

Implied Volatility Smirk in Lévy Markets

April 7, 2016

Abstract

We introduce skewed Lévy models, characterized by a symmetric jump measure multiplied by dumping exponential factor. This models exhibit a clear implied volatility pattern, where the dumping parameter controls the skew of the implied volatility curve, resulting in a measure of the skewness of the model. We show that the variation of this parameter produces the typical smirk observed in implied volatility curves. Some theoretical facts supporting this findings are proved, and some open questions are posed.

Keywords: Skewness; Lévy Processes; Implied volatility Smirk.

JEL Classification: C52; G10

1 Introduction

Since the seminal paper by Black and Scholes [1973] has appeared, many attempts have been made to capture the empirical behavior of the implied volatility surface. The best known facts are the volatility smile and the volatility smirk, where we look at it as a function of moneyness and maturity respectively. For example the fact that out-of-the money put options are more expensive than the corresponding out-of-the money call options, the fact has been extensively studied by many authors, among them we recall the work of Foresi and Wu [2005]. They establish this fact based on a large set of option prices where the underlyings are major equity indexes from twelve countries. Also, Carr and Wu [2003] analyze the characteristic pattern of implied volatility smirks across maturities using S&P500 index options. Their findings imply that the index risk neutrally has an asymmetric distribution.

On the other hand, there is a well-established relationship between the symmetry of the implied volatility and the market symmetry, which is equivalent to the put-call symmetry. This has been derived by Fajardo and Mordecki [2006] and Carr and Lee [2009] for Lévy process and local and stochastic volatility models respectively. Also, Fajardo and Mordecki [2014] showed the relationship between the skewness premium and the market symmetry parameter.

In this paper focusing on a subclass of Lévy processes, where exponential dampening controls the skewness, we use duality techniques to obtain a result that allows us to relate the implied volatility skew with a market skewness parameter. Although, more general processes have been studied in the literature which include stochastic volatility models, we focus on a particular class. For this class we get a deeper insight into how this particular process generates the skew. More exactly, there is an intrinsic relationship between the market skewness parameter and the risk neutral excess of kurtosis, which will allow us to relate the risk neutral skewness and kurtosis with the implied volatility skew. The main result obtained shows that all skewed models with continuous density have a positive cross partial derivative w.r.t. x (the log-moneyness) and β at the point $x = 0$ and $\beta = -1/2$, giving, as a consequence, a precise monotonicity behavior of the implied volatility in a neighborhood of this point.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3, through put-call duality, we obtain a symmetry property of the implied volatility which extends the result obtained by Fajardo and Mordecki [2006]. In Section 4 we obtain our main result. In Section 5 we analyze our finding across different models, and present a complementary example that shows that the obtained monotonicity behavior can not be extended beyond a certain neighborhood. The last section concludes.

2 The Model

Consider a real valued stochastic process $X = \{X_t\}_{t \geq 0}$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{Q})$. We assume that the process is adapted, starts at zero, has càdlàg paths, and is such that for $0 \leq s < t$ the random variables $X_t - X_s$ are independent of the σ -field \mathcal{F}_s , with a distribution that depends only on the difference $t - s$. The process X is a Lévy process. For general references on Lévy processes see Jacod and Shiryaev [1987],

Skorokhod [1991], Bertoin [1996] or Sato [1999]. For Lévy processes used in financial models see Boyarchenko and Levendorskii [2002], Schoutens [2003] or Cont and Tankov [2004].

In order to characterize the law of X under \mathbb{Q} , consider the Lévy-Khinchine formula, that, for $z \in i\mathbb{R}$, states

$$\mathbb{E} e^{zX_t} = e^{t\psi(z)},$$

where

$$\psi(z) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zh(y))\Pi(dy), \quad (1)$$

with $h(y) = y\mathbf{1}_{\{|y|<1\}}$, a fixed truncation function, γ and $\sigma \geq 0$ real constants, and Π a measure on $\mathbb{R} \setminus \{0\}$ ¹ such that $\int (1 \wedge y^2)\Pi(dy) < \infty$, called the *Lévy measure*. The triplet (γ, σ, Π) is the *characteristic triplet* of the process, and completely determines its law.

By a *Lévy market* we mean a model for a financial market with two assets: a savings account B , given by the deterministic process:

$$B_t = e^{rt}, \quad t \geq 0,$$

for a fixed interest rate $r \geq 0$ and a stock S , with price process $S = \{S_t\}_{t \geq 0}$ modelled by:

$$S_t = S_0 e^{X_t + rt}, \quad S_0 > 0,$$

where $X = \{X_t\}_{t \geq 0}$ is a Lévy process.

In this approach we assume that the probability measure \mathbb{Q} is already the risk neutral martingale measure. In other words, prices for derivatives written on the underlying process $S = \{S_t\}_{t \geq 0}$ are computed as expectations with respect to \mathbb{Q} , and the discounted process $\{e^{-rt}S_t\}_{t \geq 0}$ is a \mathbb{Q} -martingale.

