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MATEUS DE LIMA SANTOS

AN ESSAY ON SELF-ENFORCING DEBT

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Dissertação apresentada à Escola de Economia de São Paulo da Fundação Getulio Vargas como requisito para obtenção do título de Mestre em Economia de Empresas

Campo de Conhecimento:
Teoria Econômica

Orientador: Prof. Dr. Victor Filipe Martins-da-Rocha

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ABSTRACT

We analyze repayment incentives in an infinite horizon competitive economy where agents cannot commit to financial contracts. We follow Bulow and Rogoff (1989) by assuming that a defaulting agent is excluded from borrowing forever but keeps the ability to save. Hellwig and Lorenzoni (2009) provide an important characterization result by showing that endogenous debt limits are self-enforcing and not-too-tight if, and only if, they form a rational bubble in the sense that they can be exactly rolled over at infinity. Our contribution is technical. We provide a rigorous and correct proof of this result without imposing any *ad-hoc* assumption on the endogenous debt limits. In that respect, we extend the result in Bidian and Bejan (2015).

Keywords: limited commitment, self-enforcing debt, rational bubbles.

RESUMO

Nós analisamos incentivos de repagamento em uma economia competitiva de horizonte infinito onde os agentes não podem se comprometer com contratos financeiros. Nós seguimos Bulow and Rogoff (1989) ao assumir que um agente que deu default é excluído da possibilidade de empréstimo para sempre mas mantém a possibilidade de poupança. Hellwig and Lorenzoni (2009) fornecem um importante resultado de caracterização ao mostrarem que limites à dívida endógenos são auto-sustentados e não muito restritos se, e somente se, eles formam uma bolha racional no sentido que podem ser exatamente rolados até o infinito. Nossa contribuição é técnica. Nós provemos uma prova rigorosa e correta desse resultado sem impor nenhuma condição *ad-hoc* nos limites à dívida endógenos. Nesse sentido, nós estendemos o resultado de Bidian and Bejan (2015).

Palavras-chave: compromisso limitado, dívida auto-sustentada, bolhas racionais.

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1 INTRODUCTION

It is well established that when agents are allowed to borrow in an environment of competitive dynamic economies with infinite horizon and sequential trading, technicalities require the imposition of some sort of constraint to the agent's portfolio choice, e.g., debt limits, borrowing limits and short sales limits. On the one hand, these restrictions preclude Ponzi schemes, therefore are necessary conditions to the existence of optimal solutions. On the other hand, they should be set in such a way that do not restrict agents more than required, that is, they should not impose any unjustified friction.

In a full commitment environment, the only role of debt limits should be to preclude Ponzi schemes. Then, they should be chosen in order to never bind in equilibrium. Following standard arguments, we can show this implies that the agent's wealth, defined as the present value of her endowments, is finite. A direct consequence is that consumption also has finite present value. We deduce that an agent's equilibrium debt is bounded from above by her natural debt limit, defined as her wealth at each period. Since natural debt limits never bind at equilibrium, we can conclude that there is no loss of generality in restricting debt to be lower than the natural debt limits.

However, the assumption that agents can commit to financial contracts is rather unrealistic. When commitment is limited, a debtor may not honor her commitments and choose to default. This decision depends on the default punishment. When the consequence of default is independent of the default level, a debtor's choice is either to fully repay her debt (and avoid the punishment) or to fully default. If markets are complete and potential lenders perfectly anticipate this rational behavior, they only accept to lend if they understand that the borrower will have incentives to repay.[1] This point was formally modeled in a partial equilibrium setting by the seminal contribution of Eaton and Gersovitz (1981). It was extended to a general equilibrium environment by Kehoe and Levine (1993) who introduced the notion of constrained Arrow–Debreu equilibrium where consumption plans were restricted to satisfy participation constraints.

Alvarez and Jermann (2000) realized that the participation constraints imposed on consumption plans could be replaced by adequate (and therefore endogenously determined) debt limits on bond holdings. In other words, under limited commitment, debt limits could be set not only to prevent Ponzi schemes but also to guarantee repayment incentives. They introduced the important notion of self-enforcing (and not-too-tight) debt limits, the loosest debt limits that are consistent with the borrower's incentives to repay.

[1] This argument is not valid anymore when markets are incomplete. Indeed, a lender may accept to buy the non-contingent debt issued by a borrower if she anticipates that there will be repayment with some positive probability.

Eaton and Gersovitz (1981), Kehoe and Levine (1993) and Alvarez and Jermann (2000) consider permanent reversion to autarky (exclusion from credit markets) as the default punishment. In a seminal contribution, Bulow and Rogoff (1989) argue that this strong punishment may not be adequate to sovereigns. They show that if a sovereign's default induces exclusion from borrowing but not from saving (in the sense that after default the sovereign can purchase foreign assets, like debt issued by another sovereign country), then no positive lending can be sustained.

Formally, Bulow and Rogoff (1989) proved that if the sovereign's wealth is finite and not-too-tight debt limits are tighter than the natural debt limits (as it is the case in the standard full commitment environment), then debt limits must be zero and the sovereign cannot borrow. This result hinges on two assumptions on endogenous variables: (1) interest rates are high enough such that the agent's wealth is finite, and (2) debt limits are tighter than natural debt limits. These two properties are satisfied in a competitive equilibrium with full commitment. However, Hellwig and Lorenzoni (2009) show that this is not necessarily the case when all agents cannot commit to financial promises. They construct a simple example to show that equilibrium interest rates can be so low that each agent's wealth is infinite. They also show that, in contrast to the classic result by Bulow and Rogoff (1989), not-too-tight debt limits can be positive.

In the example proposed by Hellwig and Lorenzoni (2009), the equilibrium not-too-tight debt limits form a rational bubble in the sense that the agent can exactly roll-over existing debt period after period.[2] This property was already pointed out by Kocherlakota (2008) who proved that if debt limits form a bubble then they are not-too-tight. Hellwig and Lorenzoni (2009) claimed that the converse is true, that is, debt limits are not-too-tight if, and only if, they form a rational bubble. This characterization allows to get an equivalence result, whereby the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with unbacked public debt.

To prove that not-too-tight debt limits must form a rational bubble, Hellwig and Lorenzoni (2009) show that it is possible to construct an auxiliary process of debt limits that forms a rational bubble and satisfies the following conditions: the auxiliary debt limits coincide with the initial debt limits when they bind and are weakly looser when they do not bind. To get the desired result, it is sufficient to show that the auxiliary debt limits actually coincide with the initial debt limits. To prove this coincidence, Hellwig and Lorenzoni (2009) claimed that the optimal allocation under the initial debt limits is also optimal under the auxiliary debt limits. This seems natural since the Euler equations are unchanged under both debt limits. The only technical issue is related to the Transversality condition.

Recently Bidian and Bejan (2015) identified several technical shortcomings in the arguments proposed by Hellwig and Lorenzoni (2009). One of them is related to the Transver-

[2] Equivalently, the agent is able to exactly refinance outstanding obligations by issuing new claims.

sality condition. Indeed, Hellwig and Lorenzoni (2009) implicitly assumed that the present value of non-binding not-too-tight debt limits asymptotically vanishes. This may be true when they are tighter than natural debt limits, but when the agent's wealth is infinite, natural debt limits are infinite and there is *a priori* no reason for non-binding not-too-tight debt limits to asymptotically vanish in present value terms.[3] One of the contributions of this thesis is to provide an example of non-binding not-too-tight debt limits that do not asymptotically vanish.

Another shortcoming pointed out by Bidian and Bejan (2015) is related to switching the order of expectations and limits. To fix the arguments in the proof of Hellwig and Lorenzoni (2009), they restrict attention to not-too-tight debt limits that satisfy a uniform integrability condition. This is *a priori* an *ad-hoc* assumption since not-too-tight debt limits are endogenously determined. A second contribution of this thesis is to provide another example to illustrate that this uniform integrability condition is not necessarily satisfied.

The third and main contribution of this work is to show that the characterization result conjectured by Hellwig and Lorenzoni (2009) is actually true. We provide an alternative proof that does not require any assumption on endogenous variables. Our approach is to observe that the optimal consumption process does not depend on the process of not-too-tight debt limits, whenever the agent initial bond holding is binding. To conclude the desired result we present a recursive argument in which we note that not-too-tight debt limits do form a bubble.

The organization of this work is as follows: in the second chapter we describe the set-up of our model: a dynamic stochastic competitive economy with limited commitment where a defaulting agent is excluded from borrowing but keeps the ability to purchase assets. Chapter 3 reviews the existing results of the literature. Chapter 4 contains the proof of the general characterization result. After the conclusion, we provide three appendices. The first one contains omitted proofs, the second one presents the two examples showing that the assumptions imposed by Hellwig and Lorenzoni (2009) and Bidian and Bejan (2015) are not satisfied by all not-too-tight debt limits and the last one discusses the standard full commitment environment.

[3] Even if some agent's wealth is finite, Martins-da-Rocha and Vailakis (2017) show that not-too-tight debt limits may exceed natural debt limits.

2 THE MODEL

2.1 Fundamentals and markets

We consider an infinite horizon economy with limited commitment and self-enforcing debt limits. Time and uncertainty are discrete. Time is indexed by $t \in \mathbb{N} \cup \{0\}$ and, at each period t, there exists a finite set of events. There is a single perishable good. We analyze the optimal decision of an infinitely lived agent who transfers resources over time and across events by trading a complete set of one-period contingent securities in a competitive environment.

Uncertainty is modeled by an event tree, hereafter Σ, that characterizes the disclosure of information regarding the economy over time. Formally, let a tree be a sequence of sets $(S^t)_{t\geq 0}$ satisfying the following properties:

- the set S^0 is a singleton denoted by $\{s^0\}$;
- for each $t \geq 1$, the set S^t is finite and we pose $\Sigma = \bigcup_{t\geq 0} S^t$;
- there exists a function $\sigma : \Sigma \setminus \{s^0\} \to \Sigma$ such that for each $t \geq 0$, we have $\sigma(S^{t+1}) = S^t$.[1]

Let an element s^t of S^t be called a date-t event, or more generally an event. Define the truncated economy, consisting of all date-t events up to period T, as

$$\Sigma^T := \bigcup_{t=0}^{T} S^t.$$

When $\sigma(s^{t+1}) = s^t$, we say that s^{t+1} is an immediate successor of s^t and write $s^{t+1} \succ s^t$. The function σ maps every event $s^t \succ s^0$ to its unique predecessor (we assume that s^0 has no predecessor). Observe that every event s^t has a finite number of successors, elements of S^{t+1}. Let $\tau > t$, and $s^t \in S^t$, $s^\tau \in S^\tau$. If $\sigma^{\tau-t}(s^\tau) = s^t$ then s^τ is called a successor of s^t and we also write $s^\tau \succ s^t$.[2] We also use the notation $s^\tau \succeq s^t$ whenever $s^\tau \succ s^t$ or $s^\tau = s^t$. For any $\tau > t$, the set $S^\tau(s^t) := \{s^\tau \in S^\tau \;:\; s^\tau \succeq s^t\}$ denotes the collection of all date-τ events that are successors of s^t.

Remark 2.1. A event s^t is uniquely determined by its history from s^0, that is,

$$(s^0 = \sigma^t(s^t), \sigma^{t-1}(s^t), ..., \sigma(s^t), s^t) \in \prod_{\tau=0}^{t} S^\tau.$$

For all $s^t \in \Sigma$, we denote by $\Sigma(s^t)$ the subtree starting at s^t and formally defined by

$$\Sigma(s^t) := \{s^\tau \in \Sigma \;:\; s^\tau \succeq s^t\}.$$[3]

[1] Note that the function σ is surjective, but it is not necessarily injective.

[2] Any immediate successor is in particular a successor, though the converse is not true in general.

[3] Observe that $\Sigma = \Sigma(s^0)$ and $\Sigma(s^t) = \bigcup_{\tau\geq 0} S^{t+\tau}(s^t)$.

For every date $T \geq t$, we let $\Sigma^T(s^t) := \Sigma(s^t) \cap \Sigma^T$ denote the subtree starting at event s^t and truncated at date T.

The agent's endowments are stochastic. We denote by $e(s^t)$ the agent's endowment at event s^t in units of the consumption good. It is assumed to be strictly positive, i.e., $e(s^t) > 0$. The process of initial endowments is denoted by $e = (e(s^t))_{s^t \in \Sigma} \in (\mathbb{R}_{++})^{\Sigma}$.

The agent's preference relation is represented by the lifelong discounted and expected utility function,

$$U(c|s^0) := \lim_{T \to \infty} \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t)),$$

or, equivalently,

$$U(c|s^0) := \sum_{s^t \in \Sigma} \beta^t \pi(s^t) u(c(s^t)), \tag{2.1}$$

where $\beta \in (0,1)$ is the intertemporal discount factor, $\pi(s^t)$ is the unconditional probability of s^t, and $u : \mathbb{R}_+ \to \mathbb{R}$ is the instantaneous (or Bernoulli) function.[4] We assume that u is continuous, bounded, strictly increasing and strictly concave on $\mathbb{R}_+$. We also assume that it is continuously differentiable on $(0, \infty)$ and satisfies Inada's condition at the origin.[5]

Given any event s^t, we denote by $U(c|s^t)$ the continuation utility along the subtree $\Sigma(s^t)$ defined by

$$U(c|s^t) := \sum_{\tau=t}^{\infty} \sum_{s^\tau \in S^\tau(s^t)} \beta^{\tau-t} \pi(s^\tau|s^t) u(c(s^\tau)). \tag{2.2}$$

The agent has access to a complete set of one-period contingent bonds, in the sense that, for every s^t, there is a spot market for s^{t+1}-contingent bonds paying one unit of consumption good contingent to s^{t+1}, with $s^{t+1} \succ s^t$. Let $a(s^{t+1})$ denote the agent's holding of the bond contingent to s^{t+1}, and let $a = (a(s^t))_{s^t \in \Sigma}$ be the corresponding process of bond holdings.[6]

For every event $s^t \in \Sigma$ and strict successor event $s^{t+1} \succ s^t$, we let $q(s^{t+1})$ denote the price, in units of s^t-consumption, of one unit of the bond contingent to s^{t+1}. The process of bond prices is then defined by $q = (q(s^t))_{s^t \succ s^0}$. We restrict attention to strictly positive prices $q(s^t) > 0$ since the Bernoulli function u is assumed to be strictly increasing.

