FUNDAÇÃO GETULIO VARGAS
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CAIO PAES LEME LORECCHIO

# BAD REPUTATION WITH RATING SYSTEMS

SÃO PAULO
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Dissertação apresentada à Escola de Economia de São Paulo da Fundação Getulio Vargas como requisito para obtenção do título de Mestre em Economia de Empresas.

Campo de Conhecimento:
Microeconomia - Teoria dos Jogos

Orientador: Prof. Dr. Daniel Monte

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*"I honestly believe it is better to know nothing than to know what ain't so."*

*(Josh Billings, Everybody's Friend)*

# RESUMO

Este trabalho analisa um modelo de má reputação com sistemas de *rating* como uma forma particular de memória limitada. Em cada período, um cliente preocupado apenas com ganhos correntes escolhe se contrata ou não um especialista. O cliente compreende as regras de transição do sistema, mas observa apenas a realização de um *rating* (uma nota) que carrega informação sobre o provável tipo de especialista para tomar a decisão de contrato. Um especialista do tipo estratégico escolhe prover ou não o tratamento correto quando contratado e um especialista do tipo ruim sempre oferece o tratamento mais caro, independentemente do problema observado. Quando clientes observam todo o histórico de interacões, um especialista estratégico ou tem fortes incentivos para oferecer o tratamento barato (quando o tratamento correto seria o mais caro) ou eventualmente a crença no mercado de que ele é do tipo ruim é suficientemente grande para que deixe de ser contratado. Quando clientes possuem apenas o sistema de *rating* como fonte de informação, este trabalho demonstra que não apenas é possível evitar esse efeito negativo, como também é possível aumentar os ganhos de equilíbrio em comparação à ausência de qualquer sistema informacional. Ademais, este trabalho desenha os sistemas ótimos do ponto de vista tanto do cliente quando do especialista para todas as crenças iniciais, discutindo como eles diferem em um sistema de dois estados e quando há ganhos de eficiência.

Palavras-chave: jogos de má reputação, sistemas de *rating*, persuasão Bayesiana.

# ABSTRACT

We study a bad reputation model with rating system as a special form of limited memory. At each period, a myopic customer knowing the rules of the system but observing only a current public realization of a finite set of states uses this information to infer expert's type and take hiring decisions. A strategic expert chooses whether or not to provide correct treatment whenever hired and a bad (committed) expert always proposes an expensive treatment. With full memory, a patient expert cannot refrain from gaining reputation of being bad or lying to separate herself from a bad type. With rating systems, we show that it is possible not only to overcome bad reputation effect, but generate higher equilibrium outcomes relative to trivial information censoring (no memory at all). We characterize optimal systems from customer and strategic expert's point of view in a two-state setting for all prior beliefs and show how they differ and when a rating system can bring efficiency to experts' markets.

Keywords: bad reputation games, rating systems, Bayesian persuasion.

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# 1 INTRODUCTION

Social interaction is mainly about trust. As Rotter (1970, p. 443) puts it “the entire fabric of our day-to-day living rests on trust – buying gasoline, paying taxes, going to the dentist, flying to a convention – almost all our decisions involve trusting someone else”. In some markets, trust is specially required for proper functioning. This is the case with experts’ markets, where a provider might know more about the needs of a customer than the customer himself.

Trust can be achieved through reputation, which is acquired on the basis of observed behavior through time. A customer repeatedly interacting with the same provider can use own past experience to anticipate this provider’s actions. But it is usually the case that customers rely only on publicly available information to decide whether or not to hire an expert for the first time and the result of this interaction might add new information for future customers. The expert as well relies on the flow of customers to change public perceptions of her.

In the reputation literature, including its application on repeated games between a long-run agent and a sequence of short-run principals, it is well known that introducing a small uncertainty about agent’s type might change significantly equilibrium behavior and payoffs. Fudenberg and Levine (1989; 1992) have shown that introducing a public prior belief that long-run agent is credibly committed to principals’ preferred action assures her a payoff arbitrarily close to what she would gain if she was the committed type indeed. In other words, reputational concern can benefit both short-run principals and long-run strategic agents imitating good (behavioral) types.

However, reputation might harm players as well. Ely and Välimäki (2003) (henceforth EV) constructed a model where a long-run expert privately observes in each period a costumer’s need upon being hired and chooses the appropriate treatment, which cannot be verified by customers. If it is believed that this expert is a commitment type playing undesired principals’ actions, there is major change in equilibrium payoffs and behavior as well, but now they show that reputation has a negative effect, a phenomenon they call “bad reputation”.

EV model assumes short-run customers get to see full history of past outcomes. They define beliefs and take hiring decisions conditioned to the entire history of past interactions, no matter how long or complex those histories might be (full memory setting). In reality, modern interaction between experts and customers is mediated through web-based review platforms that gather many different sources of information into simple rating systems (numbers, stars, etc.). It is natural to raise the following questions: it is possible to avoid bad reputation by censoring information through some simple processes (finite time-independent rating systems)? Under what conditions can these simple processes improve upon extreme settings, i.e, full memory and no memory at all?

In this dissertation, we study EV bad reputation environment with a special type of limited memory setting: rating systems, consisting of (i) a finite set of states; (ii) time-independent transition rules conditioned on realized signals from customer-expert interactions; (iii) an initial distribution over states. We show that simple rating systems (indeed, a two-state system deterministic system) not only overcomes the bad reputation effect and generates an equilibrium where expert avoids never being hired and is not tempted to lie to customers, but might give players higher outcomes than under the full memory game and the no memory game (the infinite repetition of the one-shot interaction) as well. Therefore, in some sense, rating systems bring back the positive effect of reputational concerns.

We also show how to design optimal rating policies to achieve highest possible *ex-ante* expected payoffs in a two-state system for each player. This requires some randomization of transition rules. We argue that there might exist some conflict of interests over what an optimal rating policy should be from each player's point of view, since customer always wants to minimize chances of interacting with a possible bad type and strategic expert always wants to maximize the chances of staying in a rating where hiring takes place. Nevertheless, it is possible to design a system that brings efficiency to expert-customer interaction. Specifically, it is possible to give strategic expert higher equilibrium outcome relative to extreme memory settings for every prior belief without harming customer relative to the absence of any information mechanism.

Finally, we consider the problem of maximizing the long-run hiring frequency as an extension of our model and show that the optimal rating policy to do so is equal to expert's optimal policy. The interesting conclusion is that a rating system might persuade customer to accept higher probabilities of hiring a bad type as well as to keep hiring experts even if prior market belief is high enough not to do so.

To see why discussing bad reputation effect in experts' markets through rating systems is interesting, consider the market of auto repairs. An honest mechanic can evaluate whether a car needs an engine replacement (high cost treatment) or a tune-up (low cost treatment) and a customer can only trust on mechanic's diagnosis on the basis of what he believes an honest mechanic would do. Providing an engine replacement when the real problem requires a tune-up generates an unnecessary cost and providing a tune-up when the real problem requires an engine replacement does not solve customer's needs. When it is believed that greedy mechanics always propose an engine replacement, an honest one might want to propose an incorrect tune-up to distinguish herself, harming the customer. Even if she refrains from doing so, the customer always takes the interaction with a grain of salt, since he knows that honest expert's incentives for separation by providing inappropriate treatment are strong.

There are many other social interactions where this bad reputation effect arises. For example, a high skilled social scientist advising a decision-maker on efficiency of some affirmative action by race might have some concerns to be seen as a racist by a policy-maker when

reporting that such policy is ill-conceived to address racism (MORRIS, 2001). She might choose to lie and recommend the affirmative action, but policy maker will not believe in her advice if he knows she is lying. Another example, a patient cannot perfectly know whether the drug prescribed by a physician is the most effective treatment. If it is believed that a physician is funded by some drug company to prescribe its products, even if she is providing the correct treatment indeed, patients might choose not to comply with it[1].

Bad reputation might be particularly harmful when current customers always observe the results of past interactions. As in the auto-repair example, observing a engine-replacement is always bad news. Recently, however, web-based review platforms for experts' service have become widespread[2], complementing and even substituting word-of-mouth among customers as the main source of information transmission, since it gathers many different sources of information into a simple, easy-to-understand evaluation system (HOWARD; FEYMEN, 2017). Customers have many specific forums to review medical services (ZocDoc, HealthGrades.com, RateMDs.com, Vitals), home services (Angie's list), auto repair (MechanicRatingz.com), as well as general forums (Yelp).

The result that a rating system can bring efficiency to experts' market might have important policy implications. There has been many resistance of physicians to online rating platforms, for example. In fact, some American companies specialized in helping doctors conceal reviews by their patients and devise "anti-ratings" contracts, by asking patients to sign away their right to provide additional information of a doctor in online platforms (GOLDMAN, 2013).

## 1.1 Related literature

This dissertation joins a literature on bad reputation games. Ely, Fudenberg and Levine (2008) extend EV seminal model (which we use as the setting of our study) to a broader class of games, considering very general stage game payoffs and a richer set of commitment types. *A priori*, there is no reason not to believe that our model supports these generalizations, even though we do not deal with them in this dissertation. In the last topic of their paper, EV show that it is possible to avoid bad reputation effect under a score mechanism, but this result relies crucially on customer not being myopic. Mailath and Samuelson (2006, chapter 18) consider the possibility of a random customer being captive, i.e, a type committed to hire no matter the history. When this is the case, bad reputation is avoidable as well. We take a different approach and maintain the assumption of a myopic Bayesian customer.

[1] Surely, there might be legal punishment systems to enforce right actions, but punishment systems such as legal courts most of time cannot validate experts' decision. Furthermore, punishment can lead experts to internalize these as expected costs and deliver services accordingly, behaving in a socially inefficient way. This is case when doctors practice defensive medicine (KESSLER; MCCLELLAN, 1996).

[2] One study found that 65% of Americans reported to use web-based physician rating websites (HANAUER et al., 2014)

By discussing information censoring, our work connects with the strategic information disclosure literature. Ekmekci and Wilson (2007) and Ekmekci (2011) study a rating system in a moral hazard game and the system structure is very close to ours with some fundamental differences. For instance, in their models, there is a permanent flow of informative signals about long-run's type. In credence goods markets, expert needs to keep being hired to send the signal she wants, so there might not exist such permanent flow of informative signals.

There are few models of bad reputation and information disclosure, but all of them assume customers to have a different type of information constraint, namely bounded recall, i.e, they can only observe limited excerpts of the past. Sperisen (2015) finds that it is possible to avoid bad reputation if new customers forget some past information with a time-increasing probability: each customer observes an unbroken chain of information so a current customer might take better hiring decisions and still avoid leading subsequent market beliefs to a critical level if the game has been going on for a long time. Lillethun (2016) finds it is optimal for an information designer maximizing discounted flow of myopic customers' payoffs (thus playing the role of a long-run patient "customer") to conceal expert's history of actions with a combination of rounds of no disclosure, full disclosure and partial disclosure.

By studying optimal rating structures from principal's (customer) point of view and considering strategic and commitment types of agents (experts), our work also relates with the literature of reputation and optimal information design (SMOLIN, 2015; HORNER; LAMBERT, 2016, KOVBASYUK; SPAGNOLO, 2016). Since we analyze as well optimal mechanisms from the perspective of player responsible for sending signals to other, the systems can be viewed as trying to persuade a Bayesian receiver to take the best action for the sender. Thus, we connect with the literature on Bayesian persuasion as well (KAMENICA; GENTZKOW, 2011; ELY, 2017).

## 1.2 Outline

The rest of this dissertation proceeds as follows. First, we discuss the EV bad reputation model in extreme environments, i.e, in the one-shot interaction and in the repeated game with full memory. We then model the repeated interaction with a general (but finite and Markovian) rating system. Third, we illustrate our setting with a deterministic two-state rating system and show under what conditions it generates an equilibrium where expert avoids never being hired and always plays customer's preferred actions when hired. We also show for what range of prior beliefs this simple system improves upon the one-shot interaction expected payoffs. Fourth, we characterize optimal policies for each player, still restricted to a two-state system. Finally, we analyze as an extension from the two-state system the optimal rating policy to maximize the long-run hiring frequency in the economy. Unless explicitly stated, all proofs of lemmas and propositions are in the Appendix.

# 2 MODEL

The setting of the study is a model based on EV. In their model, a long-run player 1 (expert) interacts with a sequence of identical time $t$ myopic players 2 (customers). For convenience, we will refer to this succession of players 2 interacting with player 1 (she) simply as the short-run customer (he).

At the beginning of every stage game, Nature draws a problem $\theta \in \{\theta_H, \theta_L\}$, each happening with probability $1/2$. Without observing $\theta$, customer decides first whether to hire (*In*) or not (*Out*) the expert. If he chooses not to do so, he gets an outside option payoff, normalized to zero. If he chooses to hire, expert then perfectly observes $\theta$ and decides the level of service to provide. An appropriate treatment for $\theta_H$ would be $t_H$ and an appropriate treatment for $\theta_L$ would be $t_L$. The customer observes treatment and payoffs are realized: the right one generates $u$ and the wrong generates $-w$.

There is incomplete information regarding expert's type. With common knowledge probability $\rho$ (initial prior), expert is a type committed to a pure strategy: she always provides the high-cost treatment $t_H$. This behavioral "bad" type will be denoted by $B$. With probability (1-$\rho$), the expert is a strategic type[1], denoted by $S$, and has payoffs at the end of each period similar to customer's. We summarize the payoffs for the strategic expert and customer attached to each terminal node of the extensive form game in figure 1 below.

