Large Coalition Bargaining Games

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Outline

Motivation

- Frequently the continuum case is easier to study and understand than the finite case.
- Example: the EU council of ministers voting procedure might be considered a large finite game; the number of possible coalitions being large.
- the limit game is not that hard.

My plan for today

- The continuum model is a limit of finite games
- I present the continuum model for the simultaneous proposer coalitional bargaining solution
- the random proposer model is equally amenable to our approach, however:
- the simultaneous proposer is simpler to present.

The continuum game per se

Simultaneous proposer model

1. There are m types each with size at most 1;
2. A coalition $s = (s_1, \ldots, s_m)$ with $0 \leq s_i \leq 1$ members of type i, $1 \leqq i \leqq m$ has payoff $f(s) \geqq 0$;
3. We suppose $f(0) = 0$.

Simultaneous proposer model

Description

- The number v_i is to be the (per capita) value of the game for type $i = 1, \ldots, m$.
- If $0 \leqq s_i \leqq 1$ then $s_i v_i$ is the total value for the coalition.
- And $|s| = s_1 + \ldots + s_m$ is the size of the coalition s.
- The number $\delta \in (0, 1)$ is the discount rate.

Definition

If $v \in \mathbb{R}^m$, the average gain of a coalition $s = (s_1, ..., s_m)$ is defined as

$$\gamma(0, v) = 0$$

and if $s \neq 0$:

$$\gamma(s, v) = \frac{f(s) - \delta \langle v, s \rangle}{|s|}.$$

Continued

Define the highest average gain,

$$\gamma(v) = \sup\{\gamma(s, v) : s \in [0, \eta]\},$$

and the set of coalitions that achieve this gain:

$$\mathcal{F}(v) = \{s \in [0, \eta] : \gamma(s, v) = \gamma(v)\}.$$

Simultaneous proposer bargaining equilibrium definition

Definition

A coalition bargaining continuum equilibrium is a pair (v, μ) where $v \in \mathbb{R}_+^m$, μ a probability measure on $[0,1]^m$, such that

$$\operatorname{supp} \mu \subset \mathcal{F}(v)$$

subject for all $i = 1, \ldots m$:

$$v_i = (\delta v_i + \gamma(v)) \int s_i d\mu.$$

Non–existence example

Example

Suppose $m = 1$ and $a > 2b$, $b > 0$. Let $f(s) = as - bs^2$. This function is strictly increasing. Suppose (v, μ) is an equilibrium. If $a - \delta v > 0$,

$$\sup_s \frac{as - bs^2 - \delta vs}{s} = \sup_s a - bs - \delta v = a - \delta v > 0.$$

Thus $\gamma(v) = a - \delta v$ and $\mathcal{F}(v) = \emptyset$. If $a - \delta v \leq 0$, then $\gamma(v) = 0$ and $\mathcal{F}(v) = \{0\}$. Now this implies $\mu = \delta_{\{0\}}$ and

$$v = (\delta v + \gamma(v)) \int s \, d\mu = 0.$$

A contradiction.

Existence theorem

Theorem

There is an equilibrium if:

i. f *is uppersemicontinuous,*
ii. $f(0) = 0$,
iii. $\limsup_{s \to 0} \frac{f(s)}{|s|} = 0$.

The following is elementary:

Lemma

$\mathcal{F}(v)$ *is closed and non–empty.*

Lemma

The function $v \to \gamma(v)$ is convex and continuous.

1. For each s, $v \to \frac{f(s)}{|s|} - \delta \left\langle v, \frac{s}{|s|} \right\rangle$ is a linear function. Thus $\gamma(v)$ is convex, being the supremum of a family of linear functions.
2. And is continuous since a convex function is continuous on the interior of its domain.

Lemma

If (v, μ) *is an equilibrium then*

$$0 \leq v_i \leq \frac{1}{1-\delta} \sup_s \frac{f(s)}{|s|}. \tag{1}$$

Let $\gamma = \gamma(v)$. From

$$v_i = (\gamma + \delta v_i) \int s_i \, d\mu \leq \gamma + \delta v_i$$

we get $v_i \leq \frac{\gamma}{1-\delta} \leq \frac{1}{1-\delta} \sup_s \frac{f(s)}{|s|}$.

