the better performance of the baseline model is the higher volatility of returns compared to the heterogeneous-CRRA case.

In turn, this higher volatility is due to the fact that the variation of the interest rate reinforces the respective variation of the equity premium, while in the heterogeneous CRRA

case it largely offsets it. Specifically, as Figure 6 shows, both the Sharpe ratio and the interest rate decline with X_t . In other words, the lower compensation required for risk associated with a positive endowment shock is reinforced by a declining interest rate. Discount rates, and hence returns, are consequently more volatile.

By contrast, in the CRRA case the interest rate is a hump-shaped function of X_t : As we discussed earlier (Section 3.3), the sign of the reaction of the interest rate in response to an increase in X_t depends on the interaction of two effects: a drop in precautionary savings and a change in savings motivated by intertemporal smoothing. The first effect tends to raise the interest rate, whereas the second effect tends to lower the interest rate as long as agents with the higher risk aversion also have higher IES (as is the case with CRRA preferences). With CRRA preferences the first effect dominates for low values of X_t and the second effect dominates for higher values of X_t . As a result, over a portion of the state space, the interest rate counteracts the negative relation between the Sharpe ratio and consumption shocks. The result is a more muted variation in discount rates and a lower volatility than in our baseline calibration, where the larger difference in IES between the two agents (compared to the CRRA case) ensures an interest rate that decreases with X_t .

The fact that in the baseline parametrization the interest rate decreases, while the price-to-dividend ratio increases, with X_t implies a negative relation between the log-price-dividend ratio and the real interest rate. In the next section, we show that both in the data and in the baseline case of the model regressions of the real interest rate on the log-price-dividend ratio yield coefficients that are very small in magnitude and negative.

Our second remark pertains to the type of population required to match asset-pricing facts. A potential concern with choosing preference parameters to match asset-pricing moments is that such parameters may lead to an implausible concentration of wealth and consumption in the hands of a small minority, or to excessively volatile cross-sectional consumption and wealth distributions. To address these concerns, we start by noting that an attractive feature of the OLG structure of the model is that it generates a *continuous* (rather than two-point) cross-sectional consumption distribution reflecting the history of aggregate

shocks faced by different cohorts of individuals over their life-cycle, and their different savings and portfolio decisions. This adds an important layer of discipline to the model, since its predictions can be compared to real-world cross-sectional distributions.

Adding support to the model, we find that the same assumptions on the population that generate the key asset-pricing facts are also helpful in matching a couple of salient features of real-world consumption and wealth distributions, namely *(i)* a large Gini coefficient for wealth inequality and a substantially smaller Gini coefficient for consumption inequality and *(ii)* small year-over-year changes in these Gini coefficients. We substantiate these claims by simulating artificial cross sections of individuals inside the model featuring the same household numbers and number of cross sections as found in empirical studies.²⁶

Concerning the first feature, which pertains to the level of inequality, we find that the Gini coefficient of wealth inequality is 0.67 inside the model, while it is 0.9 in the data. At the same time, the consumption Gini coefficient inside the model is 0.3, which is close to its data counterpart (0.28). The model can produce a large inequality of wealth and also a substantially smaller inequality of consumption for two reasons: a) inside the model total wealth inequality (i.e., wealth inequality measured by combining agents' financial wealth and value of human capital) is smaller than pure financial wealth inequality, and b) richer households tend to have lower consumption-to-total-wealth ratios, i.e., they tend to be better savers.

The second feature of the data that the model captures is the small *variation* in the cross-sectional distributions of consumption and wealth. In contrast to models that need large yearly movements in these distributions to be consistent with asset-return properties, our model's outcome is in line with the data. Specifically, we find that year-over-year changes in the consumption Gini coefficient²⁷ are about 0.95%, whereas in the data the respective figure is 0.70%. The reason for the model's ability to match patterns of asset returns (and

²⁶The data for the Gini coefficient for financial wealth are from Wolff (2010) and are based on the Survey of Consumer Finances. The respective data for the consumption of the Gini coefficient are from Cutler and Katz (1992) and are based on the Consumer Expenditure Survey. We explain in detail how we perform the simulation of our artificial cross sections in appendix B.

²⁷Similar conclusions hold for wealth inequality. In the data, year-over-year changes to wealth inequality are 1.02 %, whereas in the model they are slightly below 1.7%.

especially the high variability of returns), while also implying non-volatile cross-sectional consumption and wealth distributions, is quite simple.

Movements in asset prices are governed exclusively by the *current* values of X_t and the aggregate shock. By contrast, an agent's consumption is affected by the history of past values of X_t and shocks since her birth. For this reason, our OLG model produces not only inter-group inequality, but also a large consumption (and wealth) dispersion across different age cohorts of the same group. This intra-group dispersion reflects the entire path of X_t and shocks, so that — unlike with asset prices — current shocks affect it relatively little.

We note that the model's implications for the *variability* of inequality are likely to be robust to the introduction of realistic features that, in the interest of parsimony, we left out. The implications for the level of inequality, on the other hand, would be impacted by idiosyncratic shocks and bequests, which tend to increase inequality, as well as by taxes and government redistribution programs, which tend to reduce it. These features could be added, but would be distractions from the main insights of the paper. For our purposes, it suffices to note that the model — despite its abstractions — implies properties of inequality that are not at odds with the data.

We conclude this discussion by noting that the parameters, which we chose to match asset-price properties, are actually in line with the findings of empirical studies utilizing micro-level data to estimate Euler equations. A stylized fact of such studies is that estimates for the IES for poorer households tend to be close to zero, while the estimates of the IES for richer households tend to be substantially higher.²⁸ We note, however, that the precise magnitude of these estimates can vary widely depending on the instruments chosen to estimate the parameters, the definition of non-durables and services, the period over which consumption data are aggregated, etc.

²⁸These statements hold also conditional on participation. See Vissing-Jorgensen (2002).

Horizon (Years)	Data (Long Sample)		Model	
	Coefficient	R^2	Coefficient	R^2
1	-0.13	0.04	-0.09 [-0.29,0.01]	0.03 [0.00,0.10]
3	-0.35	0.09	-0.25 [-0.68,0.00]	0.08 [0.00,0.23]
5	-0.60	0.18	-0.41 [-1.03,0.01]	0.12 [0.00,0.32]
7	-0.75	0.23	-0.54 [-1.27,0.00]	0.16 [0.00,0.41]

Table 2: Long-horizon regressions of excess returns on the log P/D ratio. To account for the well-documented finite-sample biases driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 106-year long samples, where the initial condition for X_0 for each of these simulation paths is drawn from the stationary distribution of X_t . For each of these 106-year long simulated samples, we run predictive regressions of the form $\log R_{t \rightarrow t+h}^e = \alpha + \beta \log (P_t/D_t)$, where R_t^e denotes the gross excess return in period t and h is the horizon for returns in years. We report the median values for the coefficient β and the R^2 of these regressions, along with the respective [0.025, 0.975] confidence intervals.

4.2 Conditional moments and pathwise comparisons with the data

Besides matching unconditional moments of the data, the model can also match well known facts pertaining to the predictability of excess returns by such predictive variables as the price-to-dividend ratio.

Table 2 shows that the model reproduces the empirical evidence on predictability of excess returns over long horizons. This is not surprising in light of Figure 6: Since dividend growth is i.i.d. in our model, movements in the price-dividend ratio must be predictive of changes in discount rates. As Figure 6 shows, interest rate movements in our model are of much smaller magnitude than excess-return movements. As a result the price-dividend ratio predicts excess returns quite well.

Table 3 reinforces this point. In this table we use the current log-price-dividend ratio to predict real riskless returns. Similar to the data, the model produces negative and very small coefficients of predictability of the riskless rate, while the respective coefficients for excess returns (Table 2) are an order of magnitude larger. (It is useful to note here that the negative comovement between the real interest rate and the log price-to-dividend ratio

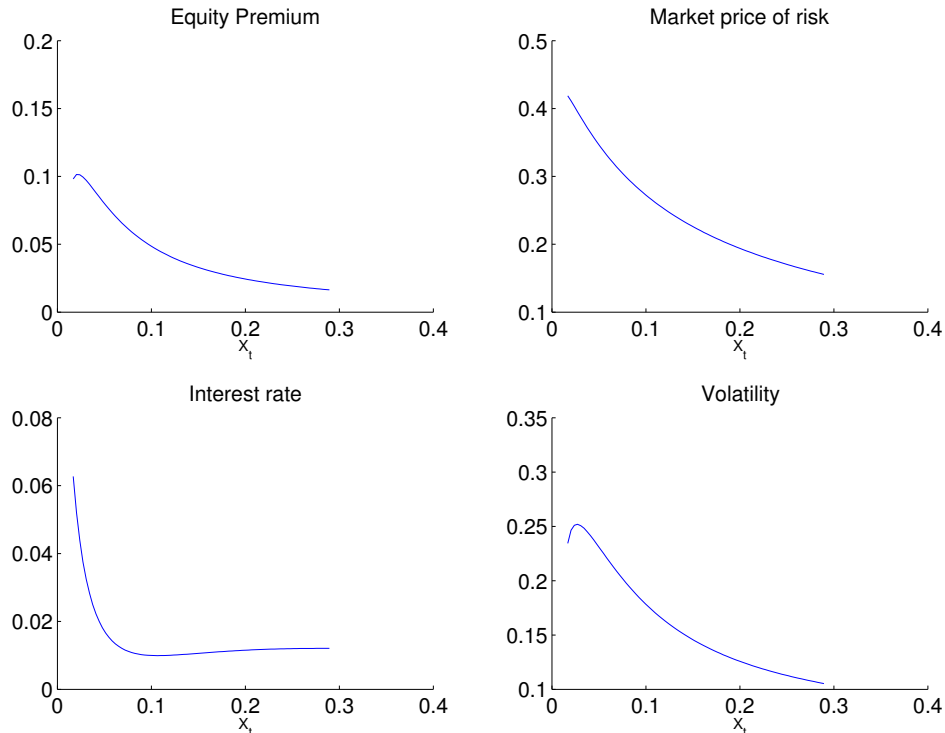


Figure 6: Equity premium, market price of risk, interest rate, and return volatility for the baseline parametrization as a function of the consumption share of type-A agents X_t . The range of values of X_t spans the interval between the bottom 0.5 and the top 99.5 percentiles of the stationary distribution of X_t . The exact stationary distribution of X_t is given in Figure 7.