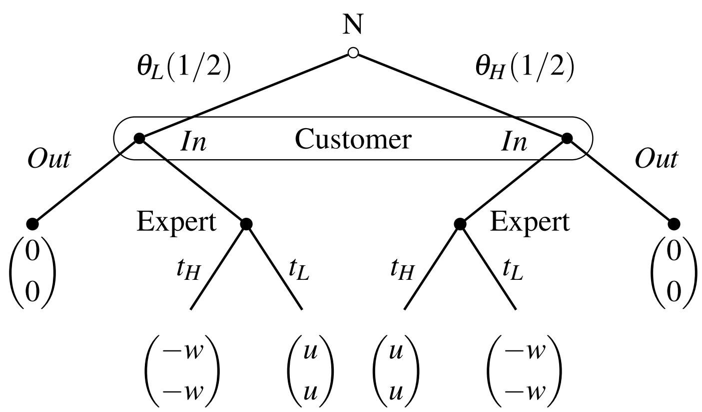



Figure 1 – Stage game payoffs

## 2.1 One-shot interaction

Assume $w > u > 0$. It is straightforward to see that a strategic expert strictly prefers to provide right treatment when hired in the stage game in absence of a repeated setting. For that

[1] Unless explicitly stated otherwise, when we refer simply to the expert, we will be referring to this strategic type.

reason, we will often refer to the strategic expert as the good one. However, our assumption implies that even if she does not face incentive problems to tell the truth, customer strictly prefers not to hire if there is a sufficiently high probability of interacting with a bad type. To see this, let $\vec{v}^0(\rho)$ denote customer's expected payoff from hiring in the stage game as a function of prior belief. This is given by:

$$v^0(\rho) = \rho\left(\frac{u-w}{2}\right) + (1-\rho)u \tag{2.1}$$

On the one hand, hiring a bad expert might lead to a right treatment ($t_H$ for $\theta_H$) and gain $u$ with probability 1/2. On the other hand, doing so might lead as well to a wrong treatment ($t_H$ for $\theta_L$) and loss $-w$ with probability 1/2. Hiring a strategic expert playing correct treatments always leads to $u$ for sure. Not hiring yields a zero payoff, so customer cannot benefit from hiring if $\vec{v}^0(\rho) < 0$. A simple manipulation of equation (2.1) shows that this is equivalent to $\rho$ being higher than $\rho^* := \frac{2u}{u+w}$. This is the highest value of a prior for which a customer would be willing to hire an expert in a one-shot interaction.

Since expert strictly prefers to provide the correct treatment in the one-shot interaction and customer's hiring decision depends on $\rho$, her expected payoff, denoted by $V^0$, is a function of prior belief as well. From this dependence and from equation (2.1), we can represent each player's expected payoff as:

$$v^0(\rho) = \begin{cases} u - \rho\left(\frac{w+u}{2}\right) \Leftrightarrow \rho \leq \rho^* \\ 0 \Leftrightarrow \rho > \rho^* \end{cases} \qquad V^0(\rho) = \begin{cases} u \Leftrightarrow \rho \leq \rho^* \\ 0 \Leftrightarrow \rho > \rho^* \end{cases} \tag{2.2}$$

Suppose now there exists a third player, an intermediary agency (it), responsible for designing some mechanism to bring the chances of hiring a bad expert and not hiring a good one to certain targets (probabilities or signal realizations) $f^B_{In}$ and $f^S_{Out}$, respectively. *Ex-ante*, this intermediary credibly commits to a public policy to achieve the desired targets and cannot deviate from it. Players observe the policy and take actions accordingly[2].

Let $f_{In} := \rho f^B_{In} + (1-\rho)(1-f^S_{Out})$ and $f_{Out} := \rho(1-f^B_{In}) + (1-\rho)f^S_{Out}$. We assume that intermediary commits to some chance of hiring or not hiring for at least one of types, i.e, $f_{In}$ and $f_{Out}$ are both positive. Given the policy, customer has some updated beliefs on what are the chances of hiring a bad type and not hiring a strategic one. These beliefs are $\rho_{In} = \rho f^B_{In}/f_{In}$ and $\rho_{Out} = \rho(1-f^B_{In})/f_{Out}$. Since expert has no reason to lie in the one-shot interaction, the expected gain from hiring depends only prior beliefs, signals and stage game payoffs. Thus, customer's expected payoff from the policy is given by:

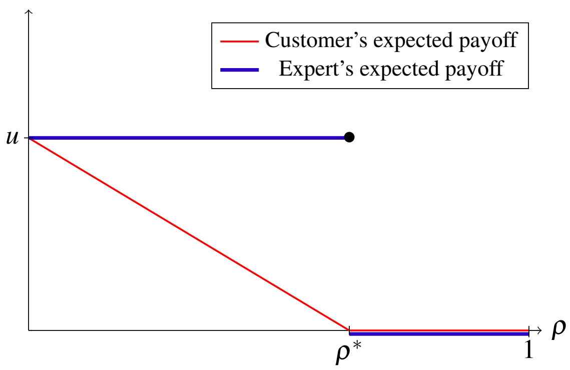
[2] We will study a rating system as a specific mechanism to achieve the targets later on. For now, it suffices to assume that these targets are exogenously given by the agency and are achievable.

$$v^*(\rho) = \rho f^B_{In}\left(\frac{u-w}{2}\right) + (1-\rho)(1-f^S_{Out})u = f_{In}\left[\rho_{In}\left(\frac{u-w}{2}\right) + (1-\rho_{In})u\right] \tag{2.3}$$

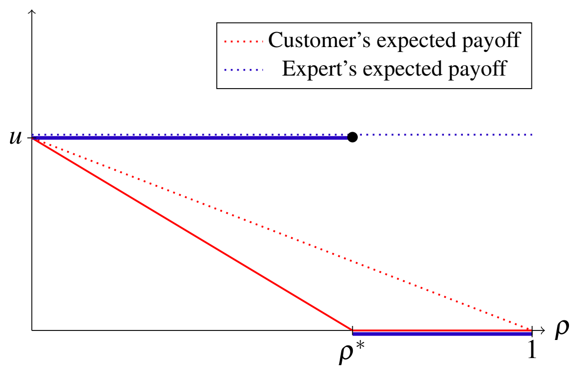
Clearly, customer hires if and only if $\rho_{In} \leq \rho^*$. It is also clear that designer's optimal target is $f^B_{In} = f^S_{Out} = 0$ if it wants to maximize customer's expected payoff. Then, customer achieves the complete information payoff $(1-\rho)u$ with maximally separated beliefs: $\rho_{In} = 0 < \rho^*$ and $\rho_{Out} = 1 > \rho^*$. Strategic expert achieves an expected payoff $V^*(\rho) = u$, since she always gets hired. Note that this is not the unique optimal target for the strategic expert: any targeting scheme that keeps $f^S_{Out} = 0$ for every $\rho$ leads to same result, since strategic expert's expected payoff in this environment is given by:

$$V^*(\rho) = (1-f^S_{Out})u \tag{2.4}$$

We give a graphic representation of one-shot interactions with and without a mechanism designer. In panel (a) of figure 2 below we represent equation (2.2). Customer hires up to the point where $\rho > \rho$, obtaining the values depicted in the red line as expected payoff. Strategic expert only obtains the upper blue line as expected payoff ($u$) for prior beliefs below or equal to the critical level and the lower blue line (zero) otherwise. In panel (b) we add representation of payoffs under a complete information setting: the red line corresponds to $(1-\rho)u$, achieved from equation (2.3) by reaching $f^B_{In} = f^S_{Out} = 0$ and the blue one corresponds to $u$, achieved from equation (2.4) by reaching $f^S_{Out} = 0$ for any value of $f^B_{In}$.




(a) One-shot interaction payoffs




(b) Complete information payoffs

Figure 2 – Expected payoffs with and without incomplete information

## 2.2 Repeated game with full memory

In this section, we discuss EV bad reputation result under full memory. They show that if customer gets to see all past treatments chosen by expert (but not the associated severities), even if the strategic expert is willing to provide correct treatment, she will rarely be hired in any

(Nash) equilibrium if she is sufficiently patient. Moreover, if incentives to play wrong treatment to avoid this zero payoff equilibrium are strong enough, she will not be hired at all. What this section highlights is that full memory and reputation effect harm both expert and customer in this game.

In the infinite repetition of the described stage game, strategic expert discounts future payoffs at a rate $\delta \in (0,1)$ and evaluate continuation payoffs under average discounted criterion. She chooses what treatment to offer to the hiring customer, potentially considering the private history of all past interactions. This naturally includes all observations of severities and treatments every time she got hired. It also includes what customer observes in every period in terms of available information to take hiring decisions and form beliefs about expert being bad.

Formally, let $Y := \{Out, t_L, t_H\}$ be the set of observable outcomes from every stage game interaction and $y$ an element of $Y$. At the beginning of a stage game in period $t$, expert observes a sequence $h^t := \{(\theta_0, y_0), ...(\theta_{t-1}, y_{t-1})\}$ of past severities and public outcomes. For notational convenience, We let $\theta = \emptyset$ whenever $y = Out$ and write the expanded set of severities as $\Theta = \{\theta_H, \theta_L, \emptyset\}$.

In this section, we assume customer gets to see all past sequence of public outcomes. More precisely, let $\hbar^t = \{y_0, ..., y_{t-1}\}$ denote a period $t$ public history and $\bar{H}^t$ the set of all such histories. The set of all possible public histories in the game is $\bar{\mathscr{H}} = \cup_{t=0}^{\infty} \bar{H}^t$. Whenever expert gets hired in $h^t \in H^t = (Y \times \Theta)^{t-1}$ and upon observing $\theta_H(\theta_L)$, she chooses right treatment $t_H(t_L)$ with probability $\beta_H(h^t)(\beta_L(h^t))$. We refer to this pair of actions as $\beta(h^t)$. Customer's hiring decision is conditioned only on the observation of some $\hbar^t \in \bar{H}^t$ and we denote it as $\alpha(\hbar^t)$. Abusing notation, we denote expert's strategy by $\beta$ and customer's strategy by $\alpha$. We refer to a strategy profile as $\sigma = (\alpha, \beta)$. Each strategy is given by:

$$\beta : \cup_{t=0}^{\infty} (H^t \times \{In, Out\} \times \Theta) \to \Delta\{t_H, t_L\}^2$$
$$\alpha : \bar{\mathscr{H}} \to \Delta\{In, Out\}$$

If customer plays according to the public information from $\bar{\mathscr{H}}$, expert has a strategy conditioned only on $\bar{\mathscr{H}}$ as a best reply (MAILATH; SAMUELSON, 2006, lemma 7.1.1). To see this is true, let $h^t$ be strategic expert's private history leading to public history $\hbar^t$ in period $t$. Since customer's hiring decision in $t$ depends only $\hbar^t$, her continuation value must be independent of private history $h^t$, so her best-reply to customer's decision must depend only on public information. Therefore, it is not troubling to focus on expert's strategies that are stationary on $\bar{\mathscr{H}}$, as we do from now on.

Let $V(\hbar|\theta)$ denote expert's expected continuation value whenever hired in $\hbar$ and conditioned on observing $\theta$. We can write it as:

$$V(\hbar|\theta_H) = (1-\delta)[\beta_H(\hbar)u - (1-\beta_H(\hbar))w] + \delta[\beta_H(\hbar)V(\hbar,t_H) + (1-\beta_H(\hbar))V(\hbar,t_L)]$$
$$V(\hbar|\theta_L) = (1-\delta)[\beta_L(\hbar)u - (1-\beta_L(\hbar))w] + \delta[\beta_L(\hbar)V(\hbar,t_L) + (1-\beta_L(\hbar))V(\hbar,t_H)]$$

The expected continuation value is:

$$V(\hbar) = \alpha(\hbar)\left[\frac{1}{2}V(\hbar|\theta_H) + \frac{1}{2}V(\hbar|\theta_L)\right] + (1-\alpha(\hbar))\delta V(\hbar, Out)$$

Denote by $\gamma(\hbar) := (1/2)\beta_H(\hbar) + (1/2)(1-\beta_L(\hbar))$ the probability of a $t_H$ signal if expert is strategic and plays $\beta(\hbar) = (\beta_H(\hbar), \beta_L(\hbar))$. For the bad type, this probability is always 1. Additionally, let $\beta_{truth}(\hbar) := (1/2)\beta_H(\hbar) + (1/2)\beta_L(\hbar)$ denote strategic expert's play of right treatment for every severity if hired in $\hbar$. Using continuation values whenever hired, we rewrite expert's continuation value as:

$$\begin{aligned} V(\hbar) = \alpha(\hbar)\{&(1-\delta)[\beta_{truth}(\hbar)u - (1-\beta_{truth}(\hbar))w] + \\ &+ \delta[\gamma(\hbar)V(\hbar,t_H) + (1-\gamma(\hbar))V(\hbar,t_L)]\} + (1-\alpha(\hbar))\delta V(\hbar, Out) \end{aligned} \tag{2.5}$$

For every $\hbar \in \bar{\mathscr{H}}$, let $f(\hbar) := \rho(\hbar) + (1-\rho(\hbar))\gamma(\hbar)$. For every accessible public history $\hbar$, i.e, every history such that $f(\hbar) > 0$, we assume that customer's (updated) belief after observing $t_H$ is given by a standard Bayesian computation:

$$\rho(\hbar, t_H) := \frac{\rho(\hbar)}{f(\hbar)} = \frac{\rho(\hbar)}{\rho(\hbar) + (1-\rho(\hbar))\gamma(\hbar)} \tag{2.6}$$

Recall our definition of $\beta_{truth}(\hbar)$ as strategic expert's expected play of correct actions in $\hbar$. Then customer's payoff if he hires in this realization of public outcome is:

$$v(\hbar) = \rho(\hbar)\left(\frac{u-w}{2}\right) + (1-\rho(\hbar))(\beta_{truth}(\hbar)u - (1-\beta_{truth}(\hbar))w) \tag{2.7}$$

Note that even if expert is known to be good for sure ($\rho(\hbar) = 0$), hiring cannot be a best-response to a good expert's strategy that plays correct actions below a minimal probability in that state (if either $\beta_H$ or $\beta_L$ is lower than $\beta^* := \frac{w-u}{w+u}$). Similarly, even if good expert's strategy implies a correct treatment in $\hbar$, i.e, $\beta_{truth}(\hbar) = 1$, hiring cannot be a best-response to a sufficiently high prior belief, i.e, $\rho(\hbar) > \rho^* = \frac{2u}{w+u}$, with $\rho^*$ as we defined in the one-shot section.

When customer has full memory, he can use his infinite information set to perform a Bayesian update of beliefs after every observable outcome in period $t$ (as in equation (2.6)) and use this posterior as the prior belief of expert being bad type at $t+1$. But every observation

of $t_H$ generates a higher public belief for the next period: whenever hiring, customer expects strategic expert to play $t_H$ with some probability strictly between 0 and 1; so every observation of it at the end of the stage game leads to an increasing belief update.

