STOCHASTIC CONTROL AND DIFFERENTIAL GAMES WITH PATH-DEPENDENT CONTROLS

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Abstract. In this paper we consider the functional Itô calculus framework to find a path-dependent version of the Hamilton-Jacobi-Bellman equation for stochastic control problems with path-dependence in the controls. We also prove a Dynamic Programming Principle for such problems. We apply our results to path-dependence of the delay type. We further study Stochastic Differential Games in this context.

Key words. Functional Itô Calculus, Path-dependence, Stochastic Control, Stochastic Games, Delay.

AMS subject classifications. 49L99, 91A15, 60H30

1. Introduction.

Stochastic optimization problems appear naturally in various areas of applications. Portfolio allocation, investment-consumption utility maximization, hedging in incomplete markets and real options are some important examples in Finance and Economics. See, for instance, Pham [2009] and Carmona [2016]. The standard case deals with a controlled diffusion

$$\begin{cases} dx_s^{t,y,A_T} = b(s, x_s, \alpha_s)ds + \sigma(s, x_s, \alpha_s)dw_s, \text{ if } s > t, \\ \\ x_t^{t,y,A_T} = y, \end{cases}$$

and a cost functional

$$J(t, y, A_T) = \mathbb{E}\left[g(x_T^{t,y,A_T}) + \int_t^T f(s, x_s^{t,y,A_T}, \alpha_s)ds\right],$$

where $A_T = (\alpha_t)_{t\in[0,T]}$ is a admissible control and g and f are suitable functions. The quantity of interest here is the value function:

$$V(t, y) = \inf_{A_T\in\mathbb{A}} J(t, y, A_T),$$

where $\mathbb{A}$ is a set of admissible controls. Differently from the usual theory, we are denoting the control as A_T instead of α. This notation is consistent with the functional Itô calculus, as we comment in Section 1.1. Moreover, it makes it explicit the time horizon on which the control is being considered.

Two very important results on Stochastic Control are the Dynamic Programming Principle (DPP) and the Verification Theorem for the related Hamilton-Jacobi-Bellman (HJB) equation. The main contribution of our paper is to extend the DPP and the HJB to controlled diffusion and cost functional that depend on the path of the control α. The main example to have in mind is the delayed diffusion

$$dx_t^{A_T} = (\alpha_t - \alpha_{t-\tau})dt + \sigma dw_t, \tag{1}$$

for a fixed τ. Stochastic Control has been extended to consider path-dependence in the state variable x, see, for example, Fournié [2010], Xu [2013] and Ji et al. [2015].

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Functional Itô calculus was also applied to the stochastic control problem of portfolio optimization with bounded memory in Pang and Hussain [2015]. Furthermore, the theory was also applied to zero-sum stochastic differential games in Pham and Zhang [2014]. However, these references do not deal with path-dependence in the control, only on the state of the system. This generalization is fundamentally different from the one pursued in our paper, which will become clear in the sections to follow, see Remark 2.4.

Path-dependent controls are still incipient in theory and applications of stochastic control and differential games. This is very likely related to the lack of theoretical tools to deal with such objects in an appropriate way. We hope this work will provide a useful framework.

For example, in Gozzi and Marinelli [2006] and Gozzi and Masiero [2015], the authors considered a class of problems that exhibit a particular type of path-dependence in the control, namely delayed controls. The method implemented there is a classical infinite dimensional analysis and they derived an infinite dimensional HJB equation. However, their method is strongly related to the delay-type of path-dependence. Additionally, we forward the reader to the following articles Alekal et al. [1971], Chen and Wu [2011], Huang et al. [2012]. These results were recently applied to stochastic games in Carmona et al. [2016].

Our approach uses the functional Itô calculus framework, introduced by Bruno Dupire in the seminal paper Dupire [2009], which allows us to consider more general path-dependent structures. Although our method could be also seen as an infinite dimensional analysis, it is rather different than the one applied in Gozzi and Marinelli [2006] and Gozzi and Masiero [2015]. Our method delivers a simpler HJB equation that can be applied to virtually any path-dependent structure in the control and it could be formulated in the deterministic case as well. Our assumptions are mainly related to the well-posedness of the optimal control problem (smoothness, measurability and integrability). Additionally, our method could be applied to delay of the type of Equation (1) with no additional difficulty, which is not the case of the method derived in Gozzi and Marinelli [2006] and Gozzi and Masiero [2015]. See Section 2.2.1 for more details.

The structure of the paper is as follows. We finish this introduction with the main definitions and results of functional Itô calculus. In Section 2.1, we introduce the problem we are considering and derive the main results of our work: the DPP in Theorem 2.1 and the Verification Theorem for the path-dependent HJB equation in Theorem 2.3. An example is analyzed in Section 2.2.1. Additionally, in Section 2.3, we briefly study stochastic differential games with path-dependent actions.

1.1. A Crash Course in Functional Itô Calculus. The important notions of the functional Itô calculus framework will be introduced in this section. For more details and results, we forward the reader to Cont and Fournié [2010], Dupire [2009].

We start by fixing a time horizon $T > 0$. Denote Λ_t^n the space of càdlàg paths in $[0,t]$ taking values in $\mathbb{R}^n$ and define $\Lambda^n = \bigcup_{t\in[0,T]} \Lambda_t^n$ and $\Lambda^{n\times k} = \bigcup_{t\in[0,T]} \Lambda_t^n \times \Lambda_t^k$. Elements of $\Lambda^{n\times k}$ are two paths taking values in $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively, with the same time interval as domain. When it is not necessary to distinguish the dimensions of these spaces, we will use the notation Λ.

Moreover, when considering examples with delay, one could consider $\bigcup_{t\in[-\tau,T]} \Lambda_t$, where τ is the largest possible value for the delay. In the examples studied here, we will assume that any path at negative time is zero. This does not increase the difficulty in our calculations and could be easily relaxed.

Capital letters will denote elements of Λ (i.e. paths) and lower-case letters will denote spot value of paths. In symbols, $Y_t \in \Lambda$ means $Y_t \in \Lambda_t$ and $y_s = Y_t(s)$, for $s \leq t$.