should not be interpreted as saying that the real interest rate is “countercyclical”. In our model all shocks are permanent and hence there is no notion of a “cycle” in the conventional, Beveridge-Nelson sense nor a notion of an “output gap”.) Finally, we also note that in Table 3 we don’t include comparisons between the R^2 of regressions in the data and the model. The reason is that, unlike regression coefficients, the R^2 depends on inflation shocks, which we do not model.

We conclude with a figure showing that the model can roughly reproduce even more detailed, pathwise properties of excess returns. Specifically, we perform the following exercise. We take annual consumption data since 1889. We pick the starting value X_0 (where year zero is 1889) to equal the model-implied stationary mean of X_t . For the realized consump-

Horizon (Years)	Data (Long Sample)	Model
1	-0.006	-0.013 [-0.060,0.005]
3	-0.039	-0.031 [-0.157,0.012]
5	-0.073	-0.047 [-0.248,0.019]
7	-0.116	-0.059 [-0.336,0.025]

Table 3: Long-horizon regressions of the real interest rate on the log P/D ratio. To account for the well-documented finite-sample biases driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 106-year long samples, where the initial condition for X_0 for each of these simulation paths is drawn from the stationary distribution of X_t . For each of these 106-year long simulated samples, we run predictive regressions of the form $\log R_{t \rightarrow t+h}^f = \alpha + \beta \log (P_t/D_t)$, where R_t^f is the gross riskless rate of return at time t and h is the horizon for returns in years. We report the median values for the coefficient β of these regressions, along with the respective $[0.025, 0.975]$ confidence intervals.

tion data between 1889 and 2010, we construct the model-implied path of X_t . Since X_t is the only state variable in the model, we can use knowledge of X_t to construct a series of model-implied excess returns $\mu_t - r_t$ for $t = 1889, 1890$, and so on. The top panel of Figure 8 compares the model-implied excess returns at time t with the realized excess return in the data. The bottom panel performs the same exercise averaging returns over rolling eight-year periods.

Clearly, the model fails to reproduce the data exactly both at the one-year and the eight-year horizon. This is hardly surprising given the stylized nature of the model. Of particular interest for our purposes is the behavior of the model for excess returns at eight-year horizons, since averaging helps isolate low-frequency movements in excess returns, which are more strongly influenced by movements in expected excess returns rather than shocks.²⁹ Focusing on the bottom panel of the figure, we note that the model performs reasonably well until 1945 and then again from the 1970's onwards. The model fails to capture the behavior

²⁹Our results are similar when we consider five- or ten-year averages, or when we isolate business-cycle frequencies using band-pass filtering.

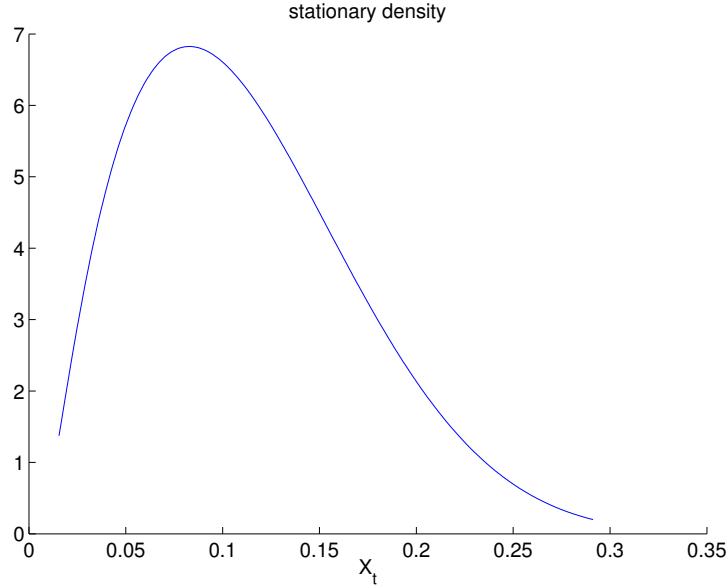


Figure 7: Stationary density of X_t . The range of values of X_t spans the interval between the bottom 0.5 and the top 99.5 percentiles of the stationary distribution of X_t .

of average returns between 1945 and 1970. Interestingly, in the data the rolling eight-year excess return peaks in 1948 and then progressively declines until 1966. In the model excess returns remain around their long-term average until 1953, while the rapid consumption gains of the 1950s and 1960s lead to a run-up in excess returns that reaches a peak in 1960, followed by a decline. One potential explanation for the failure of the model during that period is that some of the productivity and consumption gains that followed the war were not shocks, but rather gains anticipated by market participants who understood that productivity “catches up” rapidly after a war. Accordingly, the model-implied excess returns “lag” the actual data.

5 Concluding remarks

We analyze the asset-pricing implications of preference heterogeneity in a framework combining overlapping generations and recursive utilities. We express the equilibrium in terms of the solution to a system of ordinary differential equations, and characterize properties of the solution.

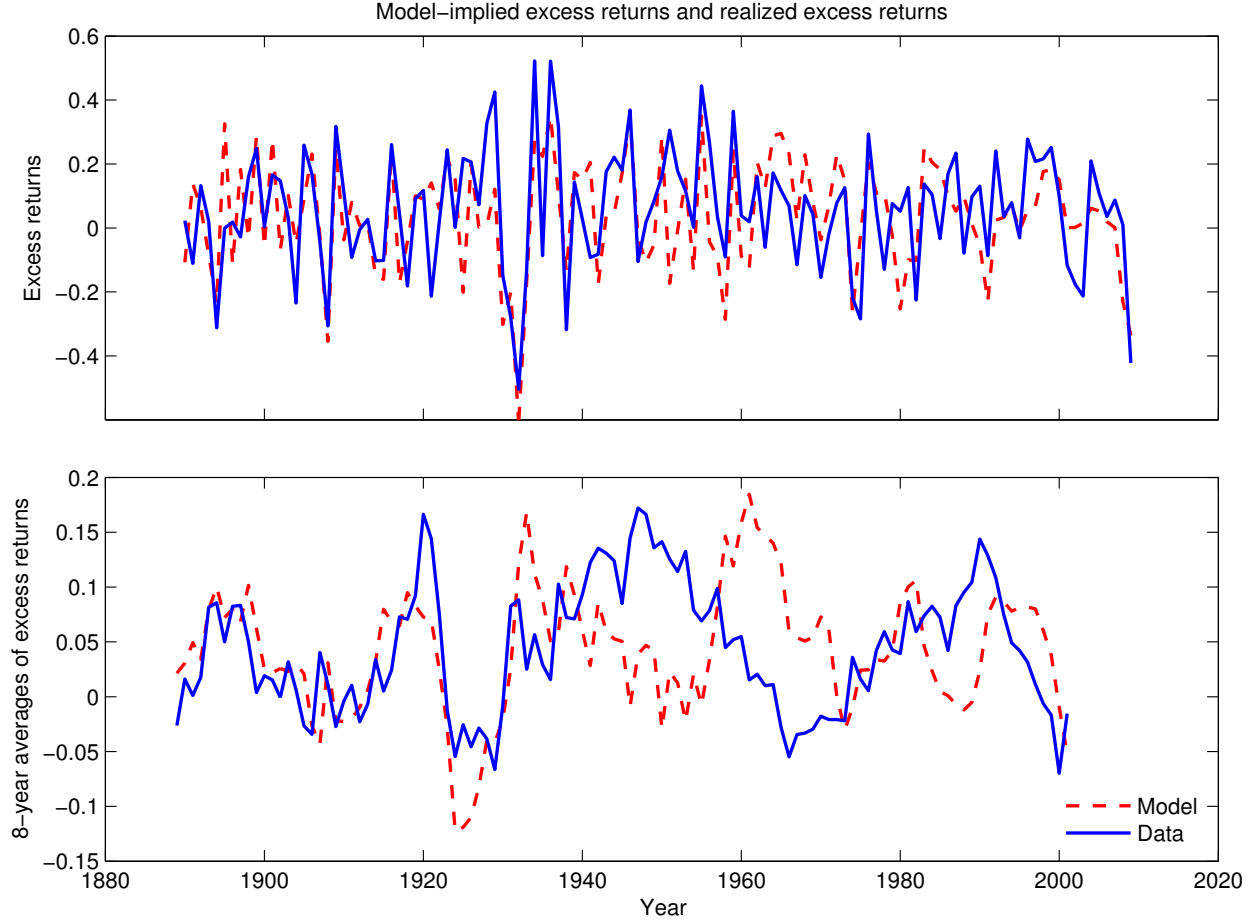


Figure 8: Excess returns according to the model and realized excess returns in the data. To compute model-implied excess returns, we set the initial value of the consumption share of type- A agents (X_t) equal to its model-implied stationary mean. Since — according to the model — consumption growth is i.i.d., we determine consumption innovations by de-meaning first differences of log consumption growth from 1889 onwards. To account for the different standard deviations of consumption growth in the pre- and post- World War II sample we divide the de-meaned first differences of log consumption by the respective standard deviations of the two subsamples. The resulting series of consumption innovations, together with the initial condition X_0 , is used to compute the model-implied path of X_t . Given X_t , the dotted line depicts the model-implied excess return for each year (top panel) and the corresponding annualized 8-year excess return (bottom panel). For comparison, the solid line depicts the data counterparts.