We say that a pair (c, a) of consumption and bond holding satisfies the flow budget constraints when

$$c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \leq e(s^t) + a(s^t), \quad \forall s^t \in \Sigma. \tag{2.3}$$

Because the Bernoulli function is assumed to be strictly increasing, any increase in consumption implies increase in utility. Therefore, if there were no limits on debt levels, the agent would have

4 For every period t, we have, $\pi(s^t) > 0$ for all $s^t \in S^t$ and $\sum_{s^t \in S^t} \pi(s^t) = 1$.

5 That is, $\lim_{c \to 0} [u(c) - u(0)]/c = \infty$.

6 The initial financial claim $a(s^0)$ represents past decisions not modelled.

incentives to keep rolling over debt in order to raise wealth and, hence, utility. Such debt holding arrangement is called a Ponzi scheme. It is therefore necessary to impose a mechanism to rule out such schemes, once they preclude the existence of an optimal solution.

Ponzi schemes are ruled out if some sort of constraint on bond holdings is imposed. We adopt restrictions on the amount of debt an agent is able to hold on each event $s^{t+1} \succ s^0$, i.e.,[7]

$$a(s^{t+1}) \geq -D(s^{t+1}), \tag{2.4}$$

where $D(s^t) \geq 0$ is called a debt limit. Let $D = (D(s^t))_{s^t \in \Sigma}$ be the agent's process of debt limits.

In the standard literature where agents can commit to financial contracts, it is commonly assumed that debt limits coincide with natural debt limits.[8] Before providing a formal definition, we need to introduce the following notations. Given a process $q = (q(s^t))_{s^t \succ s^0}$ of bond prices, we let $p(s^t)$ denote the corresponding Arrow-Debreu price at s^0, defined as the price of one unit of s^t-consumption good in units of s^0-consumption good. That is, $p(s^0) := 1$ and, for every $s^t \succ s^0$,

$$p(s^t) := q(s^t)p(\sigma(s^t)).$$

Let $p = (p(s^t))_{s^t \in \Sigma}$ be the corresponding process of Arrow-Debreu prices. Fix an arbitrary process $x = (x(s^t))_{s^t \in \Sigma} \in (\mathbb{R}_+)^\Sigma$. Given some event s^t and some period $T \geq t$, we let $\mathrm{PV}^T(x|s^t)$ denote the truncated (up to time T) present value at s^t of the process x restricted to the corresponding subtree $\Sigma(s^t)$ and defined by

$$\mathrm{PV}^T(x|s^t) := \frac{1}{p(s^t)} \sum_{\tau=t}^{T} \left[\sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)x(s^\tau) \right].$$

Passing to the limit when T tends to infinity, we obtain the present value over the whole subtree $\Sigma(s^t)$, i.e.,

$$\mathrm{PV}(x|s^t) := \frac{1}{p(s^t)} \sum_{s^\tau \in \Sigma(s^t)} p(s^\tau)x(s^\tau).$$

Observe that, *a priori*, we have $\mathrm{PV}(x|s^t) \in [0, \infty]$.

Definition 2.1 (Natural Debt Limits)**.** *The process $N = (N(s^t))_{s^t \in \Sigma}$ of natural debt limits is defined as the present value of the agent's future endowments, i.e.,*

$$N(s^t) := \mathrm{PV}(e|s^t), \quad \forall s^t \in \Sigma.$$

Remark 2.2. One can note from the above definition that

$$N(s^t) = e(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})N(s^{t+1}), \quad \forall s^t \in \Sigma. \tag{2.5}$$

[7] Once $a(s^0)$ is exogenous, the restriction $a(s^0) \geq -D(s^0)$ is assumed to be satisfied.

[8] See for instance Miao (2014, ch. 13) or Ljungqvist and Sargent (2004, ch. 8).

This implies that if $N(s^t) < \infty$ for some event s^t, then $N(s^\tau) < \infty$ for every event $s^\tau \succ s^t$. However, there is *a priori* no reason for $N(s^t)$ to be finite since this depends on the interest rates (bond prices) as we will discuss later on.[9]

The natural debt limits are known as the agent's natural ability to repay since it represents the amount she can borrow when she can commit to deliver her future endowments and consume zero forever.

We fix an initial financial claim $b \in \mathbb{R}$. The agent's budget set, denoted by $B(D,b|s^0)$, is the set of all pairs (c,a) satisfying equations (2.3) and (2.4), for every event $s^t \in \Sigma$ with the initial condition $a(s^0) = b$. A pair (c,a) is said budget feasible (given the initial financial claim b) whenever it belongs to $B(D,b|s^0)$. Let $d(D,b|s^0)$ denote the agent's demand set. It is the set of all budget feasible pairs that maximize her lifelong utility function, i.e.,

$$d(D,b|s^0) := \operatorname{argmax}\{U(c|s^0) \ : \ (c,a) \in B(D,b|s^0)\}.$$

We denote by $V(D,b|s^t)$ the largest continuation utility over the subtree $\Sigma(s^t)$, defined as

$$V(D,b|s^t) := \sup\{U(c|s^t) \ : \ (c,a) \in B(D,b|s^t)\}, \tag{2.6}$$

where $B(D,b|s^t)$ is the set of all plans (c,a) satisfying equations (2.3) and (2.4) for every $s^\tau \succeq s^t$ together with the initial condition $a(s^t) = b$.

Remark 2.3. For any process of debt limits and any initial event s^t, the value function $b \mapsto V(D,b|s^t)$ is not decreasing.

2.2 Necessary and sufficient conditions for optimality

In this section we introduce the notion of consistent debt limits. Then, we provide necessary and sufficient conditions for optimality by means of Euler equations and the Transversality condition.

Definition 2.2 (Consistency)**.** *A process $D = (D(s^t))_{s^t \in \Sigma}$ of debt limits is said to be consistent if current debt can be repaid using current endowments and borrowing against successor events, i.e.,*

$$D(s^t) \le e(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \forall s^t \in \Sigma.$$

Remark 2.4. Observe that debt limits are consistent if, and only if, for each event s^t, the budget set $B(D, -D(s^t)|s^t)$ is non-empty.

We now present conditions which turn out to be necessary and sufficient to characterize optimality. Consider a budget feasible pair $(c,a) \in B(D,a(s^0)|s^0)$ with strictly positive consumption.

[9] In Appendix C we analyze the competitive equilibrium of the benchmark environment with full commitment. We show that the aggregate wealth is always finite and that *never binding debt limits* are tighter than natural debt limits.

- It is said to satisfy the Euler equations (EE) when

$$\forall s^t \in \Sigma, \quad \forall s^{t+1} \succ s^t, \quad q(s^{t+1}) \geq \beta \pi(s^{t+1}|s^t) \frac{u'(c(s^{t+1}))}{u'(c(s^t))}, \tag{EE}$$

  with an equality when $a(s^{t+1}) > -D(s^{t+1})$.

- It is said to satisfy the Transversality condition (TC) when

$$\lim_{t \to \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t))[a(s^t) + D(s^t)] = 0. \tag{TC}$$

The following lemma identifies sufficient conditions for optimality of a budget feasible pair with strictly positive consumption.

Lemma 2.1. *Let $(c,a) \in B(D, a(s^0)|s^0)$ with $c(s^t) > 0$, for every $s^t \in \Sigma$. If the pair (c,a) satisfies the flow budget constraints with equality, the Euler equations and the Transversality condition, then $(c,a) \in d(D, a(s^0)|s^0)$.*

If debt limits are consistent, the sufficient conditions described above are actually necessary.

Lemma 2.2. *Let D be a process of consistent debt limits. If $(c,a) \in d(D, a(s^0)|s^0)$ with $a(s^0) \geq -D(s^0)$ and $c(s^t) > 0$, for every $s^t \in \Sigma$, then it satisfies the flow budget constraint with equality, the Euler equations and the Transversality condition.*

2.3 Self-enforcing debt under limited commitment

We assume that agents cannot commit to their financial contracts. Their repayment incentives depend on the default punishment. We follow Bulow and Rogoff (1989), Hellwig and Lorenzoni (2009) and Bidian and Bejan (2015) by assuming that a defaulting agent is excluded from credit markets forever but keeps the availability to save. This leads to the following formal definition of the value of the default option.

Definition 2.3 (Value of the Default Option)**.** *The agent's continuation value in case of default at event s^t is*

$$V_{\text{def}}(s^t) := V(0,0|s^t).$$

Since the default punishment is independent of the size of the default level, the agent decides either to fully default or to fully repay. Formally, if the agent 's bond holding contingent to event s^t is $a(s^t)$, then she decides to repay her debt if, and only if,

$$V(D, a(s^t)|s^t) \geq V_{\text{def}}(s^t).$$

In particular, given that markets are complete, lenders have no incentives to provide credit if they anticipate that the borrower will default. In other words, the debt hold by an agent should

be consistent with repayment incentives. As suggested by Alvarez and Jermann (2000), competition among lenders should guarantee that the maximum possible debt an agent can hold is exactly the amount that makes her indifferent between defaulting or not. We consider the following definitions.

Definition 2.4 (Self-Enforcing and Not-too-Tight)**.** *The process $D = (D(s^t))_{s^t \in \Sigma}$ is said to be self-enforcing when*

$$V(D, -D(s^t)|s^t) \geq V_{\text{def}}(s^t), \quad \forall s^t \in \Sigma.$$

It is said to be (self-enforcing and) not-too-tight when

$$V(D, -D(s^t)|s^t) = V_{\text{def}}(s^t), \quad \forall s^t \in \Sigma.$$

It follows directly from the definition that not-too-tight debt limits are (strictly) consistent.

Lemma 2.3. *If a process of debt limits D is not-too-tight, then*

$$D(s^t) \leq e(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \forall s^t \in \Sigma.$$

Proof. Fix an event s^t. In order to have $V(D, -D(s^t)|s^t) = V(0,0|s^t)$, the budget set $B(D, -D(s^t)|s^t)$ must be non-empty. This implies that

$$D(s^t) \leq e(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}).$$

□

Remark 2.5. In Chapter 4, we analyze other possible default punishments. However, when the value of default option is $V(0,0|s^t)$, we can show that the not-too-tight debt limits are actually strictly consistent. Indeed, assume, by way of contradiction, that the above inequality is actually an equality. We then get that

$$V(D, -D(s^t)|s^t) = u(0) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V(D, -D(s^{t+1})|s^{t+1}).$$

Combining the above inequality with the property that D is not-too-tight, we get that

$$V(0,0|s^t) = u(0) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V(0,0|s^{t+1}).$$

Observe that after default, the agent has the option to consume her endowment. This implies that

$$V(0,0|s^t) \geq u(e(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V(0,0|s^{t+1}).$$

Since $e(s^t) > 0$, we get a contradiction.

It is important to observe that if D is not-too-tight, then $D(s^t)$ represents the largest debt level at event s^t that is compatible with repayment incentives, given that the future debt limits are defined by the process $(D(s^\tau))_{s^\tau \succ s^t}$. In other words, a process of not-too-tight debt limits is a solution to an implicit (*a priori* non-linear) difference equation (with the process of zero debt limits as the particular solution). The objective of this thesis is to provide a complete characterization of not-too-tight debt limits.

As it is the case for linear difference equations, the linearity of the flow budget constraints implies that there is a strong connection between not-too-tight debt limits and debt limits that satisfy the following exact roll-over property.

Definition 2.5 (Exact Roll-Over)**.** *A process $D = (D(s^t)_{s^t \in \Sigma})$ of debt limits is said to allow for exact roll-over (or form a bubble) when*

$$D(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \textit{for all } s^t \succeq s^0.$$

Remark 2.6. If we let $m := p/\pi$, then a process $D = (D(s^t))_{s^t \in \Sigma}$ of debt limits satisfies exact roll-over if, and only if, $mD := (m(s^t)D(s^t))_{s^t \in \Sigma}$ is a martingale process.[10]

Following the arguments in the proof of the Bubble Equivalence Theorem in Kocherlakota (2008), we can show that if a process allows for exact roll-over, then it is not-too-tight. This follows from a simple translation invariance property of the flow budget constraint.

Lemma 2.4. *If D is a process of debt limits that allows for exact roll-over, then it is not-too-tight.*

Proof. Fix a process $D = (D(s^t))_{s^t \in \Sigma}$ of debt limits that satisfies exact roll-over. Fix an arbitrary event s^τ and let $(c, a) \in d(0, 0|s^\tau)$. For every successor event $s^t \succeq s^\tau$, we have the following (binding) flow budget constraint

$$c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) = e(s^t) + a(s^t)$$

and the corresponding debt constraints

$$a(s^{t+1}) \geq 0, \quad \forall s^{t+1} \succ s^t.$$

Adding $-D(s^t)$ in both sides of the flow budget constraint and using the exact roll-over property, we get that

$$c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \tilde{a}(s^{t+1}) = e(s^t) + \tilde{a}(s^t)$$

with $\tilde{a} := a - D$. Observe moreover that

$$\tilde{a}(s^{t+1}) \geq -D(s^{t+1}), \quad \forall s^{t+1} \succ s^t.$$

10 A process $M = (M(s^t))_{s^t \in \Sigma}$ is a martingale when $M(s^t) = \sum_{s^{t+1} \succ s^t} M(s^{t+1}) \pi(s^{t+1}|s^t)$, for every s^t.