In every (Nash) equilibrium of the full memory game, as $\delta$ approaches one, it must be that customer hires less frequently, so that expected payoff (as in equation (2.5)) approaches zero. This happens because hiring is only possible if expert is playing correct actions with positive probability for every severity $\theta$. Since the posterior is increasing in the prior, a infinite sequence of $\theta_H$ inevitably leads to posterior hitting critical level $\rho^*$, even if she frequently plays $t_L$ in $\theta_H$ with some probability.

Furthermore, if we assume that expert is hired at any history on equilibrium path at which she is revealed to be good (that is, whenever she plays $t_L$; EV refer to this assumption as renegotiation-proofness), then hiring does not take place at all if expert is sufficiently patient: the temptations to play $t_L$ are so high that she will play it before reaching critical belief $\rho^*$. But if customer knows that incentives to lie are strong enough he will not hire at the critical history in the first place. Expert knows about customer unwillingness to hire and lies whenever close to a belief slightly below critical level. A backward induction leads to the no-hiring result.

The above discussion gives an intuition for the following proposition from EV (which we do not reproduce its proof here).

**Proposition 1** (EV bad reputation effect)**.** *Let $\rho > 0$ be given. For a patient expert, any Nash equilibrium leads her to a (discounted average) payoff of at most zero. Furthermore, if it is assumed that she is hired at any history on the equilibrium path at which she reveals herself to be good, then the unique equilibrium outcome is a never hiring strategy.*

From previous discussion, censoring information in a trivial way (no memory at all, or the infinite repetition of the one-shot interaction) eliminates bad reputation effect (at least for $\rho \leq \rho^*$), but it also makes reputation concerns have no effect at all: it neither tempts expert to lie nor does it help customer take better hiring decisions. As we will show later on, censoring information in a non-trivial way might avoid the dramatic result of the zero equilibrium payoff and the strong incentives to deviate; furthermore, it might approximate equilibrium payoffs to the complete information case.

## 2.3 Repeated game with rating systems

Assume now that customer takes a hiring decision restricted to an element from a finite set of states $\mathcal{I}$ as the only source of information. Since the observable information is known to the expert in every period, her private history at the beginning of the stage game in period $t \geq 1$ is now given recursively by $h^t := (i_t, (h^{t-1}, \theta_{t-1}, y_{t-1}))$, with initial history being the initial state, i.e, $h^0 = i_0$.

Similar to the framework of previous section, whenever hired in $h^t \in H^t := \mathcal{I} \times (H^{t-1} \times Y \times \Theta)$ and upon observing $\theta_H(\theta_L)$, she chooses right treatment $t_H(t_L)$ with probability $\beta_H(h^t)(\beta_L(h^t))$. We refer to this pair of actions as $\beta(h^t)$. Customer observes an element from $\mathcal{I}$ before choosing whether to interact or not with the expert. We assume now that he does not keep track of how many times the stage game has been played. Therefore, his hiring decision is conditioned only on the observation of some $i \in \mathcal{I}$ and we denote it as $\alpha(i)$.

An information designer chooses a finite Markovian rating system, i.e, a public information set $\mathcal{I} = \{1, ..., I\}$ (rating set); an initial distribution $g_0 \in \Delta(\mathcal{I})$ and transition rules $\varphi_y(i, j)$ from any rating $i$ to any rating $j$ upon observing public outcome $y$. We will represent a rating system by the triplet $\phi = (\mathcal{I}, g_0, \varphi)$. The timing of events is as follows: at the beginning of every period, the rating designer announces a rating. Expert observes all history of the game and customer only observes the announced rating. He takes a hiring decision and if it is positive, she takes a treatment decision. The signal is realized and dictates the rating in the beginning of the next period. Figure 3 below summarizes the rating system.

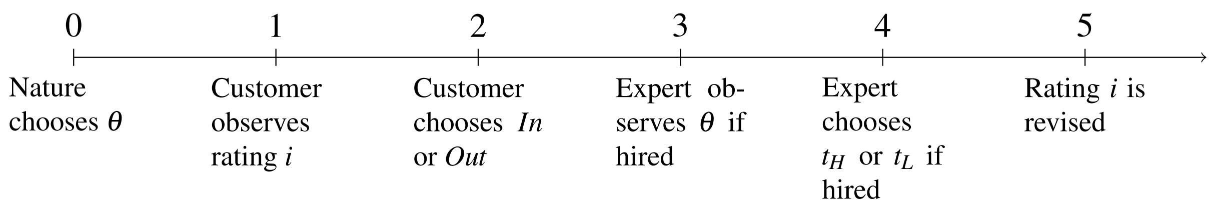



Figure 3 – Information dynamics with a rating system

We note that the full memory setting is a special case of the repeated with a rating system. To see this, suppose information set is not finite anymore, but equals the set of public histories, i.e, $\mathcal{I} = \bar{\mathcal{H}}$. The transition rules of a full-memory setting are the following: consider any period-$t$ and any history $\hbar^t$. If $y$ is observed at the end of the period, let $\tilde{\hbar}^{t+1} = (\hbar^t, y)$ be the new history formed by the concatenation of previous history and the observable signal. Then $\varphi_y(\hbar^t, \tilde{\hbar}^{t+1}) = 1$ and 0 for every other history different than $\tilde{\hbar}^t$. Ekmekci (2011) calls the full-memory setting equivalent rating as the transparent one.

### 2.3.1 Strategies

As before (and abusing notation), expert's strategy is $\beta$ and customer's strategy is $\alpha$. The strategy profile is $\sigma = (\alpha, \beta)$. Each strategy is given by:

$$\beta : \cup_{t=0}^{\infty} (H^t \times \{In, Out\} \times \Theta) \to \Delta\{t_H, t_L\}^2$$
$$\alpha : \mathcal{I} \to \Delta\{In, Out\}$$

The same discussion presented in the full memory setting about strategic expert having a strategy conditioned on public information as a best reply applies here, so we will focus on expert's strategies that are stationary on $\mathcal{J}$.

### 2.3.2 Expert's expected payoff

Let $V(i|\theta)$ denotes the expected continuation value whenever hired in $i$ and conditioned on observing $\theta$. We can write it as:

$$\begin{aligned} V(i|\theta_H) &= (1-\delta)[\beta_H(i)u-(1-\beta_H(i))w]+\delta\sum_{j\in\mathcal{J}}[\beta_H(i)\varphi_H(i,j)+(1-\beta_H(i))\varphi_L(i,j)]V(j) \\ V(i|\theta_L) &= (1-\delta)[\beta_L(i)u-(1-\beta_L(i))w]+\delta\sum_{j\in\mathcal{J}}[\beta_L(i)\varphi_L(i,j)+(1-\beta_L(i))\varphi_H(i,j)]V(j) \end{aligned} \tag{2.8}$$

The expected continuation value is:

$$V(i)=\alpha(i)\left[\frac{1}{2}V(i|\theta_H)+\frac{1}{2}V(i|\theta_L)\right]+(1-\alpha(i))\delta\sum_{j\in\mathcal{J}}\varphi_{Out}(i,j)V(j) \tag{2.9}$$

Recall that $\gamma(i) := (1/2)\beta_H(i)+(1/2)(1-\beta_L(i))$ is the probability of a $t_H$ signal if expert is strategic and plays $\beta(i)=(\beta_H(i),\beta_L(i))$. For the bad type, this probability is always 1. The total transition of each type from any $i$ to $j$ is given by:

$$\tau^S(i,j) := \alpha(i)[\gamma(i)\varphi_H(i,j)+(1-\gamma(i))\varphi_L(i,j)]+(1-\alpha(i))\varphi_{Out}(i,j) \tag{2.10}$$

$$\tau^B(i,j) := \alpha(i)\varphi_H(i,j)+(1-\alpha(i))\varphi_{Out}(i,j) \tag{2.11}$$

Moreover, $\beta_{truth}(i) := (1/2)\beta_H(i)+(1/2)\beta_L(i)$ denote strategic expert's play of right treatment for every severity if hired in $i$. Using continuation values whenever hired in (2.8) and transition probability (2.10), we rewrite good expert's continuation value as:

$$V(i)=(1-\delta)\alpha(i)[\beta_{truth}(i)u-(1-\beta_{truth}(i))w]+\delta\sum_{j\in\mathcal{J}}\tau^S(i,j)V(j) \tag{2.12}$$

The transition rules in (2.10) define a stochastic matrix $T^S=[\tau^S(i,j)]$ and for any initial distribution over states $g_0\in\Delta(\mathcal{J})$, $g_0\cdot(T^S)^t$ gives the distribution on ratings at the start of period $t$. Suppose the process starts in $i$, i.e, $g_0(i)=1$. Let $[\tau^S(i,j)]^t$ denote strategic expert's probability of transitioning from $i$ to $j$ in $t$ periods (the $(i,j)$ entry of $(T^S)^t$). Then (2.8) becomes:

$$V(i)=\sum_{j\in\mathcal{J}}\sum_{t=0}^{\infty}(1-\delta)\delta^t[\tau^S(i,j))]^t\alpha(j)[\beta_{truth}(j)u-(1-\beta_{truth}(j))w]$$

The following lemma (WILSON, 2014) simplifies the analysis of strategic expert's optimal actions and rating policy. It states that we can write her continuation value as a weighted sum of each current payoffs in every rating, with weights being the (discounted) long-run chances of reaching each rating from $i$. The implication of the lemma is that, for any initial distribution $g_0$, $\sum_{t=0}^{\infty}(1-\delta)(g_0 \cdot \delta T^S)^t$ converges to an unique stationary distribution of another auxiliary transition matrix $T^S_{\delta,g_0} = [\tau^S_{\delta,g_0}(i,j)]$ with each entry being $\tau^S_{\delta,g_0}(i,j) := (1-\delta)g_0(j) + \delta\tau^S(i,j)$.

**Lemma 1** (Discounted convergence)**.** *Consider any initial distribution $g_0$, discount factor $\delta \in (0,1)$ and transition matrix $T = [\tau(i,j)]$. Define the alternative transition matrix $T_{\delta,g_0} = [\tau_{\delta,g_0}(i,j)]$ where $\tau_{\delta,g_0}(i,j) = (1-\delta)g_0(j) + \delta\tau(i,j)$. Then $\lim_{t\to\infty}(T_{\delta,g_0})^t$ converges to a stochastic matrix with each row being its unique stationary distribution $f_{\delta,g_0}$. Moreover, $f_{\delta,g_0} = (1-\delta)g_0 \cdot \sum_{t=0}^{\infty}(\delta T)^t = (1-\delta)g_0 \cdot (I-\delta T)^{-1}$.*

Using lemma 1, we can compute the stationary probability vector as a solution to the linear system of equations $f^S_{\delta,g_0}(i) = \sum_{j\in\mathcal{J}} f^S_{\delta,g_0}(j)\tau^S_{\delta,g_0}(j,i)$ for every $i$ as well as $\sum_{i\in\mathcal{J}} f^S_{\delta,g_0}(i) = 1$. Letting $f^S_{\delta,i}$ denote the stationary distribution of expert's transition matrix with $g_0(i) = 1$, we can write continuation value in any $i$ as:

$$V(i) = \sum_{j\in\mathcal{J}} f^S_{\delta,i}(j)\alpha(j)[\beta_{truth}(j)u - (1-\beta_{truth}(j))w] \tag{2.13}$$

If strategic expert was to evaluate her *ex-ante* expected payoff from the rating system, she would take the sum of every expected payoff $V(i)$ weighted by the initial distribution $g_0$, i.e:

$$\bar{V} = \sum_{i\in\mathcal{J}} g_0(i)V(i) = \sum_{i\in\mathcal{J}} f^S_{\delta,g_0}(i)\alpha(i)[\beta_{truth}(i)u - (1-\beta_{truth}(i))w] \tag{2.14}$$

### 2.3.3 Customer's expected payoff and beliefs

Customer does not observe time, so he computes the probability distribution over states as if the game had been going on for a long time with a believed expert's strategy and a known rating system. Technically, this means that he compute beliefs according an improper uniform rule over the calendar time (or time-average convergence), as we explain in this section.

The transitions in (2.10) and (2.11) define stochastic matrices $T^S$ and $T^B$ for the good and bad expert, respectively. We assume that customer computes the long-run chances of finding type $\kappa$ in each rating given $g_0$ by the limiting time-average sum of this type visiting each rating in each period, i.e, they compute $\lim_{t\to\infty}[\frac{1}{t}\sum_{s=0}^{t-1}(g_0 \cdot T^{\kappa})^s]$. The following lemma states that the process $\lim_{t\to\infty}[\frac{1}{t}\sum_{s=0}^{t-1}(T^{\kappa})^s]$ always converges to a stochastic matrix $F^{\kappa}$ where each row is a stationary distribution. Thus, for every $g_0$, the probability vector $f^{\kappa}_{g_0} = g_0 \cdot F$ is a unique

stationary distribution over states. This stationary distribution can be computed as a solution of the system $f^{\kappa}_{g_0}(i) = \sum_{j\in\mathcal{J}} f^{\kappa}_{g_0}(j)\tau^{\kappa}(j,i)$ for every $i$[3] as well as $\sum_{i\in\mathcal{J}} f^{\kappa}_{g_0}(i) = 1$.