In particular, we show that in the case in which agents have heterogeneous CRRA preferences, the ratio of the market price of risk to the consumption-growth volatility is negatively correlated with consumption shocks and given by a stationary weighted average of agents'

risk aversions. When agents have recursive preferences, however, this ratio is also affected by the endogenous persistence in individual agents' consumption growth caused by the optimal consumption sharing rule. We analyze parameter conditions that lead to a higher market price of risk. In some cases, the resulting market price of risk may even be higher than in a world populated exclusively by the agent with the highest risk-aversion coefficient.

Finally, we separate the effects of risk-aversion heterogeneity and IES heterogeneity and find that risk-aversion heterogeneity is indispensable for the model to have interesting asset-pricing implications. However, we also find that the two dimensions of heterogeneity interact in interesting ways. Importantly, the flexibility to choose IES heterogeneity independently of risk-aversion heterogeneity allows one to control interest-rate time variation independently of the variation in the market price of risk.

This flexibility is important for the model's ability to reproduce asset-pricing facts quantitatively. Interestingly, we find that the type of assumptions on the IES and risk-aversion heterogeneity required to match asset-pricing facts are also consistent with the salient properties of real-world cross-sectional consumption and wealth distributions. Most importantly, the interaction of the OLG and preference-heterogeneity features of the model allows for both a high variability in asset prices and a low variability in cross-sectional measures of inequality.

References

- Abel, A. B. (2003). The effects of a baby boom on stock prices and capital accumulation in the presence of social security. *Econometrica* 71(2), 551–578.
- Ait-Sahalia, Y. and A. W. Lo (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics* 94, 9–51.
- Backus, D., B. R. Routledge, and S. E. Zin (2008). Who holds risky assets? mimeo, NYU and CMU.
- Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance* 59(4), 1481–1509.
- Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics* 121(3), 823–66.
- Bhamra, H. and R. Uppal (2009). The effect of introducing a non-redundant derivative on the volatility of stock-market returns when agents differ in risk aversion. *Review of Financial Studies* 22, 2303–2330.
- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy* 93(2), 223–247.
- Branger, N., I. Dumitrescu, V. Ivanova, and C. Schlag (2012). Preference heterogeneity and survival in long-run risk models. Working paper, University of Frankfurt.
- Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy* 107(2), 205–251.
- Campbell, J. Y. and N. G. Mankiw (1989). *Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence.*, pp. 185–216. Princeton U and NBER: Cambridge, Mass. and London:.
- Chan, Y. L. and L. Kogan (2002). Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices. *Journal of Political Economy* 110(6), 1255–1285.

- Constantinides, G. M., J. B. Donaldson, and R. Mehra (2002). Junior can't borrow: A new perspective of the equity premium puzzle. *Quarterly Journal of Economics* 117(1), 269–96.
- Constantinides, G. M. and D. Duffie (1996). Asset pricing with heterogeneous consumers. *Journal of Political Economy* 104, 219–40.
- Cutler, D. M. and L. F. Katz (1992). Rising inequality? changes in the distribution of income and consumption in the 1980's. *American Economic Review* 82(2), 546–551.
- Cvitanic, J. and S. Malamud (2010). Relative extinction of heterogeneous agents. *The B.E. Journal of Theoretical Economics* 10.
- Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica* 60(2), 353–394.
- Duffie, D., P.-Y. Geoffard, and C. Skiadas (1994). Efficient and equilibrium allocations with stochastic differential utility. *Journal of Mathematical Economics* 23(2), 133–146.
- Dumas, B. (1989). Two-person dynamic equilibrium in the capital market. *Review of Financial Studies* 2(2), 157–188.
- Dumas, B., R. Uppal, and T. Wang (2000). Efficient intertemporal allocations with recursive utility. *Journal of Economic Theory* 93, 240–259.
- Epstein, L. G. (1987). A simple dynamic general equilibrium model. *Journal of Economic Theory* 41(1), 68–95.
- Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica* 57(4), 937–69.
- Farmer, R. E. A. (2002). Fiscal policy, equity premia and heterogeneous agents. Working Paper, UCLA.
- Gârleanu, N., L. Kogan, and S. Panageas (2009). The demographics of innovation and asset returns. NBER Working Paper 15457.
- Gentry, W. M. and G. R. Hubbard (2002). Entrepreneurship and household saving. mimeo, Williams College and Columbia GSB.

- Gomes, F. and A. Michaelides (2008). Asset pricing with limited risk sharing and heterogeneous agents. *Review of Financial Studies* 21(1), 415–449.
- Gomes, J. F., L. Kogan, and M. Yogo (2009). Durability of output and expected stock returns. *Journal of Political Economy* 117(5), 941–986.
- Guisen, F. (2009). A parsimonious macroeconomic model for asset pricing. *Econometrica* 77(6), 1711–1740.
- Hall, R. E. (1988). Intertemporal substitution in consumption. *Journal of Political Economy* 96(2), 339–357.
- Heaton, J. and D. J. Lucas (2000). Stock prices and fundamentals. *NBER Macroeconomics Annual* 14, 213–42.
- Hubbard, R. G., J. Skinner, and S. P. Zeldes (1994). The importance of precautionary motives in explaining individual and aggregate saving. *Carnegie-Rochester Conference Series on Public Policy* 40, 59–125.
- Kan, R. (1995). Structure of Pareto optima when agents have stochastic recursive preferences. *Journal of Economic Theory* 66(2), 626–631.
- Karatzas, I. and S. E. Shreve (1998). *Methods of mathematical finance*, Volume 39. Springer-Verlag.
- Karlin, S. and H. M. Taylor (1981). *A Second Course in Stochastic Processes*. London, Oxford, Boston, New York and San Diego: Academic Press.
- Kreps, D. M. and E. L. Porteus (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica* 46(1), 185–200.
- Longstaff, F. A. and J. Wang (2008). Asset pricing and the credit market. mimeo, UCLA Anderson and MIT Sloan.
- Lucas, Robert E., J. and N. L. Stokey (1984). Optimal growth with many consumers. *Journal of Economic Theory* 32(1), 139–171.
- Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica* 46(6), 1429–1445.

- Ma, C. (1993). Market equilibrium with heterogenous recursive-utility-maximizing agents. *Economic Theory* 3(2), 243–266.
- Malloy, C., T. Moskowitz, and A. Vissing-Jorgensen (2009). Long-run stockholder consumption risk and asset returns. *Journal of Finance* 64(6), 2427–2479.
- Mehra, R. and E. C. Prescott (1985). The equity premium: A puzzle. *Journal of Monetary Economics* 15(2), 145–161.
- Papanikolaou, D. (2010). Investment-specific shocks and asset prices. Working Paper, Kellogg School of Management.
- Piazzesi, M. and M. Schneider (2009). Inflation and the price of real assets. Federal Reserve Bank of Minneapolis, Staff Report: 423.
- Piazzesi, M., M. Schneider, and S. Tuzel (2007). Housing, consumption and asset pricing. *Journal of Financial Economics* 83(3), 531–569.
- Santos, T. and P. Veronesi (2006). Labor income and predictable stock returns. *Review of Financial Studies* 19(1), 1–44.
- Schroder, M. and C. Skiadas (1999). Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory* 89(1), 68–126.
- Spear, S. E. and S. Srivastava (1986). Markov rational expectations equilibria in an overlapping generations model. *Journal of Economic Theory* 38(1), 35–62.
- Storesletten, K., C. I. Telmer, and A. Yaron (2007). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics* 10(4), 519–548.
- Tuzel, S. (2010). Corporate real estate holdings and the cross-section of stock returns. *Review of Financial Studies* 23(6), 2268–2302.
- Vissing-Jorgensen, A. (2002). Limited asset market participation and the elasticity of intertemporal substitution. *Journal of Political Economy* 110(4), 825–853.

- Wang, J. (1996). The term structure of interest rates in a pure exchange economy with heterogeneous investors. *Journal of Financial Economics* 41(1), 75–110.
- Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics* 24(3), 401–421.
- Wolff, E. N. (2010). Recent trends in household wealth in the united states: Rising debt and the middle-class squeeze - an update to 2007. Working Paper No. 589. Annandale-on-Hudson, NY: The Levy Economics Institute of Bard College.
- Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies* 32, 137–50.
- Zapatero, F. and C. Xiueros (2010). The representative agent of an economy with external habit formation and heterogeneous risk aversion. *Review of Financial Studies* 23, 3017–3047.