A functional is any function $f : \Lambda \longrightarrow \mathbb{R}$. For such objects, we define, when the limits exist, the time and space functional derivatives, respectively, as

$$\Delta_t f(Y_t) = \lim_{\delta t \to 0^+} \frac{f(Y_{t,\delta t}) - f(Y_t)}{\delta t}, \tag{2}$$

$$\Delta_x f(Y_t) = \lim_{h \to 0} \frac{f(Y_t^h) - f(Y_t)}{h}, \tag{3}$$

where

$$Y_{t,\delta t}(u) = \begin{cases} y_u, & \text{if} \quad 0 \leq u \leq t, \\ y_t, & \text{if} \quad t \leq u \leq t + \delta t, \end{cases}$$

$$Y_t^h(u) = \begin{cases} y_u, & \text{if} \quad 0 \leq u < t, \\ y_t + h, & \text{if} \quad u = t, \end{cases}$$

see Figures 1 and 2. In the case when the path Y_t lies in a multidimensional space, the path deformations above are understood as follows: the flat extension is applied to all dimension jointly and equally and the bump is applied to each dimension individually.

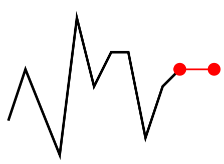

FIG. 1. *Flat extension of a path.*

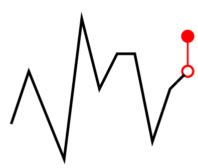

FIG. 2. *Bumped path.*

We consider here continuity in Λ as the usual continuity in metric spaces with respect to the metric:

$$d_\Lambda(Y_t, Z_s) = \|Y_{t,s-t} - Z_s\|_\infty + |s - t|,$$

where, without loss of generality, we assume $s \geq t$, and

$$\|Y_t\|_\infty = \sup_{u \in [0,t]} |y_u|.$$

The norm $|\cdot|$ is the usual Euclidean norm in the appropriate space, depending on the dimension of the path being considered. This continuity notion could be relaxed, see, for instance, Oberhauser [2016].

Moreover, we say a functional f is *boundedness-preserving* if, for every compact set $K \subset \mathbb{R}^d$, there exists a constant C such that $|f(Y_t)| \leq C$, for every path Y_t satisfying $Y_t([0,t]) = \{y \in \mathbb{R}^d \;;\; Y_t(s) = y \text{ for some } s \in [0,t]\} \subset K$.

A functional $f : \Lambda \longrightarrow \mathbb{R}$ is said to belong to $\mathbb{C}^{1,2}$ if it is Λ-continuous, boundedness-preserving and it has Λ-continuous, boundedness-preserving derivatives $\Delta_t f$, $\Delta_x f$ and $\Delta_{xx} f$. Here, clearly, $\Delta_{xx} = \Delta_x \Delta_x$.

The Itô formula can be generalized to this framework. The proof can be found in Dupire [2009]. We start by fixing a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

THEOREM 1.1 (Functional Itô Formula; Dupire [2009]). *Let x be a continuous semimartingale and $f \in \mathbb{C}^{1,2}$. Then, for any $t \in [0,T]$,*

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s)ds + \int_0^t \Delta_x f(X_s)dx_s + \frac{1}{2}\int_0^t \Delta_{xx} f(X_s)d\langle x\rangle_s \quad \mathbb{P}\text{-}a.s.$$

2. Main results.

2.1. Stochastic Control with path-dependent Controls. We suggest the reader to always keep in mind this example:

$$dx_t^{A_T} = (\alpha_t - \alpha_{t-\tau})dt + \sigma dw_t.$$

Consider a d-dimensional Brownian motion $(w_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ in this space, satisfying the usual conditions, to which the Brownian motion $(w_t)_{t\in[0,T]}$ is adapted. One could assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented natural filtration of w. The set of admissible controls $\mathbb{A}(\mathcal{F}_t)$, or just $\mathbb{A}$, is the space of $\mathcal{F}_t$-progressively measurable, càdlàg processes in $L^2(\Omega\times[0,T])$ taking value in some subset $\mathcal{A} \subset \mathbb{R}^k$. Additional restrictions on $\mathbb{A}$ will be assumed.

We will use the following notation: $A_T = (\alpha_t)_{t\in[0,T]}$, i.e. the path of the control $\alpha \in \mathbb{A}$, and

$$(Z_t \otimes A_T)(s) = \begin{cases} z_s, & \text{if } s < t, \\ \alpha_s, & \text{if } s \geq t. \end{cases} \tag{4}$$

The path $Z_t \otimes A_T$ is equal Z up to time t (excluding it) and then follows the control α.

We will consider the following path-dependent controlled diffusion dynamics for x:

$$\begin{cases} dx_s^{Y_t,A_T} = b(X_s^{Y_t,A_T}, A_s)ds + \sigma(X_s^{Y_t,A_T}, A_s)dw_s, & \text{if } s > t, \\ X_t^{Y_t,A_T} = Y_t, \end{cases} \tag{5}$$

where $b : \Lambda^{n\times k} \longrightarrow \mathbb{R}^n$ and $\sigma : \Lambda^{n\times k} \longrightarrow \mathbb{R}^{n\times d}$, with $\mathbb{R}^{n\times d}$ denoting the space of $n \times d$ matrices. Notice that we are allowing for path-dependence on the state system, x, and on the control, α. To guarantee existence and uniqueness of strong solutions, we assume there exists a constant $K > 0$ such that

$$\begin{cases} |b(Y_s, Z_s) - b(Y_s', Z_s)| \leq K\|Y_s - Y_s'\|_\infty, \\ |\sigma(Y_s, Z_s) - \sigma(Y_s', Z_s)| \leq K\|Y_s - Y_s'\|_\infty, \\ |b(Y_s, Z_s)| + |\sigma(Y_s, Z_s)| \leq K\left(1 + |s| + \|Y_s\|_\infty\right), \end{cases}$$

for all $s \geq t$, $(Y_s, Z_s), (Y_s', Z_s) \in \Lambda^{n\times k}$. These assumptions could be weaken, but it is outside the scope of this work.

Moreover, we consider the following class of cost functionals $J : \Lambda^n \times \mathbb{A} \longrightarrow \mathbb{R}$:

$$J(Y_t, A_T) = \mathbb{E}\left[g(X_T^{Y_t,A_T}) + \int_t^T f(X_s^{Y_t,A_T}, A_s)ds\right], \tag{6}$$

where $g : \Lambda^n_T \longrightarrow \mathbb{R}$ and $f : \Lambda^{n\times k} \longrightarrow \mathbb{R}$ satisfy certain measurability and integrability conditions. Notice that $J(Y_T, A_T) = g(Y_T)$. We additionally assume that the admissible controls in $\mathbb{A}$ satisfy certain straightforward integrability conditions depending on the functionals b, σ and f so that Equations (5) and (6) are well-defined.