**Lemma 2** (Time-average convergence)**.** *Consider any initial distribution $g_0$ and transition matrix $T = [\tau(i,j)]$. Then $\lim_{t\to\infty}[\frac{1}{t}\sum_{s=0}^{t-1}(T)^s]$ converges to a stochastic matrix $F$ with each row being a stationary distribution, so that $f_{g_0} = g_0 \cdot F$ is a unique stationary distribution, i.e, $f_{g_0} \cdot T = f_{g_0}$.*

Let bad type's induced distribution under time-average convergence be $f^B_{g_0} := \{f^B_{g_0}(i)\}_{i\in\mathcal{J}}$ and strategic type's induced distribution be $f^S_{g_0} := \{f^S_{g_0}(i)\}_{i\in\mathcal{J}}$. For every $i \in \mathcal{J}$, let $f(i) := \rho f^B_{g_0}(i) + (1-\rho) f^S_{g_0}(i)$. We say that state $i$ is accessible if there exists a positive chance of being reached by at least one of expert's type, i.e, $f(i) > 0$. For every accessible state $i$, We assume that (updated) belief is given by a standard Bayesian computation:

$$\rho(i) := \frac{\rho f^B_{g_0}(i)}{f(i)} = \frac{\rho f^B_{g_0}(i)}{\rho f^B_{g_0}(i) + (1-\rho) f^S_{g_0}(i)} \tag{2.15}$$

We will refer to a consistent belief distribution $\rho := \rho(i)_{i\in\mathcal{J}}$ when each accessible state has a computed belief according to (2.15). Recall our definition of $\beta_{truth}(i)$ as expert's expected play of correct actions in $i$. Then customer's payoff if he hires in this rating is:

$$v(i) = \rho(i)\left(\frac{u-w}{2}\right) + (1-\rho(i))(\beta_{truth}(i)u - (1-\beta_{truth}(i))w) \tag{2.16}$$

Note that even if expert is known to be good for sure ($\rho(i) = 0$), hiring cannot be a best-response to a good expert's strategy that plays correct actions below a minimal probability in that state (if either $\beta_H$ or $\beta_L$ is lower than $\beta^* := \frac{w-u}{w+u}$). Similarly, even if good expert's strategy implies a correct treatment in $i$, i.e, $\beta_{truth}(i) = 1$, hiring cannot be a best-response to a sufficiently high prior belief, i.e, $\rho(i) > \rho^* = \frac{2u}{w+u}$.

If customer was to evaluate his *ex-ante* expected payoff from the rating system, he would take the sum of each gain in each state from interacting with each expert, weighted by the long-run chances of each expert reaching each state. This is equivalent to a computation of the time-average geometric sum of stochastic matrices from initial periods towards infinity. Precisely, let $g^{\kappa}_t(i)$ denote the probability distribution of type $\kappa \in \{B,S\}$ at the start of period $t$ under time-average convergence. Then from lemma 2:

[3] If the transition matrix contains exactly one irreducible subset (as it will be the case for most of our discussion), in a remark following lemma 2 in the appendix, we show that the distribution is unique and independent of the initial distribution $g_0$.

$$
\begin{aligned}
\bar{v} &= \lim_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1}\sum_{i\in\mathcal{I}}\alpha(i)\left[\rho g_s^B(i)\left(\frac{u-w}{2}\right)+(1-\rho)g_s^S(i)(\beta_{truth}(i)u-(1-\beta_{truth}(i)w))\right] \\
&= \rho\sum_{i\in I}\alpha(i)f_{g_0}^B(i)\left(\frac{u-w}{2}\right)+(1-\rho)\sum_{i\in I}\alpha(i)f_{g_0}^S(i)(\beta_{truth}(i)u-(1-\beta_{truth}(i)w))
\end{aligned}
\tag{2.17}
$$

### 2.3.4 Equilibrium and optimal ratings

Given any rating system $\phi = (\mathcal{I}, g_0, \varphi)$, a strategy profile $\sigma = (\alpha, \beta)$ induces payoffs in every rating and stochastic matrices that determines the expected payoffs of the game for the good expert and the customer as well as a distribution of beliefs $\rho = \rho(i)_{i\in\mathcal{I}:f(i)>0}$ for the customer.

However, since customer only observes $i$, he must define actions and beliefs at the start of the game in terms of what he thinks expert's strategy will be, given the information mechanism. At the same time, expert must choose her strategy in terms of what she thinks customer's strategy will be, since she cannot generate a perturbation on beliefs by changing customer's actions as the game evolves.

Another issue to discuss is the states where customer does not expect to reach. If in equilibrium it turns out to be that this is not the case, due to expert's deviation, our definition of beliefs does not say anything yet on how customer will take proper actions. Similarly, if expert could deviate to reach an *a priori* non-reachable rating, it is not clear whether she would do so.

As in Lillethun (2016), we deal with the first issue by recurring to a believed strategy profile $\hat{\sigma} = (\hat{\alpha}, \hat{\beta})$. Players define strategy, compute stationary distributions and form beliefs in terms of this believed profile. In equilibrium, no deviations from it should take place, so the strategy profile equals the believed one. To deal with the second issue, note that bad expert is a behavioral type, so customer should not expect deviations from this type of player. If he eventually observes an *a priori* non-reachable rating $i$, it must be due to strategic expert's deviation, so it is reasonable to suppose in this case that $\rho(i) = 0$ (so customer hires if expert does not have reasons not to play right actions)[4].

Formally, given a rating system $\phi$, an equilibrium is a strategy profile $\sigma$, a believed profile $\hat{\sigma}$ and a belief distribution $\rho$ such that, for every $i \in \mathcal{I}$:

i $\alpha(i) > 0 \Leftrightarrow v(i|\phi, \hat{\sigma}) = \hat{\rho}(i)\left(\frac{u-w}{2}\right) + (1-\hat{\rho}(i))(\hat{\beta}_{truth}(i)u - (1-\hat{\beta}_{truth}(i))w) \geq 0$;

ii $\beta(i)$ maximizes $V(i|\phi, \hat{\sigma}) = \sum_{j\in\mathcal{I}} \hat{f}^S_{\delta,i}(j)\hat{\alpha}(j)[\beta_{truth}(j)u - (1-\beta_{truth}(j))w]$;

4 This does not generate a permanent deviating behavior towards this newly achievable rating because customer will not hire in any state $j$ that leads to $i$ if good expert's incentive to lie is strong enough.

iii $\hat{\rho}(i)$ is consistent whenever reachable (derived from Bayes' rule given $\rho$ and $\hat{\sigma}$); otherwise $\hat{\rho}(i) = 0$;

iv $\sigma = \hat{\sigma}$.

We will often refer to the strategy where expert plays right actions whenever hired as a truth-telling strategy; and the strategy where customer takes a hiring decision in at least one rating that otherwise would not do so in the one-shot interaction as an influential strategy. We will say that an equilibrium is truth-telling and influential if expert plays a truth-telling strategy and customer plays an influential one.

Since equilibrium strategy profiles are defined for any arbitrary rating system and implied beliefs affect equilibrium payoffs, we could also ask what rating systems implement some strategy profile as an equilibrium with the highest expected payoffs. Thus, the intermediary designer could choose ratings conditioned on their best equilibrium outcomes.

To do so, let $\Phi(\sigma)$ denote the set of rating systems that implement the strategy profile $\sigma$ as an equilibrium. Suppose first that the rating designer wishes to maximize strategic expert's *ex-ante* expected payoff as in (2.14). It would solve the following problem:

$$\underset{\phi \in \Phi(\sigma)}{sup} \left\{ \bar{V}(\phi, \sigma) \right\} \tag{2.18}$$

An optimal rating policy (if it exists) would be the one attaining such value. Suppose instead that the rating designer were to maximize customer's *ex-ante* expected payoff as in (2.17). It would solve the following problem:

$$\underset{\phi \in \Phi(\sigma)}{sup} \left\{ \bar{v}(\phi, \sigma) \right\} \tag{2.19}$$

# 3 RATING SYSTEMS WITH TWO STATES

To illustrate our model, we analyze rating processes with two states. This is a very special case, since it requires interaction with the smallest class of memories (one bit only). We start the analysis with a very simple system, where transition rules are deterministic. We show that even in this simple environment, censoring information can improve customer's and expert's *ex-ante* expected payoffs relative to extreme memory settings (one-shot and full memory) for a range of prior beliefs. We then characterize optimal two-state systems for both players and discuss how they differ according to each player's viewpoint.

## 3.1 Example: deterministic rating system

Consider the following rating system with two states: starting in 2, strategic expert remains there as long as no observation of $t_H$ has taken place. Every time this happens, she moves to state 1 and returns to state 2 after customer chooses not to hire in that state in one period. If she eventually gets hired in 1, she stays there. If she does not get hired in 2, she stays there as well.

Our equilibrium candidate is never hiring in 1 (but hiring in 2) and always playing a truth-telling strategy in 1 (if she were to be hired, she would tell the truth). One way of interpreting these ratings is that rating 1 is the "bad" one and rating 2 is the "good" one. This system and the equilibrium candidate are represented in figure 4 below.

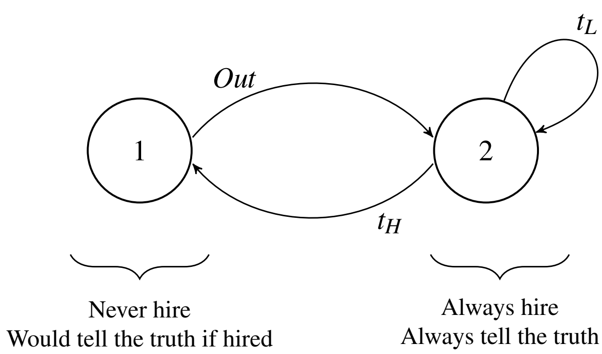



Figure 4 – A two-state rating system with equilibrium candidate

Lemmas 1 and 2 ensure that the stochastic processes both for strategic expert and myopic customer will converge to stationary distributions. Furthermore, observe that in our system, all stochastic transition matrices are itself irreducible. Thus, for the time-average convergence, the limiting distribution will be the same irrespective of the initial distribution, i.e, $f^{\kappa}_{g_0} = f^{\kappa}$ for

every $g_0 \in \Delta(\{1,2\})$ and $\kappa \in \{S,B\}$. Therefore, to compute customer's equilibrium beliefs, we simply solve $f^\kappa \cdot T^\kappa = f^\kappa$. The transition matrices are:

$$\mathbf{T}^B = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \mathbf{T}^S = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

Recall that the entries of the matrices $T^S$ and $T^B$ above are computed from equations (2.10) and (2.11), respectively. The expert moves from state 2 to state 1 with probability $1/2$ because $\theta_H$ is observed in every hiring interaction with probability $1/2$. The stationary distributions are $f^B = (1/2, 1/2)$ and $f^S = (1/3, 2/3)$. From consistency requirement, (equation (2.15)), implied beliefs are:

$$\rho(1) = \frac{3\rho}{3\rho + 2(1-\rho)} \qquad \rho(2) = \frac{3\rho}{3\rho + 4(1-\rho)}$$

Customer's strategy is indeed part of an equilibrium for this rating system if and only if $\rho(1) > \rho^*$ and $\rho(2) \leq \rho^*$ (condition (i) of our equilibrium concept). It is straightforward to verify that belief in 2 is always lower than initial prior and belief in 1 is always higher than the initial prior. Thus, if customer hires in the one-shot interaction, hiring in 2 is always optimal; not hiring in 1 is a best response for a range of initial prior values. Similarly, if customer does not hire in the one-shot interaction, not hiring in 1 is always optimal; hiring in 2 is a best response for a range of initial prior values. Specifically, from our computation of implied beliefs, this range of prior beliefs over which not hiring in 1 and hiring in 2 is optimal is:

$$\rho \in \left( \frac{2\rho^*}{3-\rho^*}, \frac{4\rho^*}{3+\rho^*} \right] \tag{3.1}$$

From equation (2.17), we find customer's *ex-ante* expected payoff for the obtained range of parameters and represent it in terms of initial prior beliefs below. It is easily verified that it is higher than the one-shot interaction payoff in (2.1) if $\rho$ is contained in the interval (3.1).

$$\bar{v}^*(\rho) = \begin{cases} \rho\left(\frac{u-w}{2}\right) + (1-\rho)u \ \text{ if } \rho \leq \frac{2\rho^*}{3-\rho^*} \\ \rho\left(\frac{u-w}{4}\right) + (1-\rho)\frac{2u}{3} \ \text{ if } \rho \in \left(\frac{2\rho^*}{3-\rho^*}, \frac{4\rho^*}{3+\rho^*}\right] \\ 0 \ \text{ if } \rho > \frac{4\rho^*}{3+\rho^*} \end{cases} \tag{3.2}$$

Expert's strategy is part of an equilibrium if and only it maximizes her expected payoff given customer's strategy (condition (ii) of our equilibrium requirement). This is equivalent not

to profiting from deviations. In state 1, it is optimal to play correct actions if she eventually gets hired: not doing so yields a negative payoff without moving her to the hiring state. In state 2, it is optimal to play $t_L$ whenever she observes $\theta_L$, since that yields a positive payoff and maintains her at the hiring state. The only reasonable deviation might be playing $t_L$ in $\theta_H$: doing so yields a negative payoff but guarantees she will stay in the hiring state in the next period. Formally, from equations in (2.8), we need to test whether the following holds:

$$V(2|t_H,\theta_H) = (1-\delta)u + \delta V(1) \geq (1-\delta)(-w) + \delta V(2) = V(2|t_L,\theta_H)$$

From lemma 1, we can compute equilibrium expected payoff $V(1)$ and $V(2)$. To do so, we need to find the probability vector satisfying $f^S_{\delta,g_0} \cdot T^S_{\delta,g_0} = f^S_{\delta,g_0}$ and $f^S_{\delta,g_0} \cdot \mathbf{1} = 1$, where the $(i,j)$-th entry of $T^S_{\delta,g_0}$ is $\tau^S_{\delta,g_0} := (1-\delta)g_0(j) + \delta\tau^S(i,j)$. Therefore, this stochastic matrix is given by:

$$\mathbf{T}^S_{\delta,g_0} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} (1-\delta)g_0(1) & (1-\delta)g_0(2)+\delta \\ (1-\delta)g_0(1)+\frac{\delta}{2} & (1-\delta)g_0(2)+\frac{\delta}{2} \end{pmatrix} \end{array}$$

That gives the stationary probabilities:

$$f^S_{\delta,g_0}(1) = \frac{(1-\delta)g_0(1)+\delta/2}{1+\delta/2} \qquad f^S_{\delta,g_0}(2) = \frac{(1-\delta)g_0(2)+\delta}{1+\delta/2}$$

Note that as $\delta \to 1$, this stationary distribution equals the one we computed for analyzing customer' s equilibrium behavior. From equation (2.13), to find $V(i)$, we use the above distribution as if the process started in $i$, i.e, as if $g_0(i) = 1$ and 0 otherwise. Therefore, expected payoffs are:

$$V(1) = f^S_{\delta,1}(2)u = \left[\frac{2\delta}{2+\delta}\right]u \qquad V(2) = f^S_{\delta,2}(2)u = \left[\frac{2}{2+\delta}\right]u$$