Main theorem proof idea

Let $M = \frac{1}{1-\delta} \sup_s \frac{f(s)}{|s|}$. Define $X = [0, M]^m$ and

$$\mathcal{F}(v) = \left\{ s \in [0,1]^m : \gamma(s, v) = \gamma(v) \right\}$$

If $v \in X$, $\mu \in \mathcal{P}$ let

$$\Phi(v, \mu) =$$

$$\left\{ x : x_i = \frac{\gamma(v) \int s_i \, d\mu}{1 - \delta \int s_i \, d\mu}, 1 \leq i \leq m \right\} \times \{\nu \in \mathcal{P} : \operatorname{supp} \nu \subset \mathcal{F}(v)\}.$$

Φ has a fixed point (v, μ). This fixed point is an equilibrium.

The sequence of finite games associated to f
Symmetric coalitional bargaining game

If n is an integer, let $N_1, \ldots, N_m$ be disjoint sets with n members each and $N = \cup_{i=1}^{m} N_i$. Define for $S \subset N$, the coalitional function,

$$F(S) = f\left(\frac{\#(S \cap N_1)}{n}, \ldots, \frac{\#(S_m \cap N_m)}{n}\right). \tag{2}$$

The simultaneous proposer model is a new bargaining procedure studied in Gomes (2016). It proceeds as follows:

1. Each $i \in N$ announces a coalition $C_i \subset N$;
2. If there is C such that $i \in C \iff C_i = C$ the coalition is formed.
3. The coalition formed divide the payoff $F(C)$ accordingly to $(v_i)_{i=1}^{m}$ and the average coalition gain $\gamma(v)$ (next slide)

Coalitional bargaining solution

Definition

Define $X(i) = \{C \subset N : i \in C\}$. The coalitional bargaining solution (v, μ) is defined by

1. $$v_i = (\delta v_i + \gamma) \int 1_{X(i)} d\mu(C), 1 \leqq i \leqq n$$
$$\gamma(v) = \max\{\gamma(C, v) : C \subset N\}.$$

2. Here $\gamma(C, v) = \frac{1}{\#C}(F(C) - \delta \langle v, 1_C \rangle)$ and
$$\gamma(C, v) < \gamma(v) \implies \mu(C) = 0.$$

Using the symmetry

Using the symmetry we obtain a solution (v, μ) such that

1. $v_i = v_j$ if $i, j \in N_k$ for some $k \leqq m$.
2. And $\mu(S) = \mu(T)$ if $\#(S \cap N_i) = \#(T \cap N_i)$ for every $i \leqq m$.

Symmetry
Continued

Thus if $s_i = \frac{k_i}{n}, k_i \in \{0, 1, \ldots, n\}$, $s = (s_1, \ldots, s_m)$,

$$\mu_s = \sum_{\substack{C \subset N, \\ \frac{\#(C \cap N_i)}{n} = s_i, 1 \le i \le m}} \mu(C)$$

there exists $(v_1, \ldots, v_m) \geqq 0$ such that, if $|s| = \sum_{i=1}^m s_i$

$$v_i^n = \left(\delta v_i^n + \gamma^n(v)\right) \int s_i \, d\mu^n(s),$$

$$\gamma^n(v) = \max_s \gamma^n(s, v),$$

$$\gamma^n(s, v) = \frac{1}{|s|} \left(f(s) - \delta \left\langle s, v^n \right\rangle\right).$$

The limit $n \to \infty$

Lemma

The sequence $(v^n)_n$ is bounded.

Thus taking a subsequence if necessary,

$$v_i^n \to v_i, 1 \leqq i \leqq m,$$
$$\mu^n \xrightarrow{w} \mu,$$
$$\gamma^n (v^n) \to \overline{\gamma}.$$

The limit of

$$v_i^n = (\delta v_i^n + \gamma^n (v^n)) \int s_i \, d\mu^n (s)$$

is

$$v_i = (\delta v_i + \overline{\gamma}) \int s_i \, d\mu (s) ,$$
$$\operatorname{supp} \mu \subset [0, 1]^m .$$

Theorem

Suppose $f : [0,1]^m \to [0,\infty)$ *is uppersemicontinuous and*

i. $f(0) = 0$,

ii. $\limsup_{s\to 0} \frac{f(s)}{|s|} = 0$.

