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Essays in Applied Microeconomics

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A meus pais e irmão.
A Natália.

Resumo

Esta tese é composta de três artigos. No primeiro artigo, “Risk Taking in Tournaments”, é considerado um torneio dinâmico no qual jogadores escolhem como alocar seu tempo em atividades que envolvem risco. No segundo artigo, “Unitização de Jazidas de Petróleo: Uma Aplicação do Modelo de Green e Porter”é analisado a factibilidade de se haver um acordo de cooperação em um ambiente de common-pool com incerteza nos custos das empresas. No terceiro artigo, “Oilfield Unitization Under Dual Fiscal Regime: The Regulator Role over the Bargaining”, por sua vez, é estudado a unitização quando existem dois regimes fiscais distintos, como os jogadores se beneficiam disso e o papel do regulador no regime de partilha.

Palavras-chave: Torneios. Tomada de Risco. Unitização. Regimes Fiscais do Petróleo.

Abstract

This dissertation comprises of three articles. The first article, “Risk Taking in Tournaments”, considers a dynamic tournament model where players choose how to spend their time in activities that involve risk. The second article, “Unitização de Jazidas de Petróleo: Uma Aplicação do Modelo de Green e Porter” analyses the feasibility of a cooperation agreement under a common-pool problem where the production cost is uncertain. The third article, “Oilfield Unitization Under Dual Fiscal Regime: The Regulator Role over the Bargaining”, by its turn, studies the unitization when there exist two different fiscal regimes and how firms are benefited by this distortion, and the role of the regulator in the profit sharing contract.

Keywords: Tournaments. Risk Taking. Unitization. Petroleum Fiscal Regimes.

Lista de Figuras

Lista de Tabelas

Sumário

Chapter 1

Risk Taking in Tournaments

Abstract

We study a tournament model where players must decide how to allocate their time in different activities, associated with different levels of risk. At first, we describe and study the static version of the model, where there is an optimal choice of variance that induces the highest expected output. Under a piece-rate, this would be the choice of the player, but under tournaments, players deviate from this choice to increase the probability of winning. In the dynamic setting, we show how a player seeks more or less risk as the games comes to the end. Finally, we presents the tie-break rule as a way the principal can improve the expected output.

1.1 Introduction

Many real-world compensation schemes grant prizes depending on agent's performance relative to others. This situation is called tournaments in economics. There are a wide variety of situations where we can find tournaments-like compensations. Students are graded relatively to their peers, and those who have good grades can be benefited directly by scholarships and grants, or indirectly through networking that became available due to the good grades. Research and development can also be thought as a tournament, where the firm that submit the most attractive proposal is rewarded by a contract. The elections, when candidates run for a position in the government by competing during their campaign. The investment flow in the financial market, where the hedge funds that perform better receive inflows of new investment money. A loan officer who choose different projects and those who perform better can be promoted. The practice of law, where attorneys compete to each other during a trial representing their clients and being evaluated by their performance in other to win a legal dispute. The large salaries of executives may provide incentives for all individuals in the firm who, with hard labor, may win one of the coveted top positions. Competition among employees within a firm is

among the most widespread ways to provide incentives and granting higher salary, bonus and promotions.

The contract theory literature has embraced tournaments as a descriptive model of incentives within firms. Workplace contests are among the most common motivating applications of the theoretical tournament literature. Tournaments among workers can mitigate incentive problems when the effort of workers is unobservable. A key idea underlying tournament theory is that there are clear winners and losers, and the rewards are usually indivisible, so the knowledge of it provides the incentives agents need to effort towards firms' goals, if the mechanism is well set. The incentive comes from the rationality of the players who seek to maximize their individual utility by increasing the chance of winning a prize. In contrast with other forms of remuneration, such as piece rates – where players are paid according to their production –, tournaments may involve lower information costs. It is generally cheaper to track the relative performance levels rather than to monitor absolute performance levels, especially when it involves few competitors. For example, PhD programs and employers skim through the information available of the applicants, and carefully evaluate only the very top of them. This is especially true when employees are compared in order to be rewarded, since it can isolate the common shocks that might occurred during the evaluation period. The piece rate could provide the wrong compensation for players' effort, or the employer would need to be informed about the market conditions before cross off the aggregate shock term from their performance. The relative performance evaluation removes the risk of common uncertainty, so it can reduce the amount of money paid to players, since it also work as an insurance. Both players and principal know the maximum and the least amount of money they will spend/earn.