In terms of the characteristic exponent of the process this means that

$$\psi(1) = 0, \quad (2)$$

based on the fact, that $\mathbb{E} e^{-rt}S_t = e^{t\psi(1)} = 1$. Observe that, as usual in financial modeling with Lévy processes, this requires that the complex function $\psi(z)$ defined in (1) can be extended at least to the strip $0 \leq \operatorname{Re}(z) \leq 1$, what is equivalent to $\int_{y \geq 1} e^y \Pi(dy) < \infty$, condition that we assume in what follows

¹ $\Pi(\{0\})$ could be defined as 0. Here we follow Cont and Tankov [2004].

(see Theorem 25.3 in Sato [1999]). Condition (2) can also be formulated in terms of the characteristic triplet (γ, σ, Π) of the process X as

$$\gamma = -\sigma^2/2 - \int_{\mathbb{R}} (e^y - 1 - h(y))\Pi(dy). \quad (3)$$

Consequently

$$\psi(z) = z(z-1)\frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} [(e^{zy} - 1) - z(e^y - 1)]\Pi(dy).$$

2.1 Skewed Models

We are particularly interested in the case where $\{X_t, \mathbb{Q}\}$ has a Lévy measure which can be represented in the form

$$\Pi_\beta(dy) = e^{\beta y}\Pi_0(dy), \quad (4)$$

where Π_0 is a symmetric measure, i.e. $\Pi_0(dy) = \Pi_0(-dy)$. We assume that Π_0 does not depend on β . In this case we denote \mathbb{Q} by \mathbb{Q}_β . As a consequence, we study the Lévy process

$$\{X_t, \mathbb{Q}_\beta\}_{t \geq 0} \quad (5)$$

with triplet $(\gamma_\beta, \sigma, \Pi_\beta)$ where γ_β is as in (3) with $\Pi = \Pi_\beta$ and Π_β is as in (4).

In this case, we denote the characteristic exponent by ψ_β and we write $f_t(x; \beta)$ for the density of X_t under \mathbb{Q}_β , when it exists. For the existence and smoothness of the density of an infinitely divisible distribution see sections 27 and 28 in Sato [1999].

This parametrization allows us to quantify the *smirkness* of the model by $\beta + 1/2$, since $\beta = -1/2$ gives the symmetry of the implied volatility as a function of the log-moneyness $x = \log K/F$, where K is the strike of a call option and $F = S_0 e^{rT}$ is the corresponding futures price of the underlying asset (see Fajardo and Mordecki [2006]).

3 Put-Call Duality and Implied Volatility.

3.1 Symmetry and Duality of the General Model

Let $\{X_t, \mathbb{Q}\}$ be a general Lévy process as in Section 2 and let $d\tilde{\mathbb{Q}}_t = e^{X_t}d\mathbb{Q}_t$ and $\tilde{X}_t = -X_t$ be the dual measure and the dual process respectively, where

$\tilde{\mathbb{Q}}_t$ and \mathbb{Q}_t are the restrictions of $\tilde{\mathbb{Q}}$ and \mathbb{Q} to \mathcal{F}_t respectively. We know that the triplet $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Pi})$ of $\{\tilde{X}_t\}$ under $\tilde{\mathbb{Q}}$ is

$$\begin{cases} \tilde{\gamma} &= -\sigma^2/2 - \int_{\mathbb{R}}(e^y - 1 - h(y))\tilde{\Pi}(dy) \\ \tilde{\sigma} &= \sigma \\ \tilde{\Pi}(dy) &= e^{-y}\Pi(-dy) \end{cases}$$

(see Fajardo and Mordecki [2006]) and for the option prices

$$Call(S_0, K_x, r, T, \psi) = \frac{K_x}{F} Put(S_0, K_{-x}, r, T, \tilde{\psi}), \quad (6)$$

where $K_x = S_0 e^{rT+x}$ and $F = S_0 e^{rT}$ (see Fajardo and Mordecki [2014]).

From (6) we have the following relation for the implied volatility

$$\sigma_{imp}(x, \Pi) = \sigma_{imp}(-x, \tilde{\Pi}), \quad (7)$$

where σ_{imp} denotes the implied volatility, that is the volatility parameter which reproduces the price obtained in the Lévy model if one applies the Black-Scholes formula.

3.2 Symmetry and Duality for Skewed Models

In the case of the skewed model given by (5), we have

$$\tilde{f}_t(x; \beta) = e^{-x} f_t(-x; \beta)$$

where $\tilde{f}_t(x; \beta)$ denotes the density of the distribution of \tilde{X}_t under $\tilde{\mathbb{Q}}_\beta$, the dual of \mathbb{Q}_β . Now for $\beta = -1/2$ we know that market symmetry property holds, i.e, the distribution of X_t under \mathbb{Q}_β and the distribution of \tilde{X}_t under $\tilde{\mathbb{Q}}_\beta$ are identical. Form here one gets

$$f_t(x; -1/2) = \tilde{f}_t(x; -1/2) = e^{-x} f_t(-x; -1/2). \quad (8)$$

In particular for any β , $\tilde{\Pi}_\beta(dy) = e^{-(\beta+1)y}\Pi_0(dy)$ and we can rewrite relationship (7) in the form

$$\sigma_{imp}(x, \beta + 1/2) = \sigma_{imp}(-x, -(\beta + 1/2)). \quad (9)$$

In Fajardo and Mordecki [2006] this result is obtained for the case $\beta = -1/2$.

We show in Figure 1 the implied volatility for the Variance Gamma model in terms of x and β . Here we immediately see the symmetry relation (9).