Using these values, we rewrite expert's incentive compatibility constraint as:

$$(u+w) \geq \left[\frac{\delta}{2+\delta}\right]2u$$

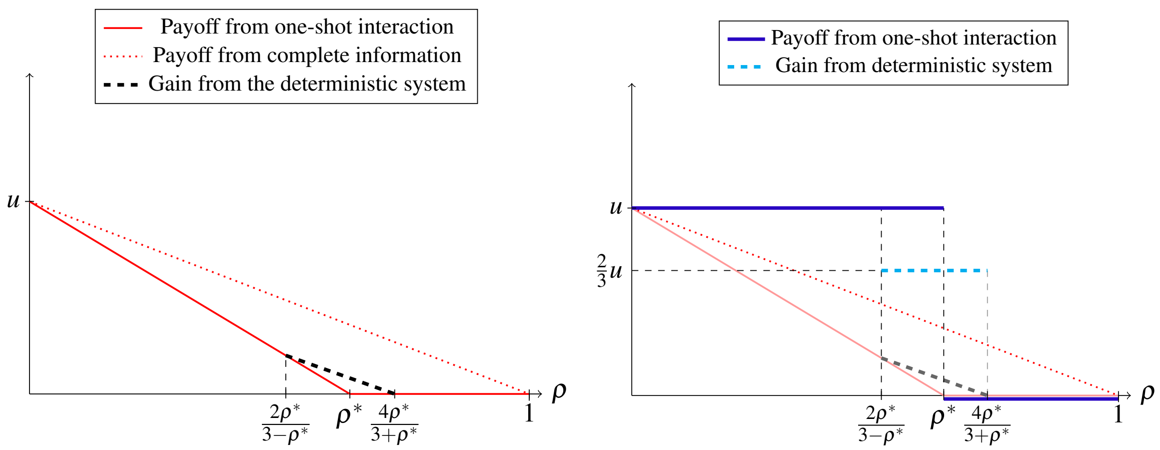
Since $u+w > 2u$ from the definition of $\rho^*$, we see that expert does not deviate from playing $t_H$ after observing $\theta_H$ in 2, no matter her discount rate. Since the process starts in 2,

expert's expected payoff as in (2.14) is V(2) for the obtained range of parameters. We depict her *ex-ante* expected payoff for every initial prior belief and for $\delta \to 1$ below. Note that for $\rho \in$ (3.1), a patient expert gets precisely the long-run probability of remaining in the hiring state, no matter where she starts:

$$\bar{V}^*(\rho) = \begin{cases} u \ \text{ if } \rho \leq \frac{2\rho^*}{3-\rho^*} \\ \frac{2u}{3} \ \text{ if } \rho \in \left(\frac{2\rho^*}{3-\rho^*}, \frac{4\rho^*}{3+\rho^*}\right] \\ 0 \ \text{ if } \rho > \frac{4\rho^*}{3+\rho^*} \end{cases} \tag{3.3}$$

The strategy profile of never hiring in 1 but hiring in 2 and always playing truthfully in 2 is indeed an equilibrium of the game for the obtained range of prior beliefs. The bad reputation effect as in proposition 1 does not hold anymore: good expert avoids never being hired in equilibrium, thus securing a positive payoff even if very patient ($\delta \to 1$). Furthermore, customer benefits from reputation for a range of parameters, since his hiring decision now allows better separation of types.

In figure 5, we represent the payoffs from the example as in equations (3.2) and (3.3) compared with the one-shot interaction and the complete information case (perfect separation). In panel (a), the black dashed line gives the gain for the range of prior beliefs over which customer *ex-ante* expected payoff is higher than the one-shot interaction. In panel (b), the solid blue line represents expert's expected payoff in the one-shot interaction and the dashed light blue line the gain for the range of prior beliefs over which the influential and truth-telling strategy profile is an equilibrium.




(a) Customer's expected payoffs

(b) Expert's expected payoffs

Figure 5 – Gains from deterministic system

Note in panel (b) of figure 5 above that for $\rho$ between $(2\rho^*/(3-\rho^*))$ and $\rho^*$, even though customer is better off relative to no system at all (one-shot interaction), expert is being

harmed by the information mechanism: she would prefer an uninformative system (does not bring new insights to change customer's hiring decision), since she would be hired for sure in this environment. As we show in the next section, there are rating systems implementing the same truth-telling and influential strategy profile for a higher range of prior beliefs that lead to higher expected payoffs.

## 3.2 Optimal systems

In order to benefit from reputation, any rating system in the two-state system must be influential for the customer. Indeed, if customer does hire in the one-shot interaction, hiring in both ratings would lead to the same *ex-ante* expected payoff; if he does not hire, not hiring in both ratings will maintain him with same payoff as well.

When this is the case, the following lemma shows that strategic expert chooses to play right actions in the hiring state and this does not depend on the rating system, nor does it depend on the discount factor. Intuitively, suppose that 1 is the non-hiring rating and 2 the hiring one. If rating 2 is absorbing, there is no reason not to play truthfully in it; if rating 1 is absorbing, a systematic deviation yielding $-w$ cannot change her probabilities of eventually reaching 1 and obtaining 0 forever. When both ratings are non-absorbing, loosing $-w$ in every deviation to keep being hired cannot be better than not being hired in next period and having a positive chance of returning to the hiring state.

**Lemma 3** (Truth-telling is optimal in a two-state rating system)**.** *Consider a two-state rating system with a non-hiring state* $(1)$ *and a hiring one* $(2)$*. Playing truthfully in the hiring state is always optimal for the strategic expert.*

Suppose that customer plays an influential strategy. Lemma 3 tells that strategic expert will play a truth-telling strategy. Without loss of generality, suppose that customer's stationary distributions does not depend on the initial distribution. From the equilibrium requirements, given customer expects expert to play a truth-telling strategy, customer's hiring decision is part of an equilibrium if and only if implied beliefs are consistent, i.e, $\rho(1) > \rho^*$ and $\rho(2) \leq \rho^*$. Manipulating equation (2.15), this is equivalent to the following relations:

$$f^B(1) > \left[\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right] f^S(1) \qquad\qquad f^B(2) \leq \left[\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right] f^S(2)$$

Since the ratio $\left[\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right]$ will often appear in evaluation of rating systems, it will convenient to refer to it as $\ell$. Note that $\ell > 0$. Furthermore, manipulating equation (2.1), we see that customer hires in the one-shot interaction if and only if $\ell \geq 1$. We can interpret this ratio as the relative prior belief of expert being bad $\left(\frac{\rho}{1-\rho}\right)$ compared with relative stage game absolute

rewards from interacting with each type $\left(\frac{\rho^*}{1-\rho^*} = \frac{u}{(1/2)|u-w|}\right)$. In that sense, it is a measure of the prior bias towards hiring in the one-shot interaction.

Computing steady-state probabilities $f^\kappa$ from each type $\kappa$ transition matrix $T^\kappa$ and using the fact that $\tau^S(1,2) = \tau^B(1,2)$ (since rating 1 does not have hiring, the total transition probability is the same for both types), we find that stationary distributions of rating 1 and 2 are somewhat connected by the following relation:

$$\frac{f^B(1)}{f^S(1)} = \left[\frac{\varphi_H(2,1)+\varphi_H(2,1)}{\varphi_H(2,1)+\varphi_L(2,1)}\right]\left(\frac{f_2^B}{f_2^S}\right) \leq 2\left(\frac{f_2^B}{f_2^S}\right)$$

Therefore, we have the restrictions on equilibrium stationary distributions below. The first and the second comes from the belief consistency requirement and the third comes from the computation of stationary probabilities. Using these inequalities, we analyze how to design the two-state rating system to maximize each player's *ex-ante* expected payoff.

$$\frac{f^B(1)}{f^S(1)} > \ell \qquad \frac{f^B(2)}{f^S(2)} \leq \ell \qquad \frac{f^B(1)}{f^S(1)} \leq 2\left(\frac{f_2^B}{f_2^S}\right) \tag{3.4}$$

### 3.2.1 Expert's optimal policy

When $\ell \geq 1$, customer hires in the one-shot interaction ($\rho \leq \rho^*$). As we noted in the deterministic example, intermediary designer could benefit expert in this case simply by devising an uninformative rating where hiring takes place irrespective of what rating customer finds himself him. Take for instance $\varphi_L(i,j) = \varphi_H(i,j) = 1$ for $i = \{1,2\}$. This yields $f^B = f^S = (1/2, 1/2)$, with implied beliefs being $\rho(1) = \rho(2) = \rho$. Clearly, hiring in every rating and playing truth-telling strategy is an equilibrium generating $\bar{V} = u$ (irrespective of the initial distribution).

When $\ell < 1$, the rating designer must solve a more interesting problem: how to persuade customer to hire in at least one rating when in absence of any system never hiring is the unique best response. If he hires in both ratings, expert must be playing right actions in both with at least some probability. In that case, there is no reason not to play a truth-telling strategy and customer's *ex-ante* expected payoff equals the one shot interaction. But this is negative, so hiring in both ratings is not optimal. We thus focus on the influential equilibrium: hiring in 2 but not in 1.

From lemma 3, expert will play a truth-telling strategy and this does not depend on the discount factor. Therefore, we can solve the problem for a patient strategic expert, i.e, for $\delta \to 1$. Using this conclusion that the the highest equilibrium outcome is a truth-telling and influential one and recalling the constraints in (3.4), the optimal rating problem as in (2.18) for $\ell < 1$ boils down to choosing the stationary distributions.

$$\begin{aligned} \underset{(f^S(1),f^B(2))}{Maximize} \quad & (1-f^S(1))u \\ \text{s.t} \quad & \frac{1-f^B(2)}{f^S(1)} > \ell, \\ & \frac{f^B(2)}{1-f^S(1)} \leq \ell, \\ \frac{1-f^B(2)}{f^S(1)} & \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right). \end{aligned} \tag{3.5}$$

When $\ell \geq 1$, expert can achieve her highest possible equilibrium payoff: $u$. When $\ell < 1$, as we show in proposition 2 below, the belief consistency requirements and the connection between long-run frequencies generates a bound on what expert can overcome in terms of equilibrium payoffs. Moreover, there are optimal values of the stationary distributions in terms of $\ell$ that generates the truth-telling and influential equilibrium if and only $\ell$ is between some lower bound and 1. Intuitively, if customer is very inclined not to hire in the one-shot interaction, two-state systems cannot persuade him otherwise. Once we find values of the distributions, we can design the transition rules to achieve such values. This requires some randomization of $\varphi_H(2,1)$ and $\varphi_{Out}(1,2)$.

**Proposition 2** (Expert's optimal rating in a two-state equilibrium)**.** *In a two-state rating system, strategic expert's maximum ex-ante expected payoff is:*

$$\bar{V}^*(\rho) = \begin{cases} u & \text{if } \rho \leq \rho^*, \\ \left[2 - \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}\right]u & \text{if } \rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

*For $\rho \leq \rho^*$, any combination of transition rules that induce hiring in every state is an optimal rating policy. For $\rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$f^B(2) = 2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right) - 1 \qquad\qquad f^S(1) = -1 + \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}$$

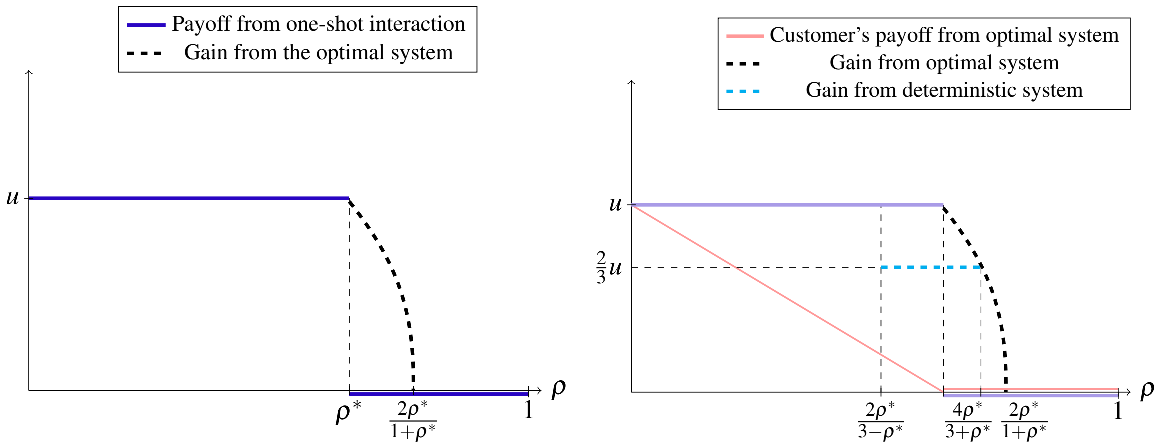
*Those probabilities are achieved by setting $\varphi_L(2,1) = 0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^* = \frac{\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - \frac{1}{2}}{1 - \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}}$$

When $\ell \leq \frac{3}{4}$, we have $r^* \leq 1$, so it is possible to set $\varphi_H(2,1) = 1$ and $\varphi_{Out}(1,2) = r^*$. That means when customer is less inclined to hire in the one-shot interaction, the rating designer must set the system to make customer more confident to be dealing with a strategic type, by having non-hiring state to be more absorbing. Conversely, when $\ell > \frac{3}{4}$, $r^* > 1$, so it is possible to set $\varphi_{Out}(1,2) = 1$ and $\varphi_H(2,1) = \frac{1}{r^*}$. That means when customer is more inclined to hire in the one-shot interaction, rating designer can set the system to make good expert leave relatively less the hiring state.

Note that this system makes customer get the same *ex-ante* expected payoff as the one-shot interaction. This happens because he does not hire in 1 but gains the same outside option payoff in 2, i.e, 0, due to $f^B(2) = \ell(1 - f^S(1))$. However, expert is better off relative to the deterministic rating system: she gets hired for sure for every $\rho \leq \rho^*$ and with positive frequency for values above the deterministic threshold as in (3.1). Therefore, a two-state rating system brings some efficiency in the expert-customer relation.

We depict this comparison in figure 6 below. In Panel (a), black dashed line gives the relative gain from the optimal system compared with the one-shot interaction as in proposition 4. Blue line represents expert *ex-ante* expected payoff in the one-shot interaction as in previous figures. In panel (b), red line gives customer's expected payoff from the mechanism, which is equal to the one-shot interaction. The light dashed blue line represents expert's payoff from the deterministic system as in (3.3). Note that the optimal rating policy generates a better payoff everywhere within the interval (3.3).