Although tournaments offers many advantages in different contexts, it can also be dangerous when it consists in a multi-period tournament. A dynamic tournament can provide the wrong incentives for players to exert effort. In many real-world situation, tournaments are played in a dynamic background. Sports is the most common example, where players competes over time and at each instant of time, they know whether they are leading or lagging in a game. We can also find examples in the labor market, where salespeople competing for a bonus or promotion are subject to a tournament that takes into account the whole months sales, and at each day or week, players are informed about his and the others sales level. The salesperson who is behind his peers can be discourage to exert effort, since it will be unlikely that he can overcome them and it will be more costly to work hard enough for reach them. The theory shows that if players are evaluated over a multi-period tournament, and at each period, they learn their relative performance, it can discourages effort by players, once the probability of winning reduces when an opponent has an advantage. In this case, the underdog may reduce his effort in the game because the marginal effect over the probability of winning is reduced and exerting effort is costly.

Nevertheless, effort may not be the only choice an agent has while competing in a tournament. In sports, a coach and his team can choose different levels of risk in a game, for example, in a soccer match the team losing the game can switch a defense player for an offensive player to increase the probability of scoring a goal. We can also observe it in other sports, like in a hockey game, when a team switch the goalie for a line player, or in car racing, where the driver can choose risky paths to overcome someone ahead of him. Not only in sports we can observe it, but also in many other situations, like labor market and financial market. An employee can exert risky tasks in order to increase his chance to be promoted, like a loan officer that selects different projects to finance and his reward depends on the return of his loans compare to the other officers. An investment fund that balances its portfolio based on expected return and risk. A CEO can affect the riskiness of the company's production and profit, based on his choice of research project, the kind of employee he selects and the market he would like to explore, in addition to deciding how hard they will work.

In the investment funds industry, for example, firms face a trade-off between expected return and risk of their portfolio. It is also possible that two different portfolios have the same expected return, but different volatility. There might be an optimal way to balance the portfolio, in terms of expected return adjusted by the risk of it. However, in a tournament players can be encouraged to seek more risk than the one that generates the highest expected return, in order to increase the probability of wining during a dynamic competition. Funds are evaluated periodically, and when it lacks performance relative to other funds, it may have incentives to deviate from the optimal choice of expected return and increase the risk of their portfolio in order to increase its probability of being the top performer and thus receiving larger amount of investment flow. We can also see this the trade-off between risk and expected output in other activities, for example, an advertisement sales, in which salespeople spend their time contacting the usual clients or they can spend time in prospecting new clients.

In this paper we study a tournament situation in which players must choose how spend their time in different activities. Each activity has an expected output and volatility that generated the trade-off between risk and expected return. In the literature of risk taking in tournament, the choice of risk is costless, so players increase the variance as much as possible in order to reduce the marginal gain from effort and thus avoid making any costly effort, the so-called risky-lazy equilibrium. Some authors work with a binary choice of variance to avoid this equilibrium. In our setting, players have a time constrain and must choose how to spend their time between two activities. Both activities have stochastic output, but one of them is riskier. In the static setting, we found that when a player has some advantage, they would deviate from the way that generates the maximum expected output. By doing it, players would be decreasing or increasing their product

volatility seeking to increase their probability of winning. Our main goal is to model the dynamic behavior during a tournament. In a dynamic setting, we found that players' behavior towards risk evolves overtime, and as the game is near the end, players will seek more risk if there are losing, and will avoid risk if they are leading. To the best of our knowledge, this behavior was not modeled yet, though it is a common wisdom.

Next section we will survey the related literature about tournament theory with risk taking. Section 1.3 we describe the model and compute the equilibrium for the static case. Sections 1.3.4 and 1.4 we describe the tournament with two and T periods, respectively. The Section 1.5 we consider the static case again, and discuss the assumption about the symmetric expected output function and what would happen if we relax this assumption. We also present a tie-break rule, so in order to win the prize they must perform much more than their opponent. We show numerically how the principal can improve players' output by setting correctly the tie-break rule.

1.2 Related Literature

The formal study of this reward structure has started with Lazear and Rosen (1981), in a labor market model. They provide an alternative model to understand why some employees earns much more after a promotion, even though they perform a similar job without increasing his marginal productivity. The idea is that players can be encouraged to effort by competing against their peers for a promotion or wage rise. Their paper shows that tournament schemes can implement first-best levels of effort, when agents are risk neutral and the prizes are carefully chosen.