We define then the value functional $V : \Lambda^{n\times k} \longrightarrow \mathbb{R}$:

$$V(Y_t, Z_t) = \inf_{A_T\in\mathbb{A}} J(Y_t, Z_t \otimes A_T).$$

THEOREM 2.1 (Dynamic Programming Principle (DPP)). *For any $u \in [t, T]$,*

$$V(Y_t, Z_t) = \inf_{A_T\in\mathbb{A}} \mathbb{E}\left[V(X_u^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_s)ds\right].$$

Proof. The proof is the same as in the path-independent case, since all the coefficients are still adapted. We follow the structure of the proof in Pham [2009].

Firstly, notice that, for any $A_T \in \mathbb{A}$ and $t \le u \le s \le T$, we have the following equivalence of paths

$$X_s^{Y_t,A_T} = X_s^{X_u^{Y_t,A_T},A_T}.$$

Then,

$$J(Y_t, A_T) = \mathbb{E}\left[g(X_T^{X_u^{Y_t,A_T},A_T}) + \int_t^u f(X_s^{Y_t,A_T}, A_s)ds + \int_u^T f(X_s^{X_u^{Y_t,A_T},A_T}, A_s)ds\right],$$

and conditioning on the path $X_u^{Y_t,A_T}$, we find

$$J(Y_t, A_T) = \mathbb{E}\left[\int_t^u f(X_s^{Y_t,A_T}, A_s)ds + J(X_u^{Y_t,A_T}, A_T)\right]. \tag{7}$$

From this and choosing the control A_T to be $Z_t \otimes A_T$, it is clear that

$$J(Y_t, Z_t \otimes A_T) \ge \mathbb{E}\left[V(X_u^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_s)ds\right].$$

Taking the infimum with respect to $A_T \in \mathbb{A}$, we find

$$V(Y_t, Z_t) \ge \inf_{A_T\in\mathbb{A}} \mathbb{E}\left[V(X_u^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_s)ds\right].$$

To prove the opposite inequality, fix $A_T \in \mathbb{A}$ and $u \in [t, T]$. Then, for any $\varepsilon > 0$, there exists $A_T^\varepsilon \in \mathbb{A}$ such that

$$V(X_u^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_u) + \varepsilon \ge J(X_u^{Y_t,(Z_t\otimes A_T)_u\otimes A_T^\varepsilon}, (Z_t \otimes A_T)_u \otimes A_T^\varepsilon).$$

It can be shown by the Measurable Selection Theorem (see, for example, Soner and Touzi [2002]) that $A_T^* = A_u \otimes A_T^\varepsilon$ belongs to $\mathbb{A}$ (i.e. it is progressively measurable). Since $Z_t \otimes A_T^* = (Z_t \otimes A_T)_u \otimes A_T^\varepsilon$, by Equation (7), we find

$$\begin{aligned}
&V(Y_t, Z_t) \le J(Y_t, Z_t \otimes A_T^*)\\
&= \mathbb{E}\left[\int_t^u f(X_s^{Y_t,Z_t\otimes A_T}, (Z_t \otimes A_T)_s)ds + J(X_u^{Y_t,Z_t\otimes A_T^*}, Z_t \otimes A_T^*)\right]
\end{aligned}$$

$$\leq \mathbb{E}\left[\int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_s)ds + V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u)\right] + \varepsilon,$$

which implies, by the fact $A_T \in \mathbb{A}$ and $\varepsilon > 0$ are arbitrary, that

$$V(Y_t, Z_t) \leq \inf_{A_T \in \mathbb{A}} \mathbb{E}\left[V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_s)ds\right],$$

from where the final result follows. ∎

2.2. The path-dependent Hamilton-Jacobi-Bellman Equation. In this section we will state the HJB equation related to our control problem and also prove a verification theorem for such equation. In the framework of the functional Itô calculus, this type of equation is called path-dependent Partial Differential Equation, PPDE. See for example, Ekren et al. [2014, 2016a,b].

We start by defining the Hamiltonian $H : \Lambda^{n \times k} \times \mathbb{R}^n \times \mathbb{S}^n \times \mathcal{A} \longrightarrow \mathbb{R}$:

$$H(Y_t, Z_t, p, \gamma, \alpha) = \frac{1}{2}\sigma\sigma^T(Y_t, Z_t^{\alpha - z_t}) : \gamma + b(Y_t, Z_t^{\alpha - z_t}) \cdot p + f(Y_t, Z_t^{\alpha - z_t}),$$

and the *modified* Hamiltonian $\widehat{H} : \Lambda^{n \times k} \times \mathbb{R}^{\Lambda^k} \times \mathbb{R}^n \times \mathbb{S}^n \longrightarrow \mathbb{R}$:

$$\widehat{H}(Y_t, Z_t, q, p, \gamma) = \inf_{\alpha \in \mathcal{A}} \left\{ q(Z_t^{\alpha - z_t}) + H(Y_t, Z_t, p, \gamma, \alpha) \right\} \tag{8}$$

The symbol $\mathbb{R}^{\Lambda^k}$ denotes the space of functionals $\Lambda^k \longrightarrow \mathbb{R}$. Notice that $Z_t^{\alpha - z_t}$ is changing the last value of the control Z_t to α.

The notation $\cdot$ and $:$ mean

$$p \cdot q = \sum_{i=1}^d p_i q_i \text{ and } \gamma : \phi = \mathrm{trace}(\gamma\phi),$$

where $p, q \in \mathbb{R}^n$ and $\gamma, \phi \in \mathbb{S}^n$, where $\mathbb{S}^n$ is the space of $n \times n$ symmetric matrices.

As we will conclude, the HJB equation in this case is given by the following PPDE:

$$\begin{cases} \widehat{H}(Y_t, Z_t, \Delta_t V(Y_t, \cdot), \Delta_x V(Y_t, Z_t), \Delta_{xx} V(Y_t, Z_t)) = 0, \\ \\ V(Y_T, Z_T) = g(Y_T), \end{cases} \tag{9}$$

for any $Z_t \in \Lambda$.