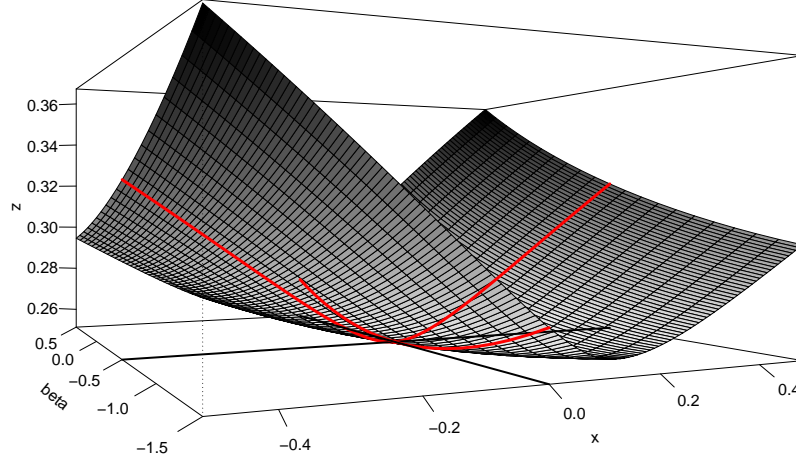


Figure 1: Variance Gamma Implied volatility as function of x and β . Other parameters: $\alpha = 5$, $\lambda = 1$, $T = 1$, $r = 0.05$.

4 Implied Volatility Behavior in a Neighborhood of $\beta = -1/2$.

In this section we present our main result. We describe the implied volatility behavior in a neighborhood of $\beta = -1/2$.

We obtain two results involving the implied volatility. The first one: the implied volatility as a function of β , in a neighborhood of $\beta = -1/2$, is increasing when x is in a right-neighborhood of $x = 0$ and is decreasing when x is in a left-neighborhood of $x = 0$. The second one: the implied volatility as a function of x , in a neighborhood of $x = 0$, is increasing when β is in a right-neighborhood of $-1/2$ and it is decreasing when β is in a left-neighborhood of $-1/2$.

In order to formulate the following results, write the Black-Scholes formula in terms of x as

$$BS(K_x, \sigma_{imp}(x, \beta)) = S_0 N(d_1) - K_x e^{-rT} N(d_2),$$

where

$$d_1 = \frac{-x + \sigma_{imp}^2(x, \beta) \frac{T}{2}}{\sigma_{imp}(x, \beta) \sqrt{T}} \quad \text{and} \quad d_2 = \frac{-x - \sigma_{imp}^2(x, \beta) \frac{T}{2}}{\sigma_{imp}(x, \beta) \sqrt{T}}.$$

Now, differentiating with respect to x , we obtain

$$\begin{aligned} \frac{\partial BS(K_x, \sigma_{imp}(x, \beta))}{\partial x} &= \frac{\partial BS(K_x, \sigma_{imp})}{\partial K_x} \frac{\partial K_x}{\partial x} + \frac{\partial BS(K_x, \sigma_{imp})}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}(x, \beta)}{\partial x} \\ &= -S_0 e^x N(d_2) + \sqrt{T} \phi(d_1) \frac{\partial \sigma_{imp}(x, \beta)}{\partial x}. \end{aligned}$$

where N and ϕ denote the standard normal cdf and density, respectively. On the other hand, the value of a call option under the Lévy process (5) is denoted by V , and given by

$$V(x, \beta) = e^{-rT} \mathbb{E}_\beta (S_T - K_x)^+ = S_0 \int_x^{+\infty} (e^y - e^x) d\mathbb{Q}_\beta. \quad (10)$$

In order to obtain the differentiability of the implied volatility with respect to x , we need the differentiability of the function $V(x, \beta)$ with respect to x . In view of (10), the existence of a continuous density of the measure \mathbb{Q}_β is a sufficient condition for the differentiability of V . According to Proposition 28.1 in Sato [1999], the condition

$$\int_{\mathbb{R}} |e^{t\psi_\beta(iz)}| dz < \infty \quad (11)$$

implies the existence of a continuous density. Throughout this section we assume that condition (11) holds. We then have

$$\frac{\partial V(x, \beta)}{\partial x} = -S_0 e^x \mathbb{Q}_\beta(X_T > x),$$

and from the equality $BS(K_x, \sigma_{imp}(x, \beta)) = V(x, \beta)$, we obtain

$$\frac{\partial \sigma_{imp}(x, \beta)}{\partial x} = \frac{S_0 e^x (N(d_2) - \mathbb{Q}_\beta(X_T > x))}{\phi(d_1) \sqrt{T}} = S_0 \frac{N(d_2) - \mathbb{Q}_\beta(X_T > x)}{\phi(d_2) \sqrt{T}}. \quad (12)$$

Next, we present our main result.