(a) one-shot vs. optimal system

(b) deterministic vs. optimal system

Figure 6 – Expert's gain from optimal two-state system

### 3.2.2 Customer's optimal policy

From customer's viewpoint, for every prior belief $\rho$ (even when $\ell \geq 1$), an informative rating system implementing an influential (thus truth-telling, from lemma 3) equilibrium is beneficial relative to the one-shot interaction, as we discussed at the beginning of the section.

Using the constraints in (3.4), the optimal rating problem as in (2.19) is now given by:

$$
\begin{aligned}
&\underset{(f^S(1),f^B(2))}{Maximize}\ \rho f^B(2)\left(\frac{u-w}{2}\right)+(1-\rho)(1-f^S(1))u \\
&\text{s.t} \qquad \frac{1-f^B(2)}{f^S(1)} > \ell, \\
&\qquad\quad \frac{f^B(2)}{1-f^S(1)} \leq \ell, \\
&\qquad\quad \frac{1-f^B(2)}{f^S(1)} \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right).
\end{aligned} \tag{3.6}
$$

Note that customer's problem implies minimizing $f^B(2)$ and $f^S(1)$, while expert's problem implies minimizing only $f^S(1)$. That means the last constraint must be binding as we show in the next proposition. Furthermore, the choice of $f^S(1)$ (thus $f^B(2)$) leading to maximum *ex-ante* expected payoff is a point satisfying both first and second constraints, but it holds if and only if $\ell$ is between some upper bound and the lower bound we computed in previous proposition. The intuition is that if customer is very inclined to hire in the one-shot interaction, he will do so in both ratings no matter the two-sate system. As before, the transition rules to achieve optimal distributions require some randomization of $\varphi_H(2,1)$ and $\varphi_{Out}(1,2)$.

**Proposition 3** (Customer's optimal rating in a two-state equilibrium)**.** *In a two-state rating system, customer's maximum ex-ante expected payoff is:*

$$
\bar{v}^*(\rho) = \begin{cases}
\rho\left(\frac{u-w}{2}\right)+(1-\rho)u & \text{if } \rho \leq \frac{\rho^*}{2-\rho^*}, \\
\rho\left(\frac{w-u}{2}\right)\left[\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}-1\right]^2 & \text{if } \rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right], \\
0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}.
\end{cases}
$$

*For $\rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$
f^B(2) = \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}-1 \qquad\qquad f^S(1) = \frac{2-\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}
$$

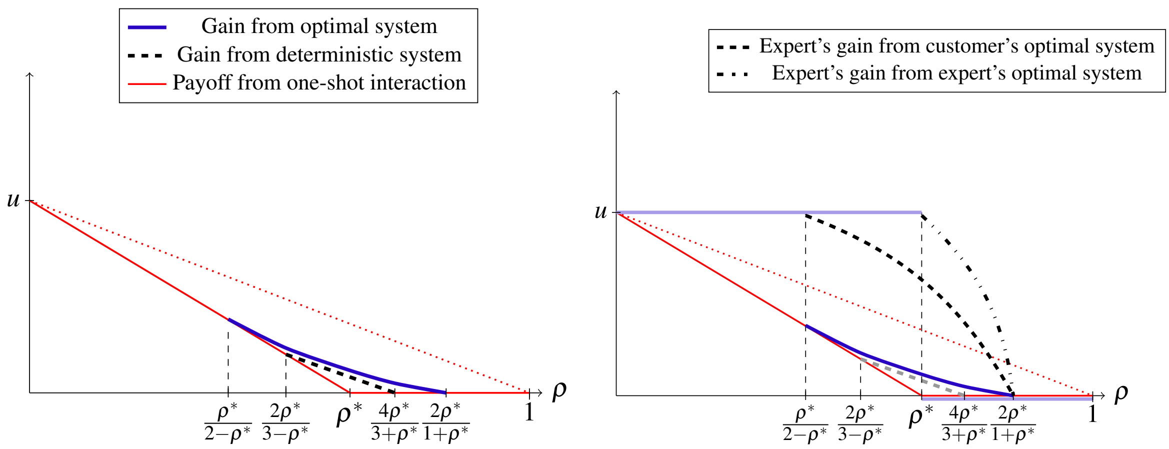
*Those probabilities are achieved by setting $\varphi_L(2,1)=0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^{**} = \frac{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1}{2 - \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}$$

When $\ell \leq \frac{9}{8}$, it is possible to set $\varphi_H(2,1) = 1$ and $\varphi_{Out}(1,2) = r^{**}$, since this value will be at most 1. That means when customer is less inclined towards hiring in the one-shot interaction, a rating designer must set the non-hiring state to be more absorbing, thus making customer more confident to be dealing with a good type in the hiring state. When $\ell > \frac{9}{8}$, it is possible to set the opposite: $\varphi_H(2,1) = \frac{1}{r^{**}}$ and $\varphi_{Out}(1,2) = 1$. That means when customer is more inclined towards hiring in the one-shot interaction, a rating designer must set the non-hiring state to be more transient and the hiring state more absorbing. That way, customer is more confident not to be dealing with good type in the non-hiring state.

In this system, customer gets a higher expected payoff than the one-shot interaction and the deterministic rating system for prior beliefs between $\frac{\rho^*}{2-\rho^*}$ and $\frac{2\rho^*}{1+\rho^*}$. But the optimal stationary distribution $f^S(1)$ is higher than the optimal one from expert's point of view as in proposition 4. Indeed, for values below $\rho^*$, expert is worse off than the one-shot interaction.

Therefore, differently than what we found when maximizing expert's *ex-ante* expected payoff, maximizing customer payoffs implies harming expert and there is some conflict of interest between the players. Intuitively, customer is more inclined towards sending strategic types to non-hiring state to reduce the chances of hiring a bad type, while a strategic expert is more inclined to accept some bad types in the hiring state if this could reduce her chances of being more frequently in the non-hiring state.




(a) one-shot, det. and optimal system

(b) Expert's gain from each player's system

Figure 7 – Customer's gain from optimal two-state system

We give a visual representation of this comparison in figure 7 above. In Panel (a), black

dashed line gives the relative gain from the deterministic system compared with the one-shot interaction; the blue line gives the gain from the optimal system as in proposition 3; solid red line gives the one-shot expected payoff and dashed red line the complete information payoff. In panel (b), light blue line gives expert's expected payoff from the one-shot interaction; dashed black line gives the gain from optimal system from customer's viewpoint as in proposition 3 and dashed-dotted black line the gain from optimal system from expert's viewpoint as in proposition 4. Note that expert gain is lower under customer's optimal system relative to her own optimal system in every $\rho \in \left[\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right)$.

# 4 EXTENSIONS

We have shown optimal two-state systems from each player's viewpoint. One immediate question that our model can bring new insights to is how to design an optimal system if the rating designer wants to maximize hiring frequency in the economy. In this chapter, we deal with this question. What is interesting here is that there is no conflict of interests between the rating designer and the expert if the former were to maximize hiring: the optimal rating policy coincide with expert's optimal policy as in proposition 4.

## 4.1 Rating systems to maximize hiring

Recall our environment of an intermediary agency under an static framework designing some mechanism to bring the chances of hiring a bad expert and not hiring a strategic one to certain probability targets ($f^B_{In}$ and $f^S_{Out}$), as we discussed in section 2.1 of chapter 2. Given the policy, customer has some updated beliefs $\rho_{In} = \rho f^B_{In}/f_{In}$ and $\rho_{Out} = \rho(1-f^B(In))/f_{Out}$ on what are the chances of hiring a bad type and not hiring a strategic one, respectively. As we discussed in that section, strategic expert has no reason to lie in the static framework. Customer hires if and only if $\rho_{In} \leq \rho^*$.

When is possible to set $f^B_{In} = f^S_{Out} = 0$, players get their complete information payoff and hiring frequency is $f_{In} = 1-\rho$. However, it is possible to do better if designer's objective is to maximize this hiring frequency $f_{In}$. Since customer always hires when $\rho \leq \rho^*$, it could set $f^B_{In} = 1$ for prior beliefs below the critical level and setting this target to a minimum level when beliefs are above $\rho^*$. This minimum level is given by the constraint $\rho_{In} \leq \rho^*$, which is binding when $\rho > \rho^*$. That leads to the following benchmark regarding the optimal policy to maximize hiring:

**Lemma 4** (Optimal policy for maximum hiring chances in static framework)**.** *Maximum hiring frequency $f^*_{In}$ in the one-shot interaction with an intermediary agency offering recommendation signals $f^B_{In}$ and $f^S_{Out}$ is:*

$$f^*_{In}(\rho) = \begin{cases} 1 & \textit{if } \rho \leq \rho^* \\ \left[\frac{1-\rho}{1-\rho^*}\right] & \textit{if } \rho > \rho^* \end{cases}$$

*For $\rho \leq \rho^*$, the optimal recommendation signals are $f^B_{In} = 1$ and $f^S_{Out} = 0$. For $\rho > \rho^*$, the optimal recommendation signals are:*

$$f^B_{In} = \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - 1 \qquad\qquad f^S_{Out} = 0$$

Under these signals, customer gets the one-shot payoff for every prior and good expert gets $u$. The hiring chances are now $1 > (1-\rho)$ when $\rho \leq \rho^*$ and $\frac{1-\rho}{1-\rho^*} > (1-\rho)$ when $\rho > \rho^*$. The information designer benefits from the fact that hiring takes place under low prior beliefs by setting $\rho_{In} = 0$ if $\rho \leq \rho^*$ and from the fact that indifference favors hiring by setting $\rho_{In} = \rho^*$ if $\rho > \rho^*$. Hiring takes place more often than under the one-shot interaction and the optimal policy from customer's point of view.

There are many reasons why designer's objective would be maximizing the hiring frequency. First, a strategic expert (or a firm of experts) could be interested in designer's service to maximize her chances of being hired, but the designer cannot know *ex-ante* what type of expert it is dealing with (or what are the composition of bad and good experts in a firm) or it has incentives to manipulate information disclosure in favor of its client. This is in line with a literature on information intermediaries inflating signals such as credit rating agencies (LIZZERI, 1999; FULGHIERI; STROBL; XIA, 2014 and DAVID; ISAKIN, 2015).

Second, the designer could be an online review platform interested in maximizing the income flow from hiring interactions and at the same time being beneficial both to customers and experts (or at least not harming players relative to the lack of any platform) and without knowing expert's type. This might be the case of Yelp services, for example. We could interpret the designer as an intermediary whose objective is to persuade, through an information structure, a Bayesian receiver to take the best action for its client (sender), as in Kamenica and Gentzkow (2011). It can design how to reveal information, but it cannot distort it. This is in line with a literature on optimal information disclosure when the information designer has itself strategic interests (RAYO; SEGAL, 2010; PEI, 2016).

Suppose now we are in a two-state rating system framework just as in chapter 3. Consider first the deterministic two-state example we presented in first section of that chapter. Recall the interval (3.1) for which the system implements a truth-telling and influential equilibrium. For $\rho \leq \frac{2\rho^*}{3-\rho^*}$, hiring takes place in every rating; for $\rho > \frac{4\rho^*}{3+\rho^*}$ hiring is never a best response; for $\rho$ between these values, $f_{In} = \rho(1/2) + (1-\rho)(2/3)$. Just as the case with expert's payoff under this system, for some range of prior beliefs, the hiring frequency is now lower than the one-shot interaction and for the other range, hiring frequency is better than the case without any system.

If rating designer's objective is maximizing the hiring chances in the two-state system, it would design any uninformative rating system for $\ell \geq 1$ to ensure a hiring frequency of probability 1 and for $\ell < 1$, it would solve the following problem:

$$\begin{aligned}
&\underset{(f^S(1),f^B(2))}{Maximize} \quad \rho f^B(2)+(1-\rho)(1-f^S(1)) && (4.1)\\
&\text{s.t} \qquad \frac{1-f^B(2)}{f^S(1)} > \ell,\\
&\qquad \frac{f^B(2)}{1-f^S(1)} \leq \ell,\\
&\qquad \frac{1-f^B(2)}{f^S(1)} \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right).
\end{aligned}$$

The first and second constraints come from the hiring strategy and belief consistency and the third one comes from the computation of stationary probabilities and the induced stochastic matrix of each type. Note that, differently from previous problems ((3.5) and (3.6)), rating designer now wants to maximize $f^B(2)$ and minimize $f^S(1)$. Since $\ell < 1$ implies the first constraint to be loose, that means the second one must be binding in equilibrium, just as it is the case in lemma 4. But different from that environment, the rating system imposes the third constraint, which affects how low $f^S(1)$ can be. The solution requires this constraint to bind as well. That gives the same optimal rating policy from expert's point of view. Therefore, interests between expert and a rating designer maximizing hiring chances coincide. Furthermore, as we discussed in section 3.2.1 of chapter 3, this policy gives customer the same payoff as the one from the one-shot interaction, thus it does not harm (neither benefits) customer relative to the one-shot framework.

**Proposition 4** (Optimal rating to maximize hiring in a two-state equilibrium)**.** *In a two-state rating system, maximum hiring frequency (for $f_{In} = f(2)$) is:*

$$f^*_{In}(\rho) = \begin{cases} 1 & \text{if } \rho \leq \rho^*, \\ \left[2\frac{(1-\rho)}{(1-\rho^*)} - \frac{\rho}{\rho^*}\right] & \text{if } \rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

*For $\rho \leq \rho^*$, any combination of transition rules that induce hiring in every state is an optimal rating policy. For $\rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$f^B(2) = 2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right) - 1 \qquad\qquad f^S(1) = -1 + \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}$$

*Those probabilities are achieved by setting $\varphi_L(2,1) = 0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^* = \frac{\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - \frac{1}{2}}{1 - \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}}$$

The same discussion following proposition 4 regarding when to set either $\varphi_{Out}(1,2)$ or $\varphi_H(2,1)$ equal 1 applies here. The optimal policy in a two-state rating system generates a hiring frequency lower than the benchmark policy for prior beliefs above critical prior $\rho^*$. Nevertheless, it is possible to support hiring in a rating system when prior belief is so high that in absence of any rating system hiring would never be a best response.