Here, the time derivative Δ_t is with respect to both variable Y and Z:

$$\Delta_t V(Y_t, Z_t) = \lim_{\delta t \to 0^+} \frac{V(Y_{t,\delta t}, Z_{t,\delta t}) - V(Y_t, Z_t)}{\delta t},$$

and the space derivative Δ_x is with respect to Y:

$$\Delta_x V(Y_t, Z_t) = \lim_{h \to 0} \frac{V(Y_t^h, Z_t) - V(Y_t, Z_t)}{h}.$$

In a less compact notation, we could write the path-dependent HJB equation (9) as

$$\begin{cases} \inf_{\alpha \in \mathcal{A}} \left\{ \Delta_t V(Y_t, Z_t^{\alpha - z_t}) + H(Y_t, Z_t, \Delta_x V(Y_t, Z_t), \Delta_{xx} V(Y_t, Z_t), \alpha) \right\} = 0, \\ \\ V(Y_T, Z_T) = g(Y_T), \end{cases}$$

Remark 2.2. This remark will be the cornerstone of the proof of the Verification Theorem that follows. Notice that $V(Y_t, Z_t^h) = V(Y_t, Z_t)$, by the definition of $\otimes$ given in (4). Denoting the functional derivatives with respect to Y and Z by Δ_x and Δ_α, respectively, we conclude $\Delta_\alpha V(Y_t, Z_t) = 0$, $\Delta_{\alpha\alpha} V(Y_t, Z_t) = 0$ and $\Delta_{x\alpha} V(Y_t, Z_t) = 0$. Hence, the dynamics of the control A_T will not impact the computations in the proof of the following theorem. This is similar to what Cont and Fournié assumed in order to consider functionals depending on the quadratic variation, see Cont and Fournié [2010]. These authors called such property *predictability*.

Moreover, if a smooth functional is predictable in a variable, then *any* space functional derivative will predictable in that variable. However, the time functional derivative will not be predictable, in general. For example, the running integral functional $f(Y_t) = \int_0^t y_u du$ is predictable, but $\Delta_t f(Y_t) = y_t$ is not.

Theorem 2.3 (Verification Theorem).
Suppose $V \in \mathbb{C}^{1,2}$ solves the HJB equation (9). Under mild integrability conditions,

$$V(Y_t, Z_t) \leq J(Y_t, Z_t \otimes A_T),$$

for any $A_T \in \mathbb{A}$. Moreover, if there exists $\widehat{A}_T \in \mathbb{A}$ such that, for any $u \in [t, T]$,

$$\begin{aligned}
&\widehat{H}(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, (Z_t \otimes \widehat{A}_T)_u, \Delta_t V(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, \cdot), \Delta_x V, \Delta_{xx} V) \\
&= \Delta_t V(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, (Z_t \otimes \widehat{A}_T)_u) \\
&+ H(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, (Z_t \otimes \widehat{A}_T)_u, \Delta_x V, \Delta_{xx} V, \widehat{\alpha}_u),
\end{aligned} \tag{10}$$

then $V(Y_t, Z_t) = J(Y_t, Z_t \otimes \widehat{A}_T)$. All the functional derivatives in (10) are computed at $(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, (Z_t \otimes \widehat{A}_T)_u)$.

Proof. Let us apply the Functional Itô Formula, Theorem 1.1, to $V(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_s)$, for fixed $A_T \in \mathbb{A}$. Notice that the path Z is frozen and that we are considering the control $Z_t \otimes A_T$, which means we follow the path Z_t as the control up to time t (excluding it) and then A_T from t to T. Moreover, since the functional derivatives of V with respect to the control α are zero, it is not required to consider the dynamics of the control α, see Remark 2.2. Furthermore, the time derivative is with respect to both variables. In the computation that follows we suppress the superscript of $X_s^{Y_t, Z_t \otimes A_T}$ for a cleaner exposition.

$$\begin{aligned}
&g(X_T) = V(X_T, Z_t \otimes A_T) = V(Y_t, Z_t) + \int_t^T \Delta_t V(X_u, (Z_t \otimes A_T)_u) du \\
&+ \int_t^T \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot b(X_u, (Z_t \otimes A_T)_u) du \\
&+ \int_t^T \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_u) dw_u \\
&+ \frac{1}{2} \int_t^T \Delta_{xx} V(X_u, (Z_t \otimes A_T)_u) : \sigma\sigma^T(X_u, (Z_t \otimes A_T)_u) du \\
&= V(Y_t, Z_t) + \int_t^T (\Delta_t V(X_u, (Z_t \otimes A_T)_u) + H(X_u, (Z_t \otimes A_T)_u, \Delta_x V, \Delta_{xx} V, \alpha_u))\, du \\
&+ \int_t^T \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_u) dw_u - \int_t^T f(X_u, (Z_t \otimes A_T)_u) du
\end{aligned}$$

$$\geq V(Y_t, Z_t) + \int_t^T \widehat{H}(X_u, (Z_t \otimes A_T)_u, \Delta_t V(X_u, \cdot), \Delta_x V, \Delta_{xx} V) du$$
$$+ \int_t^T \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_u) dw_u - \int_t^T f(X_u, (Z_t \otimes A_T)_u) du$$
$$= V(Y_t, Z_t) + \int_t^T \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_u) dw_u$$
$$- \int_t^T f(X_u, (Z_t \otimes A_T)_u) du.$$

Under integrability conditions and applying localization techniques, we might assume, without loss of generality, that the Itô integral above is a martingale. Therefore, taking expectation on both sides, we conclude:

$$V(Y_t, Z_t) \leq \mathbb{E}\left[g(X_T^{Y_t, Z_t \otimes A_T}) + \int_t^T f(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) du \right] = J(Y_t, Z_t \otimes A_T).$$

Taking the control $\widehat{A}_T$ satisfying Equation (10), we find

$$V(Y_t, Z_t) = \mathbb{E}\left[g(X_T^{Y_t, Z_t \otimes \widehat{A}_T}) + \int_t^T f(X_u^{Y_t, Z_t \otimes \widehat{A}_T}, (Z_t \otimes \widehat{A}_T)_u) du \right] = J(Y_t, Z_t \otimes \widehat{A}_T).$$

as desired. □

Remark 2.4. We would like to stress the difference between the case of path-dependent controls and state variables we are dealing with here and the case of only path-dependent state variables. In this case, it is not necessary to consider as variable of V the path of the control, Z_t. It is enough to define

$$J(Y_t, A_{t,T}) = \mathbb{E}\left[g(X_T^{Y_t, A_{t,T}}) + \int_t^T f(X_s^{Y_t, A_{t,T}}, \alpha_s) ds \right],$$
$$V(Y_t) = \inf_{A_{t,T} \in \mathbb{A}[t,T]} J(Y_t, A_{t,T}),$$

where $A_{t,T} = (\alpha_s)_{s \in [t,T]}$ and $\mathbb{A}[t,T]$ is the space of admissible controls on $[t,T]$. See, for example, Fournié [2010], Xu [2013] or Ji et al. [2015].