Then (v,μ) *obtained above is such that*

$$\operatorname{supp}\mu \subset \mathcal{F}(v) = \left\{ s \in [0,1]^m : \gamma(s,v) = \gamma(v) \right\}$$

and for all $i = 1, ...m$:

$$v_i = (\delta v_i + \gamma(v)) \int s_i \, d\mu.$$

A reciprocal result

Theorem

Any equilibrium (v, μ) such that μ has finite support, is the limit of a subsequence of the large discrete coalitional bargaining game.

Degenerated and non–degenerated equilibria

Definition

We say that the equilibrium (v, μ) is:

- degenerated if the support of μ is a singleton.
- non–degenerated if the support has more than one point.

In some cases we may ensure that $\mathcal{F}$ is a singleton.

Degenerated support condition

Lemma

If f is concave then $\frac{f(s)-\delta\langle v,s\rangle}{|s|}$ is quasi–concave. Moreover, if f is strictly concave then $\frac{f(s)-\delta\langle v,s\rangle}{|s|}$ is strictly quasi–concave.

Corollary

Suppose f is strictly concave. Then any equilibrium is degenerated.

If f is only concave we have the weaker result:

Proposition

If f is concave there is a degenerated equilibrium.

Robustness

Let $\mathcal{C}$ be the space of continuous functions $f : [0,1]^m \to [0,\infty)$. We define on $\mathcal{C}$ the metric topology of uniform convergence: The distance between f, g in $\mathcal{C}$ is

$$d(f,g) = \sup\{|f(s) - g(s)| : 0 \le s \le \eta\}.$$

Let C^2 be the set of value functions f twice continuously differentiable with the topology of uniform convergence up to the second order derivatives.

Robustness
Continued

One might conjecture (and we did) that being degenerated is a robust property: I.e. it is true in an open dense set of value functions f. After all we may perturb a function with several optima to have a single optima. The next few results shows that this is not true.

We begin with degenerated equilibrium.

Theorem

Suppose

1. *the value function* f^0 *has a unique equilibrium* $\left(v^0, \delta_{\{s^0\}}\right)$, $s^0 \in (0,1)^m$.
2. *there exists an* $\epsilon > 0$ *such that for every* $h \in \mathbb{R}^m$,

$$\left\langle D^2 f^0\left(s^0\right) h, h\right\rangle \leq -\epsilon\left|h\right|^2.$$

Then every f *sufficiently near* f^0 *in the* C^2 *topology, also has a degenerated equilibrium.*

Proof

Suppose to obtain a contradiction that there exists $f^n \to f^0$ in the C^2 topology with no degenerated equilibrium. If n is large enough, $\left\langle D^2 f^n(s)\, h, h\right\rangle \leq -\frac{\epsilon}{2}|h|^2$ for every s in a compact neighborhood U of s^0. Outside of U,

$$\sup_{s\in U^c} \frac{f^0(s) - \delta\left\langle v^0, s\right\rangle}{|s|} < \frac{f^0\left(s^0\right) - \delta\left\langle v^0, s^0\right\rangle}{|s^0|}.$$

Thus if n is large enough,

$$\sup_{s\in U^c} \frac{f^n(s) - \delta\left\langle v^0, s\right\rangle}{|s|} < \frac{f^n\left(s^0\right) - \delta\left\langle v^0, s^0\right\rangle}{|s^0|}.$$

Proof

If (v^n, μ^n) is an equilibrium for f^n, taking a subsequence if necessary, we have $(v^n, \mu^n) \to (v, \mu)$. Thus

$$v_i = (\delta v_i + \gamma) \int s_i \, d\mu,$$
$$\gamma = \lim_n \gamma^n.$$

If $\gamma^n (s^n) = \gamma^n$ and $s^n \to s^*$ then

$$\gamma = \lim_n \frac{f^n (s^n) - \delta \langle v^n, s^n \rangle}{|s^n|} = \frac{f^0 (s^*) - \delta \langle v, s^* \rangle}{|s^*|}.$$

Proof

Since $\operatorname{supp}\mu \subset \left\{s : \gamma^0(s) = \gamma^0\right\}$ and f^0 has a unique equilibrium we have $v = v^0$ and $\mu = \delta_{\left\{s^0\right\}}$. Now for every $s \in U^c$,

$$\frac{f^n(s) - \delta\langle v^n, s\rangle}{|s|} < \frac{f^n\left(s^0\right) - \delta\left\langle v^n, s^0\right\rangle}{\left|s^0\right|},$$

thus $\operatorname{supp}\mu^n \subset U$. Since in U, f^n is strictly concave, $\operatorname{supp}\mu^n$ is a singleton.