Theorem 1. *For any maturity T , if $\{X_t, \mathbb{Q}_\beta\}$ is a skewed model with continuous density, then*

1. *there exists $\varepsilon > 0$ such that:*

$$(a) \text{ if } \beta \in (-\frac{1}{2} - \varepsilon; -\frac{1}{2}) \text{ then } \frac{\partial \sigma_{imp}(0, \beta)}{\partial x} < 0,$$

$$(b) \text{ if } \beta \in (-\frac{1}{2}; -\frac{1}{2} + \varepsilon) \text{ then } \frac{\partial \sigma_{imp}(0, \beta)}{\partial x} > 0.$$

2. *There exists $\varepsilon > 0$ such that:*

$$(a) \text{ if } x \in (0, \varepsilon), \text{ then } \frac{\partial \sigma_{imp}(x, -1/2)}{\partial \beta} > 0,$$

$$(b) \text{ if } x \in (-\varepsilon, 0), \text{ then } \frac{\partial \sigma_{imp}(x, -1/2)}{\partial \beta} < 0.$$

For the proof of Theorem 1 we introduce some previous results.

Proposition 1. *Let $\{X_t, \mathbb{Q}_\beta\}$ a skewed model. Then,*

$$\psi_\beta(z) = \psi_0(z + \beta) - \psi_0(\beta) - z\mu_\beta,$$

where $\mu_\beta = \beta\sigma^2 + \int_{\mathbb{R}}(e^y - 1)(e^{\beta y} - 1)\Pi_0(dy)$.

If there exists continuous density we have

$$f_t(x; \beta) = e^{\beta(x + t\mu_\beta) - t\psi_0(\beta)} f_t(x + t\mu_\beta; 0),$$

Proof. We define the measure $\hat{\mathbb{Q}}$ such that

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}_0} = e^{\beta X_t - t\psi_0(\beta)}. \quad (13)$$

Then, from the Esscher transform, $\{X_t, \hat{\mathbb{Q}}\}$ is a Lévy process with triplet $(\hat{\gamma}, \sigma, e^{\beta y}\Pi_0(dy))$ with $\hat{\gamma} = \beta\sigma^2 + \gamma_0 + \int_{\mathbb{R}}(e^{\beta y} - 1)h(y)\Pi_0(dy)$.

We observe that $\{X_t, \hat{\mathbb{Q}}\}$ and $\{X_t, \mathbb{Q}_\beta\}$ are both Lévy processes that differ only in the drift term, therefore

$$\{X_t, \mathbb{Q}_\beta\} \stackrel{\mathcal{L}}{=} \{X_t - t\mu_\beta, \hat{\mathbb{Q}}\}, \quad (14)$$

where $\mu_\beta = \beta\sigma^2 + \gamma_0 - \gamma_\beta + \int_{\mathbb{R}}(e^{\beta y} - 1)h(y)\Pi_0(dy)$.

From (13) and (14) we obtain:

$$\mathbb{E}_\beta(e^{zX_t}) = \hat{\mathbb{E}}(e^{z[X_t - t\mu_\beta]}) = \mathbb{E}_0(e^{(z+\beta)X_t - t[\psi_0(\beta) + z\mu_\beta]}), \quad (15)$$

and from (15) the result follows.

On the other hand, if a continuous density exists we have

$$\begin{aligned} f_t(x; \beta) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izx} e^{t\psi_\beta(iz)} dz \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izx} e^{t[\psi_0(iz+\beta) - \psi_0(\beta) - iz\mu_\beta]} dz \\ &= e^{-t\psi_0(\beta)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iz(x+t\mu_\beta)} e^{t\psi_0(iz+\beta)} dz \\ &= e^{-t\psi_0(\beta)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-(iw-\beta)(x+t\mu_\beta)} e^{t\psi_0(iw)} dw \\ &= e^{\beta(x+t\mu_\beta) - t\psi_0(\beta)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iw(x+t\mu_\beta)} e^{t\psi_0(iw)} dw \\ &= e^{\beta(x+t\mu_\beta) - t\psi_0(\beta)} f_t(x + t\mu_\beta; 0). \end{aligned}$$

□

We observe that for $\beta = 0$ in (5), the process $X_t - t\gamma_0$ is symmetric because of the symmetry of the jump measure. If there exists a density, we have

$$f_t(x + t\gamma_0; 0) = f_t(-x + t\gamma_0; 0).$$

Corollary 1. *If $\{X_t, \mathbb{Q}_\beta\}$ is a skewed model with continuous density, then*

$$f_t(x; -1/2) \leq f_t(0; -1/2)e^{-1/2x} \quad \forall x \in \mathbb{R}.$$

Proof. Observe that

$$\begin{aligned} \mu_{-1/2} &= -\sigma^2/2 + \gamma_0 - \gamma_{-1/2} + \int_{\mathbb{R}} h(y)(e^{-y/2} - 1)\Pi_0(dy) \\ &= \gamma_0 + \int_{\mathbb{R}} \left((e^y - 1)e^{-y/2} - h(y) \right) \Pi_0(dy) = \gamma_0, \end{aligned} \quad (16)$$

because $(e^y - 1)e^{-y/2} - h(y)$ is a odd function. On the other hand,

$$\begin{aligned}
f_t(x + t\gamma_0; 0) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iz(x+t\gamma_0)} e^{t\psi_\beta(iz)} dz \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izx} e^{t[-iz\gamma_0 + \psi_\beta(iz)]} dz \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \cos(zx) e^{t[-iz\gamma_0 + \psi_0(iz)]} dz \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}} e^{t[-iz\gamma_0 + \psi_0(iz)]} dz \\
&= f_t(t\gamma_0; 0),
\end{aligned} \tag{17}$$

where in (17) we use that the imaginary part of $-iz\gamma_0 + \psi_0(iz) = -z^2\sigma^2/2 + \int_{\mathbb{R}} (e^{izy} - 1 - izh(y))\Pi_0(dy)$ vanishes.