We have thus proved that $(c,\tilde{a}) \in B(D,-D(s^\tau)|s^\tau)$ which implies that $V(D,-D(s^\tau)|s^\tau) \geq V(0,0|s^\tau)$.

Symmetrically, if $(c,a) \in d(D,-D(s^\tau)|s^\tau)$, then $(c,\hat{a}) \in B(0,0|s^\tau)$ where $\hat{a} := a + D$. This implies that $V(0,0|s^\tau) \geq V(D,-D(s^\tau)|s^\tau)$ and we get the desired result. $\square$

The natural question is whether all not-too-tight debt limits must form a bubble. The answer is yes.

Theorem 2.1. *A non-negative process of debt limits is not-too-tight if, and only if, it allows for exact roll-over.*

Before providing the detailed arguments of the proof in Chapter 4, we first present in Chapter 3 the existing results of the literature.

Remark 2.7. As pointed out by Kocherlakota (2008), the bubble component of the price of a long-lived asset (defined as the difference between the price of the asset and its fundamental value) must satisfy the exact roll-over property. This is why we use the expression "debt limits form a bubble" as a substitute for "debt limits allows for exact roll-over". It is worth pointing out that Kocherlakota (2008) proved that the exact roll-over property of not-too-tight debt limits allows to inject bubbles into the price of long-lived assets. Following similar intuition, Hellwig and Lorenzoni (2009) proved the equivalence between equilibrium allocations with self-enforcing private debt and allocations that are sustained with unbacked public debt or rational bubbles.[11]

[11] In their competitive equilibrium with unbacked public debt, they portray a government whose securities issued are not financed by any kind of government income, that is, the government pays the debt at each period issuing new debt.

In this chapter we present the existing results of the literature that provide necessary conditions satisfied by not-too-tight debt limits.

Bulow and Rogoff (1989) presented the first major contribution to a characterization of not-too-tight debt limits. They proved that debt limits cannot be simultaneously self-enforcing and tighter than natural debt limits. Formally, they proved the following result.[1]

Theorem 3.1 (Bulow and Rogoff (1989)). *Let D be a process of not-too-tight debt limits. If the agent's wealth is finite and debt limits are tighter than natural debt limits, i.e.,*

$$D(s^t) \leq N(s^t), \quad \textit{for all } s^t \succeq s^0,$$

then there is no debt, i.e., $D = 0$.

The above result is obtained under *ad-hoc* assumptions on endogenous variables. Indeed, it has been shown by Hellwig and Lorenzoni (2009) that in a general equilibrium environment with limited commitment, equilibrium bond prices can be so low that the agent's wealth is infinite. Moreover, even if the agent's wealth is finite, Martins-da-Rocha and Vailakis (2017) show that self-enforcing debt limits may exceed natural debt limits.

Remark 3.1. Observe that if the agent's wealth is finite and the debt limits $(D(s^t))_{s^t \in \Sigma}$ are tighter than natural debt limits, then the process D satisfies the following Transversality condition:

$$\lim_{t \to \infty} \sum_{s^t \in S^t} p(s^t) D(s^t) = 0. \tag{TC$_{\text{BR}}$}$$

Hellwig and Lorenzoni (2009) (see also Bidian and Bejan (2015)) extended Theorem 3.1 by characterizing an agent's repayment incentives without assuming *a priori* that her wealth is finite. In order to present their results, we need to introduce the following notations.

Fix an arbitrary process $D = (D(s^t))_{s^t \in \Sigma}$ of not-too-tight debt limits. Given some event s^τ, we let $(c(\cdot|s^\tau), a(\cdot|s^\tau))$ be the optimal plan in $B(D, -D(s^\tau)|s^\tau)$, i.e.,

$$(c(\cdot|s^\tau), a(\cdot|s^\tau)) \in d(D, -D(s^\tau)|s^\tau).$$

We denote by $\Sigma_{\text{NB}}(s^\tau) \subseteq \Sigma$ the set of all events $s^t \succeq s^\tau$ such that either $s^t = s^\tau$ or debt limits (satisfied by the optimal plan $(c(\cdot|s^\tau), a(\cdot|s^\tau))$) never bind along the history of events from s^τ up to s^t, i.e.,

$$\Sigma_{\text{NB}}(s^\tau) := \{s^\tau\} \cup \{s^t \succ s^\tau \,:\, a(\sigma^k(s^t)|s^\tau) > -D(\sigma^k(s^t)), \quad \forall k \in \{0, 1, \ldots, t - \tau - 1\}\}.$$

[1] Theorem 3.1 is different from the theorem presented in Bulow and Rogoff (1989), yet, Martins-da-Rocha and Vailakis (2016) proved that it is in fact a corollary from the original one.

For any period $t > \tau$, we let $S^t_{\mathrm{NB}}(s^\tau) := \Sigma_{\mathrm{NB}}(s^\tau) \cap S^t$ denote the set of all date-t events s^t that are successors of s^τ such that debt limits never bind along the history from s^τ up to s^t.

We denote by $\Sigma_{\mathrm{FB}}(s^\tau) \subseteq \Sigma$ the set of all events $s^t \succ s^\tau$ for which the debt constraint (satisfied by the optimal plan $(c(\cdot|s^\tau), a(\cdot|s^\tau))$) binds for the first time since s^τ, i.e.,

$$\Sigma_{\mathrm{FB}}(s^\tau) := \{s^t \succ s^\tau \ : \ a(s^t|s^\tau) = -D(s^t) \quad \text{and} \quad \sigma(s^t) \in \Sigma_{\mathrm{NB}}(s^\tau)\}.$$

Hellwig and Lorenzoni (2009) sketched a proof of Theorem 2.1 but Bidian and Bejan (2015) identified several technical shortcomings. The first one is related to the Transversality condition necessary for optimality. Indeed, Hellwig and Lorenzoni (2009) assumed that the optimal plan $(c(\cdot|s^0), a(\cdot|s^0))$ satisfies the following property

$$\lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t) a(s^t|s^0) = 0.$$

This does not correspond to the Transversality condition unless we have

$$\lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t) D(s^t) = 0. \tag{TC$_{\mathrm{HL}}$}$$

Observe that (TC$_{\mathrm{HL}}$) is weaker than the Transversality condition (TC$_{\mathrm{BR}}$) imposed by Bulow and Rogoff (1989). However, if the agent's wealth is not finite, the Transversality condition (TC$_{\mathrm{HL}}$) need not be satisfied. In the appendix we provide Example B.1 to illustrate this.

The incorrect Transversality condition is not the only shortcoming pointed out by Bidian and Bejan (2015). They also identified several problems related to switching the order of expectations and limits. They succeeded to fix these issues by imposing an uniform integrability condition on debt limits. Formally, they proved the following result.

Theorem 3.2 (Bidian and Bejan (2015))**.** *Let D be a process of not-too-tight debt limits. If, for every event $s^\tau \in \Sigma$,*

$$\lim_{\gamma\to\infty} \sup_{t\geq\tau} \left\{ \sum_{s^t \in S^t_{\mathrm{NB}}(s^\tau)} p(s^t) D(s^t) \mathbb{1}_{\{p(s^t)D(s^t) > \gamma\pi(s^t)\}} \right. \\ \left. + \sum_{s^r \in \Sigma^t_{\mathrm{FB}}(s^\tau)} p(s^r) D(s^r) \mathbb{1}_{\{p(s^r)D(s^r) > \gamma\pi(s^r)\}} \right\} = 0, \tag{TC_{BB}}$$

then D allows for exact roll-over.

Bidian and Bejan (2015) argue that assumption (TC_{BB}) may not be necessary. In the appendix we provide Example B.2 to illustrate this.

In this section, we follow Bidian and Bejan (2015) and consider a more general environment by allowing the value $V_{\text{def}}(s^t)$ of the default option at event s^t to be associated with other default punishments.

In the previous chapters, we followed Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009) (see also Martins-da-Rocha and Vailakis (2016) and Martins-da-Rocha and Vailakis (2017)) by assuming that $V_{\text{def}}(s^t) = V(0,0|s^t)$, nevertheless, our analysis does not depend on this specification of the value of the default option. We can also follow Kehoe and Levine (1993) and Alvarez and Jermann (2000) (see also Alvarez and Jermann (2001), Kocherlakota (2008), Bloise, Reichlin and Tirelli (2013) and Martins-da-Rocha and Vailakis (2015)) and consider the case where the default punishment is permanent reversion to autarky, i.e., $V_{\text{def}}(s^t) = U(e|s^t)$.

In our result, the only assumption we impose on the value of the default option is the following recursive property.

Assumption 4.1. *For every event $s^t \in \Sigma$, there exists $\hat{c}(s^t) > 0$ such that*

$$V_{\text{def}}(s^t) \geq u(\hat{c}(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V_{\text{def}}(s^{t+1}).$$

Remark 4.1. The two standard cases ($V_{\text{def}}(s^t) = V(0,0|s^t)$ and $V_{\text{def}}(s^t) = U(e|s^t)$) mentioned above satisfy this assumption.

Kocherlakota (2008) proved that if D is not-too-tight, then for any process M satisfying exact roll-over, the process $D+M$ is also not-too-tight.[1] The main contribution of this thesis is to show that the converse is true.

Theorem 4.1. *If D and $\widetilde{D}$ are two processes of not-too-tight debt limits, then the process $D - \widetilde{D}$ allows for exact roll-over.*

Remark 4.2. Theorem 2.1 is a particular case of the above result since we can choose $\widetilde{D} = 0$ when $V_{\text{def}}(s^t) = V(0,0|s^t)$, as the zero process is not-too-tight by construction.

Before proceeding to the proof, we provide a brief explanation regarding the keystone of our proof, our Uniqueness result, presented in Section A.4.

Under the assumption that

$$p(s^t) = \beta^t \pi(s^t) \frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}$$

[1] The arguments of the proof Lemma 2.4 can be adapted in a straightforward way to provide a proof of this result. Although Kocherlakota (2008) assumed permanent reversion to autarky as the default punishment, the proofs are valid for any default punishment satisfying Assumption 4.1.

for all $s^t \succ s^0$, one can see that Euler equations are sufficient to determine the agent's whole consumption process, that is, her consumption at all events is uniquely determined by $c(s^0)$.

Under full commitment we have that the above assumption is satisfied.[2] Then, taking $(c(\cdot\,|s^0), a(\cdot\,|s^0)) \in d(D, -D(s^0)|s^0)$ and $(\tilde{c}(\cdot\,|s^0), \tilde{a}(\cdot\,|s^0)) \in d(\widetilde{D}, -\widetilde{D}(s^0)|s^0)$, one can easily obtain that the $D - \widetilde{D}$ allows for exact roll-over.[3] However, under limited commitment, debt limits may bind and we may not have the assumption satisfied.

Our main contribution to the literature is to observe that although debt limits may bind, we still have some sort of uniqueness for the consumption process. From our Uniqueness result, we have that if $(c,a) \in d(D, -D(s^0)|s^0)$ and $(\tilde{c}, \tilde{a}) \in d(\widetilde{D}, -\widetilde{D}(s^0)|s^0)$, with D and $\widetilde{D}$ not-too-tight debt limits processes, then $c = \tilde{c}$. We obtain a connection between the real and financial sides of our economy, as all not-too-tight debt limits are associated with the same consumption process.

Proof of Theorem 4.1. Let $M := D - \widetilde{D}$. We show that $M(s^0) = \sum_{s^1 \succ s^0} q(s^1)M(s^1)$. This is sufficient to prove the desired result. Indeed, for any arbitrary event s^t, we can replace the whole tree Σ by the subtree $\Sigma(s^t)$ and obtain that $M(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})M(s^{t+1})$.

Fix some arbitrary event $s^\tau \in \Sigma$ and consider the following optimal plans:

$$(c(\cdot|s^\tau), a(\cdot|s^\tau)) \in d(D, -D(s^\tau)|s^\tau) \quad \text{and} \quad (\tilde{c}(\cdot|s^\tau), \tilde{a}(\cdot|s^\tau)) \in d(\widetilde{D}, -\widetilde{D}(s^\tau)|s^\tau).$$

Following standard arguments, we can show that the plan $(c(\cdot|s^\tau), a(\cdot|s^\tau))$ satisfies the following properties:

(i) consumption is strictly positive, i.e., $c(s^t|s^\tau) > 0$ for every $s^t \succeq s^\tau$;[4]

(ii) the flow budget constraints in $B(D, -D(s^\tau)|s^\tau)$ are satisfied with equality, i.e.,

$$c(s^t|s^\tau) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})a(s^{t+1}|s^\tau) = e(s^t) + a(s^t|s^\tau), \quad \text{for all } s^t \succeq s^\tau;$$

(iii) the Euler equations are satisfied, i.e.,

$$q(s^t) \geq \beta\pi(s^t|\sigma(s^t))\frac{u'(c(s^t|s^\tau))}{u'(c(\sigma(s^t)|s^\tau))}, \quad \text{for all } s^t \succ s^\tau,$$

with an equality when $a(s^t|s^\tau) > -D(s^t)$;

(iv) the individual Transversality condition is satisfied, i.e.,

$$\lim_{t\to\infty} \sum_{s^t \in S^t(s^\tau)} \beta^t \pi(s^t)u'(c(s^t|s^\tau))[a(s^t|s^\tau) + D(s^t)] = 0.$$

[2] Refer to the Appendix C for a discussion about this.

[3] The result follows from consolidating the flow budget constraint for both optimal pairs and subtracting one from the other.

[4] The crucial assumption to show that optimal consumption must be positive is the Inada's condition of the Bernoulli function. However, because of the possibly binding constraints, the arguments are not exactly the same as the standard argument of the full commitment environment. We refer to Appendix A.3 for details.

It is also important to observe that the debt constraint $a(s^t|s^\tau) \geq -D(s^t)$ binds if, and only if, the participation constraint $U(c(\cdot|s^\tau)|s^t) \geq V_{\text{def}}(s^t)$ also binds.