A Proofs

We start with the proof of Proposition 2. We then provide the proofs of Proposition 1 and Corollary 1 as special cases.

Proof of Proposition 2. We start by defining the constants

$$\Xi_1^i = -\frac{\alpha^i + \gamma^i - 1}{\alpha^i}, \quad \Xi_2^i = \frac{\alpha^i}{(1 - \alpha^i)(1 - \gamma^i)}, \quad \Xi_3^i = -\frac{\rho + \pi}{\alpha^i}(1 - \gamma^i), \quad \Xi_4^i = -\frac{\alpha^i + \gamma^i - 1}{(1 - \alpha^i)(1 - \gamma^i)},$$

where $i \in A, B$. Furthermore, we let

$$\frac{d\xi_t}{\xi_t} \equiv -r_t dt - \kappa_t dB_t \tag{A.1}$$

$$\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \equiv e^{-\pi(u-t)} \frac{\xi_u}{\xi_t}. \tag{A.2}$$

Because existing agents have access to a frictionless annuity market and can trade dynamically in stocks and bonds without constraints, the results in Duffie and Epstein (1992) and Schroder and Skiadas (1999) imply that the optimal consumption process for agents with recursive preferences of the form (2) is given by

$$\frac{c_{u,s}^i}{c_{t,s}^i} = e^{\frac{1}{1-\alpha^i} \int_t^u f_V^i(w) dw} \left(\frac{(1-\gamma)V_{u,s}^i}{(1-\gamma)V_{t,s}^i} \right)^{\Xi_4^i} \left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \right)^{\frac{1}{\alpha^i-1}}. \tag{A.3}$$

From this point onwards, we proceed by employing a “guess and verify” approach. First, we guess that both r_t and κ_t are functions of X_t and that X_t is Markov. Later we verify these conjectures.

Under the conjecture that both r_t and κ_t are functions of X_t and that X_t is Markov, the homogeneity of the recursive preferences in Equation (2) implies that there exist a pair of appropriate functions $g^i(X_t)$, $i \in \{A, B\}$, such that the time- t value function of an agent of type i born at time $s \leq t$ is given by

$$V_{t,s}^i = \frac{\left(\widetilde{W}_{t,s}^i\right)^{1-\gamma^i}}{1-\gamma^i} g^i(X_t)^{\frac{(1-\gamma^i)(\alpha^i-1)}{\alpha^i}}. \tag{A.4}$$

$\widetilde{W}_{t,s}^i$ denotes the total wealth of the agent given by the sum of her financial wealth and the net present value of her earnings: $\widetilde{W}_{t,s}^i \equiv W_{t,s}^i + E_t \int_t^\infty \frac{\tilde{\xi}_u}{\tilde{\xi}_t} y_{u,s} du$. Using (A.4) along with the first order

condition for optimal consumption $V_W = f_c$ gives $c_{t,s}^i = \widetilde{W}_{t,s}^i g^i(X_t)$. Using this last identity inside (A.4) and re-arranging gives

$$V_{t,s}^i = \frac{c_{t,s}^{1-\gamma^i}}{1-\gamma^i} g(X_t)^{-\frac{1-\gamma^i}{\alpha^i}}. \quad (\text{A.5})$$

Combining Equations (A.5) and (A.3) gives

$$\left(\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i} \right)^{\frac{1}{1-\gamma^i}} \left(\frac{g(X_u)}{g(X_t)} \right)^{\frac{1}{\alpha^i}} = e^{\frac{1}{1-\alpha^i} \int_t^u f_V^i(w) dw} \left(\frac{(1-\gamma)V_{u,s}^i}{(1-\gamma)V_{t,s}^i} \right)^{\Xi_4^i} \left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \right)^{\frac{1}{\alpha^i-1}}. \quad (\text{A.6})$$

Using the definition of f and Equation (A.5), we obtain

$$f_V^i(t) = \Xi_1^i g^i(X_t) + \Xi_3^i \quad (\text{A.7})$$

Equation (A.7) implies that $f_V^i(t)$ is exclusively a function of X_t . Hence Equation (A.6) implies that $\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i}$ is independent of s . In turn, Equation (A.3) implies that $\frac{c_{u,s}^i}{c_{t,s}^i}$ is independent of s . Motivated by these observations, henceforth we use the simpler notation $\frac{(1-\gamma^i)V_u^i}{(1-\gamma^i)V_t^i}$ and $\frac{c_u^i}{c_t^i}$ instead of $\frac{(1-\gamma^i)V_{u,s}^i}{(1-\gamma^i)V_{t,s}^i}$ and $\frac{c_{u,s}^i}{c_{t,s}^i}$, respectively.

Solving for $\frac{(1-\gamma^i)V_u^i}{(1-\gamma^i)V_t^i}$ from Equation (A.6) and applying Ito's Lemma to the resulting equation gives

$$d((1-\gamma^i)V_u^i) = \mu_V^i du + \sigma_V^i(1-\gamma^i)V_u^i dB_u, \quad (\text{A.8})$$

where³⁰

$$\sigma_V^i \equiv \frac{1-\gamma^i}{\gamma^i} \kappa_t - \frac{1}{\gamma^i \Xi_2^i} \frac{g^{i'}}{g^i} \sigma_X \quad (\text{A.9})$$

$$\mu_V^i \equiv -(1-\gamma^i) f^i(c_u^i, V_u^i). \quad (\text{A.10})$$

³⁰Equation (A.10) follows from the definition of V , which implies

$$(1-\gamma^i) V_{t,s}^i + \int_s^t (1-\gamma^i) f^i(c_{u,s}, V_{u,s}) du = E_t \int_s^\infty (1-\gamma^i) f^i(c_{u,s}, V_{u,s}) du.$$

From the definition of X_t we obtain

$$\begin{aligned} X_t Y_t &= \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds \\ &= \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{s,s}^A e^{\frac{1}{1-\alpha^A} \int_s^t f_v^A(w) dw} \left(\frac{(1-\gamma^A) V_t^A}{(1-\gamma^A) V_s^A} \right)^{\Xi_4^A} \left(\frac{\tilde{\xi}_t}{\tilde{\xi}_s} \right)^{\frac{1}{\alpha^A-1}} ds \end{aligned} \quad (\text{A.11})$$

Applying Ito's Lemma to both sides of Equation (A.11), using (A.8), equating the diffusion and drift components on the left- and right-hand side, and simplifying gives

$$\frac{\sigma_X}{X_t} + \sigma_Y = \Xi_4^A \sigma_V^i - \frac{\kappa_t}{\alpha^A - 1} \quad (\text{A.12})$$

$$\mu_X + X_t \mu_Y + \sigma_X \sigma_Y = \nu^A \pi \beta_t^A - \pi X_t + \frac{r_t - \rho}{1 - \alpha^A} X_t + n^A X_t, \quad (\text{A.13})$$

where

$$n^i \equiv \left(\frac{q^i(q^i - 1)}{2} \kappa_t^2 + \frac{\Xi_4^i(\Xi_4^i - 1)}{2} (\sigma_V^i)^2 - q^i \Xi_4^A \kappa_t \sigma_V^i \right), \quad (\text{A.14})$$

and $q^i \equiv \frac{1}{\alpha^i - 1}$, for $i \in \{A, B\}$. Similarly, applying Ito's Lemma to both sides of the good-market clearing equation

$$Y_t = \int_{-\infty}^t \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds + \int_{-\infty}^t \nu^B \pi e^{-\pi(t-s)} c_{t,s}^B ds, \quad (\text{A.15})$$

using (A.8), equating the diffusion and drift components on the left- and right-hand sides and combining with (A.12) and (A.13) leads to Equations (25) and (26). Using Equation (25) inside (A.12) gives (23), while (24) follows from (A.13). Finally, Equation (27) follows from (A.14) after simplifying.

The remainder of the proof shows that $\beta_{t,s}^i$, $g_{t,s}^i$ are indeed functions of X_t and shows how to obtain these functions after solving appropriate ordinary differential equations. To this end, we use the parametric specification $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$, and define

$$\phi^j(X_t) \equiv B_j \omega E_t \int_t^\infty e^{-(\pi + \delta_j)(u-t)} \frac{\xi_u}{\xi_t} \frac{Y_u}{Y_t} du, \quad (\text{A.16})$$

so that an agent's net present value of earnings at birth is given by $Y_s \left(\sum_{j=1}^2 \phi^j(X_s) \right)$. Next notice

that equation (A.16) implies

$$e^{-(\pi+\delta_j)t} Y_t \xi_t \phi^j(X_t) + B_j \omega \int_s^t e^{-(\pi+\delta_j)u} \xi_u Y_u du = B_j \omega E_t \int_s^\infty e^{-(\pi+\delta_j)u} \xi_u Y_u du. \quad (\text{A.17})$$