We now consider the robustness of non–degenerated equilibrium.

Theorem

The set of $f \in \mathcal{C}$ such that its equilibrium are non–degenerated is an open set.

Proof omitted. The next two theorems shows that this open set is non–empty.

Robustness of non–degenerated equilibrium

Theorem

Let $f(s_1, s_2) = f(s_2, s_1)$ and suppose that $\frac{f(s)}{|s|}$ has only interior maximum and

$$\max_{s\in[0,1]^2} \frac{f(s)}{|s|} > \max_{s_1=s_2} \frac{f(s)}{|s|}. \tag{*}$$

Then there is no degenerated equilibrium.

A concrete example

It remains to show that there is such a value function as above.

Theorem

There is a C^∞ value, f^0, satisfying

$$\max_{s\in[0,1]^2} \frac{f^0(s)}{|s|} > \max_{s_1=s_2} \frac{f^0(s)}{|s|}. \tag{*}$$

Given $0 < 2\epsilon < w < 1$ take a C^∞ increasing function $\phi : \mathbb{R} \to [0,1]$ such that ϕ is null on $(-\infty, w-\epsilon]$ and equal to 1 on $[w,\infty)$. Let $g(s_1,s_2) = \phi(s_1) + \phi(s_2) - \phi(s_1)\phi(s_2)$. Note that g is C^∞, symmetric, $g(0) = 0$, and

$$g(s_1,s_2) = \phi(s_1) + \phi(s_2)(1-\phi(s_1)) \le \phi(s_1) + 1 - \phi(s_1) = 1.$$

Moreover g is monotonic since

$$\frac{\partial g}{\partial s_1} = \phi'(s_1)(1 - \phi(s_2)) \geq 0, \frac{\partial g}{\partial s_2} = \phi'(s_2)(1 - \phi(s_1)) \geq 0.$$

However g may have frontier maximum. To fix this let ψ be an increasing C^∞ function such that $\psi(x) = 0$ if $x \leq \epsilon$ and $\psi(x) = 1$ if $x \geq 2\epsilon$. Now $f = g(s_1, s_2)\,\psi(s_1)\,\psi(s_2)$ satisfy all requirements.

Weighted majority game

Suppose we give weights w_i to types $i = 1, \ldots, m$. If

$$f(s) = 1 \text{ if } \langle w, s \rangle \geq q$$

we say that we have a weighted majority game.

EU example

For example, in 1995, the EU Council of Ministers had 15 members and its voting rules could be represented as the weighted majority game

$$[q = 62; w_1 = 10; 10; 10; 10; 8; 5; 5; 5; 5; 4; 4; 3; 3; 3; w_{15} = 2].$$

In the table (in the next slide) we report the value, v_i, of each country. We denote by Baron-Ferejohn the RP-coalition bargaining solution with equal probabilities.

EU value solutions

Power Distribution in 1995 (Quota=62)						
Country	**# Votes**	**Shapley-Shubik**	**Banzhaf**	**Baron-Ferejohn**	**Nucleolus**	**CBS**
Germany	10	11.7	11.2	11.5	11.5	13.4
Italy	10	11.7	11.2	11.5	11.5	13.4
France	10	11.7	11.2	11.5	11.5	13.4
UK	10	11.7	11.2	11.5	11.5	13.4
Spain	8	9.5	9.2	9.2	9.2	10.2
Belgium	5	5.5	5.9	5.7	5.7	5.3
Netherlands	5	5.5	5.9	5.7	5.7	5.3
Greece	5	5.5	5.9	5.7	5.7	5.3
Portugal	5	5.5	5.9	5.7	5.7	5.3
Sweden	4	4.5	4.8	4.6	4.6	3.8
Austria	4	4.5	4.8	4.6	4.6	3.8
Denmark	3	3.5	3.6	3.4	3.4	2.2
Ireland	3	3.5	3.6	3.4	3.4	2.2
Finland	3	3.5	3.6	3.4	3.4	2.2
Luxembourg	2	2.1	2.3	2.3	2.3	0.5