Remark 2.5. It is obvious that if the dynamics of x and the functionals g and f are path-independent in the state variable and control, we find the classical HJB equation. Moreover, if the path-dependence is only in the control, meaning that

$$h(Y_t, Z_t) = h(t, y_t, Z_t) \text{ and } g(Y_T) = g(y_T),$$

for $h = b, \sigma, f$, the path-dependent HJB Equation (9) becomes

$$\begin{cases} \widehat{H}(t, y, Z_t, \Delta_t V(t, y, \cdot), \partial_x V(t, y, Z_t), \partial_{xx} V(t, y, Z_t)) = 0, \\ V(T, y, Z_T) = g(y), \end{cases} \tag{11}$$

where ∂_x is the usual derivative with respect to the state variable and

$$\widehat{H}(t, y, Z_t, q, p, \gamma) = \inf_{\alpha \in \mathcal{A}} \Big\{ q(Z_t^{\alpha - z_t}) + \frac{1}{2} \sigma\sigma^T(t, y, Z_t^{\alpha - z_t}) : \gamma$$

$$+ b(t, y, Z_t^{\alpha - z_t}) \cdot p + f(t, y, Z_t^{\alpha - z_t})\Big\}.$$

It is worth noticing that Δ_t is still a functional derivative. More precisely, it is giving by

$$\Delta_t V(t, y, Z_t) = \lim_{\delta t \to 0^+} \frac{V(t + \delta t, y, Z_{t,\delta t}) - V(t, y, Z_t)}{\delta t}.$$

2.2.1. Delayed Control. We will exemplify the results derived in the section above, mainly the path-dependent HJB equation, by considering the delay type of path-dependence in the control as in Gozzi and Marinelli [2006], see also Alekal et al. [1971], Chen and Wu [2011], Gozzi and Masiero [2015], Huang et al. [2012]. Namely, we will assume that the drift is given by

$$b(t, y, Z_t) = a_0 y + b_0 z_t + \int_{-\tau}^{0} b_1(u) z_{t+u} du,$$

where $b_1 \in L^2([-\tau, 0]; \mathbb{R})$ or, the more complicated case, dealt in Gozzi and Masiero [2015], where b_1 is a measure. A very important example being the Dirac mass at $-\tau$. As we will see below, differently than the aforesaid references, the framework proposed here can deal with both these situations without additional difficulty.

In order to get a complete characterization of the value functional (up to computing the solution of a system of PDEs), we consider the following linear-quadratic example:

$$\begin{aligned} &b(t, y, Z_t) = z_t - z_{t-\tau}, \quad \sigma(t, y, Z_t) = \sigma, \\ &f(t, y, Z_t) = \frac{z_t^2}{2} + q z_t y + \frac{\varepsilon}{2} y^2 \text{ and } g(y) = c\frac{y^2}{2}. \end{aligned}$$

Hence

$$H(t, y, Z_t, p, \gamma, \alpha) = \frac{\sigma^2}{2}\gamma + (\alpha - z_{t-\tau})p + \frac{\alpha^2}{2} + q\alpha y + \frac{\varepsilon}{2} y^2.$$

We consider the following ansatz for the value functional, as it was examined, for instance, in Huang et al. [2012]:

$$\begin{aligned} V(t, y, Z_t) = {} & F_0(t)\frac{y^2}{2} + y \int_{t-\tau}^{t} F_1(t, \theta - t) z_\theta d\theta \\ & + \int_{t-\tau}^{t}\int_{t-\tau}^{t} F_2(t, \theta_1 - t, \theta_2 - t) z_{\theta_1} z_{\theta_2} d\theta_1 d\theta_2 + F_3(t), \end{aligned}$$

where we assume that F_2 is symmetric in the last two variables as it is usually done in these problems:

$$F_2(t, \theta_1, \theta_2) = F_2(t, \theta_2, \theta_1).$$

We can compute the derivatives of V explicitly. Δ_t would be more complicated, but for this ansatz, it may be verified that it is equivalent to taking derivative with respect to t:

$$\partial_x V = F_0(t) y + \int_{t-\tau}^{t} F_1(t, \theta - t) z_\theta d\theta, \tag{12}$$

$$\partial_{xx}V = F_0(t), \tag{13}$$

$$\begin{aligned}\Delta_t V &= F_0'(t)\frac{y^2}{2} + y\Big(F_1(t,0)z_t - F_1(t,-\tau)z_{t-\tau} \\ &+ \int_{t-\tau}^{t}\left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta}\right)(t,\theta-t)z_\theta d\theta\Big) \\ &+ 2z_t\int_{t-\tau}^{t} F_2(t,\theta-t,0)z_\theta d\theta - 2z_{t-\tau}\int_{t-\tau}^{t} F_2(t,\theta-t,-\tau)z_\theta d\theta \\ &+ \int_{t-\tau}^{t}\int_{t-\tau}^{t}\left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2}\right)(t,\theta_1-t,\theta_2-t)z_{\theta_1}z_{\theta_2}d\theta_1 d\theta_2 + F_3'(t).\end{aligned} \tag{14}$$

Combining all derivatives into the modified Hamiltonian (8), we find that the terms that depend on the current control α are:

$$yF_1(t,0)\alpha + 2\alpha\int_{t-\tau}^{t} F_2(t,\theta-t,0)z_\theta d\theta + \alpha p + \frac{\alpha^2}{2} + qy\alpha. \tag{15}$$