Recall that $\Sigma_{\text{NB}}(s^\tau)$ denotes the set of all event $s^t \in \Sigma(s^\tau)$ such that either $s^t = s^\tau$ or debt limits (satisfied by the optimal plan $(c(\cdot|s^\tau), a(\cdot|s^\tau))$) never bind along the history from s^τ up to s^t, i.e.,

$$\Sigma_{\text{NB}}(s^\tau) := \{s^\tau\} \cup \{s^t \succ s^\tau \ : \ a(\sigma^k(s^t)|s^\tau) > -D(\sigma^k(s^t)), \quad \forall k \in \{0, 1, \ldots, t-\tau-1\}\}.$$

Recall also that $\Sigma_{\text{FB}}(s^\tau)$ denotes the set of all events $s^t \succ s^\tau$ for which the debt constraint (satisfied by the optimal plan $(c(\cdot|s^\tau), a(\cdot|s^\tau))$) binds for the first time since s^τ, i.e.,

$$\Sigma_{\text{FB}}(s^\tau) := \{s^t \succ s^\tau \ : \ a(s^t|s^\tau) = -D(s^t) \quad \text{and} \quad \sigma(s^t) \in \Sigma_{\text{NB}}(s^\tau)\}.$$

For each $\kappa \in \{\text{FB}, \text{NB}\}$ and $t > \tau$, we let

$$\Sigma^t_\kappa(s^\tau) := \Sigma_\kappa(s^\tau) \cap \Sigma^t \quad \text{and} \quad S^t_\kappa(s^\tau) := \Sigma_\kappa(s^\tau) \cap S^t.^{5}$$

For any arbitrary non-binding event $s^t \in \Sigma_{\text{NB}}(s^\tau)$, we let

$$\Sigma_\kappa(s^t|s^\tau) := \Sigma_\kappa(s^\tau) \cap \Sigma(s^t).$$

Consider the optimal plan $(c(\cdot|s^0), a(\cdot|s^0)) \in d(D, -D(s^0)|s^0)$.

Proposition 4.1. *The present value of the consumption plan $c(\cdot|s^0)$ over the subtree $\Sigma_{\text{NB}}(s^0)$ is finite, i.e.,*

$$\sum_{s^t \in \Sigma_{\text{NB}}(s^0)} p(s^t) c(s^t|s^0) < \infty. \tag{4.1}$$

The following series converge in $\mathbb{R}$:

$$\sum_{s^t \in \Sigma_{\text{NB}}(s^0)} p(s^t) e(s^t) < \infty \quad \text{and} \quad \sum_{s^t \in \Sigma_{\text{FB}}(s^0)} p(s^t) D(s^t) < \infty,$$

and the following limit exists in $\mathbb{R}$

$$\text{L}_D(s^0) := \frac{1}{p(s^0)} \lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau_{\text{NB}}(s^0)} p(s^\tau) D(s^\tau). \tag{4.2}$$

Moreover, the flow budget constraints can be consolidated as follows:

$$p(s^0)D(s^0) + \sum_{s^t \in \Sigma_{\text{NB}}(s^0)} p(s^t)[c(s^t|s^0) - e(s^t)] = p(s^0)\text{L}_D(s^0) + \sum_{s^t \in \Sigma_{\text{FB}}(s^0)} p(s^t) D(s^t). \tag{4.3}$$

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[5] Observe that $\Sigma^t_\kappa(s^\tau) = \{s^\tau\} \cup \bigcup_{r=\tau+1}^t S^r_\kappa(s^\tau)$.

Proof of Proposition 4.1. Optimality implies that the flow budget constraint at any $s^t \in \Sigma$ is satisfied with equality. Then, for any $t > 1$, summing the present value of the flow budget constraints over $\Sigma_{\mathrm{NB}}^{t-1}(s^0)$, one obtain[6]

$$\sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^0)} p(s^\tau)c(s^\tau) = -p(s^0)D(s^0) + \sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^0)} p(s^\tau)e(s^\tau) + \sum_{s^\tau \in \Sigma_{\mathrm{FB}}^{t}(s^0)} p(s^\tau)D(s^\tau) - \sum_{s^0 \in S_{\mathrm{NB}}^{0}(s^0)} p(s^t)a(s^t). \tag{4.4}$$

It follows from the necessity of the Euler equations that

$$q(s^\tau) = \beta\pi(s^\tau|\sigma(s^\tau))\frac{u'(c(s^\tau))}{u'(c(\sigma(s^\tau))}, \qquad \text{for all } s^\tau \in \Sigma_{\mathrm{NB}}(s^0).$$

Then, because we sum over a subset of $\Sigma_{\mathrm{NB}}(s^0)$, we have

$$\sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^t)} p(s^\tau)c(s^\tau) = \frac{p(s^0)}{u'(c(s^0))} \sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^0)} \beta^\tau \pi(s^\tau)u'(c(s^\tau))c(s^\tau).$$

Concavity of implies that $u'(c(s^\tau))c(s^\tau) \leq u(c(s^\tau)) - u(0)$. Boundedness of u implies that $U(c(s^\tau))$ and $U(0)$ are both finite. We can use these properties to deduce that

$$\sum_{s^\tau \in \Sigma_{\mathrm{NB}}(s^t)} p(s^\tau)c(s^\tau) < \infty.$$

Since $S_{\mathrm{NB}}^t(s^0) \subseteq S^t(s^0)$, necessity of the Transversality condition implies that

$$\lim_{t\to\infty} \sum_{s^t \in S_{\mathrm{NB}}^t(s^0)} p(s^t)[a(s^t) + D(s^t)] = 0. \tag{4.5}$$

Rearranging terms in (4.4), we have

$$\underbrace{\sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^t)} p(s^\tau)c(s^\tau)}_{=:p(s^0)C_{\mathrm{NB}}^{t-1}(s^0)} + \sum_{s^t \in S_{\mathrm{NB}}^t(s^0)} p(s^t)[a(s^t) + D(s^t)] = -p(s^0)D(s^0) + \underbrace{\sum_{s^\tau \in \Sigma_{\mathrm{NB}}^{t-1}(s^0)} p(s^\tau)e(s^\tau)}_{=:p(s^0)W_{\mathrm{NB}}^{t-1}(s^0)} + \underbrace{\sum_{s^\tau \in \Sigma_{\mathrm{FB}}^{t}(s^t)} p(s^\tau)D(s^\tau)}_{=:p(s^0)D_{\mathrm{FB}}^{t}(s^0)} + \underbrace{\sum_{s^t \in S_{\mathrm{NB}}^t(s^0)} p(s^t)D(s^t)}_{=:p(s^0)\mathrm{L}_D^t(s^0)}. \tag{4.6}$$

We have already proved that the left-side of the above equation has a finite limit, therefore the non-negative sequences $(W_{\mathrm{NB}}^t(s^0))_{t>0}$ and $(D_{\mathrm{FB}}^t(s^0))_{t>0}$ are bounded, as the terms on the right-side are non-negative. Since they are increasing, they must converge to real numbers.

[6] We abuse notation and write $(c(s^t), a(s^t))$ instead of $(c(s^t|s^0), a(s^t|s^0))$.

Moreover, we can conclude from the above that the sequence $(\mathrm{L}_D^t(s^0))_{t>0}$ admits a real limit denoted $\mathrm{L}_D(s^0)$, that is,

$$\mathrm{L}_D(s^0) := \lim_{t\to\infty} \frac{1}{p(s^0)} \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t)D(s^t).$$

Then, taking limits when t tends to infinite in (4.6) implies the desired consolidated flow budget constraint. □

Remark 4.3. Hellwig and Lorenzoni (2009) obtain a similar equation but they implicitly assume that $\mathrm{L}_D(s^0) = 0$ since they claim that

$$\lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t)a(s^t|s^0) = 0.$$

We provide an example in Appendix B to show that the above condition is not necessarily satisfied.

Recall that the consumption process $(c(s^t|s^0))_{s^t\succeq s^0}$ is strictly positive, satisfies participation constraints and the Euler equations. The consumption process $(\tilde{c}(s^t|s^0))_{s^t\succeq s^0}$ also satisfies the same properties. Moreover, we have

$$U(c(\cdot|s^0)|s^0) = V_{\mathrm{def}}(s^0) = U(\tilde{c}(\cdot|s^0)|s^0).$$

We can then apply Proposition A.1 (see Appendix A.4) and conclude that the two consumption processes must coincide: $c(\cdot|s^0) = \tilde{c}(\cdot|s^0)$.[7] Replacing $(c(\cdot|s^0), a(\cdot|s^0), D)$ by $(\tilde{c}(\cdot|s^0), \tilde{a}(\cdot|s^0), \widetilde{D})$ in the definition of $\Sigma_{\mathrm{NB}}(s^0)$ and $\Sigma_{\mathrm{FB}}(s^0)$, we can define the sets $\widetilde{\Sigma}_{\mathrm{NB}}(s^0)$ and $\widetilde{\Sigma}_{\mathrm{FB}}(s^0)$. Even if the pairs $(a(\cdot|s^0), D)$ and $(\tilde{a}(\cdot|s^0), \widetilde{D})$ are different, the debt constraints must bind simultaneously. This is because, for every $s^t \succ s^0$, we have

$$\begin{aligned} a(s^t|s^0) > -D(s^t) &\iff U(c(\cdot|s^0)|s^t) > V_{\mathrm{def}}(s^t) \\ &\iff U(\tilde{c}(\cdot|s^0)|s^t) > V_{\mathrm{def}}(s^t) \\ &\iff \tilde{a}(s^t|s^0) > -\widetilde{D}(s^t). \end{aligned}$$

In particular, we must have

$$\Sigma_{\mathrm{NB}}(s^0) = \widetilde{\Sigma}_{\mathrm{NB}}(s^0) \quad \text{and} \quad \Sigma_{\mathrm{FB}}(s^0) = \widetilde{\Sigma}_{\mathrm{FB}}(s^0). \tag{4.7}$$

We can apply the arguments of the proof of Proposition 4.1 to show that Equation (4.3) is also satisfied when the pair $(c(\cdot|s^0), D)$ is replaced by $(\tilde{c}(\cdot|s^0), \widetilde{D})$, i.e.,

$$p(s^0)\widetilde{D}(s^0) + \sum_{s^t \in \widetilde{\Sigma}_{\mathrm{NB}}(s^0)} p(s^t)[\tilde{c}(s^t|s^0) - e(s^t)] = p(s^0)\mathrm{L}_{\widetilde{D}}(s^0) + \sum_{s^t \in \widetilde{\Sigma}_{\mathrm{FB}}(s^0)} p(s^t)\widetilde{D}(s^t). \tag{4.8}$$

[7] This is the main technical novelty of our proof.

Using (4.7), we can subtract (4.8) from (4.3) and get that

$$p(s^0)M(s^0) = p(s^0)\mathrm{L}_M(s^0) + \sum_{s^t \in \Sigma_{\mathrm{FB}}(s^0)} p(s^t)M(s^t) \tag{4.9}$$

where

$$L_M(s^0) := \frac{1}{p(s^0)} \lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t)M(s^t).$$

We are almost done. Indeed, since

$$S^t_{\mathrm{NB}}(s^0) = \bigcup_{s^1 \in S^1_{\mathrm{NB}}(s^0)} S^t_{\mathrm{NB}}(s^1|s^0)$$

we can deduce that[8]

$$p(s^0)L_M(s^0) = \sum_{s^1 \in S^1_{\mathrm{NB}}(s^0)} p(s^1)L_M(s^1|s^0)$$

where

$$L_M(s^1|s^0) := \frac{1}{p(s^1)} \lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^1|s^0)} p(s^t)M(s^t).$$

Moreover, since

$$\Sigma_{\mathrm{FB}}(s^0) = S^1_{\mathrm{FB}}(s^0) \cup \bigcup_{s^1 \in S^1_{\mathrm{NB}}(s^0)} \Sigma_{\mathrm{FB}}(s^1|s^0),$$

we have that

$$\sum_{s^t \in \Sigma_{\mathrm{FB}}(s^0)} p(s^t)M(s^t) = \sum_{s^1 \in S^1_{\mathrm{FB}}(s^0)} p(s^1)M(s^1) + \sum_{s^1 \in S^1_{\mathrm{NB}}(s^0)} \sum_{s^t \in \Sigma_{\mathrm{FB}}(s^1|s^0)} p(s^t)M(s^t).$$

Combining the above decompositions, we get that

$$\begin{aligned} p(s^0)M(s^0) = {} & \sum_{s^1 \in S^1_{\mathrm{FB}}(s^0)} p(s^1)M(s^1) \\ & + \sum_{s^1 \in S^1_{\mathrm{NB}}(s^0)} \left[p(s^1)L_M(s^1|s^0) + \sum_{s^t \in \Sigma_{\mathrm{FB}}(s^1|s^0)} p(s^t)M(s^t) \right]. \end{aligned} \tag{4.10}$$

To get the desired result, it is sufficient to show that, for every non-binding event $s^1 \in S^1_{\mathrm{NB}}(s^0)$, the term inside the brackets coincides with $p(s^1)M(s^1)$. To prove this, we fix an arbitrary event $s^1 \in S^1_{\mathrm{NB}}(s^0)$ and consider the following optimal plan:

$$(c(\cdot|s^1), a(\cdot|s^1)) \in d(D, -D(s^1)|s^1).$$

By the Principle of Optimality, the plan $(c(\cdot|s^0), a(\cdot|s^0))$ is also optimal in the budget set $B(D, a(s^1|s^0)|s^1)$. Since s^1 is a non-binding event of the plan $(c(\cdot|s^0), a(\cdot|s^0))$, we have $a(s^1|s^0) >$

[8] Provided we show that $L_M(s^1)$ is well-defined.