From Proposition 1 and (16) we have

$$\begin{aligned}
f_t(x; -1/2) &= e^{-1/2(x+t\gamma_0) - t\psi_0(-1/2)} f_t(x + t\gamma_0; 0) \\
&= f_t(0; -1/2) e^{-x/2} \frac{f_t(x + t\gamma_0; 0)}{f_t(t\gamma_0; 0)}.
\end{aligned} \tag{19}$$

Observe that from (18), $f_t(t\gamma_0; 0) \neq 0$.

From (18) and (19) we obtain the result. \square

Lemma 1. *If $\{X_t, \mathbb{Q}_\beta\}$ is a skewed model with continuous density, then*

$$\left. \frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta} \right|_{-1/2} < 0$$

Proof. For simplicity, we denote $f_t(y) := f_t(y; -1/2)$, the density of the distribution of X_t at $\beta = -1/2$. From the Lewis formula for a digital call option (see Lewis [2001]), for some $0 < v < 1$, we have

$$\mathbb{Q}_\beta(X_t > x) = -\frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{izx} e^{t\psi_\beta(-iz)}}{iz} dz.$$

Then, differentiating with respect to β we obtain

$$\begin{aligned}
& \left. \frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta} \right|_{-1/2} \\
&= -\frac{t}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{t\psi_\beta(-iz)}}{iz} \frac{\partial \psi_\beta(-iz)}{\partial \beta} \Big|_{-1/2} dz \\
&= -\frac{t}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{t\psi_\beta(-iz)}}{iz} \left[\int_{\mathbb{R}} (e^{-izy} - 1 + iz(e^y - 1)) y e^{-y/2} \Pi_0(dy) \right] dz \quad (20)
\end{aligned}$$

$$\begin{aligned}
&= t \int_{\mathbb{R}} y e^{-y/2} \left[\mathbb{Q}_{-1/2}(X_t > -y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^y - 1) f_t(0) \right] \Pi_0(dy) \quad (21)
\end{aligned}$$

$$\begin{aligned}
&= t \int_0^{+\infty} y e^{-y/2} \left[\mathbb{Q}_{-1/2}(X_t > -y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^y - 1) f_t(0) \right] \Pi_0(dy) \\
&\quad - t \int_0^{+\infty} y e^{y/2} \left[\mathbb{Q}_{-1/2}(X_t > y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^{-y} - 1) f_t(0) \right] \Pi_0(dy) \quad (22)
\end{aligned}$$

$$\begin{aligned}
&= t \int_0^{+\infty} y e^{-y/2} \left[\mathbb{Q}_{-1/2}(-y < X_t < 0) - (e^y - 1) f_t(0) \right] \Pi_0(dy) \\
&\quad - t \int_0^{+\infty} y e^{y/2} \left[-\mathbb{Q}_{-1/2}(0 < X_t < y) - (e^{-y} - 1) f_t(0) \right] \Pi_0(dy) \\
&= t \int_0^{+\infty} y e^{-y/2} \left[\mathbb{Q}_{-1/2}(-y < X_t < 0) + e^y \mathbb{Q}_{-1/2}(0 < X_t < y) \right. \\
&\quad \left. - 2(e^y - 1) f_t(0) \right] \Pi_0(dy),
\end{aligned}$$

where in (20) we use that Π_0 does not depend on β , in (21) we use the Fubini Theorem and in (22) we use the symmetry property of Π_0 .

Let $g(y) := \mathbb{Q}_{-1/2}(-y < X_t < 0) + e^y \mathbb{Q}_{-1/2}(0 < X_t < y) - 2(e^y - 1) f_t(0)$. From (8), we have that

$$\begin{aligned}
g'(y) &= f_t(-y) + e^y \mathbb{Q}_{-1/2}(0 < X_t < y) + e^y f_t(y) - 2e^y f_t(0) \\
&= e^y \left[\mathbb{Q}_{-1/2}(0 < X_t < y) + 2[f_t(y) - f_t(0)] \right]. \quad (23)
\end{aligned}$$

From Corollary 1 and (23) we have

$$\begin{aligned}
e^{-y}g'(y) &= \int_0^y f_t(s)ds + 2[f_t(y) - f_t(0)] \\
&\leq f_t(0) \int_0^y e^{-1/2s}ds + 2[f_t(y) - f_t(0)] \\
&= -2f_t(0)e^{-1/2y} + 2f_t(y) \leq 0.
\end{aligned}$$

Observe that $f_t(0)e^{-1/2y}$ cannot be equal to $f_t(y)$ for every $y > 0$ because in these case, from (8), we get the absurd $f_t(-y) = f_t(0)e^{1/2y}$ for $y > 0$. Then, there exist an open interval $I \subset \mathbb{R}^+$ such that $g'(y) < 0$ for $y \in I$.