This type of contract is commonly used in the labor market, so economists have been trying to understand why. Green and Stokey (1983) provide us two initial reasons why tournament contracts are so common, besides the fact that using tournaments requires less information. In their model, risk averse players competes in a rank order contract, where the principal optimally determines the prizes. They show that under a sufficiently diffuse common shock, the use of tournaments dominates the optimal independent contracts, since tournaments allow the principal to filter out the common shock. In addition, when the number of agents increase and the principal cannot observe the common shock and uses a rank order contract can do as well as a principal could do when he observes the common shock and uses an independent optimal contract. Although these cases seems to be extreme, Nalebuff and Stiglitz (1983) show that in a contest with two risk neutral players, the first best optimum can be obtained by choosing appropriately the prizes, and also show that even a two-player contest can dominate the individual linear payment schedule when players are independent. The authors also present more properties about the second best solution. Tournament-like contracts can induce players to effort more

than they do in the first best solution, but it would require the principal to compensate them in order encourages them to participate in the contest. Furthermore, introducing a tie-break rule, it is possible to improve the tournament solution with respect to the principal outcome.

However, in the real world the effort is not the only possible choice that workers have. Sometimes agents can influence both the spread of distribution of the output and the distribution mean. For example, a loan officer chooses projects with different expected returns and risks, managers can choose innovative or conservative technology strategies and fund managers can choose the riskiness and expected returns of their portfolio. To the best of our knowledge, Bronars (1987) was the first to consider the idea of player manipulating the risk to increase the probability of winning. The first published paper about risk taking in tournament is Hvide (2002), which studies the incentives for risk seeking by the identical players in a symmetric equilibrium. Since the choice of risk is costless in his model, players choose the maximum variance in order to reduce the costly effort they must do to overcome the opponent. Increasing variance in a symmetric case will reduce the incentives to do a costly effort, since the marginal increase in the probability of winning by increasing effort is low. Therefore, when players are identical, this compensation scheme induces a "risky-lazy" equilibrium, where both players choose infinity variance and zero effort.

Two papers that studied the asymmetry of player are Kräkel and Sliwka (2004), and Kräkel (2008), by considering different players' abilities. They introduce a two-stage model, where players choose the risk level, which is costless, and then the costly effort. According to them, the choice of variance has two effects: (i) the effort effect, which is the reduction of the importance of the effort due to the larger level of noise in the output, and hence, players can reduce their effort level; and (ii) likelihood effect, which is the change in the probability of winning related to the choice of risk. They show that if the leader has a large enough advantage, he would chose low-risk strategy, while the underdog's choice depends on the cost of effort. When the cost is high, the underdog will choose high-risk strategy. If the advantage is not so large, the underdog will always choose high-risk strategy, while the leader would choose high-risk strategy when the effort cost is low. Gilpatric (2009) presents a model more closely related to ours, in the sense he introduces a costly choice of risk, and propose a penalty that the last placed player must bear, and thus it reduces the excessive risk taking by players in a tournament and control the productive effort, but he does not explore the dynamic effect of risk taking.

Tournament in real life are usually dynamic, in the sense that players compete against each other in more than just one period, and usually at each period they learn their position before taking an action. Ederer (2010) and Aoyagi (2010) studied how the

midterm information affects players' decision in the last period of a two-stage tournament, where the risk choice is not available for the players. Since the output is stochastic, in the beginning of the second period one of the players will have an advantage, and if the information about the advantage is fully disclosed, then both players will have incentives to reduce effort. For the underdog, the marginal gain in the probability of winning is reduced and the cost to exert effort to tie the leader's advantage is high. However, if the principal does not reveal the information about the advantage, players would be less heterogeneous, and thus they would exert more effort than in the full disclose information tournament.

Cabral (2003) and Anderson and Cabral (2007) are the first to study a dynamic tournament where players' decision involve choosing the variance of an output. They consider an infinite time horizon, where players receive a payoff for each instant of time depending on their position. The background for their model is the research and development competition in an oligopolistic industry. Two players choose the volatility of their product quality at each instant, and receive a payoff according to the quality difference. Since the game has no end, players choice depends only on the previous quality difference and their patience. The linearity of their model results in a bang-bang solution, and their findings are that for sufficiently patient player, the underdog (or laggard) chooses high risk, while the leader will choose low-variance strategy.