$-D(s^1) = a(s^1|s^1)$. Monotonicity of the optimal bond and consumption processes in initial wealth then implies that[9]

$$(c(s^t|s^1), a(s^t|s^1)) \le (c(s^t|s^0), a(s^t|s^0)), \quad \text{for all } s^t \succeq s^1. \tag{4.11}$$

This implies that

$$\sum_{s^t \in \Sigma_{\text{NB}}(s^1|s^0)} p(s^t)c(s^t|s^1) \le \sum_{s^t \in \Sigma_{\text{NB}}(s^1|s^0)} p(s^t)c(s^t|s^0) \le \sum_{s^t \in \Sigma_{\text{NB}}(s^0)} p(s^t)c(s^t|s^0).$$

Combining the above inequality together with (4.1), we deduce that the present value of the consumption process $c(\cdot|s^1)$ over the subtree $\Sigma_{\text{NB}}(s^1|s^0)$ is finite,

$$\sum_{s^t \in \Sigma_{\text{NB}}(s^1|s^0)} p(s^t)c(s^t|s^1) < \infty.\text{[10]} \tag{4.12}$$

The monotonicity property (4.11) implies that

$$\begin{aligned}\sum_{s^t \in S^t_{\text{NB}}(s^1|s^0)} p(s^t)[a(s^t|s^1) + D(s^t)] \quad &\le \sum_{s^t \in S^t_{\text{NB}}(s^1|s^0)} p(s^t)[a(s^t|s^0) + D(s^t)] \\ &\le \sum_{s^t \in S^t_{\text{NB}}(s^0)} p(s^t)[a(s^t|s^0) + D(s^t)].\end{aligned}$$

Combining the above inequality together with (4.5), we deduce that

$$\lim_{t\to\infty} \sum_{s^t \in S^t_{\text{NB}}(s^1|s^0)} p(s^t)[a(s^t|s^1) + D(s^t)] = 0. \tag{4.13}$$

The monotonicity property (4.11) also implies that

$$\forall s^t \succ s^1, \quad s^t \in \Sigma_{\text{FB}}(s^1|s^0) \Longrightarrow a(s^t|s^1) = -D(s^t).$$

Using the above property, we can consolidate the (binding) flow budget constraints satisfied by $(c(\cdot|s^1), a(\cdot|s^1))$ and use (4.12)–(4.13) to get that

$$\mathrm{L}_D(s^1|s^0) := \frac{1}{p(s^1)} \lim_{t\to\infty} \sum_{s^t \in S^t_{\text{NB}}(s^1|s^0)} p(s^t)D(s^t)$$

is well-defined and

$$p(s^1)D(s^1) + \sum_{s^t \in \Sigma_{\text{NB}}(s^1|s^0)} p(s^t)[c(s^t|s^1) - y(s^t)] = p(s^1)\,\mathrm{L}_D(s^1|s^0) + \sum_{s^t \in \Sigma_{\text{FB}}(s^1|s^0)} p(s^t)D(s^t). \tag{4.14}$$

Applying again Proposition A.1 (see Appendix A.4), we get that the two consumption processes $c(\cdot|s^1)$ and $\tilde{c}(\cdot|s^1)$ coincide. Since we have

$$\Sigma_{\text{NB}}(s^1|s^0) = \widetilde{\Sigma}_{\text{NB}}(s^1|s^0) \quad \text{and} \quad \Sigma_{\text{FB}}(s^1|s^0) = \widetilde{\Sigma}_{\text{FB}}(s^1|s^0),$$

---

9 See Lemma 2 in Bidian and Bejan (2015) for details.

10 Condition (4.12) does not follow from the same arguments as those used to prove to Condition (4.1). Indeed, since we only know that $a(s^t|s^1) \le a(s^t|s^0)$ for every $s^t \succeq s^1$, we do not necessarily have $\Sigma_{\text{NB}}(s^1|s^0) \subseteq \Sigma_{\text{NB}}(s^1)$.

we can apply similar arguments as above to show that

$$\mathrm{L}_{\widetilde{D}}(s^1|s^0) := \frac{1}{p(s^1)} \lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^1|s^0)} p(s^t)\widetilde{D}(s^t)$$

is well-defined and that the consumption process $\tilde{c}(\cdot|s^1)$ satisfies

$$p(s^1)\widetilde{D}(s^1) + \sum_{s^t \in \Sigma_{\mathrm{NB}}(s^1|s^0)} p(s^t)[\tilde{c}(s^t|s^1) - y(s^t)] = p(s^1)\,\mathrm{L}_{\widetilde{D}}(s^1|s^0) + \sum_{s^t \in \Sigma_{\mathrm{FB}}(s^1|s^0)} p(s^t)\widetilde{D}(s^t). \tag{4.15}$$

Subtracting Equation (4.15) from Equation (4.14), we get that

$$p(s^1)M(s^1) = p(s^1)\,\mathrm{L}_M(s^1|s^0) + \sum_{s^t \in \Sigma_{\mathrm{FB}}(s^1|s^0)} p(s^t)M(s^t), \quad \text{for all } s^1 \in S^1_{\mathrm{NB}}(s^0) \tag{4.16}$$

where

$$L_M(s^1|s^0) := \frac{1}{p(s^1)} \lim_{t\to\infty} \sum_{s^t \in S^t_{\mathrm{NB}}(s^1|s^0)} p(s^t)M(s^t).$$

Combining (4.16) together with (4.10), we deduce the desired result. □

5 CONCLUSION

Under the limited commitment hypothesis, debt limits must be imposed not only to preclude Ponzi schemes but also to generate incentives such that a debtor is willing to meet her commitments. Following Bulow and Rogoff (1989), we analyze the properties satisfied by the loosest debt limits (not-too-tight) that are self-enforcing when a defaulting agent is excluded from credit markets but keeps the ability to save. It is well established in the literature (see for instance Kocherlakota (2008)) that if debt limits form a bubble, then they are not-too-tight. Recently Hellwig and Lorenzoni (2009) and Bidian and Bejan (2015) identified conditions on the debt limits such that the converse is true. We contribute to the literature by showing that the aforementioned conditions need to be satisfied by all not-too-tight debt limits. More importantly, we prove that there is no need to impose any *ad-hoc* condition on endogenous variables, that is, debt limits are self-enforcing and not-too-tight if, and only if, they form a bubble. For this we prove an uniqueness relation between the real side of the economy (the consumption process) and the financial one (the not-too-tight debt limits), which is the major departure from our approach to the existing literature.

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APPENDIX A – Technical results

In this appendix we provide the proofs that have been omitted in the main text and we present some technical results.

A.1 Proof of Lemma 2.1

Our arguments follow those in the proof of Proposition 3.9 in Dal Forno and Montrucchio (2003). Fix an arbitrary budget feasible plan $(\hat{c},\hat{a}) \in B(D,b|s^0)$. We would like to show that

$$U(\hat{c}|s^0) - U(c|s^0) \leq 0.$$

Fix $T > 0$. Using the utility function, define the difference truncation as

$$\Delta U^T := U^T(\hat{c}|s^0) - U^T(c|s^0) = \sum_{t=0}^{T-1} \sum_{s^t \in S^t} \beta^t \pi(s^t)\big(u(\hat{c}(s^t)) - u(c(s^t))\big).$$

Concavity of u implies that

$$\Delta U^T \leq \sum_{t=0}^{T-1} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t))\big(\hat{c}(s^t) - c(s^t)\big).$$

Because both allocations are budget feasible and (c,a) is assumed to satisfy the flow budget constraint with equality, we have

$$\hat{c}(s^t) - c(s^t) \leq \hat{a}(s^t) - a(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1})\big(\hat{a}(s^{t+1}) - a(s^{t+1})\big).$$

Substituting this in the inequality above, we get that

$$\Delta U^T \leq \sum_{t=0}^{T-1} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t))\big(\hat{a}(s^t) - a(s^t)\big) \\ - \sum_{t=1}^{T} \sum_{s^t \in S^t} \beta^{t-1} \pi(\sigma(s^t)) u'(c(\sigma(s^t))) q(s^t)\big(\hat{a}(s^t) - a(s^t)\big).$$

Using the Euler equations at event s^t we know that

$$\left[q(s^t) - \beta \pi(s^t|\sigma(s^t)) \frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}\right][a(s^t) + D(s^t)] = 0,$$

therefore, either $a(s^t) > D(s^t)$, and

$$\left[q(s^t) - \beta \pi(s^t|\sigma(s^t)) \frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}\right][\hat{a}(s^t) - a(s^t)] = 0,$$

or $a(s^t) = D(s^t)$, implying that $\hat{a}(s^t) \geq a(s^t)$, and

$$\left[q(s^t) - \beta\pi(s^t|\sigma(s^t))\frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}\right][\hat{a}(s^t) - a(s^t)] \geq 0.$$

Then, we obtain[1]

$$\begin{aligned}\Delta U^T \leq \sum_{t=0}^{T-1}\sum_{s^t\in S^t}\beta^t\pi(s^t)u'(c(s^t))[\hat{a}(s^t)-a(s^t)] - \sum_{t=1}^{T}\sum_{s^t\in S^t}\beta^t\pi(s^t)u'(c(s^t))[\hat{a}(s^t)-a(s^t)]\\ \leq \sum_{s^T\in S^T}\beta^T\pi(s^T)u'(c(s^T))[a(s^T)+D(s^T)].\end{aligned}$$

Passing to the limit in the above inequality and using boundedness of u, we get that

$$U(\hat{c}|s^0) - U(c|s^0) = \lim_{T\to\infty}\Delta U^T \leq \liminf_{T\to\infty}\sum_{s^T\in S^T}\beta^T\pi(s^T)u'(c(s^T))[a(s^T)+D(s^T)].$$

To get the desired result it is sufficient to assume TC.[2]

A.2 Proof of Lemma 2.2

Since the Bernoulli function is assumed to be strictly positive, the flow budget constraints must be satisfied with equality.

We now show that (c,a) satisfies the Euler equations. Fix $s^t \succeq s^0$ and $s^{t+1} \succ s^t$. Because $c(s^t) > 0$, for any $\varepsilon \in (0, c(s^t))$ we can define the pair $(\hat{c},\hat{a})$ as follows:

- $\hat{c}(s^t) := c(s^t) - \varepsilon > 0$ and $\hat{a}(s^t) := a(s^t)$,
- $\hat{c}(s^{t+1}) := c(s^{t+1}) + \frac{\varepsilon}{q(s^{t+1})}$ and $\hat{a}(s^{t+1}) := a(s^{t+1}) + \frac{\varepsilon}{q(s^{t+1})}$,
- $\hat{c}(s^\tau) := c(s^\tau)$ and $\hat{a}(s^\tau) := a(s^\tau)$ for any $s^\tau \notin \{s^t, s^{t+1}\}$.

Observe that $(\hat{c},\hat{a})$ belongs to the budget set $B(D,b|s^0)$. We then get that $U(c|s^0) - U(\hat{c}|s^0) \geq 0$, which implies that

$$u(c(s^t)) - u(c(s^t)-\varepsilon) \geq \beta\pi(s^{t+1}|s^t)\left(u\left(c(s^{t+1}) + \frac{\varepsilon}{q(s^{t+1})}\right) - u(c(s^{t+1}))\right).$$

Dividing both sides by ε and taking the limit when ε tends to zero, we obtain

$$u'(c(s^t)) \geq \beta\pi(s^{t+1}|s^t)\frac{u'(c(s^{t+1}))}{q(s^{t+1})},$$

---

1 Recall that $\hat{a}(s^0) = a(s^0)$.

2 Observe that assuming it satisfies the Tranversality condition is stronger than needed. It suffices to assume that

$$\liminf_{T\to\infty}\sum_{s^T\in S^T}\beta^T\pi(s^T)u'(c(s^T))\big(a(s^T)+D(s^T)\big) = 0,$$

however, it turns out to be a necessary condition.

which is the first part of the Euler equation.

Assume now that $a(s^{t+1}) > -D(s^{t+1})$. Since $c(s^{t+1}) > 0$, for any $\varepsilon \in (0, c(s^{t+1}))$ we can define the pair $(\hat{c}, \hat{a})$ as follows:

- $\hat{c}(s^{t+1}) := c(s^{t+1}) - \varepsilon$ and $\hat{a}(s^{t+1}) := a(s^{t+1}) - \varepsilon$,
- $\hat{c}(s^t) := c(s^t) + q(s^{t+1})\varepsilon$ and $\hat{a}(s^t) := a(s^t)$,
- $\hat{c}(s^\tau) := c(s^\tau)$ and $\hat{a}(s^\tau) := c(s^\tau)$ for any $s^\tau \notin \{s^t, s^{t+1}\}$.