The infimum is then attained at

$$\widehat{\alpha}(t,y,Z_t,p) = -qy - p - yF_1(t,0) - 2\int_{t-\tau}^{t} F_2(t,\theta-t,0)z_\theta d\theta,$$

and the minimum value of the expression (15) is given by $-\widehat{\alpha}(t,y,Z_t,p)^2/2$. The HJB equation in this example becomes:

$$\begin{cases} \Delta_t V(t,y,Z_t^{-z_t}) + \dfrac{\sigma^2}{2}\partial_{xx}V(t,y,Z_t) - z_{t-\tau}\partial_x V(t,y,Z_t) \\ -\dfrac{1}{2}\widehat{\alpha}^2(t,y,Z_t,\partial_x V(t,y,Z_t)) + \dfrac{\varepsilon}{2}y^2 = 0, \\ V(T,y,Z_T) = c\dfrac{y^2}{2}. \end{cases} \tag{16}$$

Notice that $\Delta_t V(t,y,Z_t^{-z_t})$ removes the terms that depend on z_t in Equation (14). Additionally, the optimal control is given by

$$\begin{aligned}\widehat{\alpha}(t,y,Z_t,\partial_x V(t,y,Z_t)) &= -(F_0(t) + F_1(t,0) + q)y \\ &\quad - \int_{t-\tau}^{t}\left(F_1(t,\theta-t) + 2F_2(t,\theta-t,0)\right)z_\theta d\theta.\end{aligned}$$

Combining all derivatives into HJB Equation (16), we find

$$\begin{aligned}&\frac{1}{2}y^2\left(F_0'(t) - (F_0(t) + F_1(t,0) + q)^2 + \varepsilon\right) \\ &+ y\,(-(F_1(t,-\tau) - F_0(t))z_{t-\tau} \\ &- \frac{1}{2}(F_0(t) + F_1(t,0) + q)\int_{t-\tau}^{t}\left(F_1(t,\theta-t) + 2F_2(t,\theta-t,0)\right)z_\theta d\theta \\ &+ \int_{t-\tau}^{t}\left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta}\right)(t,\theta-t)z_\theta d\theta\Big)\end{aligned}$$

$$
\begin{aligned}
&+ F_3'(t) + \frac{\sigma^2}{2} F_0(t) - z_{t-\tau} \int_{t-\tau}^{t} (2F_2(t, \theta - t, -\tau) - F_1(t, \theta - t)) z_\theta d\theta \\
&+ \int_{t-\tau}^{t} \int_{t-\tau}^{t} \left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2} \right) (t, \theta_1 - t, \theta_2 - t) z_{\theta_1} z_{\theta_2} d\theta_1 d\theta_2 \\
&- \frac{1}{2} \int_{t-\tau}^{t} \int_{t-\tau}^{t} (F_1(t, \theta_1 - t) + 2F_2(t, \theta_1 - t, 0)) \\
&(F_1(t, \theta_2 - t) + 2F_2(t, \theta_2 - t, 0)) \, z_{\theta_1} z_{\theta_2} d\theta_1 d\theta_2 = 0,
\end{aligned}
$$

with the following final conditions:

$$
\begin{cases}
F_0(T) = c, \\
F_1(T, \theta - T) = 0, \ \forall \ \theta \in (T - \tau, T), \\
F_2(T, \theta_1 - T, \theta_2 - T) = 0, \ \forall \ \theta_1, \theta_2 \in (T - \tau, T), \\
F_3(T) = 0.
\end{cases}
$$

Therefore, we find that, for any $t \in [0, T]$ and $\theta, \theta_1, \theta_2 \in (-\tau, 0)$,

$$
\begin{cases}
F_0'(t) - (F_0(t) + F_1(t, 0) + q)^2 + \varepsilon = 0, \\
F_0(T) = c,
\end{cases} \tag{17}
$$

$$
\begin{cases}
\left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta} \right) (t, \theta) \\
-\frac{1}{2} (F_0(t) + F_1(t, 0) + q) \, (F_1(t, \theta) + 2F_2(t, \theta, 0)) = 0, \\
F_1(t, -\tau) = -F_0(t), \\
F_1(T, \theta) = 0,
\end{cases} \tag{18}
$$

$$
\begin{cases}
\left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2} \right) (t, \theta_1, \theta_2) \\
-\frac{1}{2} (F_1(t, \theta_1) + 2F_2(t, \theta_1, 0))(F_1(t, \theta_2) + 2F_2(t, \theta_2, 0)) = 0, \\
F_2(T, \theta_1, \theta_2) = 0, \\
F_2(t, \theta, -\tau) = F_2(t, -\tau, \theta) = -\frac{1}{2} F_1(t, \theta),
\end{cases} \tag{19}
$$

$$
\begin{cases}
F_3'(t) + \frac{\sigma^2}{2} F_0(t) = 0, \\
F_3(T) = 0.
\end{cases} \tag{20}
$$

In Figure 3, we show the numerical solution of the PDE system above for the following parameters: $q = 1$, $\varepsilon = 2$, $c = 0$, $T = 1$, $\tau = 0.05$ and $\sigma = 1$.

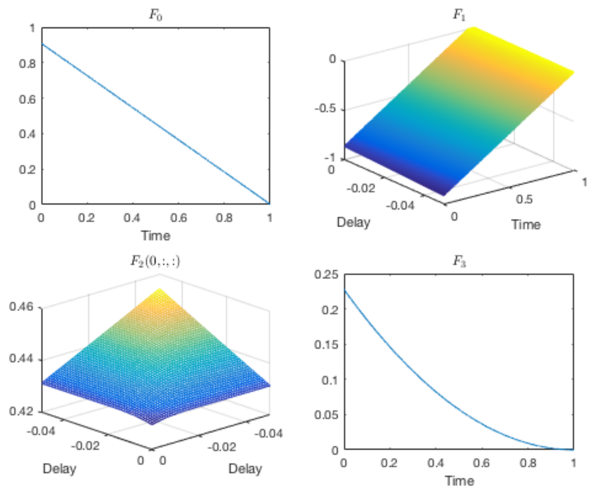

FIG. 3. *Numerical solution of the system PDEs (17)-(20)*

2.3. Stochastic Differential Games. In this section, we will briefly analyze Stochastic Differential Games. Firstly, we present the general theory relating the game value function and a version of the HJB equation when there is path-dependence in the control. Then, we exemplify the theory using the delayed stochastic differential game proposed in Carmona et al. [2016].