Applying Ito's Lemma to both sides of equation (A.17), setting the drifts equal to each other, and using the fact that the right hand side of the above equation is a local martingale with respect to t (so that its drift is equal to zero) results in the differential equation

$$0 = \frac{\sigma_X^2}{2} \frac{d^2 \phi^j}{dX^2} + \frac{d\phi^j}{dX} (\mu_X + \sigma_X(\sigma_Y - \kappa)) + \phi^j (\mu_Y - r - \pi - \delta_j - \sigma_Y \kappa) + B_j \omega, \quad (\text{A.18})$$

where $j = 1, 2$. A similar reasoning allows us to obtain an expression for $g^i(X_t)$, where $i \in A, B$. Since at each point in time, an agent's present value of consumption has to equal her total wealth, we obtain

$$\frac{1}{g^i(X_t)} = \frac{\widetilde{W}_{t,s}^i}{c_{t,s}^i} = E_t \int_t^\infty \frac{\tilde{\xi}_u}{\tilde{\xi}_t} \frac{c_u^i}{c_t^i} du = E_t \int_t^\infty \frac{\tilde{\xi}_u}{\tilde{\xi}_t} \frac{c_u^i}{c_t^i} du. \quad (\text{A.19})$$

Using (A.3) and (A.5) gives

$$\frac{c_u^i}{c_t^i} = e^{\frac{1}{\gamma^i} \int_t^u (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} \left(\frac{g^i(X_u)}{g^i(X_t)} \right)^{-\frac{\Xi_1^i}{\gamma^i}} \left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \right)^{-\frac{1}{\gamma^i}}. \quad (\text{A.20})$$

Using (A.20) inside (A.19) and re-arranging implies that

$$g^i(X_t)^{-1 - \frac{\Xi_1^i}{\gamma^i}} \left(\tilde{\xi}_t \right)^{\frac{\gamma^i - 1}{\gamma^i}} e^{\frac{1}{\gamma^i} \int_0^t (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} + \int_s^t \left(\tilde{\xi}_u \right)^{\frac{\gamma^i - 1}{\gamma^i}} e^{\frac{1}{\gamma^i} \int_s^u (\Xi_1^i g^i(X_w) + \Xi_3^i) dw} g^i(X_u)^{-\frac{\Xi_1^i}{\gamma^i}} du$$

is a local martingale. Applying Ito's Lemma and setting the drift of the resulting expression equal to zero gives

$$0 = \frac{\sigma_X^2}{2} M_1^i \left((M_1^i - 1) \left(\frac{\frac{dg^i}{dX_t}}{g^i} \right)^2 + \frac{\frac{d^2 g^i}{dX_t^2}}{g^i} \right) + M_1^i \frac{\frac{dg^i}{dX_t}}{g^i} (\mu_X - M_2^i \sigma_X \kappa) + \left(\frac{\kappa^2(X_t)}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) - M_1^i g^i + \frac{\Xi_3^i}{\gamma^i} \right), \quad (\text{A.21})$$

where $M_1^i = -1 - \frac{\Xi_1^i}{\gamma^i}$ and $M_2^i = \frac{\gamma^i - 1}{\gamma^i}$.

Concerning boundary conditions for the ODEs (A.18) and (A.21), we note that at the boundaries $X_t = 0$ and $X_t = 1$ the value of $\sigma_X(X_t)$ is zero. Accordingly, we require that all terms in equations (A.18) and (A.21) that contain σ_X as a multiplicative factor vanish.³¹ Accordingly, we are left with the following conditions when $X_t = 0$ or $X_t = 1$:

$$0 = \frac{d\phi^j}{dX} \mu_X + \phi^j (\mu_Y - r - \pi - \delta_j - \sigma_Y \kappa) + B_j \omega, \quad (\text{A.22})$$

$$0 = M_1^i \frac{1}{g^i} \frac{dg^i}{dX_t} \mu_X + \left(\frac{\kappa^2(X_t)}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) - M_1^i g^i + \frac{\Xi_3^i}{\gamma^i} \right). \quad (\text{A.23})$$

We conclude the proof by noting that since both $g_{s,t}^i$ for $i \in \{A, B\}$ and ϕ^j for $j = 1, 2$ are functions of X_t , so is $\beta^i(X_t)$, which is by definition equal to $\beta^i(X_t) = g^i(X_t) \left[\sum_{j=1}^2 \phi^j(X_s) \right]$. The fact that $g_{s,t}^i$ and β_t^i are functions of X_t verifies the conjecture that X_t is Markovian and that r_t and κ_t are functions of X_t , which implies that the value functions of agents $i \in A, B$ take the form (A.4). ■

Lemma 1 *The price-to-output ratio is given by*

$$s(X_t) \equiv \frac{S_t}{Y_t} = \left[\frac{X_t}{g^A(X_t)} + \frac{1 - X_t}{g^B(X_t)} \right] - \sum_{j=1}^2 \frac{\pi}{\pi + \delta_j} \phi^j(X_t), \quad (\text{A.24})$$

while the volatility of returns is given by

$$\sigma_t = \frac{s'(X_t)}{s(X_t)} \sigma_X(X_t) + \sigma_Y. \quad (\text{A.25})$$

Proof of Lemma 1. Using (8), applying Ito's lemma to compute $d(e^{-\pi s} \xi_s W_s^i)$, integrating, and using a transversality condition we obtain

$$W_{t,s}^i = E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} (c_{u,s}^i - y_{u,s}) du. \quad (\text{A.26})$$

³¹Specifically, we require that the terms $\sigma_X^2 \frac{d^2 \phi^j}{dX^2}$, $\sigma_X^2 \frac{d^2 g^i}{dX^2}$, $\sigma_X \frac{d\phi^j}{dX}$, and $\sigma_X \frac{dg^i}{dX}$ tend to zero as X_t tends to either zero or one. These conditions ensure that any economic effects at these boundaries (on the consumption-to-wealth ratio, the present-value-of-income-to-current-output ratio, etc.) of the type of agents with zero consumption weight at the boundary occurs *only* through the anticipated (deterministic) births of new generations of such agents.

The market-clearing equations for stocks and bonds imply

$$S_t = \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i W_{t,s}^i ds. \quad (\text{A.27})$$

Substitution of (A.26) into (A.27) gives

$$\begin{aligned} S_t &= \sum_{i \in \{A, B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} v^i \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\ &\quad - \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du \right] ds. \end{aligned} \quad (\text{A.28})$$

We can compute the first term in (A.28) as

$$\begin{aligned} &\sum_{i \in \{A, B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du \right] ds \\ &= \sum_{i \in \{A, B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \frac{c_{u,s}^i}{c_{t,s}^i} du \right] ds \\ &= \sum_{i \in \{A, B\}} v^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \frac{c_{t,s}^i}{g(X_t)} ds = Y_t \left[\frac{X_t}{g^A(X_t)} + \frac{1 - X_t}{g^B(X_t)} \right] \end{aligned} \quad (\text{A.29})$$

Similarly, using (4) and (5) we can compute the second term in (A.28) as

$$\begin{aligned} &\int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du \right] ds \\ &= \omega \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} Y_u \left(\sum_{j=1}^2 B_j e^{-\delta_j(u-s)} \right) du \right] ds \\ &= Y_t \times \sum_{j=1}^2 \int_{-\infty}^t \pi e^{-(\pi+\delta_j)(t-s)} \left[B_j \omega E_t \int_t^\infty e^{-(\pi+\delta_j)(u-t)} \frac{\xi_u}{\xi_t} \frac{Y_u}{Y_t} du \right] ds \\ &= Y_t \times \sum_{j=1}^2 \int_{-\infty}^t \pi e^{-(\pi+\delta_j)(t-s)} \phi^j(X_t) ds \\ &= Y_t \times \sum_{j=1}^2 \frac{\pi}{\pi + \delta_j} \phi^j(X_t). \end{aligned} \quad (\text{A.30})$$

Combining (A.29) and (A.30) inside (A.28) gives (A.24). Equation (A.25) follows upon applying Ito's Lemma to S_t and matching diffusion terms. ■

Proof of Corollary 1 and Proposition 1. Proposition 1 is a special case of Proposition 2 given by $\gamma^A = \gamma^B$.³² Since $\sigma_X(X_t) = 0$ when $\gamma^A = \gamma^B$, Lemma 1 implies that $\sigma_t = \sigma_Y$ when $\gamma^A = \gamma^B$. Turning to the proof of Proposition 1, the fact that agents have the same preferences implies that $\mu_X = \sigma_X = 0$ for all X_t , $g^i(X_t)$ and $\phi^j(X_t)$ for $i \in \{A, B\}$, $j = 1, 2$ are constants, and hence equation (26) becomes (9), while (25) becomes $\kappa = \gamma\sigma_Y^2$. Furthermore equation (A.21) becomes an algebraic equation with solution

$$g = \pi + \frac{\rho}{1-\alpha} - \frac{\alpha}{1-\alpha} \left(r + \frac{\gamma}{2} \sigma_Y^2 \right), \quad (\text{A.31})$$

while the present value of an agent's earnings at birth, divided by Y_s , is equal to

$$\omega E_s \int_s^\infty G(u-s) \frac{\xi_u Y_u}{\xi_s Y_s} du = \omega \left(\int_0^\infty G(u) e^{-(r+\pi+\gamma\sigma_Y^2-\mu_Y)u} du \right). \quad (\text{A.32})$$

Combining (A.31) with (A.32) and the definition of β leads to (10). ■

Lemma 2 *Assume that all agents have the same preferences. Let $\chi \equiv \rho + \pi - \alpha(\mu_Y - \frac{\gamma}{2}\sigma^2)$ and assume that $\chi > \pi$, $\mu_Y - \gamma\frac{\sigma_Y^2}{2} > 0$, and*

$$\frac{1}{\omega} < \frac{\chi \int_0^\infty G(u) e^{-\chi u} du}{\pi \int_0^\infty G(u) e^{-\pi u} du}. \quad (\text{A.33})$$

Then $\beta > 1$ and hence the interest rate given by (9) is lower than the respective interest rate in an economy featuring an infinitely-lived representative agent.