Observe that $(\hat{c}, \hat{a})$ belongs to the budget set $B(D, b|s^0)$. We get that $U(c|s^0) - U(\hat{c}|s^0) \geq 0$. Following a limiting argument similar to the previous one, we get the reverse inequality:

$$u'(c(s^t)) \leq \beta\pi(s^{t+1}|s^t)\frac{u'(c(s^{t+1}))}{q(s^{t+1})}.$$

To conclude the lemma we still have to show that (c, a) satisfies the Transversality condition. Let $\varepsilon \in (0, 1)$ and fix $t > 0$. Consider the allocation $(\hat{c}, \hat{a})$ defined as follows:

- for every $\tau < t$ and every $s^\tau \in S^\tau$,
$$\hat{c}(s^\tau) := c(s^\tau) \quad \text{and} \quad \hat{a}(s^\tau) := a(s^\tau);$$
- for every $s^t \in S^t$,
$$\hat{c}(s^t) := c(s^t) + \varepsilon \sum_{s^{t+1} \succ s^t} q(s^{t+1})[a(s^{t+1}) + D(s^{t+1})] \quad \text{and} \quad \hat{a}(s^t) := a(s^t);$$
- for every $\tau > t$ and every $s^\tau \in S^\tau$,
$$\hat{c}(s^\tau) := (1-\varepsilon)c(s^\tau) \quad \text{and} \quad \hat{a}(s^\tau) := (1-\varepsilon)a(s^\tau) + \varepsilon(-D(s^\tau)).$$

The process of bond holdings satisfies the debt constraints since

$$(1-\varepsilon)a(s^\tau) + \varepsilon(-D(s^\tau)) \geq -D(s^\tau).$$

For any event s^τ with $\tau < t$, the flow budget constraints are unchanged. At any event $s^t \in S^t$, the new bond holding $\hat{a}(s^{t+1})$ contingent to every successor event s^{t+1} involves more debt since

$$\hat{a}(s^{t+1}) = a(s^t) - \varepsilon[a(s^t) + D(s^t)].$$

This new issuance finances the increase of consumption $\hat{c}(s^t) - c(s^t)$. For every event s^τ with $\tau > t$, we have

$$\begin{aligned}
&\hat{c}(s^\tau) + \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})\hat{a}(s^{\tau+1}) \\
&\qquad = (1-\varepsilon)c(s^\tau) + (1-\varepsilon)\sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})a(s^{\tau+1}) - \varepsilon \sum_{s^{\tau+1} \succ s^\tau} q(s^{\tau+1})D(s^{\tau+1}) \\
&\qquad\qquad \leq e(s^\tau) + (1-\varepsilon)a(s^\tau) - \varepsilon D(s^\tau) = e(s^\tau) + \hat{a}(s^\tau).
\end{aligned}$$

The above inequality is true because D is a process of consistent debt limits.

We have proved that $(\hat{c},\hat{a})$ belongs to the budget set $B(D,b|s^0)$. Using the optimality of (c,a), we have

$$0 \geq U(\hat{c}|s^0) - U(c|s^0) = \sum_{s^t \in S^t} \beta^t \pi(s^t)\big(u(\hat{c}(s^t)) - u(c(s^t))\big) + \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \beta^\tau \pi(s^\tau)\big(u(\hat{c}(s^\tau)) - u(c(s^\tau))\big).$$

Concavity of u implies that

$$0 \geq \sum_{s^t \in S^t} \beta^t \pi(s^t)\big(u(\hat{c}(s^t)) - u(c(s^t))\big) + \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \beta^\tau \pi(s^\tau)\big([(1-\varepsilon)u(c(s^\tau)) + \varepsilon u(0)] - u(c(s^\tau))\big).$$

Rearranging terms, and using the utility function, we obtain

$$\sum_{s^{t+1} \in S^{t+1}} \beta^{t+1} \pi(s^{t+1})[U(c|s^{t+1}) - U(0|s^{t+1})] \geq \sum_{s^t \in S^t} \beta^t \pi(s^t) \frac{u\Big(c(s^t) + \varepsilon \sum_{s^{t+1} \succ s^t} q(s^{t+1})[a(s^{t+1}) + D(s^{t+1})]\Big) - u(c(s^t))}{\varepsilon},$$

which is true for all $\varepsilon \in (0,1)$. Passing to the limit when ε tends to zero

$$\sum_{s^{t+1} \in S^{t+1}} \beta^{t+1} \pi(s^{t+1})[U(c|s^{t+1}) - U(0|s^{t+1})] \geq \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t)) \sum_{s^{t+1} \succ s^t} q(s^{t+1})[a(s^{t+1}) + D(s^{t+1})].$$

Now passing to the limit when t tends to infinity, boundedness of u implies that

$$0 \geq \limsup_{t \to \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t)) \sum_{s^{t+1} \succ s^t} q(s^{t+1})[a(s^{t+1}) + D(s^{t+1})].$$

Using the Euler equations [3]

$$0 \geq \limsup_{t \to \infty} \sum_{s^{t+1} \in S^{t+1}} \beta^{t+1} \pi(s^{t+1}) u'(s^{t+1})[a(s^{t+1}) + D(s^{t+1})].$$

The necessity of the Transversality condition follows from the above inequality.

[3] Recall that

$$q(s^{t+1})[a(s^{t+1}) + D(s^{t+1})] = \beta \pi(s^{t+1}|\sigma(s^{t+1})) \frac{u'(c(s^{t+1}))}{u'(c(\sigma(s^{t+1})))}[a(s^{t+1}) + D(s^{t+1})].$$

A.3 Strictly positive consumption

Lemma A.1. *Fix a process D of not-too-tight debt limits. Let $(c(\cdot|s^\tau), a(\cdot|s^\tau))$ be the optimal plan in $B(D, -D(s^\tau)|s^\tau)$ at some event s^τ. For every successor event $s^t \succeq s^\tau$ we have $c(s^t|s^\tau) > 0$.*

Proof. We only prove the result for $s^\tau = s^0$. The general case can be obtained by replacing the whole tree Σ by the subtree $\Sigma(s^\tau)$.

Denote by (c, a) the optimal plan in $B(D, -D(s^0)|s^0)$.[4] Because D is not-too-tight, the consumption process c satisfies

$$\forall s^t \succeq s^0, \quad U(c|s^t) \geq V_{\text{def}}(s^t).$$

Claim A.1. We must have $c(s^0) > 0$.

Proof. Assume by way of contradiction that $c(s^0) = 0$. We let $\Delta(s^t) := U(c|s^t) - V_{\text{def}}(s^t)$ for any event $s^t \succeq s^0$. Observe that

$$\Delta(s^0) \leq \left[u(0) - u(\hat{c}(s^0))\right] + \beta \sum_{s^1 \in S^1} \pi(s^1|s^0)[U(c|s^1) - V_{\text{def}}(s^1)].^5$$

Then, we get that

$$\Delta(s^0) \leq \left[u(0) - u(\hat{c}(s^0))\right] + \beta \sum_{s^1 \in S^1_{\text{NB}}(s^0)} \pi(s^1|s^0)\Delta(s^1) \tag{A.1}$$

where $S^1_{\text{NB}}(s^0) = \Sigma_{\text{NB}}(s^0) \cap S^1$ is the set of date-1 events s^1 where the participation constraint for the optimal plan (c, a) is not binding, i.e., $U(c|s^1) > V_{\text{def}}(s^1)$.

Fix any $s^1 \in S^1_{\text{NB}}(s^0)$. Applying the same arguments as above, we have

$$\Delta(s^1) \leq \left[u(c(s^1)) - u(\hat{c}(s^1))\right] + \beta \sum_{s^2 \in S^2_{\text{NB}}(s^1|s^0)} \pi(s^2|s^1)\Delta(s^2) \tag{A.2}$$

where $S^2_{\text{NB}}(s^1|s^0) := \Sigma_{\text{NB}}(s^1|s^0) \cap S^2$ is the set of date-2 events s^2 successors of s^1 for which the participation constraint for the optimal plan (c, a) is not binding, i.e., $U(c|s^2) > V_{\text{def}}(s^2)$. Since

$$S^2_{\text{NB}}(s^0) = \bigcup_{s^1 \in S^1_{\text{NB}}(s^0)} S^2_{\text{NB}}(s^1|s^0),$$

we can combine (A.1) and (A.2) to get that

$$\Delta(s^0) \leq \left[u(0) - u(\hat{c}(s^0))\right] + \beta \sum_{s^1 \in S^1_{\text{NB}}(s^0)} \pi(s^1|s^0) \left[u(c(s^1)) - u(\hat{c}(s^1))\right] + \beta^2 \sum_{s^2 \in S^2_{\text{NB}}(s^0)} \pi(s^2|s^0)\Delta(s^2).$$

[4] We abuse notation and write $(c(s^t), a(s^t))$ instead of $(c(s^t|s^0), a(s^t|s^0))$.

[5] Recall that $\hat{c}(s^0) > 0$ is such that $V_{\text{def}}(s^0) \geq u(\hat{c}(s^0)) + \beta \sum_{s^1 \succ s^0} \pi(s^1|s^0) V_{\text{def}}(s^1)$.

Repeating the above argument, we can prove that for every $t \geq 1$

$$\begin{aligned}
\Delta(s^0) \quad &\leq \quad \left[u(0) - u(\hat{c}(s^0))\right] \\
&+ \quad \beta \sum_{s^1 \in S^1_{\mathrm{NB}}(s^0)} \pi(s^1|s^0)\left[u(c(s^1)) - u(\hat{c}(s^1))\right] \\
&\vdots \\
&+ \quad \beta^t \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} \pi(s^t|s^0)\left[u(c(s^t)) - u(\hat{c}(s^t))\right] \\
&+ \quad \beta^{t+1} \sum_{s^{t+1} \in S^{t+1}_{\mathrm{NB}}(s^0)} \pi(s^{t+1}|s^0)\Delta(s^{t+1}).
\end{aligned}$$

Since the Bernoulli function u is assumed to be bounded, we have

$$\lim_{t\to\infty} \beta^{t+1} \sum_{s^{t+1} \in S^{t+1}_{\mathrm{NB}}(s^0)} \pi(s^{t+1}|s^0)\Delta(s^{t+1}) = 0.$$

Then, we get that

$$\begin{aligned}
\Delta(s^0) \leq \left[u(0) - u(\hat{c}(s^0))\right] + \beta \sum_{s^1 \in S^1_{\mathrm{NB}}(s^0)} \pi(s^1|s^0)\left[u(c(s^1)) - u(\hat{c}(s^1))\right] \\
+ \sum_{t\geq 2} \beta^t \sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} \pi(s^t|s^0)\left[u(c(s^t)) - u(\hat{c}(s^t))\right].
\end{aligned}$$

Since $\Delta(s^0) \geq 0$, there must exist some date $t \geq 1$ and some date-t event $s^t \in S^t_{\mathrm{NB}}(s^0)$ such that $c(s^t) > 0$. We propose to replace $(c(s^t), a(s^t))$ by

$$(\tilde{c}(s^t), \tilde{a}(s^t)) := (c(s^t) - \varepsilon, a(s^t) - \varepsilon)$$

for some $\varepsilon > 0$ small enough (the way we choose ε is explained below). At the predecessor event $s^{t-1} = \sigma(s^t)$, we replace $a(s^{t-1})$ by $\tilde{a}(s^{t-1}) := a(s^{t-1}) - q(s^t)\varepsilon$ and we let the consumption unchanged, i.e., $\tilde{c}(s^{t-1}) := c(s^{t-1})$. For any other predecessor event s^r satisfying $s^{t-1} \succ s^r \succ s^1$ we pose

$$\tilde{a}(s^r) := a(s^r) - q(s^{r+1}) \ldots q(s^t)\varepsilon$$

and $\tilde{c}(s^r) := c(s^r)$. For the initial event event s^0, the consumption $c(s^0)$ (which is assumed to be zero) is replaced by

$$\tilde{c}(s^0) := c(s^0) + q(s^1) \ldots q(s^t)\varepsilon.$$

Since $s^t \in S^t_{\mathrm{NB}}(s^0)$, for every event s^r satisfying $s^t \succeq s^r \succ s^0$, we have $a(s^r) > -D(s^r)$.[6] This implies that we can choose $\varepsilon > 0$ small enough such that $\tilde{a}(s^r) \geq -D(s^r)$ for any event s^r satisfying $s^t \succeq s^r \succ s^0$. In particular, the new plan $(\tilde{c}, \tilde{a})$ belongs to the budget set $B(D, -D(s^0)|s^0)$.[7] Recall that $c(s^0) = 0$. Since u satisfies Inada's property at the origin, we can choose ε small enough such the marginal gain at event s^0 compensates the marginal loss at event s^t. We obtain a contradiction, that is, $U(\tilde{c}|s^0) > U(c|s^0)$. □

6 By the Principle of Optimality, for any event $s^\tau \succ s^0$, the plan (c, a) is optimal in the budget set $B(D, a(s^\tau)|s^\tau)$. Since the process D is not-too-tight, it then follows that $U(c|s^\tau) > V_{\mathrm{def}}(s^\tau)$ if, and only if, $a(s^\tau) > -D(s^\tau)$.

7 For events s^r that do not satisfy $s^t \succeq s^r \succ s^0$, we pose $(\tilde{c}(s^r), \tilde{a}(s^r)) := (c(s^r), a(s^r))$.

We have proved that $c(s^0) > 0$. Fix an arbitrary event $s^t \succ s^0$. By the Principle of Optimality, the plan (c,a) is optimal in the budget set $B(D,a(s^t)|s^t)$. Since $a(s^t) \geq -D(s^t)$, monotonicity of the optimal consumption process in initial wealth implies that $c(s^t) \geq c(s^t|s^t)$ where $(c(\cdot|s^t),a(\cdot|s^t))$ is the optimal plan in $B(D,-D(s^t)|s^t)$. Applying the previous claim to the subtree $\Sigma(s^t)$, we can conclude that $c(s^t|s^t) > 0$, which implies the desired result: $c(s^t) > 0$. □

A.4 Uniqueness result

Consider the general default option $V_{\text{def}}(s^t)$ of Chapter 4. A strictly positive consumption process $c = (c(s^t))_{s^t \in \Sigma}$ is said to satisfy

(i) the participation constraints when $U(c|s^t) \geq V_{\text{def}}(s^t)$, for each $s^t \succeq s^0$;

(ii) the Euler equations when, for each $s^t \succ s^0$, we have

$$q(s^t) \geq \beta\pi(s^t|\sigma(s^t))\frac{u'(c(s^t))}{u'(c(\sigma(s^t)))}$$

with an equality when $U(c|s^t) > V_{\text{def}}(s^t)$.

Proposition A.1. *If c and $\tilde{c}$ are two strictly positive consumption processes satisfying the participation constraints, the Euler equations and such that $U(c|s^0) = U(\tilde{c}|s^0)$, then we must have $c = \tilde{c}$.*

Proof. Let c and $\tilde{c}$ be two strictly positive consumption processes satisfying the participation constraints, the Euler equations and such that $U(c|s^0) = U(\tilde{c}|s^0)$. Without any loss of generality, we can assume that $\tilde{c}(s^0) \geq c(s^0)$.