Consider N agents indexed by $i = 1, \dots, N$. These agents will act on a system whose state is described below:

$$\begin{cases} dx_s^{i,Y_t,A_T} = b^i(X_s^{Y_t,A_T}, A_s)ds + \sigma^i(X_s^{Y_t,A_T}, A_s)dw_s^i, \text{ if } s > t, \\ \\ X_t^{i,Y_t,A_T} = Y_t^i, \end{cases}$$

for $i = 1, \dots, N$, where w^i is a d_i-dimensional standard Brownian motion, $A_T = (A_T^1, \dots, A_T^N)$ with A_T^i being the k_i-dimensional control chosen by agent i. These Brownian motions could be correlated. Moreover,

$$(b^i, \sigma^i) : \Lambda^{n\times k} \longrightarrow \mathbb{R}^{n_i} \times \mathbb{R}^{n_i \times d_i},$$

and $\mathcal{A}^i$ is the set of actions of the agent i, with $n = n_1 \times \dots \times n_N$ and $k = k_1 \times \dots \times k_N$. We will use the notation $x_s^{Y_t,A_T} = (x_s^{1,Y_t,A_T}, \dots, x_s^{N,Y_t,A_T})$. The set of admissible controls of agent i is denoted by $\mathbb{A}^i$ and $\mathbb{A} = \mathbb{A}^1 \times \dots \times \mathbb{A}^N$. The agent i chooses its own control α^i to minimize its own cost functional:

$$J^i(Y_t, A_T) = \mathbb{E}\left[g^i(X_T^{Y_t,A_T}) + \int_t^T f^i(X_s^{Y_t,A_T}, A_s)ds\right],$$

where $g^i : \Lambda_T^n \longrightarrow \mathbb{R}$ and $f^i : \Lambda^{n\times k} \longrightarrow \mathbb{R}$ are his/hers terminal and running costs. In what follows, we will seek a *closed-loop Nash equilibrium.*

Assuming that the other $N-1$ agents have already chosen their actions, denoted by $A_T^{-i} = (A_T^1, \dots, A_T^{(i-1)},\ A_T^{(i+1)}, \dots, A_T^N)$, the value functional for agent i will be then given by

$$V^i(Y_t, Z_t, A_T^{-i}) = \inf_{A_T^i \in \mathbb{A}^i} J^i(Y_t, Z_t \otimes (A_T^{-i}, A_T^i)),$$

where $(A_T^{-i}, A_T^i) = (A_T^1, \dots, A_T^{(i-1)}, A_T^i, A_T^{(i+1)}, \dots, A_T^N)$. Therefore, under the assumptions of Theorem 2.3, we have a verification theorem for the following HJB Equation

$$\begin{cases} \widehat{H^i}(Y_t, Z_t, \Delta_t V^i(Y_t, \cdot, A_T^{-i}), \Delta_x V^i(Y_t, Z_t, A_T^{-i}), \Delta_{xx} V^i(Y_t, Z_t, A_T^{-i}), \alpha_t^{-i}) = 0, \\ V^i(Y_T, Z_T, A_T^{-i}) = g^i(Y_T), \end{cases}$$

where

$$\begin{aligned} \widehat{H^i}(Y_t, Z_t, q, p, \gamma, \alpha^{-i}) = \inf_{\alpha^i \in \mathcal{A}^i} \Big\{ & q(Z_t^{(\alpha^{-i}, \alpha^i) - z_t}) + \frac{1}{2}\sigma\sigma^T(Y_t, Z_t^{(\alpha^{-i}, \alpha^i) - z_t}) : \gamma \\ & + b(Y_t, Z_t^{(\alpha^{-i}, \alpha^i) - z_t}) \cdot p + f^i(Y_t, Z_t^{(\alpha^{-i}, \alpha^i) - z_t}) \Big\}. \end{aligned}$$

Notice that $Z_t^{(\alpha^{-i}, \alpha^i) - z_t}$ changes the control at time t to (α^{-i}, α^i).

2.4. Delayed Games. In this section, we will study the model introduced in Carmona et al. [2016], where the authors proposes a stochastic differential game with delay in the control to analyze the systemic risk within a bank system.

Fix $n_i = d_i = k_i = 1$ and

$$\begin{aligned} b^i(Y_t, Z_t) &= z_t^i - z_{t-\tau}^i, \\ \sigma^i(Y_t, Z_t) &= \sigma, \\ f^i(Y_t, Z_t) &= \frac{(z_t^i)^2}{2} - q z_t^i(\bar{y}_t - y_t^i) + \frac{\varepsilon}{2}(\bar{y}_t - y_t^i)^2, \\ g^i(Y_T) &= \frac{c}{2}(\bar{y}_T - y_T^i)^2, \end{aligned}$$

where $\bar{x} = \dfrac{1}{N}\sum_{i=1}^N x_i$. Let us consider the same ansatz for the value functional as in Carmona et al. [2016],

$$\begin{aligned} V^i(Y_t, Z_t) = {} & \frac{1}{2}E_0(t)(\bar{y}_t - y_t^i)^2 + (\bar{y}_t - y_t^i)\int_{t-\tau}^t E_1(t, \theta - t)(\bar{z}_\theta - z_\theta^i)d\theta \\ & + \int_{t-\tau}^t \int_{t-\tau}^t E_2(t, \theta_1 - t, \theta_2 - t)(\bar{z}_{\theta_1} - z_{\theta_1}^i)(\bar{z}_{\theta_2} - z_{\theta_2}^i)d\theta_1 d\theta_2 + E_3(t). \end{aligned}$$

Assuming that α^j has been chosen, for $j \neq i$, the optimal control for the player i is given by

$$\widehat{\alpha}^i(t, y, Z_t, p) = q(\bar{y} - y^i) - p_i - (\bar{y} - y^i)F_1(t, 0) - 2\int_{t-\tau}^t F_2(t, \theta - t, 0)(\bar{z}_\theta - z_\theta^i)d\theta.$$

This is the same optimal control found in the aforesaid reference.