Since g is decreasing, $g(y) \leq g(0) = 0$ for $y > 0$ and $g(y) < 0$ for $y \in I$, this implies

$$\frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta} \Big|_{-1/2} = t \int_0^{+\infty} ye^{-y/2}g(y)\Pi_0(dy) < 0.$$

□

Proof of Theorem 1. From (12) we obtain:

$$\begin{aligned}
\frac{\partial^2 \sigma_{imp}(x, \beta)}{\partial \beta \partial x} &= \frac{\partial}{\partial \beta} \frac{\partial \sigma_{imp}(x, \beta)}{\partial x} \\
&= S_0 \frac{[\phi(d_2) \frac{\partial d_2}{\partial \beta} - \frac{\partial \mathbb{Q}_\beta(X_T > x)}{\partial \beta}] - [N(d_2) - \mathbb{Q}_\beta(X_T > x)](-d_2) \frac{\partial d_2}{\partial \beta}}{\phi(d_2)\sqrt{T}}.
\end{aligned}$$

We observe that

$$\frac{\partial d_2}{\partial \beta} = - \frac{T^{3/2} \sigma_{imp}^2(x, \beta) \frac{\partial \sigma_{imp}(x, \beta)}{\partial \beta} - (x + \sigma_{imp}^2(x, \beta) \frac{T}{2}) \frac{\partial \sigma_{imp}(x, \beta)}{\partial \beta}}{\sigma_{imp}^2(x, \beta) T}.$$

From (9), we have $\partial_\beta \sigma_{imp}(0, -1/2) = 0$ and then $\partial_\beta d_2(0, -1/2) = 0$ where we conclude that, from Lemma 1

$$\frac{\partial^2 \sigma_{imp}(0, -1/2)}{\partial \beta \partial x} = - \frac{S_0}{\phi(\sigma_{imp}(0, -1/2) \frac{\sqrt{T}}{2}) \sqrt{T}} \frac{\partial \mathbb{Q}_\beta(X_T > 0)}{\partial \beta} \Big|_{\beta=-1/2} > 0. \tag{24}$$

From (24) we have

$$\frac{\partial}{\partial \beta} \frac{\partial \sigma_{imp}(0, \beta)}{\partial x} \Big|_{-1/2} > 0,$$

then, there exists $\varepsilon > 0$ such that $\partial_x \sigma_{imp}(0, \beta)$ is increasing as a function of $\beta \in (-\frac{1}{2} - \varepsilon; -\frac{1}{2} + \varepsilon)$. The results 1(a) and 1(b) are obtained from $\partial_x \sigma_{imp}(0, -1/2) = 0$.

Now we prove 2(a). The result 2(b) is obtained from the implied volatility symmetry (9).

Under regularity conditions we have $\frac{\partial^2 \sigma_{imp}(x, \beta)}{\partial \beta \partial x} = \frac{\partial^2 \sigma_{imp}(x, \beta)}{\partial x \partial \beta}$, then from (24) we have

$$\frac{\partial}{\partial x} \frac{\partial \sigma_{imp}}{\partial \beta}(0, -1/2) = \frac{\partial^2 \sigma_{imp}(0, -1/2)}{\partial x \partial \beta} > 0. \quad (25)$$

From (25) and the symmetry (9) we have that there exists $\varepsilon > 0$ such that, if $x \in (0, \varepsilon)$ then

$$\frac{\partial \sigma_{imp}}{\partial \beta}(x, -1/2) > \frac{\partial \sigma_{imp}}{\partial \beta}(0, -1/2) = 0.$$

This conclude the proof of the Theorem. \square

5 Examples

We discuss the applicability of our Theorem to infinite activity models, jump-diffusion models and show a complementary example that shows why the obtained monotonicity results of the implied volatility is only local.

5.1 Infinite Activity Models.

The Generalized Hyperbolic (GH), including the Variance Gamma (VG) and Normal Inverse Gaussian (NIG) and the Meixner models, are skewed, as follows from Fajardo and Mordecki [2006]. Particular instances of the CGMY model is also skewed (see Fajardo and Mordecki [2006]). As they all have a continuous density, they verify the hypothesis of Theorem 1.

In Figure 2 we show the shape of the implied volatility for the VG model in terms of β .