Fix some arbitrary event s^τ. Denote by $\mathscr{S}_{\text{NB}}(s^\tau) \subseteq \Sigma(s^\tau)$ the set of all event $s^t \succeq s^\tau$ such that either $s^t = s^\tau$ or participation constraints for the consumption plan c never bind along the history of events from s^τ up to s^t, i.e.,

$$\mathscr{S}_{\text{NB}}(s^\tau) := \{s^\tau\} \cup \{s^t \succ s^\tau \;:\; U(c|\sigma^k(s^t)) > V_{\text{def}}(\sigma^k(s^t)), \quad \forall k \in \{0,1,\ldots,t-\tau-1\}\}.$$

We denote by $\mathscr{S}_{\text{FB}}(s^\tau) \subseteq \Sigma(s^\tau)$ the set of all events $s^t \succ s^\tau$ for which the participation constraints for the consumption plan c binds for the first time since s^τ, i.e.,

$$\mathscr{S}_{\text{FB}}(s^\tau) := \{s^t \succ s^\tau \;:\; U(c|s^t) = V_{\text{def}}(s^t) \quad \text{and} \quad \sigma(s^t) \in \mathscr{S}_{\text{NB}}(s^\tau)\}.$$

Since the consumption plan $\tilde{c}$ satisfies the participation constraints, we have

$$U(\tilde{c}|s^0) - U(c|s^0) = \underbrace{\sum_{s^t \in \mathscr{S}_{\text{FB}}(s^0)} \beta\pi(s^t)\left[U(\tilde{c}|s^t) - U(c|s^t)\right]}_{\geq 0} + \sum_{s^t \in \mathscr{S}_{\text{NB}}(s^0)} \beta^t\pi(s^t)\left[u(\tilde{c}(s^t)) - u(c(s^t))\right]. \quad \text{(A.3)}$$

For every $s^t \in \mathscr{S}_{\text{NB}}(s^0)$, we have

$$\beta^t \pi(s^t)\frac{u'(c(s^t))}{u'(c(s^0))} = p(s^t) \geq \beta^t \pi(s^t)\frac{u'(\tilde{c}(s^t))}{u'(\tilde{c}(s^0))} \geq \beta^t \pi(s^t)\frac{u'(\tilde{c}(s^t))}{u'(c(s^0))},$$

where the first equality follows from the Euler equations satisfied with equality by the consumption plan c, the second inequality follows from the Euler equations satisfied by the consumption plan $\tilde{c}$, and the last inequality follows from the assumption that $\tilde{c}(s^0) \geq c(s^0)$ and the strict concavity of u. We can then deduce that, for each $s^t \in \mathscr{S}_{\text{NB}}(s^0)$, we have $u'(c(s^t)) \geq u'(\tilde{c}(s^t))$ which implies that $\tilde{c}(s^t) \geq c(s^t)$. It then follows from (A.3) that

$$U(\tilde{c}|s^\tau) = U(c|s^\tau), \quad \text{for all } s^\tau \in \mathscr{S}_{\text{FB}}(s^0)$$

and

$$\tilde{c}(s^t) = c(s^t), \quad \text{for all } s^t \in \mathscr{S}_{\text{NB}}(s^0).$$

Fix an arbitrary event $s^\tau \in \mathscr{S}_{\text{FB}}(s^0)$. Since $U(\tilde{c}|s^\tau) = U(c|s^\tau)$, we can apply the above argument to the subtree $\Sigma(s^\tau)$ and deduce that

$$U(\tilde{c}|s^t) = U(c|s^t), \quad \text{for all } s^t \in \mathscr{S}_{\text{FB}}(s^\tau)$$

and

$$\tilde{c}(s^t) = c(s^t), \quad \text{for all } s^t \in \mathscr{S}_{\text{NB}}(s^\tau).$$

After successive iterations of the above arguments, we deduce that $c(s^t) = \tilde{c}(s^t)$ for every event $s^t \in \Sigma$. $\square$

APPENDIX B – Examples

In this appendix we provide the examples that have been mentioned in main text concerning the *ad-hoc* assumptions imposed by the existing literature on debt limits.

B.1 Example B.1

In the following we provide an example such that the Transversality condition (TC_{HL}) is not satisfied.

Example B.1. *Consider a deterministic environment and let $e = (e_t)_{t\geq 0}$ be the agent's endowment sequence defined at the initial date as*

$$e_0 := \bar{e} + \varepsilon,$$

and at all successor dates $t \geq 1$

$$e_t := \bar{e} - (1-\beta)\varepsilon,$$

where $\bar{e}$ is arbitrary and $\varepsilon > 0$ is chosen small enough such that $\bar{e} > (1-\beta)\varepsilon$. The sequence $(q_t)_{t\geq 1}$ of bond prices is defined by $q_t := \beta$ for each $t > 1$.

Proposition B.1. *For the economy described in Example B.1, there exists a sequence $(D_t)_{t\geq 0}$ of not-too-tight debt limits that does not satisfy (TC_{HL}).*

Proof. Let the debt limits sequence $D = (D_t)_{t\geq 0}$ be defined by

$$D_t := \frac{2\varepsilon}{\beta^{t-1}}, \quad \text{for all } t \geq 0.$$

Observe D satisfies exact roll-over by construction, therefore, it is not-too-tight. Let $(c_t)_{t\geq 0}$ be the consumption sequence defined by

$$c_t := \bar{e} + \varepsilon(1-\beta)^2, \quad \text{for all } t \geq 0,$$

and let $(a_t)_{t\geq 0}$ be the bond holding sequence defined by

$$a_0 := -2\beta\varepsilon \quad \text{and} \quad a_t := \frac{-2\varepsilon}{\beta^{t-1}} + 2\varepsilon - \beta\varepsilon, \quad \forall t \geq 1.$$

We claim that $(c,a) \in d(D, -D_0|0)$. To show that (c,a) belongs to the budget set $B(D, -D_0|0)$, we first observe that $a_0 = -D_0$, and because $2\varepsilon - \beta\varepsilon > 0$, it follows that $a_t > -D_t$, for all $t > 0$. Moreover,

$$c_0 + \beta a_1 = \bar{e} + \varepsilon - 2\beta\varepsilon + \beta^2\varepsilon + \beta(-\beta\varepsilon) = \bar{e} + \varepsilon - 2\beta\varepsilon = e_0 + a_0,$$

and

$$c_t + \beta a_{t+1} = \bar{e} + \varepsilon - 2\beta\varepsilon + \beta^2\varepsilon + \beta\left(\frac{-2\varepsilon}{\beta^t} + 2\varepsilon - \beta\varepsilon\right) =$$

$$\bar{e} - (1-\beta)\varepsilon + \left(\frac{-2\varepsilon}{\beta^{t-1}} + 2\varepsilon - \beta\varepsilon\right) = e_t + a_t,$$

for all $t \geq 1$. One can see that the flow budget constraint are satisfied with equality. Because consumption is constant and the debt limits never bind after time 0, the Euler equation is satisfied. To show that Transversality condition is satisfied,

$$\lim_{T\to\infty} \beta^T u'(c_T)[D_T + a_T] = \lim_{T\to\infty} \beta^T u'(c)[2\varepsilon - \beta\varepsilon] = 0.$$

Therefore, $(c,a) \in d(D, -D_0|0)$. To finish, we just need to show that the Transversality condition ($\mathrm{TC_{HL}}$) is not zero. First observe that since debt limits never bind after time 0, we have that $S^t_{\mathrm{NB}}(0) = \{t\}$. Then, we get

$$\lim_{t\to\infty} p_t D_t = \lim_{t\to\infty} \beta^t \frac{2\varepsilon}{\beta^t} = 2\varepsilon > 0.$$

□

B.2 Example B.2

In the following we provide an example for which the Transversality condition ($\mathrm{TC_{BB}}$) is not satisfied.

Example B.2. *The primitives $(\beta, u(\cdot))$ together with $\alpha \in (0,1)$ are chosen such that there exists a pair $(\underline{c}, \overline{c})$ satisfying*

$$0 < \underline{c} < \overline{c}, \quad \underline{c} + \overline{c} = 1 \quad \textit{and} \quad 1 - \beta\alpha = \beta(1-\alpha)\frac{u'(\underline{c})}{u'(\overline{c})}.$$

We let $(q^{\mathrm{C}}, q^{\mathrm{NC}})$ be defined by

$$q^{\mathrm{C}} := \beta(1-\alpha)\frac{u'(\underline{c})}{u'(\overline{c})} \quad \textit{and} \quad q^{\mathrm{NC}} := \beta\alpha.$$

Let this economy be periodically subject to positive and negatives shocks, we assume that in each period the chance of maintaining the shock type for the next period is equal for both types. More formally, in this economy uncertainty is captured by the Markov process s^t with state space $Z = \{H, L\}$, with $S^0 = \{H\}$, and transition probabilities

$$\pi(s_{t+1} = H | s_t = H) = \pi(s_{t+1} = L | s_t = L) = \alpha.$$

Let an event s^t to be defined as a history of shocks such that

$$s^t = (H, s_1, s_2, ..., s_t), \textit{ with } s_i \in Z,\ i \in \{1, ..., t\}.$$

We call λ^t the event such that the history until time t is $(H,L,L,...,L)$. Moreover, fix $\delta > 0$ such that $q^{\mathrm{C}}\delta < \underline{c}$ and let $(\underline{e},\overline{e})$ be the pair defined by

$$\underline{e} := \underline{c} - q^{\mathrm{C}}\delta \quad \textit{and} \quad \overline{e} := \overline{c} + q^{\mathrm{C}}\delta.$$

Let the endowment process $e = (e(s^t))_{s^t\in\Sigma}$ be defined by

$$e(s^t) := \begin{cases} \overline{e}, & \textit{if} \quad s_t = H \\ \underline{e}, & \textit{if} \quad s_t = L. \end{cases}$$

The sequence $(q(s^t))_{s^t\in\Sigma}$ of bond prices is defined by

$$q(s^t) := \begin{cases} q^{\mathrm{NC}}, \textit{if} \quad s_t = s_{t-1} \\ q^{\mathrm{C}}, \textit{otherwise}. \end{cases}$$

Proposition B.2. *For the economy described in Example B.2, there exists a process of not-too-tight debt limits that does not satisfy (TC_{BB}).*

Proof. Fix $M > 0$. Let $(D(s^t))_{s^t\in\Sigma}$ be a process of debt limits defined by

$$D(s^t) := \begin{cases} \dfrac{M}{p(s^t)}, \text{ if } s^t = \lambda^t \\ 0, \text{ otherwise}. \end{cases}$$

Observe that the process D is not-too-tight since it satisfies exact roll-over. Consider the pair (c,a) defined by

$$c(s^t) := \begin{cases} \overline{c}, & \text{if} \quad s_t = H \\ \underline{c}, & \text{if} \quad s_t = L, \end{cases}$$

and $a(s^0) = -M$, and for all $s^t \succ s^0$

$$a(s^t) := \begin{cases} -\frac{M}{p(s^t)} + \delta, & \text{if} \quad s^t = \lambda^t \\ \delta, \quad \text{if} \quad s^t \neq \lambda^t \quad \text{and} \quad s_t = L \\ 0, \quad \text{otherwise}. \end{cases}$$

We claim that $(c,a) \in d(D, -D(s^0)|s^0)$ and that the corresponding condition (TC_{BB}) is not satisfied.

We start by proving the optimality of (c,a). Fix an event s^t and let $(s_0, s_1, ..., s_t)$ be its history. Note that $a(s^0) = -D(s^0)$ and the flow budget constraint is satisfied by construction. Euler equations are also satisfied because

$$\text{if } (s_{t-1}, s_t) = \begin{cases} (H,H), & \text{then} \quad \beta\pi(s^t)\frac{u'(c(s_t))}{u'(c(s_{t-1}))} = \beta\alpha = q^{\mathrm{NC}}, \\ (L,H), & \text{then} \quad \beta\pi(s^t)\frac{u'(c(s_t))}{u'(c(s_{t-1}))} = \beta(1-\alpha)\frac{u'(\overline{c})}{u'(\underline{c})} < q^{\mathrm{C}}, \\ (H,L), & \text{then} \quad \beta\pi(s^t)\frac{u'(c(s_t))}{u'(c(s_{t-1}))} = \beta(1-\alpha)\frac{u'(\underline{c})}{u'(\overline{c})} = q^{\mathrm{C}}, \\ (L,L), & \text{then} \quad \beta\pi(s^t)\frac{u'(c(s_t))}{u'(c(s_{t-1}))} = \beta\alpha = q^{\mathrm{NC}}, \end{cases}$$

and whenever $(s_{t-1}, s_t) = (L, H)$, the debt constraint binds since $a(s^t) = 0 = D(s^t)$. We also have that the Transversality condition is satisfied. Indeed, since $a(s^t) + D(s^t) \leq \delta$ for every event s^t, we have

$$\lim_{t \to \infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c(s^t))[a(s^t) + D(s^t)] \leq u'(\underline{c}) \delta \lim_{t \to \infty} \beta^t = 0.$$

We can conclude that $(c, a) \in d(D, -D(s^0)|s^0)$.