Proof of Lemma 2. Let $\bar{r} \equiv \rho + (1-\alpha)\mu_Y - \gamma(2-\alpha)\frac{\sigma_Y^2}{2}$ denote the interest rate in the economy featuring an infinitely lived agent and also let $r^* \equiv \mu_Y - \gamma\sigma_Y^2$. Note that $\chi > \pi$ is equivalent to $\bar{r} > r^*$. We first show that condition (A.33) and $\chi > \alpha\pi$ imply, respectively, the two inequalities

$$0 > \rho + (1-\alpha)[\mu_Y + \pi(1-\beta(\bar{r}))] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - \bar{r} \quad (\text{A.34})$$

$$0 < \rho + (1-\alpha)[\mu_Y + \pi(1-\beta(r^*))] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - r^*. \quad (\text{A.35})$$

Since preferences are homogeneous, equation (A.31) holds. Using the definition of \bar{r} and $\beta(\bar{r})$ on the right hand side of (A.34) and simplifying gives $0 > (1-\alpha)\pi[1 - \chi\omega(\int_0^\infty G(u)e^{-\chi u} du)]$, which

³²Assuming existence of an equilibrium, the existence of a steady state follows from $\sigma_X = 0$ for all X_t , along with $\mu_X(0) = \pi\nu^A\beta^A(0) > 0$, $\mu_X(1) = -\pi\nu^B\beta^B(1) < 0$, and the intermediate value theorem.

is implied by (A.33). Similarly, using the definition of r^* and $\beta(r^*)$ inside (A.35) and simplifying gives $0 < \left[\rho + \pi - \alpha(\mu_Y + \pi - \gamma \frac{\sigma_Y^2}{2}) \right] (1 - \omega)$, which is implied $\chi > \alpha\pi$. Given the inequalities (A.34) and (A.35), the intermediate value theorem implies that there exists a root of equation (9) in the interval (r^*, \bar{r}) . Accordingly, $\beta > 1$. ■

Remark 5 *In an influential paper, Weil (1989) pointed out that the standard representative-agent model cannot account for the low level of the risk-free rate observed in the data. Motivated by the low estimates of the IES in Hall (1988) and Campbell and Mankiw (1989), Weil’s reasoning was that such values of the IES would lead to interest rates that are substantially higher than the ones observed in the data. Weil referred to this observation as the “low risk-free rate puzzle”. Lemma 2 gives sufficient conditions so that the interest rate in our OLG economy is lower than in the respective infinitely-lived, representative-agent economy. Whether the key condition (A.33) of the Lemma holds or not depends crucially on the life-cycle path of earnings. The easiest way to see this is to assume that $\omega\chi > \pi$ and restrict attention to the parametric case $G(u) = e^{-\delta u}$, so that condition (A.33) simplifies to $\delta > \frac{\chi\pi(1-\omega)}{\omega\chi-\pi}$. Hence, the interest rate is lower in the overlapping generations economy as long as the life cycle path of earnings is sufficiently downward-sloping. The intuition for this finding is that agents who face a downward-sloping path of labor income need to save for the latter years of their lives. The resulting increased supply of savings lowers the interest rate. This insight is due to Blanchard (1985), who considered only the deterministic case and exponential specifications for $G(u)$. Condition (A.33) generalizes the results in Blanchard (1985). In particular, it allows $G(u)$ to have any shape, potentially even sections where the life-cycle path of earnings is increasing.*

Proof of Proposition 3. We start by fixing arbitrary values α^A , α^B , and γ , and throughout we let $\gamma^B = \gamma$ and $\gamma^A = \gamma^B - \varepsilon$, $\varepsilon \geq 0$. For $\kappa(X_t)$ given by (25), we let $Z(X_t; \varepsilon) \equiv \kappa(X_t) - \Gamma(X_t)\sigma_Y$. Clearly $Z(\bar{X}; 0) = 0$ (by Proposition 1). To prove the proposition, it suffices to show that $\left. \frac{dZ(\bar{X}; \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} \geq 0$. Direct differentiation, along with the fact that $\sigma_X = 0$ when $\varepsilon = 0$, gives

$$\left. \frac{dZ(\bar{X}; \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \sum_{i \in \{A, B\}} \omega^i(\bar{X}) \left(\frac{1 - \gamma^i - a^i}{\alpha^i} \right) \frac{g^{i'}(\bar{X}, 0)}{g^i(\bar{X}, 0)} \times \left(\left. \frac{d\sigma_X(\bar{X}; \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} \right). \quad (\text{A.36})$$

To determine the sign of $\frac{g^{i'}(\bar{X},0)}{g^i(\bar{X},0)}$, we write equation (A.21) — for $\varepsilon = 0$ — as

$$0 = M_1^i \frac{g^{i'}}{g^i} \mu_X + \left[\frac{(\gamma\sigma)^2}{2} M_2^i (M_2^i - 1) - M_2^i (r(X_t) + \pi) + \frac{\Xi_3^i}{\gamma} - M_1^i g^i \right]. \quad (\text{A.37})$$

Differentiating (A.37) with respect to X and evaluating the result at $X = \bar{X}$ using $\mu_X = 0$ yields

$$\frac{g^{i'}}{g^i} M_1^i (\mu'_X - g^i) = M_2^i r', \quad (\text{A.38})$$

where primes denote derivatives with respect to X . Re-arranging (A.38) and using the definitions of M_1^i and M_2^i gives

$$\frac{g^{i'}}{g^i} = \frac{M_2^i}{M_1^i} \left(\frac{r'}{\mu'_X - g^i} \right) = \frac{\alpha^i}{1 - \alpha^i} \frac{r'}{\mu'_X - g^i}. \quad (\text{A.39})$$

A similar computation starting from equation (A.18) and using $\phi^j(\bar{X}) = \frac{B_j \omega}{r + \pi + \delta_j + \gamma \sigma^2 - \mu_Y}$ yields

$$\frac{\phi^{j'}}{\phi^j} = - \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X} r'. \quad (\text{A.40})$$

The fact that \bar{X} is a stable steady state³³ implies that $\mu'_X \leq 0$, and accordingly equation (A.39) implies that $\frac{1}{\alpha} \frac{g^{i'}}{g^i}$ has the opposite sign from r' . Similarly, equation (A.40) implies that $\frac{\phi_j'}{\phi_j}$ has the opposite sign³⁴ from r' .

We next show that if $\gamma^A = \gamma^B = \gamma$ and $\alpha^B > \alpha^A$, then $r'(\bar{X}) > 0$. We proceed by supposing the contrary: $\gamma^A = \gamma^B = \gamma$ and $\alpha^B > \alpha^A$, but $r' \leq 0$. Differentiating equation (26) with respect to X_t and evaluating the resulting expression at $X_t = \bar{X}$, we obtain

$$r' = \frac{1}{\Theta(\bar{X})} \left\{ \left(\frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) [r - \rho] - n^A + n^B \right\} - \pi \frac{1}{\Theta(\bar{X})} \sum_i v^i (\beta^i)'. \quad (\text{A.41})$$

The definition of $n^i(X_t)$ in equation (27) along with $\gamma^A = \gamma^B = \gamma$ and $\sigma_X = 0$ gives

$$n^A - n^B = \left[\frac{2 - \alpha^A}{(1 - \alpha^A)} - \frac{2 - \alpha^B}{(1 - \alpha^B)} \right] \frac{\kappa^2(\bar{X})}{2\gamma} = \gamma \left(\frac{1}{1 - \alpha^A} - \frac{1}{1 - \alpha^B} \right) \frac{\sigma_Y^2}{2}. \quad (\text{A.42})$$

³³Note that when $\varepsilon = 0$, the evolution of X_t is deterministic.

³⁴Note that B_j and ϕ^j have the same sign.