# 5 CONCLUSION

Rating systems are becoming a common information channel for customers to seek experts' services. In the market of experts, a small uncertainty on an expert's intention can dramatically reduce customers' willingness to hire and even generate market disruption, as argued in EV's bad reputation effect. A crucial assumption for their results is that customers never forget and always relies on every public information to learn about expert's type. We show that simple rating systems, where customers get to observe only a current draw from finite set of states (or summary statistics) and know the rules of system, not only can overcome bad reputation, but generates higher equilibrium outcomes relative to extreme memory settings.

Specifically, we illustrate the rating model with a two-state system and show that it is possible to generate an equilibrium where good expert keeps being hired in equilibrium; never refrains from playing right actions (truth-telling equilibrium) and customer takes an action which he would not do so under the one-shot interaction or full memory (influential equilibrium). With two ratings, we obtain optimal payoffs for customer and expert for all prior beliefs and consider as an optimal hiring rate if this was the objective of the information designer.

The optimal policies reveal that there might be some conflict of interests between customer and strategic expert when designing the system each player's viewpoint. A customer always wants to minimize his chances of ever interacting with a bad type and an expert always wants to maximize her chances of being hired. That means customer might be more willing to send good types no non-hiring states to reduce chances of hiring a bad expert, while strategic expert would accept some bad types in a hiring state if this could reduce her chances of visiting a non-hiring state. Nevertheless, there is room for efficiency improvement: it is possible to design an optimal system from expert s perspective that gives customer the same *ex-ante* expected payoff as in the one-shot interaction but improves expert's *ex-ante* expected payoff in equilibrium.

The main next step is to characterize optimal ratings for general systems with many states. For instance, one interesting open question is whether expanding the number of ratings can generate truth-telling and influential equilibria across a far-reaching range of prior beliefs or how close can a general rating system get to complete information payoffs. We leave these questions to further research.

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# APPENDICES

# APPENDIX A – Proofs

**Lemma A.1** (Discounted convergence)**.** *Consider any initial distribution $g_0$, discount factor $\delta \in (0,1)$ and transition matrix $T = [\tau(i,j)]$. Define the alternative transition matrix $T_{\delta,g_0} = [\tau_{\delta,g_0}(i,j)]$ where $\tau_{\delta,g_0}(i,j) = (1-\delta)g_0(j) + \delta\tau(i,j)$. Then $\lim_{t\to\infty}(T_{\delta,g_0})^t$ converges to a stochastic matrix with each row being its unique stationary distribution $f_{\delta,g_0}$. Moreover, $f_{\delta,g_0} = (1-\delta)g_0 \cdot \sum_{t=0}^{\infty}(\delta T)^t = (1-\delta)g_0 \cdot (I-\delta T)^{-1}$.*

*Proof.* Consider the stochastic matrix $T_{\delta,g_0}$ with transition rules for each pair $(i,j)$ given by:

$$\tau_{\delta,g_0}(i,j) = (1-\delta)g_0(j) + \delta\tau(i,j)$$

For some initial state $i_0$ with positive probability, $\tau_{\delta,g_0}(i,i_0) \geq (1-\delta)g(i_0) > 0$ for every $i$. Now for any $j \in \mathcal{J}$ and every $t = 1,2,..$ define $\varepsilon_j^{(t)} := \underset{k}{min}\,[\tau_{\delta,g_0}(k,j)]^t$ and $\varepsilon^{(t)} := \sum_{j\in\mathcal{J}} \varepsilon_j^{(t)}$, where $[\tau_{\delta,g_0}(k,j)]^t$ is the $(k,j)$-th element of $T^t_{\delta,g_0}$. Note that $\varepsilon^t_{i_0} > 0$ for any $t$, which implies $\varepsilon^t > 0$. Theorem 11.4 from Stokey and Lucas (1989) states that $T_{\delta,g_0}$ has a unique irreducible subset (aperiodic as well) if and only for some $N \geq 1$, $\varepsilon^t_{i_0} > 0$. In that case, $(T_{\delta,g_0})^t$ converges to a stochastic matrix with each row being its unique stationary distribution $f_{\delta,g_0}$.

For any rating $i$, each period $t$-probability distribution is given by $g_t(i|\delta,g_0) = \sum_{j\in\mathcal{J}} g_{t-1}(j|\delta,g_0)\tau_{\delta,g}$ and satisfies $g_t(i|\delta,g_0) = (1-\delta)g_0(i) + \delta\sum_{j\in J} g_{t-1}(j|\delta,g_0)\tau(j,i)$. Iterating this expression leads to:

$$g_t(\delta,g_0) = (1-\delta)g_0 \cdot \sum_{s=0}^{t-1}(\delta T)^s + g_0 \cdot (\delta T)^t$$

Since $g_t(\delta,g_0)$ converges to $f_{\delta,g_0}$ and $(\delta T)^t$ converges to 0 as $t$ goes to infinity, it follows that:

$$\begin{aligned} f_{\delta,g_0} &= \lim_{t\to\infty} g_t(\delta,g_0) = (1-\delta)g_0 \cdot \lim_{t\to\infty}\left(\sum_{s=0}^{t-1}(\delta T)^s\right) = (1-\delta)g_0 \cdot \sum_{t=0}^{\infty}(\delta T)^t \\ &= (1-\delta)g_0 \cdot (I-\delta T)^{-1} \end{aligned}$$

That $\sum_{t=0}^{\infty}(\delta T)^t = (I-\delta T)^{-1}$ comes from the fact that:

$$
\begin{aligned}
(I-\delta T)\left(\sum_{t=0}^{N}(\delta T)^t\right) = \left(\sum_{t=0}^{N}(\delta T)^t\right)(I-\delta T) = \left(\sum_{t=0}^{N}(\delta T)^t\right) - \left(\sum_{t=0}^{N}(\delta T)^{t+1}\right) \\
= I-(\delta T)^{N+1}
\end{aligned}
$$

Passing to the limit, $(I-\delta T)\left(\sum_{t=0}^{\infty}(\delta T)^t\right) = I$ or $\sum_{t=0}^{\infty}(\delta T)^t = (I-\delta T)^{-1}$.

□

**Lemma A.2** (Time-average convergence)**.** *Consider any initial distribution $g_0$ and transition matrix $T = [\tau(i,j)]$. Then $\lim_{t\to\infty}[\frac{1}{t}\sum_{s=0}^{t-1}(T)^s]$ converges to a stochastic matrix $F$ with each row being a stationary distribution, so that $f_{g_0} = g_0 \cdot F$ is a unique stationary distribution, i.e, $f_{g_0} \cdot T = f_{g_0}$.*

*Proof.* We can always partition the transition matrix into a finite class $\mathscr{C}$ of $k \geq 1$ irreducible subsets and a set $T_r$ of transient ones, so that $\mathscr{I} = C_1 \cup C_2 ... \cup C_k \cup Tr$. Abusing notation, We use the same label to denote each set of associated transition rules and define $R_k$ as the set of transition rules from the transient class to an irreducible subset $C_k$. Thus, $T$ can be written as:

$$
T = \begin{pmatrix}
C_1 & & & & \\
& C_2 & & & \\
& & \ddots & & \\
& & & C_k & \\
R_1 & R_2 & \dots & R_c & T_r
\end{pmatrix}
$$

The blank spaces represent a matrix of zero-valued entries. Every state within some irreducible set is recurrent and communicates with every other state within the same set. The $t^{th}$ power of $T$ is given by:

$$
T^t = \begin{pmatrix}
C_1^t & & & & \\
& C_2^t & & & \\
& & \ddots & & \\
& & & C_k^t & \\
A_{1,t} & A_{2,t} & \dots & A_{c,t} & Tr^t
\end{pmatrix}
$$

Each $A_{i,t}$ is a function of irreducible and transient sets. Summing over each power of $T$ up to $t$ and dividing by $t$ we get:

$$\frac{1}{t}\sum_{s=0}^{t-1} T^s = \begin{pmatrix} \frac{1}{t}\sum_{s=0}^{t-1}(C_1)^s & & & & \\ & \frac{1}{t}\sum_{s=0}^{t-1}(C_2)^s & & & \\ & & \ddots & & \\ & & & \frac{1}{t}\sum_{s=0}^{t-1}(C_k)^t & \\ \frac{1}{t}B_{1,t} & \frac{1}{t}B_{2,t} & \dots & \frac{1}{t}B_{k,t} & \frac{1}{t}\sum_{s=0}^{t-1} Tr^t \end{pmatrix}$$

Each $B_{i,t}$ is $\sum_{s=0}^{t-1} A_{i,t}$. Now, for a finite state space, it can be shown that each time-average geometric series of an irreducible set converges to an unique distribution satisfying $f^k \cdot C_k = f^k$ (see Stokey and Lucas, 1989, theorem 11.1 or Puterman, 1994, appendix A). That means all rows of $\bar{C}_k = \lim_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t-1}(C_k)^s$ are equal to $f^k$. Furthermore, it can be shown that $\lim_{s\to\infty}(Tr)^s = 0$ since all states within this set are transient (thus, $\lim_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t-1}(Tr)^s = 0$ as well). So the limiting matrix is:

$$F = \begin{pmatrix} \bar{C}_1 & & & & \\ & \bar{C}_2 & & & \\ & & \ddots & & \\ & & & \bar{C}_k & \\ \bar{B}_1 & \bar{B}_1 & \dots & \bar{B}_1 & 0 \end{pmatrix}$$

Note that if $F$ is a limiting matrix, it satisfies $F \cdot T = T \cdot F = F$. We still need to say something about the convergence of $\bar{B}_i = \lim_{t\to\infty} B_{i,t}$. To do so, we group the collection of irreducible matrices into $Ir$ and the collection of transitions from transient states to irreducible sets into $D$. In that way, the transition matrix $T$ and the corresponding limiting matrix $F$ are further simplified to:

$$T = \begin{pmatrix} Ir & 0 \\ D & Tr \end{pmatrix} \qquad\qquad F = \begin{pmatrix} \bar{Ir} & 0 \\ \bar{D} & 0 \end{pmatrix}$$

Since $\lim_{t\to\infty}(Tr)^t = 0$, $(I - Tr)$ has an inverse and the sum $\sum_{s=0}^{\infty}(Tr)^s$ converges to $N = (I - Tr)^{-1}$. Moreover, $T \cdot F = F$ implies $D \cdot \bar{I}_r + Tr \cdot \bar{D} = \bar{D}$ or $\bar{D} = N \cdot D \cdot \bar{I}_r$. We conclude that the limiting matrix is given by:

$$F = \begin{pmatrix} \bar{C}_1 & & & & \\ & \bar{C}_2 & & & \\ & & \ddots & & \\ & & & \bar{C}_k & \\ (I-Tr)^{-1}R_1\bar{C}_1 & (I-Tr)^{-1}R_2\bar{C}_2 & \dots & (I-Tr)^{-1}R_k\bar{C}_k & 0 \end{pmatrix}$$

For any $g_0$, letting $f = g_0 \cdot F$, we have $f \cdot T = (g_0 \cdot F) \cdot T = g_0 \cdot (F \cdot T) = g_0 \cdot F = f$, thus $f$ is a stationary distribution over states.

**Remark.** Suppose now $T$ has exactly one irreducible subset $C$. Then $C$ has a unique stationary distribution with $f^c(i) > 0$ for all $i \in C$. Defining $\tilde{f}$ as a $|Tr|$ vector of zeros, $f = (f^c, \tilde{f})$ is a stationary distribution of $T$. Conversely, suppose $f = (f^c, \hat{f})$ is a stationary distribution of $T$. Then $\hat{f} \cdot Tr = \hat{f}$ as well as $\hat{f} \cdot (Tr)^t = \hat{f}$ for every period $t$. However, since $\lim_{t\to\infty}(Tr)^t = 0$, it follows that $\hat{f} = 0$. That implies $f^c$ is a stationary distribution of $C$ and $f = (f^c, 0)$. Since $f = g_0 \cdot F$ and $f$ is a distribution not dependent on $g_0$, the solution to the steady-state problem does not depend on $g_0$.

□

**Lemma A.3** (Truth-telling is optimal in a two-state rating system)**.** *Consider a two-state rating system with a non-hiring state* (1) *and a hiring one* (2)*. Playing truthfully in the hiring state is always optimal for the strategic expert.*

*Proof.* From lemma 1, we can compute strategic expert's chances of reaching state $i$ as a solution to the steady-state equation $f^S_{\delta,g_0}(i) = \sum_{j=1,2} f^S_{\delta,g_0}(j)\tau^S_{\delta,g_0}(j,i)$ and expected payoff as $V(i) = \sum_{j=1,2} f^S_{\delta,i}(j)\alpha(j)[\beta_{truth}(j)u - (1-\beta_{truth}(j))w)$. We wish to implement an equilibrium where hiring takes place only in 2 and expert tells the truth whenever hired. Thus, expected payoffs become:

$$V(1) = f^S_{\delta,1}(2)u = \left[\frac{\delta\varphi_{Out}(1,2)}{1-\delta+\delta(1/2)[\varphi_H(2,1)+\varphi_L(2,1)]+\delta\varphi_{Out}(1,2)}\right]u$$
$$V(2) = f^S_{\delta,2}(2)u = \left[\frac{1-\delta+\delta\varphi_{Out}(1,2)}{1-\delta+\delta(1/2)[\varphi_H(2,1)+\varphi_L(2,1)]+\delta\varphi_{Out}(1,2)}\right]u$$

From equation (2.8), playing $t_H$ in $\theta_H$ is optimal if the following holds:

$$\begin{aligned} V(2|t_H,\theta_H) =& (1-\delta)u + \delta[\varphi_H(2,1)V(1) + \varphi_H(2,2)V(2)] \\ & \geq (1-\delta)(-w) + +\delta \sum_{j=1,2} \varphi_L(2,i)V(i) = V(2|t_L,\theta_H) \end{aligned}$$

Substituting $V(1)$ and $V(2)$, this is simplified to:

$$(u+w) \geq 2u\left[\frac{\delta(\varphi_H(2,1)-\varphi_L(2,1))}{2(1-\delta)+\delta[\varphi_H(2,1)+\varphi_L(2,1)]+2\delta\varphi_{Out}(1,2)}\right]$$

By assumption, $(u+w) > 2u$, so it remains to check whether the term in brackets is at most 1. Indeed, this is always the case, since:

$$\delta(\varphi_H(2,1)-\varphi_L(2,1)) \leq 2(1-\delta)+\delta[\varphi_H(2,1)+\varphi_L(2,1)]+2\delta\varphi_{Out}(1,2)$$
$$\Rightarrow 1-\delta+\delta[\varphi_L(2,1)+\varphi_{Out}(1,2)] \geq 0$$

□

**Proposition A.2** (Expert's optimal rating in a two-state equilibrium)**.** *In a two-state rating system, strategic expert's maximum ex-ante expected payoff is:*

$$\bar{V}^*(\rho) = \begin{cases} u & \text{if } \rho \leq \rho^*, \\ \left[2-\frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}\right]u & \text{if } \rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

*For $\rho \leq \rho^*$, any combination of transition rules that induce hiring in every state is an optimal rating policy. For $\rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$f^B(2) = 2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)-1 \qquad f^S(1) = -1+\frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}$$

*Those probabilities are achieved by setting $\varphi_L(2,1)=0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^* = \frac{\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}-\frac{1}{2}}{1-\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}}$$

*Proof.* When $\ell \geq 1$ ($\rho \leq \rho^*$), customer hires in the one-shot interaction, so the rating designer could benefit strategic expert simply by devising an uninformative rating where hiring takes place irrespective of what rating customer finds himself him. Take for instance $\varphi_L(i,j) = \varphi_H(i,j) = 1$ for $i = \{1,2\}$. This yields $f^B = f^S = (1/2,1/2)$, with implied beliefs being $\rho(1) = \rho(2) = \rho$. Clearly, hiring in every rating and playing truth-telling strategy is an equilibrium generating $\bar{V} = u$ (irrespective of the initial distribution).