It remains to show that (TC_{BB}) is not satisfied. We first observe that for every $t > 0$ debt limits never bind at event λ^t since

$$a(\lambda^t) = -\frac{M}{p(\lambda^t)} + \delta > -\frac{M}{p(\lambda^t)} = -D(\lambda^t).$$

This implies that $\lambda^t \in S^t_{\mathrm{NB}}(s^0)$ for every $t > 0$. Moreover, we have $p(\lambda^t)D(\lambda^t) = M$ for every $t > 0$. We can then deduce that

$$\sum_{s^t \in S^t_{\mathrm{NB}}(s^0)} p(s^t) D(s^t) \mathbb{1}_{\{p(s^t)D(s^t) > \gamma \pi(s^t)\}} \geq M \mathbb{1}_{\{M > \gamma \pi(\lambda^t)\}}.$$

Finally, observe that

$$\pi(\lambda^t) = (1 - \alpha) \alpha^{t-1} \overset{t \to \infty}{\longrightarrow} 0.$$

This implies that, for any $\gamma > 0$, we have

$$\sup_{t \geq 0} \sum_{s^t \in S^t_{\mathrm{NB}}} p(s^t) D(s^t) \mathbb{1}_{\{p(s^t)D(s^t) > \gamma \pi(s^t)\}} \geq M > 0.$$

The above inequality implies that (TC_{BB}) cannot hold. $\square$

APPENDIX C – Competitive equilibrium with full commitment

In this appendix we analyze the benchmark environment where agents can commit to their financial promises. In that respect, there is no reason to impose debt limits consistent with repayment incentives. If the only role that we want the debt limits to perform is to discourage agents from entering into Ponzi schemes, then they should not per se introduce imperfections into the model. In other words, debt limits should permit all justified transfers of income, that is, they should never bind at equilibrium.

C.1 Never binding debt limits

In this section, we analyze the necessary conditions satisfied by debt limits that never bind. These results will be applied in the next section when we study the properties of a competitive equilibrium.

In the following lemma we show that if debt limits are never binding, then the agent's wealth and the present value of the optimal consumption are finite.

Lemma C.1. *Fix a process D of consistent debt limits and an arbitrary event s^t. Consider the optimal plan $(c,a) \in d(D,b|s^t)$ for some initial financial claim b. If $a(s^\tau) > D(s^\tau)$, for all $s^\tau \succ s^t$, then*

$$\mathrm{PV}(c|s^t) < \infty \quad \textit{and} \quad \mathrm{PV}(e|s^t) < \infty.$$

Proof. We first show that the present value of consumption is finite. Fix $T > t$. Given that debt limits do not bind, we have the following Euler equations:

$$q(s^\tau) = \beta\pi(s^\tau|\sigma(s^\tau))\frac{u'(c(s^\tau))}{u'(c(\sigma(s^\tau)))}, \quad \forall s^\tau \succ s^t.$$

Using the definition of Arrow-Debreu prices,

$$\frac{p(s^\tau)}{p(s^t)} = \beta^{\tau-t}\pi(s^\tau|s^t)\frac{u'(c(s^\tau))}{u'(c(s^t))}, \quad \forall s^\tau \succ s^t,$$

we deduce that

$$\frac{u'(c(s^t))}{p(s^t)}\sum_{s^\tau\in\Sigma^T(s^t)} p(s^\tau)c(s^t) \le \sum_{s^\tau\in\Sigma^T(s^t)} \beta^{\tau-t}\pi(s^\tau|s^t)u(c(s^t)) - \sum_{s^\tau\in\Sigma^T(s^t)} \beta^{\tau-t}\pi(s^\tau|s^t)u(0)$$

where the inequality follows from the concavity of u. Passing to the limit when T tends to infinite and using the boundedness of u, we get that

$$\mathrm{PV}(c|s^t) \le \frac{1}{u'(c(s^t))}[U(c|s^t) - U(0|s^t)] < \infty.$$

Now we prove that the present value of the endowment process is also finite. Optimality of the allocation implies that the flow budget constraint binds at every event. Fix $T > 0$. Summing the present value at s^t of the flow budget constraint over all events in $\Sigma^T(s^t)$, we obtain

$$\mathrm{PV}^T(c|s^t) + \underbrace{\sum_{s^{T+1}\in S^{T+1}(s^t)} \frac{p(s^{T+1})}{p(s^t)} a(s^{T+1})}_{L(s^t):=} = \mathrm{PV}^T(e|s^t) + a(s^t). \tag{C.1}$$

Note that

$$L(s^t) = \frac{1}{p(s^t)}\left[\sum_{s^{T+1}\in S^{T+1}(s^t)} p(s^{T+1})[a(s^{T+1}) + D(s^{T+1})] - \underbrace{\sum_{s^{T+1}\in S^{T+1}(s^t)} p(s^{T+1}) D(s^{T+1})}_{\geq 0}\right]. \tag{C.2}$$

Therefore, it follows that

$$\mathrm{PV}^T(c|s^t) + \frac{1}{p(s^t)} \sum_{s^{T+1}\succ s^t} p(s^{T+1})[a(s^{T+1}) + D(s^{T+1})] \geq \mathrm{PV}^T(e|s^t) + a(s^t).$$

Using the first part of the proof and the Transversality condition (TC), we deduce that the left hand side converges to a finite number when T tends to infinity. This is sufficient to get the desired result.[1] □

In the above lemma the main relevant aspect is the fact that debt limits never bind. Euler equations then imply that prices coincide with the agent's intertemporal marginal rate of substitution at every event. In the case of limited commitment, as debt limits may bind, the above result do not hold in general.

We have proved that if debt limits never bind at an optimal allocation, then the agent's wealth is finite. Reciprocally, if the agent's wealth is finite, there is a natural candidate for debt limits that never binds.

Lemma C.2. *If the agent's wealth is finite then the natural debt limits never bind at optimal allocation. Formally, for every event s^t and any initial financial claim $b > -N(s^t)$, if $(c,a) \in d(N,b|s^t)$ then $a(s^\tau) > -N(s^\tau)$ for every $s^\tau \succ s^t$.*

The standard proof makes use of the Inada condition to ensure strictly positive consumption. We omit the standard arguments.

In the next lemma we show that a process of debt limits that never binds does not necessarily satisfy a Transversality condition, but it satisfies an interesting limiting property.

Lemma C.3. *Fix a process D of consistent debt limits and an arbitrary event s^t. Consider the optimal plan $(c,a) \in d(D,b|s^t)$ for some initial financial claim b and assume that debt limits*

[1] Recall that $(\mathrm{PV}^T(e|s^t))_{T\geq t}$ is a positive increasing sequence.

never bind, i.e., $a(s^\tau) > D(s^\tau)$, for all $s^\tau \succ s^t$. For every event $s^\tau \succeq s^t$, the following series converge in $\mathbb{R}$

$$E(s^\tau) := \frac{1}{p(s^\tau)} \lim_{T\to\infty} \sum_{s^T \in S^T(s^\tau)} p(s^T)D(s^T).$$

Moreover, the process $(E(s^\tau))_{s^\tau \succeq s^t}$ satisfies exact roll-over and we have

$$\mathrm{PV}(c|s^\tau) - E(s^\tau) = \mathrm{PV}(e|s^\tau) + a(s^\tau), \quad \forall s^\tau \succeq s^t.$$

Proof. Fix $T \geq t$. We already saw that

$$\mathrm{PV}^T(c|s^t) + \frac{1}{p(s^t)} \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})[a(s^{T+1}) + D(s^{T+1})] - \frac{1}{p(s^t)} \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})D(s^{T+1}) = \mathrm{PV}^T(e|s^t) + b.$$

Passing to the limit when T tends to the infinity and using the Transversality condition, we get that

$$E(s^t) := \frac{1}{p(s^t)} \lim_{T\to\infty} \sum_{s^T \in \Sigma^T(s^t)} p(s^T)D(s^T)$$

is finite and satisfies

$$\mathrm{PV}(c|s^t) - E(s^t) = \mathrm{PV}(e|s^t) + b. \tag{C.3}$$

From the optimality principle, we have $(c,a) \in d(D, a(s^{t+1})|s^{t+1})$, for all $s^{t+1} \succ s^t$. Applying the same argument as above, we get that $E(s^{t+1})$ is well-defined for every $s^{t+1} \succ s^t$. Moreover, since

$$S^T(s^t) = \bigcup_{s^{t+1} \succ s^t} S^T(s^{t+1}),$$

we deduce that

$$E(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})E(s^{t+1}).$$

Repeating the above argument for any successor event $s^\tau \succeq s^t$, we get that $(E(s^\tau))_{s^\tau \succeq s^t}$ satisfies exact roll-over. □

C.2 General equilibrium

In what follows we introduce the definition of a competitive equilibrium with full commitment and show that never binding debt limits can be set to be the natural debt limits without loss of generality.

There is a finite set I of agents with identical preference relation (for simplicity). Each agent i's initial endowment is represented by a process $e^i = (e^i(s^t))_{s^t \in \Sigma}$ where each $e^i(s^t)$ is strictly positive. We fix an allocation $(a^i(s^0))_{i \in I}$ of initial financial claims satisfying the following market clearing condition

$$\sum_{i \in I} a^i(s^0) = 0.$$

An allocation is a family $(c^i, a^i)_{i\in I}$ where each (c^i, a^i) is a pair of consumption and bond holdings processes. An allocation satisfies the market clearing condition whenever for every event $s^\tau \succeq s^0$, we have

$$\sum_{i\in I} c^i(s^t) = \sum_{i\in I} e^i(s^t), \tag{C.4}$$

$$\sum_{i\in I} a^i(s^t) = 0. \tag{C.5}$$

for every event $s^t \succeq s^0$.

We say that the consumption allocation $(c^i)_{i\in I}$ is resource feasible if it satisfies (C.4). Equation (C.5) means that bonds are in zero net supply.

Definition C.1 (Competitive Equilibrium with Full Commitment)**.** *A competitive equilibrium with full commitment*

$$(q, (c^i, a^i, D^i)_{i\in I})$$

consists of a process of state-contingent bond prices q, an allocation $(c^i, a^i)_{i\in I}$ satisfying the market clearing conditions and an allocation $(D^i)_{i\in I}$ of debt limits processes such for each agent i,

(a) the pair (c^i, a^i) is optimal in the budget set $B^i(D^i, a^i(s^0)|s^0)$, i.e.,[2]

$$(c^i, a^i) \in d^i(D^i, a^i(s^0)|s^0);$$

(b) the debt limits D^i are consistent and never bind, i.e.,

$$a^i(s^t) > -D^i(s^t), \quad \forall s^t \succeq s^0.$$

The objective of this section is to show that if $(q, (c^i, a^i, D^i)_{i\in I})$ is a competitive equilibrium with full commitment, then we can replace agents' debt limits by the natural debt limits in the sense that $(q, (c^i, a^i, N^i)_{i\in I})$ is also a competitive equilibrium. Following this result, there is no loss of generality in imposing the natural debt limits when agents can commit to their financial promises. This result stands in contrast with the limited commitment environment since natural debt limits can be infinite and when they are finite, debt limits should usually be strictly tighter than natural debt limits.

Proposition C.1. *Let $(q, (c^i, a^i, D^i)_{i\in I})$ be a competitive equilibrium with full commitment. For each agent i, the natural debt limits N^i are finite and the process of debt limits D^i can be replaced by N^i in the sense that $(q, (c^i, a^i, N^i)_{i\in I})$ is also a competitive equilibrium with full commitment.*

[2] The definitions of the budget set $B^i(D^i, b|s^t)$ and the demand $d^i(D^i, b|s^t)$ correspond to the definition of $B(D^i, b|s^t)$ and $d(D^i, b|s^t)$ where the endowment process e is replaced by e^i.

Proof. Since debt limits never bind, we can apply the lemmas of the previous section to deduce that each agent i's wealth $N^i(s^0)$ is finite. Moreover, we have

$$\mathrm{PV}(c^i|s^0) - E^i(s^0) = \mathrm{PV}(e^i|s^0) + a^i(s^0)$$

where $(E^i(s^t))_{s^t \succeq s^0}$ is the process defined by

$$E^i(s^t) := \frac{1}{p(s^t)} \lim_{\tau\to\infty} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)D^i(s^\tau).$$

Summing over i and using the market clearing conditions, we deduce that

$$\sum_{i\in I} E^i(s^0) = 0.$$

Since the process $(E^i(s^t))_{s^t \succeq s^0}$ is non-negative and satisfies exact roll-over, we get that $E^i(s^t) = 0$ for every s^t.

Consistency of D^i implies that

$$D^i(s^t) \le e^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})D^i(s^{t+1}), \quad s^t \in \Sigma.$$

We deduce that for every $T > t$,

$$D^i(s^t) \le \mathrm{PV}^T(e^i|s^t) + \frac{1}{p(s^t)} \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})D^i(s^{T+1}).$$

Using the result of Proposition C.1, we can pass to the limit when T tends to infinity and get that

$$D^i(s^t) \le N^i(s^t) + E^i(s^t).$$

Since we proved that $E^i(s^t) = 0$, we get that the equilibrium never binding debt limits D^i must be tighter than the natural debt limits N^i. It then follows that $(c^i, a^i) \in B^i(N^i, a^i(s^0)|s^0)$. We still have to show that $(c^i, a^i) \in d^i(N^i, a^i(s^0)|s^0)$. Optimality of (c^i, a^i) in the budget set $B^i(D^i, a^i(s^0)|s^0)$, where D^i never binds, implies that the flow budget constraints always bind, the Euler equations are satisfied and the following Transversality condition is satisfied

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0.$$ [3]

Since

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)D^i(s^t) = p(s^0)E^i(s^0) = 0$$

we deduce that

$$\lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + N^i(s^t)] = \lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t)a^i(s^t) = 0.$$

[3] Recall that $p(s^t) = \beta\pi(s^t)u'(c^i(s^t))/u'(c^i(s^0))$ since debt limits never bind.

This implies that the Transversality condition associated to the budget set $B^i(N^i, a^i(s^0)|s^0)$ is satisfied. Finally, observe that the debt limits N^i never bind since

$$a^i(s^t) > -D^i(s^t) \geq -N^i(s^t), \quad \forall s^t \succeq s^0.$$

We proved that $(q, (c^i, a^i, N^i)_{i \in I})$ is also a competitive equilibrium with full commitment. $\square$