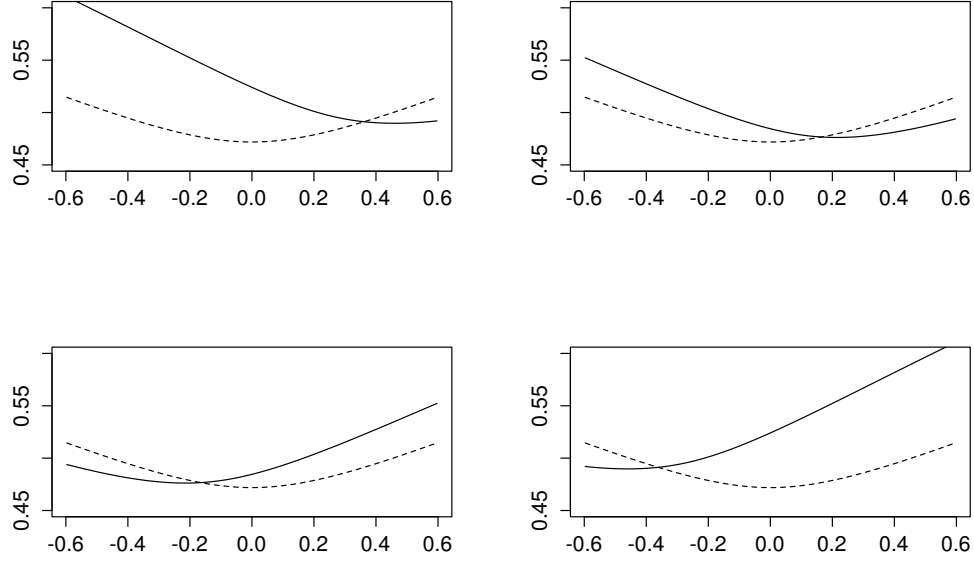


Figure 2: Variance Gamma implied volatility in terms of x . Doted line $\beta = -0.5$. Continuous line: top left $\beta = -1.5$, top right $\beta = -1$, bottom left $\beta = 0$, bottom right $\beta = 0.5$. Other parameters: $\alpha = 4$, $\lambda = 2$, $T = 1$, $r = 0.05$.

5.2 Jump-Diffusion Models.

Observe that any jump-diffusion model with positive diffusion component ($\sigma > 0$) has a differentiable density of any order (use Theorem 4 V.4 in Feller [1971] and Theorem 2.27 in Folland [1999]).

The Merton model has a Lévy measure of the type $\Pi(dy) = e^{\beta y} \Pi_0(dy)$ with Π_0 symmetric, but depending on β :

$$\begin{aligned} \Pi(dy) &= \lambda \frac{1}{\sqrt{2\pi}\sigma_J} \exp \left\{ -\frac{1}{2} \left(\frac{y - \mu_J}{\sigma_J} \right)^2 \right\} dy \\ &= \left(\lambda e^{-\frac{1}{2} \left(\frac{\mu_J}{\sigma_J} \right)^2} \right) e^{\beta y} \frac{1}{\sqrt{2\pi}\sigma_J} \exp \left\{ -\frac{y^2}{2\sigma_J^2} \right\} dy, \end{aligned}$$

where $\beta = \mu_J/\sigma_J^2$ and μ_J depends on β . For this reason we adapt the Merton model re-parameterizing its Lévy measure.

Definition 1 (Skewed Merton model). *We define the Skewed Merton model as a Lévy process with triplet $(\gamma, \sigma, \Pi(dy))$, where*

$$\Pi(dy) = \lambda e^{\beta y} \frac{1}{\sqrt{2\pi}\sigma_J} \exp\left\{-\frac{y^2}{2\sigma_J^2}\right\} dy.$$

In Figure 3 we show the behavior of the implied volatility in a neighborhood of $\beta = -1/2$.

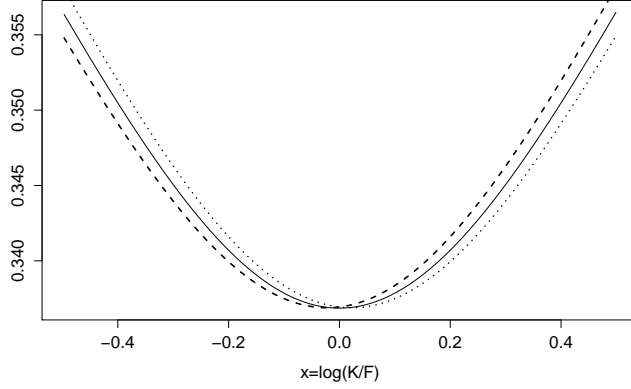


Figure 3: Skewed Merton Implied volatility with: $T = 1$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 2$, $\sigma_J = 0.2$. Continuous line $\beta = -0.5$, dotted line: $\beta = -0.6$, dashed line: $\beta = -0.4$.

The Kou model has a jump density given by $g(y) = p\theta_1 e^{\theta_1 y} \mathbf{1}_{\{y \leq 0\}} + (1 - p)\theta_2 e^{-\theta_2 y} \mathbf{1}_{\{y > 0\}}$, where $p \in (0, 1)$ and $\theta_1, \theta_2 > 0$. The following particular case constitutes a skewed model.

Definition 2 (Skewed Kou Model). *We define the Skewed Kou model as a Lévy process with triplet $(\gamma, \sigma, \Pi(dy))$, where*

$$\Pi(dy) = \lambda e^{\beta y - \alpha|y|} dy.$$

The relation between the parameters of the general and skewed models are $\lambda = p\theta_1 = (1 - p)\theta_2$, $\theta_1 = \beta + \alpha$ and $\theta_2 = \beta - \alpha$.

5.3 A Complementary Example.

In this section we show an example of a Lévy process where the corresponding model verifies the regularity conditions of Theorem 1, but does not verify $\partial\sigma_{imp}(0, \beta)/\partial\beta > 0$ for all $\beta > -1/2$ neither $\partial\sigma_{imp}(x, -1/2)/\partial\beta > 0$ for all $x > 0$.

Definition 3 (Diffusion with a two-sided Poisson jump process). *We define the diffusion with a two-sided Poisson jump process as a Lévy process where the jump measure is given by*

$$\Pi_\beta(dy) = \lambda e^{\beta y} (\delta_a(y) + \delta_{-a}(y)) dy.$$

If the diffusion with a two-sided Poisson jump process has positive diffusion part, then it verify the Theorem 1. However, we show in Figure 5 that $\partial_x\sigma_{imp}(0, \beta)$ is not positive, neither negative² for all $\beta > -1/2$. And we show in Figure 4 that σ_{imp} has local minima at $x = a$ and $x = -a$, where the results of Theorem 1 cannot be extended to the entire real line.