When $\ell < 1$ ($\rho > \rho^*$), customer does not hire in the one-shot interaction, so the rating designer must persuade customer to hire in state 2. It thus solves the following maximization problem:

$$\begin{aligned}
&\underset{(f^S(1),f^B(2))}{Maximize} \quad (1-f^S(1))u \\
&\text{s.t} \qquad \frac{1-f^B(2)}{f^S(1)} > \ell, \\
&\qquad\qquad \frac{f^B(2)}{1-f^S(1)} \leq \ell, \\
&\frac{1-f^B(2)}{f^S(1)} \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right).
\end{aligned}$$

The first constraint comes from customer not hiring in state 1; the second from persuading customer to hire in 2 and the third from the stochastic matrix of each type and the computation of the stationary distributions. From the first and second constraints respectively, we have:

$$f^B(2) < 1-\ell f^S(1) \qquad\qquad f^B(2) \leq \ell(1-f^S(1))$$

When $\ell < 1$, $\ell(1-f^S(1)) < 1-\ell f^S(1)$, so we can ignore the first inequality. Manipulating the third constraint yields:

$$f^B(2) \geq \frac{1-f^S(1)}{1+f^S(1)}$$

Combining the constraints we need that:

$$\frac{1-f^S(1)}{1+f^S(1)} \leq f^B(2) \leq \ell(1-f^S(1)) \Rightarrow f^S(1) \geq \frac{1-\ell}{\ell}$$

When $\ell < 1/2$, this cannot be satisfied, since $f^S(1)$ is at most 1. Therefore, our problem requires $\ell \geq 1/2$. Recalling that $\ell = [\rho^*(1-\rho)/\rho(1-\rho^*)]$, this is equivalent to:

$$\ell = \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} \geq \frac{1}{2} \Rightarrow \rho \leq \frac{2\rho^*}{1+\rho^*}$$

Given that expert's payoff is maximized by minimizing $f^S(1)$, the solution of the problem is:

$$f^S(1) = \frac{1-\ell}{\ell} = -1 + \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}$$

Substituting this value in the constraints for $f^B(2)$, we find:

$$2\ell - 1 \leq f^B(2) \leq 2\ell - 1 \Rightarrow f^B(2) = 2\ell - 1 = 2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right) - 1$$

Applying this optimal value of $f^S(1)$ in expert's *ex-ante* expected payoff and recalling the bounds for $\rho$, we achieve the maximum payoff function:

$$\bar{V}^*(\rho) = \begin{cases} u & \text{if } \rho \leq \rho^*, \\ \left[2 - \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}\right] u & \text{if } \rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

Once we have the optimal value $f^S(1)$, we can find transition rules to achieve it. Start by setting $\varphi_L(2,1) = 0$ and define $r = \frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)}$. Then $f^S(1)$ can be rewritten as:

$$f^S(1) = \frac{\tau^S(2,1)}{\tau^S(1,2) + \tau^S(2,1)} = \frac{(1/2)\varphi_H(2,1)}{\varphi_{Out}(1,2) + (1/2)\varphi_H(2,1)} = \frac{1}{1+2r}$$

To achieve optimal value, we need:

$$\frac{1}{1+2r} = -1 + \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)} \Rightarrow r = r^* := \frac{\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - \frac{1}{2}}{1 - \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}}$$

$\square$

**Proposition A.3** (Customer's optimal rating in a two-state equilibrium)**.** *In a two-state rating system, customer's maximum ex-ante expected payoff is:*

$$\bar{v}^*(\rho) = \begin{cases} \rho\left(\frac{u-w}{2}\right) + (1-\rho)u & \text{if } \rho \leq \frac{\rho^*}{2-\rho^*}, \\ \rho\left(\frac{w-u}{2}\right)\left[\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1\right]^2 & \text{if } \rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

*For $\rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$f^B(2) = \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1 \qquad\qquad f^S(1) = \frac{2 - \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}$$

*Those probabilities are achieved by setting $\varphi_L(2,1) = 0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^{**} = \frac{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1}{2 - \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}$$

*Proof.* As we showed in lemma 3, expert plays a truth-telling strategy when customer plays an influential one. Thus, the rating designer is left with the following maximization problem as in (4.1):

$$\begin{aligned}
&\underset{(f^S(1), f^B(2))}{Maximize} \ \rho f^B(2)\left(\frac{u-w}{2}\right) + (1-\rho)(1-f^S(1))u \\
&\text{s.t} \qquad\qquad \frac{1-f^B(2)}{f^S(1)} > \ell, \\
&\qquad\qquad\qquad \frac{f^B(2)}{1-f^S(1)} \leq \ell, \\
&\qquad\qquad\qquad \frac{1-f^B(2)}{f^S(1)} \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right).
\end{aligned}$$

The first constraint comes from customer not hiring in state 1; the second from persuading customer to hire in 2 and the third from the stochastic matrix of each type and the computation of the stationary distributions. Manipulating the third constraint yields:

$$f^B(2) \geq \frac{1-f^S(1)}{1+f^S(1)}$$

Customer's payoff is maximized by minimizing $f^B(2)$ and $f^S(1)$. Thus, this constraint must be binding in equilibrium. Substituting this equality in customer's payoff, we have:

$$\rho f^B(2)\left(\frac{u-w}{2}\right) + (1-\rho)(1-f^S(1))u = (1-f^S(1))\left[\rho\left(\frac{1}{1+f^S(1)}\right)\left(\frac{u-w}{2}\right) + (1-\rho)u\right]$$

It can be shown that this function is continuous and strictly concave in $f^S(1)$. Therefore, it admits an unique interior solution, given by:

$$f^S(1) = \frac{2-\sqrt{2\ell}}{\sqrt{2\ell}} = \frac{2-\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}$$

Therefore, optimal $f^B(2)$ is:

$$f^B(2) = \sqrt{2\ell} - 1 = \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1$$

Note that our problem requires $\ell \leq 2$, otherwise the optimal stationary probabilities are higher than 1. Indeed, it requires $\ell < 2$ to satisfy the first constraint of the maximization, since:

$$\ell f^S(1) = \sqrt{\frac{\ell}{2}}(2-\sqrt{2\ell}) = \sqrt{\frac{\ell}{2}}(1-f^B(2)) < 1 - f^B(2) \Leftrightarrow 0 \leq \ell < 2$$

The solution also requires $\ell \geq 1/2$, otherwise the optimal stationary probabilities are negative. When $\ell \geq 1/2$, the solution satisfies the second constraint, since:

$$\ell(1-f^S(1)) = \sqrt{2\ell}(\sqrt{2\ell}-1) = \sqrt{2\ell} f^B(2) \geq f^B(2) \Leftrightarrow \ell \geq \frac{1}{2}$$

Therefore, our problem requires $\ell \in [1/2, 2)$. In terms of of $\rho$, this is equivalent to:

$$\ell = \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} \in \left[\frac{1}{2}, 2\right) \Rightarrow \rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right]$$

Applying these optimal values of $f^S(1)$ and $f^B(2)$ in customer's *ex-ante* expected payoff and recalling the bounds for $\rho$, we achieve the maximum payoff function:

$$\bar{v}^*(\rho) = \begin{cases} \rho\left(\frac{u-w}{2}\right) + (1-\rho)u & \text{if } \rho \leq \frac{\rho^*}{2-\rho^*}, \\ \rho\left(\frac{w-u}{2}\right)\left[\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1\right]^2 & \text{if } \rho \in \left(\frac{\rho^*}{2-\rho^*}, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \text{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

Once we have the optimal values $f^S(1)$ and $f^B(2)$, we can find transition rules to achieve them. Start by setting $\varphi_L(2,1) = 0$ and define $r = \frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)}$. Then $f^B(2)$ can be rewritten as:

$$f^B(2) = \frac{\tau^B(1,2)}{\tau^B(1,2) + \tau^B(2,1)} = \frac{\varphi_{Out}(1,2)}{\varphi_{Out}(1,2) + \varphi_H(2,1)} = \frac{r}{1+r}$$

To achieve optimal value, we need:

$$\frac{r}{1+r} = -1 + \sqrt{2\left(\frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}\right)} \Rightarrow r = r^{**} := \frac{\sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)} - 1}{2 - \sqrt{2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right)}}$$

□

**Lemma A.4** (Optimal policy for maximum hiring chances in static framework)**.** *Maximum hiring frequency $f^*_{In}$ in the one-shot interaction with an intermediary agency offering recommendation signals $f^B_{In}$ and $f^S_{Out}$ is:*

$$f^*_{In}(\rho) = \begin{cases} 1 & \text{if } \rho \le \rho^* \\ \left[\frac{1-\rho}{1-\rho^*}\right] & \text{if } \rho > \rho^* \end{cases}$$

*For $\rho \le \rho^*$, the optimal recommendation signals are $f^B_{In} = 1$ and $f^S_{Out} = 0$. For $\rho > \rho^*$, the optimal recommendation signals are:*

$$f^B_{In} = \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - 1 \qquad\qquad f^S_{Out} = 0$$

*Proof.* Designer's objective for $\ell < 1$ is:

$$\begin{aligned} \underset{(f^S_{Out}, f^B_{In})}{Maximize} \quad & \rho f^B_{In} + (1-\rho)(1 - f^S_{Out}) \\ \text{s.t} \quad & \frac{1 - f^B_{In}}{f^S_{Out}} > \ell, \\ & \frac{f^B_{In}}{1 - f^S_{Out}} \le \ell. \end{aligned}$$

If the second constraint is satisfied, the first one will be as well, so we need only to consider $f^B_{In} \leq \ell(1 - f^S_{Out})$. But since the maximization occurs by maximizing $f^B_{In}$ and minimizing $f^S_{Out}$, this must be binding. Since there is no additional constraint on $f^S_{Out}$, we can set it to be zero and $f^B_{In} = \ell$. Substituting this value in the hiring frequency for $\rho > \rho^*$, we get:

$$f^*_{In}(\rho) = \begin{cases} 1 & \text{if } \rho \leq \rho^* \\ \left[\frac{1-\rho}{1-\rho^*}\right] & \text{if } \rho > \rho^* \end{cases}$$

□

**Proposition A.4** (Optimal rating to maximize hiring in a two-state equilibrium). *In a two-state rating system, maximum hiring frequency (for $f_{In} = f(2)$) is:*

$$f^*_{In}(\rho) = \begin{cases} 1 & \textit{if } \rho \leq \rho^*, \\ \left[2\frac{(1-\rho)}{(1-\rho^*)} - \frac{\rho}{\rho^*}\right] & \textit{if } \rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right], \\ 0 & \textit{if } \rho > \frac{2\rho^*}{1+\rho^*}. \end{cases}$$

*For $\rho \leq \rho^*$, any combination of transition rules that induce hiring in every state is an optimal rating policy. For $\rho \in \left(\rho^*, \frac{2\rho^*}{1+\rho^*}\right]$, the optimal rating policy is the one leading to following stationary probabilities:*

$$f^B(2) = 2\left(\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}\right) - 1 \qquad\qquad f^S(1) = -1 + \frac{\rho(1-\rho^*)}{\rho^*(1-\rho)}$$

*Those probabilities are achieved by setting $\varphi_L(2,1) = 0$ and positive transitions $\varphi_{Out}(1,2), \varphi_H(2,1)$ such that:*

$$\frac{\varphi_{Out}(1,2)}{\varphi_H(2,1)} = r^* = \frac{\frac{\rho^*(1-\rho)}{\rho(1-\rho^*)} - \frac{1}{2}}{1 - \frac{\rho^*(1-\rho)}{\rho(1-\rho^*)}}$$

*Proof.* If rating designer's objective is maximizing the hiring frequency in the two-state system, it designs any uninformative rating system for $\ell \geq 1$ to ensure a hiring frequency of probability 1 and for $\ell < 1$, it solves the following problem:

$$\underset{(f^S(1),f^B(2))}{Maximize} \quad \rho f^B(2)+(1-\rho)(1-f^S(1))$$

$$\text{s.t} \qquad \frac{1-f^B(2)}{f^S(1)} > \ell,$$

$$\frac{f^B(2)}{1-f^S(1)} \leq \ell,$$

$$\frac{1-f^B(2)}{f^S(1)} \leq 2\left(\frac{f^B(2)}{1-f^S(1)}\right).$$

Since $\ell < 1$, the second inequality subsumes the first. Since the maximization takes place for high values of $f^B(2)$ and low values of $f^S(1)$, this second inequality must be binding. Combining this with the third constraint, we find the minimum value for $f^S(1)$. Then $f^B(2)$ and $f^S(1)$ are equal to the values derived for expert's optimal rating policy. Substituting those values in the hiring frequency yields the desired results.

□