Furthermore, noting that $\beta^i = g^i \sum_{j=1,2} \phi^j$ and using (A.39) and (A.40) leads to

$$\begin{aligned} \pi \sum_{i \in \{A,B\}} v^i (\beta^i)' &= -r' \left[\pi \sum_{i \in \{A,B\}} v^i \left(\frac{\alpha^i}{1 - \alpha^i} \frac{1}{g^i - \mu'_X} g^i \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X} \right) \right] \\ &\leq -r' \left[\pi \sum_{i \in \{A,B\}} v^i \left(\frac{(\alpha^i)^+}{1 - \alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega} \right) \right]. \end{aligned} \quad (\text{A.43})$$

Inequality (A.43) follows from the facts that (a) r' has been assumed to be non-positive, (b) $\mu'_X \leq 0$, and (c) $\sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega}{\phi^j} - \mu'_X}$ is positive and increasing in μ'_X .³⁵

Let

$$\eta \equiv \pi \sum_{i \in \{A,B\}} v^i \left(\frac{(\alpha^i)^+}{1 - \alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega} \right). \quad (\text{A.44})$$

Let $\alpha^A = \alpha^B = \alpha$. Using $\phi^j(\bar{X}) = \frac{B_j \omega}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$ and equation (A.31), we obtain that

$$\eta = \pi \left(\frac{\alpha^+}{1 - \alpha} \sum_{j=1,2} \frac{\omega B_j}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y} + \sum_{j=1,2} \frac{\omega B_j \left(\pi + \frac{\rho}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \left(r + \frac{\gamma}{2} \sigma_Y^2 \right) \right)}{(r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y)^2} \right),$$

where r is given in equation (9). We shall assume that

$$\eta < 1 - \alpha, \quad (\text{A.45})$$

which is the case, for instance, when ω is sufficiently small.³⁶ Using (A.43) and (A.42) inside (A.41) gives

$$r' \geq \frac{1}{\Theta(\bar{X})} \left(\frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) \left[r - \rho + \gamma \frac{\sigma_Y^2}{2} \right] - \frac{\eta}{\Theta(\bar{X})} r', \quad (\text{A.46})$$

³⁵To see that $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$ is positive, note that $\frac{1}{(\phi^1)^{-1} B_1 \omega - \mu'_X} = \frac{1}{r + \pi + \delta_1 + \sigma_Y \kappa - \mu_Y - \mu'_X} > \frac{1}{r + \pi + \delta_2 + \sigma_Y \kappa - \mu_Y - \mu'_X}$ since $\delta_1 < \delta_2$. Furthermore, since $B_1 > -B_2$, it follows that $\phi^1 > -\phi^2$. Treating μ'_X as a variable and differentiating $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$ with respect to μ'_X shows that $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega - \mu'_X}$ is increasing in μ'_X .

³⁶Note that an implication of Lemma 2 is that the interest rate satisfies $r > \mu_Y - \gamma \sigma_Y^2$ for any value of ω , and hence the expressions $\frac{1}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$ and $\frac{\pi + \frac{\rho}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \left(r + \frac{\gamma}{2} \sigma_Y^2 \right)}{r + \pi + \delta_j + \gamma \sigma_Y^2 - \mu_Y}$ must approach finite limits as ω goes to zero.

or

$$[\Theta(\bar{X}) - \eta] r' \geq \left(\frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A} \right) \left[r - \rho + \gamma \frac{\sigma_Y^2}{2} \right]. \quad (\text{A.47})$$

The term $r - \rho + \gamma \frac{\sigma_Y^2}{2}$ is positive if³⁷

$$\mu_Y - \frac{\gamma \sigma_Y^2}{2} + \pi > \omega(\rho + \pi) \frac{\sum_{j=1,2} \frac{B_j}{\delta_j + \rho - \mu + \frac{\gamma}{2} \sigma_Y^2 + \pi}}{\sum_{j=1,2} \frac{B_j}{\delta_j + \pi}}, \quad (\text{A.48a})$$

$$\rho + \pi(1 - \alpha) > \alpha \left(\mu_Y - \gamma \frac{\sigma_Y^2}{2} \right). \quad (\text{A.48b})$$

We note that condition (A.48b) is automatically satisfied when $\alpha < 0$ and $\mu_Y \geq \gamma \frac{\sigma_Y^2}{2}$, while condition (A.48a) holds when ω is small.

When $|\alpha^A - \alpha^B|$ is small (but not zero), continuity implies that the right hand side of (A.47) has the same sign as $\frac{1}{1 - \alpha^B} - \frac{1}{1 - \alpha^A}$, which is positive. However, given the supposition that r' is non-positive and assumption (A.45), the left-hand side of (A.47) is non-positive, which is a contradiction. We therefore conclude that, when $\alpha^B > \alpha^A$, $r' > 0$. A symmetric argument implies that, when $\alpha^B < \alpha^A$, $r' < 0$. Recalling that r' and $\frac{1}{\alpha^i} \frac{g^{i'}}{g^i}$ have the same sign, it follows that the term $\sum_{i \in \{A, B\}} \omega^i(\bar{X}) \left(\frac{1 - \gamma^i - a^i}{\alpha^i} \right) \frac{g^{i'}(\bar{X}, 0)}{g^i(\bar{X}, 0)}$ is positive when either condition 1 or condition 2 of the Lemma holds.

Consider now the term $\left. \frac{d\sigma_X(\bar{X}; \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}$, which we want positive. Direct differentiation of (23) gives

$$\left. \frac{d\sigma_X(\bar{X}; \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{\bar{X} \left(\frac{d\Gamma(\bar{X}; \varepsilon)}{d\varepsilon} + 1 \right)}{\bar{X} (1 - \bar{X}) \left[\frac{1 - \gamma - a^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1 - \gamma - a^B}{\alpha^B} \frac{g^{B'}}{g^B} \right] + \gamma^B}. \quad (\text{A.49})$$

³⁷When the economy is populated by a single agent, we can replicate the steps of the proof of Lemma 2 to prove this fact. Specifically, letting $r^{**} = \rho - \frac{\gamma}{2} \sigma_Y^2$ we arrive at the conclusion that

$$0 < \rho + (1 - \alpha) [\mu_Y + \pi (1 - \beta(r^{**}))] - \gamma(2 - \alpha) \frac{\sigma_Y^2}{2} - r^{**},$$

as long as condition (A.48a) holds. Similarly, setting $\bar{r} = \rho + (1 - \alpha) (\mu_Y + \pi) - \gamma(2 - \alpha) \frac{\sigma_Y^2}{2}$ gives

$$0 > \rho + (1 - \alpha) [\mu_Y + \pi (1 - \beta(\bar{r}))] - \gamma(2 - \alpha) \frac{\sigma_Y^2}{2} - \bar{r}.$$

as long as equation (A.48b) holds. Hence there must exist a root in the interval (r^{**}, \bar{r}) .

The numerator on the right hand side of equation (A.49) is positive, and as long as α^A and α^B are close enough, $\frac{1-\gamma-\alpha^A}{\alpha^A} \frac{g^{A'}}{g^A} - \frac{1-\gamma-\alpha^B}{\alpha^B} \frac{g^{B'}}{g^B}$ is arbitrarily close to zero, so that the denominator is also positive.

The proof can now be concluded by invoking uniform continuity on compact sets. Specifically, the right-hand side of equation (A.47) is weakly positive, for every γ , if $\alpha^B \geq \alpha^A$ and $\alpha^B - \alpha^A$ is sufficiently small. For $\alpha^B - \alpha^A$ sufficiently small, the right-hand-side of (A.49) is also positive. It follows that, for any value γ , (A.36) is positive for $\alpha^B - \alpha^A$ and ε sufficiently small. Since γ is restricted to a compact set, the admissible maximum size of $\alpha^B - \alpha^A$ and ε is strictly positive. ■

Proof of Proposition 4. We start by letting $\tilde{\sigma}_X(X_t) \equiv \frac{\sigma_X(X_t)}{\sigma_Y}$ and $\tilde{\gamma}(X_t) = \frac{\kappa(X_t)}{\sigma_Y}$. Expressing equations (23)–(26) in terms of $\tilde{\sigma}_X(X_t)$ and $\tilde{\gamma}(X_t)$ (rather than $\sigma_X(X_t)$ and $\kappa(X_t)$) and inspecting the resulting equations shows that there are no first-order σ_Y terms in the equations for $r(X_t)$ and $\tilde{\gamma}(X_t)$. Rather, the lowest (non-zero) order is two. A similar observation applies for the differential equations (A.18) and (A.21) for ϕ^j and g^i , respectively.

To establish (28)–(29) it therefore suffices to show that there exist parameters $\alpha^A, \alpha^B, \gamma^A, \gamma^B$, and v^A such that

$$\frac{\kappa(X_t)}{\sigma_Y} = \Gamma(X_t) + \sum_{i \in \{A, B\}} \omega^i(X_t) \left(\frac{1 - \gamma^i - \alpha^i}{\alpha^i} \right) \frac{g^{i'}}{g^i} \frac{\sigma_X(X_t)}{\sigma_Y} \quad (\text{A.50})$$

is equal to D_1 when $\sigma_Y = 0$, while $\frac{\kappa'(X_t)}{\sigma_Y}$ is equal to $-D_2$. To establish these facts, we proceed in two steps. First we show that, when $\alpha^A = \alpha^B = \alpha$ and $\sigma_Y = 0$, the second term on the right-hand side of (A.50) is zero, and so is its derivative with respect to X_t . In the second step we show that there exist γ^A, γ^B and small enough v^A such that $\Gamma(\bar{X}) = D_1$ and $\Gamma'(\bar{X}) = -D_2$. In a third step we address the qualification of equation (30) that $\lim_{x \rightarrow 0} \frac{\epsilon_3(x)}{x} = 0$ and its implication.

Step 1. Let $\alpha^A = \alpha^B \equiv \alpha$ and $\sigma_Y^2 = 0$. Then $\Theta(X)$ is constant and $n^i(X) = 0$, and equation (26) gives

$$r(X_t) = \rho + (1 - \alpha) \left[\mu_Y - \pi \left(\sum_{i \in \{A, B\}} v^i \beta^i(X_t) - 1 \right) \right]. \quad (\text{A.51})$$

The consumption ratio $\beta^i(X)$, in turn, is given by $\beta^i(X) = g^i(X)(\phi^1(X) + \phi^2(X))$. Note that, if g^i and ϕ^j are constant (i.e., independent of X_t), then so is r . On the other hand, equations (A.21)

and (A.18), under the standing assumption $\sigma_Y^2 = 0$, are satisfied by constant functions g^i and ϕ^j as long as r is constant. We conclude that the system of equations (A.18), (A.21), and (A.51) admit solutions r , g^i , and ϕ^j that are independent of X_t and therefore $g^{i'} = g^{i''} = 0$. Accordingly, the second term on the right-hand side of (A.50) and its derivative with respect to X_t are both zero.