It is clear that this setting is not interesting as a realistic model, however it is a important example when we try to derive results on implied volatility in the context of Lévy process.

²Observe that here we consider any maturity, in contrast with the results obtained in Gerhold and Gülüm [2014] for small-maturity options.

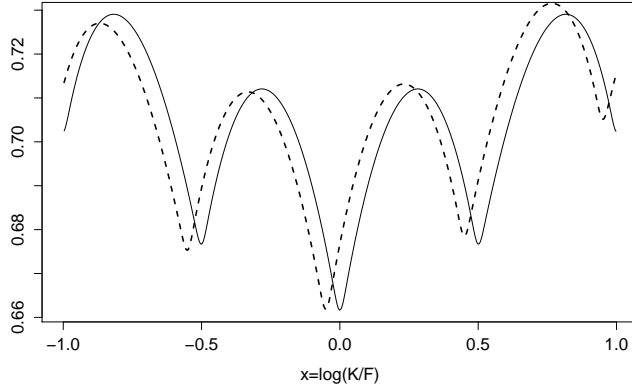


Figure 4: Implied Volatility under diffusion with a two-sided Poisson jump process with $\sigma = 0.01$, $a = 0.5$ and $\lambda = 2$. Continuous line: $\beta = -0.5$, dashed line: $\beta = -0.4$.

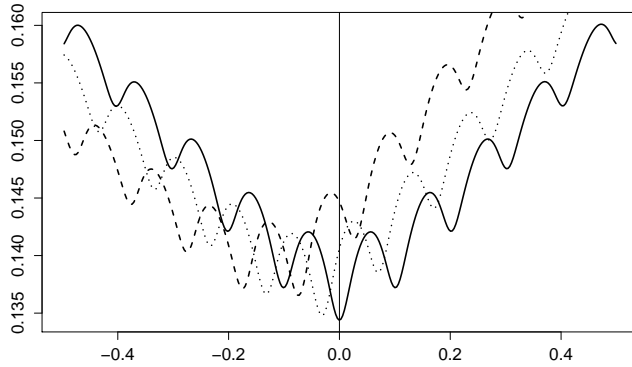


Figure 5: Implied Volatility under diffusion with a two-sided Poisson jump process with $\sigma = 0.01$, $a = 0.1$, $\lambda = 2$, $T = 1$ and $r = 0.05$. Continuous line: $\beta = -0.5$, dotted line: $\beta = 1$ and dashed line: $\beta = 3$.

6 Conclusions

The skewed Lévy models introduced in this paper are characterized by a symmetric jump measure multiplied by a dumping exponential factor that

depends on a parameter β . This parameter quantifies the skewness of the model, and its variation captures the typical smirk feature observed in implied volatility curves. Our main result (see Theorem 1) shows that skewed models with continuous density exhibit a precise monotonicity behavior of the implied volatility surface, as a function of the log-moneyness and the skewness parameter. These results are independent of the maturity of the options, and apply to many main models in the literature. Other popular models can be easily adapted to be skewed. Finally, we present a simple example that shows that the obtained monotonicity behavior can not be further extended.

Our proposal establishes a link between the visual analysis of the implied volatility curves and its mathematical modelling, through the skewness parameter β . It also raises several questions regarding the other second derivatives of the implied volatility surface.

References

- J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, 1996.
- F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81:637–659, 1973.
- S. Boyarchenko and S. Levendorskiĭ. *Non-Gaussian Merton-Black-Scholes Theory*. World Scientific, River Edge, NJ, 2002.
- P. Carr and R. Lee. Put Call Symmetry: Extensions and Applications. *Math. Finance.*, 19(4):523–560, 2009.
- P. Carr and L. Wu. Finite Moment Log Stable Process and Option Pricing. *Journal of Finance*, 58(2):753–777, 2003.
- R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall /CRC Financial Mathematics Series, 2004.
- J. Fajardo and E. Mordecki. Symmetry and Duality in Lévy Markets. *Quantitative Finance*, 6(3):219–227, 2006.
- J. Fajardo and E. Mordecki. Skewness Premium with Lévy Processes. *Quantitative Finance*, 14(9):1619–1626, 2014.

- W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.
- G. Folland. *Real analysis: modern techniques and their applications.* Pure and applied mathematics. Wiley, 1999. ISBN 9780471317166.
- S. Foresi and L. Wu. Crash-O-Phobia: A Domestic Fear or A Worldwide Concern? *Journal of Derivatives*, 13(2), 2005.
- S. Gerhold and I. C. Gülüm. The Small-Maturity Implied Volatility Slope for Lévy Models. Preprint, available at <http://arxiv.org/abs/1310.3061>, 2014.
- J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, Berlin, Heidelberg, 1987.
- A. L. Lewis. A Simple Option Formula for General Jump-diffusion and Other Exponential Lévy processes. Working paper. Envision Financial Systems and OptionCity.net Newport Beach, California. Available at <http://www.optioncity.net>, 2001.
- K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge., 1999.
- W. Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives.* Wiley, New York, 2003.
- A. V. Skorokhod. *Random Processes with Independent Increments.* Kluwer Academic Publishers, Dordrecht, 1991.