Assuming each player is following this strategy and noticing that p_j in the formula for $\widehat{\alpha}_j$ should be replaced by $\partial_{x_j} V^j$ (and not $\partial_{x_j} V^i$), we find that the HJB equation turns into

$$
\begin{cases}
\Delta_t V^i(t, y, Z_t^{-z_t}) + \displaystyle\sum_{j=1}^{N} \left(\frac{\sigma^2}{2} \partial_{x_j x_j} V^i + (\widehat{\alpha}_j(t, y, Z_t, \partial_{x_j} V^i) - z^j_{t-\tau}) \partial_{x_j} V^i \right) \\
+ \dfrac{1}{2} \widehat{\alpha}^i(t, y, Z_t, \partial_{x_i} V^i)^2 - q \widehat{\alpha}^i(t, y, Z_t, \partial_{x_i} V^i)(\bar{y}_t - y^i_t) + \dfrac{\varepsilon}{2} (\bar{y}_t - y^i_t)^2 = 0, \\
V(Y_T, Z_T) = \dfrac{c}{2} (\bar{y}_t - y^i_T)^2.
\end{cases}
\tag{21}
$$

Following the same rationale as in Section 2.2.1, it is straightforward to find the same system of PDEs as in Carmona et al. [2016]. Indeed, the system of PDEs in Section 2.2.1 is the limit as $N \to +\infty$ of the system in the aforesaid reference, as one would expect.

3. Conclusions. In this paper, we have studied stochastic control and differential games when there exists path-dependence in the control (or action) of the agent. We have analyzed the important example of delayed dependence. The framework used was the relatively recent functional Itô calculus, which has been proven to be an excellent tool to deal with complicated path-dependence structures, see Jazaerli and Saporito [2013]. Although we have focused on delayed dependence, because of practical importance, there are no major impediments to examine more interesting structures. We hope this work will allow the consideration of different path-dependent structures in other applications.

Compared to the theory of Gozzi and Marinelli [2006] and Gozzi and Masiero [2015], that deals with just the delayed case, the method proposed here allows in principle very general path-dependence in the controls. Moreover, the HJB found here is significantly simpler than the one of the previous reference and it could be directly applied to (Dirac) measures, as it was done in Section 2.2.1.

Future research will be conducted to analyze viscosity solutions (existence and uniqueness) of the path-dependent HJB derived here. Viscosity solution of similar PPDEs have been extensively studied in recent years, see for example Ekren et al. [2014, 2016a,b]. Moreover, it would be interesting to apply the theory developed here to Stackelberg games, Bensoussan et al. [2015].

Acknowledgements. I would like to thank J.P. Fouque for bringing such a interesting problem to my attention and for all the insightful conversations. I also thank J. Zhang and M. Mousavi for the innumerous helpful discussions and comments.

References.

Y. Alekal, P. Brunovsky, D. Chyung, and E. Lee. The Quadratic Problem for Systems with Time Delays. *IEEE Trans. Autom. Control*, 16(6):673–687, 1971.

A. Bensoussan, M. H. M. Chau, and S. C. P. Yam. Mean Field Stackelberg Games: Aggregation of Delayed Instructions. *SIAM J. Control Optim.*, 53(4):2237–2266, 2015.

R. Carmona. *Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications*. SIAM, 2016.

R. Carmona, J.-P. Fouque, M. Mousavi, and L.-H. Sun. Systemic Risk and Stochastic Games with Delay. *Preprint*, 2016.

L. Chen and Z. Wu. The Quadratic Problem for Stochastic Linear Control Systems with Delay. In *Proceedings of the 30th Chinese Control Conference*, pages 1344–1349, July 2011.

R. Cont and D.-A. Fournié. Change of Variable Formulas for Non-Anticipative Functional on Path Space. *J. Funct. Anal.*, 259(4):1043–1072, 2010.

B. Dupire. Functional Itô Calculus. 2009. Available at SSRN: http://ssrn.com/abstract=1435551.

I. Ekren, C. Keller, N. Touzi, and J. Zhang. On Viscosity Solutions of Path Dependent PDEs. *Ann. Probab.*, 42(1):204–236, 2014.

I. Ekren, N. Touzi, and J. Zhang. Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part I. *Ann. Probab*, 44(2):1212–1253, 2016a.

I. Ekren, N. Touzi, and J. Zhang. Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part II. *Ann. Probab*, 44(4):2507–2553, 2016b.

D.-A. Fournié. *Functional Itô Calculus and Applications*. PhD thesis, Columbia University, 2010.

F. Gozzi and C. Marinelli. Stochastic Optimal Control of Delay Equations Arising in Advertising Models. In G. Da Prato and L. Tubaro, editors, *Stochastic Partial Differential Equations and Applications - VII*, pages 133–148. Chapman & Hall/CRC, 2006.

F. Gozzi and F. Masiero. Stochastic Optimal Control with Delay in the Control: Solution Through Partial Smoothing. *Submitted*, 2015. Available at arXiv: http://arxiv.org/abs/1506.06013.

J. Huang, X. Li, and J. Shi. ForwardBackward Linear Quadratic Stochastic Optimal Control Problem with Delay. *Sys. Control Lett.*, 61:623–630, 2012.

S. Jazaerli and Y. F. Saporito. Functional Itô Calculus, Path-dependence and the Computation of Greeks. *Submitted*, 2013. Available at arXiv: http://arxiv.org/abs/1311.3881.

S. Ji, L. Wang, and S. Yang. Path-Dependent Hamilton-Jacobi-Bellman Equations Related to Controlled Stochastic Functional Differential Systems. *Optim. Control Appl. Meth.*, 36:109–120, 2015.

H. Oberhauser. An extension of the Functional Itô Formula under a Family of Non-dominated Measures. *Stoch. Dyn.*, 16(4), 2016.

T. Pang and A. Hussain. An Application of Functional Itô's Formula to Stochastic Portfolio Optimization with Bounded Memory. *Proceedings of the Conference on Control and its Applications*, 2015.

H. Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer, 2009.

T. Pham and J. Zhang. Two Person Zero-sum Game in Weak Formulation and Path Dependent Bellman-Isaacs Equation. *SIAM J. Control Optim.*, 52(4):2090–2121, 2014.

H. M. Soner and N. Touzi. Dynamic Programming for Stochastic Target Problems and Geometric Flows. *J. Eur. Math. Soc.*, 4(3):6201–236, 2002.

Y. Xu. Probabilistic Solutions for a Class of Path-Dependent Hamilton-Jacobi-Bellman Equations. *Stoch. Anal. Appl.*, 31:440–459, 2013.