Step 2. Now, fixing $\alpha_A = \alpha_B = \alpha$, suppose that we want to find γ^A and γ^B to achieve

$$\Gamma(\bar{X}) = D_1 \tag{A.52}$$

$$\Gamma'(\bar{X}) = -\left(\frac{1}{\gamma^A} - \frac{1}{\gamma^B}\right) \Gamma(\bar{X})^2 = -D_2. \tag{A.53}$$

With $\sigma_Y = 0$, $\bar{X} = v^A$ is a constant.³⁸ We compute γ^A and γ^B explicitly by solving the following linear system in $1/\gamma^A$ and $1/\gamma^B$:

$$\frac{\bar{X}}{\gamma^A} + \frac{1 - \bar{X}}{\gamma^B} = D_1^{-1} \tag{A.54}$$

$$\frac{1}{\gamma^A} - \frac{1}{\gamma^B} = D_1^{-2} D_2. \tag{A.55}$$

The solution is

$$\gamma^B = D_1^{-1} (1 - \bar{X} D_1^{-1} D_2) \tag{A.56}$$

$$\gamma^A = D_1^{-1} + D_1^{-2} D_2 (1 - \bar{X}). \tag{A.57}$$

We note that γ^B is automatically positive, while $\gamma^A > 0$ as long as $\bar{X} = v^A$ is small enough.

For future reference we note that steps 1 and 2 (adapted in a straightforward way) continue to imply equations (28)–(29) not only in the case $\alpha^A = \alpha^B$, but more generally when $|\alpha^A - \alpha^B| = O(\sigma_Y^2)$.

The next step addresses the variability of the interest rate.

³⁸ $\bar{X} = v^A$ because, at $\alpha^A = \alpha^B$ and $\sigma_Y = 0$, $\beta^A = \beta^B$ and thus equation (24) implies that $\mu_X(v^A) = 0$.

Step 3. Fix now $\alpha^B = \alpha$ at the value chosen in Step 1, let $\Delta\alpha = \alpha^A - \alpha^B$, and consider the Taylor expansion

$$r(X; \Delta\alpha, \sigma_Y^2) = r_0(X; \Delta\alpha) + r_1(X; \Delta\alpha)\sigma_Y^2 + o(\sigma_Y^2) \quad (\text{A.58})$$

$$r'(X; \Delta\alpha, \sigma_Y^2) = r'_0(X; \Delta\alpha) + r'_1(X; \Delta\alpha)\sigma_Y^2 + o(\sigma_Y^2). \quad (\text{A.59})$$

We want to show that there exists $\Delta\alpha$ such that $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$. We note that — as established in step 1 — $r'_0(\bar{X}; 0) = 0$ and that $\frac{\partial}{\partial \Delta\alpha} r'_0(\bar{X}; 0) \neq 0$ generically. Now, if $r'_1(\bar{X}; 0) = 0$, then clearly $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$. If $r'_1(\bar{X}; 0) \neq 0$, the mapping $\Delta\alpha \mapsto \frac{r'_0(\bar{X}; \Delta\alpha)}{r'_1(\bar{X}; \Delta\alpha)}$ equals zero at zero and has non-zero derivative at zero, so its range contains a neighborhood of zero. Consequently, for σ_Y^2 small enough, $\Delta\alpha$ exists such that $r'_0(\bar{X}; \Delta\alpha) + r'_1(\bar{X}; \Delta\alpha)\sigma_Y^2 = 0$.

Furthermore, $\Delta\alpha$ is in $O(\sigma_Y^2)$, which means — in light of our remark at the end of step 2 — that by choosing $\Delta\alpha$ so as to ensure that ϵ_3 is of order $o(\sigma_Y^2)$, the remainder terms ϵ_1 and ϵ_2 remain of $O(\sigma_Y^2)$.³⁹

Now the fact that for the parameter choices made above $\frac{|\kappa(\bar{X})\sigma(\bar{X})'|_{\sigma_X(\bar{X})}}{|r'(\bar{X})|_{\sigma_X(\bar{X})}}$ approaches infinity as σ_Y approaches zero is immediate in light of (28)–(30), along with (A.25).⁴⁰ ■

B Details on the calibration

For all quantitative exercises, we set $\mu_Y = 0.02$ and $\sigma_Y = 0.041$, so that time-integrated data from our model can roughly reproduce the first two moments of annual consumption growth. We note that, due to time integration, a choice of instantaneous volatility $\sigma_Y = 0.041$ corresponds to a volatility of 0.033 for model-implied, time-integrated, yearly consumption data, consistent with the long historical sample of Campbell and Cochrane (1999). The parameter π controls the birth-and-death rate, and we set $\pi = 0.02$ so as to approximately match the birth rate (inclusive of the net immigration rate) in the US population. (Source: U.S. National Center for Health Statistics, Vital Statistics of the United States, and National Vital Statistics Reports (NVSR), annual data, 1950–2006.) We note that since the death and birth rate are the same in the model, $\pi = 0.02$ implies that

³⁹Under common regularity conditions, the solutions of differential equations depend smoothly on the parameters.

⁴⁰Specifically, using the observations we made in steps 1–3 imply that for the parameter choices we made $\sigma'(\bar{X}) = o(\sigma_Y)$.

on average agents live for 50 years after they start making economic decisions. Assuming that this age is about 20 years in the data, $\pi = 0.02$ implies an average lifespan of 70 years. We set $\omega = 0.92$ to match the fact that dividend payments and net interest payments to households are $1 - \omega = 0.08$ of personal income. (Source: Bureau of Economic Analysis, National income and product accounts, Table 2.1., annual data, 1929-2010.) To arrive at this number, we combine dividend and net interest payments by the corporate sector (i.e., we exclude net interest payments from the government and net interest payments to the rest of the world), in order to capture total flows from the corporate to the household sector. We note that the choice of $1 - \omega = 0.08$ is consistent with the gross profit share of GDP being about 0.3, since the share of output accruing to capital holders is given by the gross profit share *net of the investment share*. As a result, in our endowment economy, which features no investment, it would be misleading to match the gross profit share, since this would not appropriately deduct investment, which does not constitute “income” for the households. Instead it seems appropriate to match the parameter ω directly to the fraction of national income that accrues in the form of dividends and net interest payments from the domestic corporate sector.

For the specification of the life-cycle earnings $G(t - s)$ we use the life-cycle profile of earnings estimated in Hubbard et al. (1994). We estimate the parameters B_1 , δ_1 , B_2 , and δ_2 using non-linear least squares to project (5) on the life-cycle profile of earnings estimated in Hubbard et al. (1994). These parameters are $B_1 = 30.72$, $B_2 = -30.29$, $\delta_1 = 0.0525$, and $\delta_2 = 0.0611$.

To obtain simulated cross sections of households in Section 4, we simulate 1,000 independent paths of aggregate shocks over 4,000 years starting with X_0 at its stationary mean and isolate the last 300 years of each path to ensure stationarity. Fixing each of the 1,000 paths in turn, we draw from the population age distribution the ages of 4,000 artificial households. We then repeat this exercise for $T - k$, where k are numbers chosen to reflect the number of years between the cross sections considered in Wolff (2010) and Cutler and Katz (1992) respectively. As a result, we obtain 1,000 independent, simulated versions with the same cross-sectional and time-series characteristics as Wolff (2010) and Cutler and Katz (1992).

Parameter	Data	Baseline	CRRA	$v^A=0.08$
Risk aversion of type- A agents		1.5	1.5	1.5
Risk aversion of type- B agents		10	10	10
IES of type- A agents		0.7	$\frac{1}{\gamma^A}$	0.6
IES of type- B agents		0.05	$\frac{1}{\gamma^B}$	0.03
Population share of type- A agents		0.01	0.01	0.08
Equity premium	5.2 %	5.1 %	2.1 %	3.7 %
Volatility of returns	18.2%	17.0 %	8.5 %	16.1 %
Average (real) interest rate	2.8%	1.4%	2.1 %	2.3 %
Volatility of the real interest rate	0.92%	0.93%	0.13 %	0.86%

Table B.1: This table repeats the quantitative exercise of Table 1, except that we include an additional column where we restrict $v^A = 0.08$ (in addition to $\gamma^B \leq 10$) and determine the rest of the parameters to match (as closely as possible) the asset pricing moments at the bottom of the table. The choice $v^A = 0.08$ is based on the fact that entrepreneurs account for 8% of the population. A potential motivation for matching the fraction of entrepreneurs is that entrepreneurs are likely to be more risk tolerant. In fact, an extended version of the model can generate this outcome endogenously. In a previous version of the paper, we included a time-zero occupational choice for an agent. We modeled an entrepreneur as someone who accepts a multiplicative shock (with mean 1) on her time-zero endowment of earnings in exchange for a non-pecuniary benefit of being self-employed. It followed that the relatively more risk-tolerant agents “self-select” into entrepreneurship. We also note that setting $v^A = 0.1$ produces similar results to $v^A = 0.08$.