Many authors have empirically studied the problem of risk taking in tournaments. Brown et al. (1996) and Chevalier and Ellison (1997) study the mutual fund industry and associate the risk taking by the managers of investment portfolios to the relative performance evaluation. Funds charge investors a percentage of their investment, called the management fee, and thus mutual fund companies seeks increase the total amount of asset. Chevalier and Ellison (1997) first study the relationship between the inflow of new investment into a mutual fund and the funds historic performance. They estimate how the market implicitly compensates funds for increasing the riskiness of their portfolios toward the end of the year as a function of the funds' performance from January to September year-to-date, age, and other characteristics. The expected inflow is shown to be non-decreasing function of the previous excess return of the funds, and it presents a convex shape when the returns exceed the market by 15 to 22 points. Therefore, this pattern shows that very large returns bring higher flows as a fund starts to make annual "best fund" lists, and gain more attention from potential investors. Secondly, they show that funds that are somewhat behind the market have an incentive to gamble and try to catch up, while funds that are somewhat ahead of the market have an incentive to lock in their gains. However, at the extreme points the incentive reverse. Finally, they show that mutual funds alter their portfolios between September and December in a manner consistent with the September incentives to take risk.

Genakos and Pagliero (2012) empirically study the impact of interim rank on risk taking using data from weightlifting competitions. In this sport, players must announce the weight they will attempt to lift at the beginning of each stage. The authors consider the relative increase of the announcements as a measure of risk, that is, how much a player increased his announcement from the previous one relative to the player placed first. In order to evaluate the players' behavior towards the risk, they estimate three different models: (i) Announcement vs Ranking; (ii) Announcement vs Absolute distance from competitors; and (iii) Announcement vs Intensity of the Competition. In the first model, they found an inverted-U relationship between the announcement and interim ranking, that is, the announcements increase from the first to sixth place but then decrease for further decreases in rank. One possible reason for riskier strategies for the third and fourth rank is the prize discontinuity between them, which provides incentives to take risk. In the model (ii), they found the expected relationship between risk taking and distance from competitors. A player choose riskier strategy when the competitor in front is close and the trailing player is far. On the other hand, a player reduces the risk when the competitor in front is far and the one trailing is close. Finally, in the model (iii), they measure the intensity of competition by the number of athletes that are close to each other, and they found that more intense competitions stimulates more risk taking. However, Genakos and Pagliero (2012) did not consider how risk taking by the competitors evolves along the time, since their main concern was to evaluate how ranking affects the risk taking.

Although it is a common wisdom players' behavior in a dynamic tournament, to the best of our knowledge, no theoretical study has modeled how players seek more or less risk as the game is near the end. The main purpose of this paper is to evaluate players' behavior in the dynamic tournaments. At first, we will introduce the static model, in which one player has an exogenous advantage at the beginning of the period, and evaluate how players seek risk. In Section 1.4 we present the dynamic model, where players start the tournament at the same position, and thus the advantage becomes endogenous. Next section we describe the basic model in a static setting, where the advantage is exogenous. We show the tournament incentives leads to inefficiency, and show how players behave in the equilibrium.

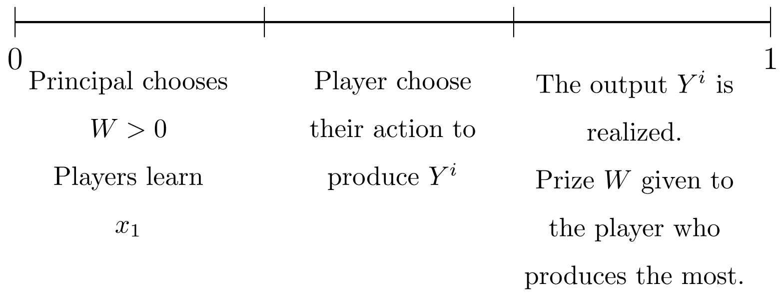
1.3 The Static Model

In this section, we study a tournament model in which one player may have an advantage relative to his opponent, in terms of production level, and how it affects the risk taking by players.

1.3.1 Environment

We consider two players ($i = A,B$) competing for a prize $W > 0$, which is granted to the player who produces the most, while the other player receives zero payoff. The tournament setting has the following timing: First, players learn their initial (dis)advantage, represented by x_1 as the initial advantage of player A, and choose their action. The output is realized and the principal gives the prize to the player with highest output, after observing it. The loser does not receive anything.

Figure 1.1: Tournament Timing.



The production is a stochastic function of how players spend their time. There are two activities that players can engage in order to produce output, denoted by $k \in \{r,s\}$. Each of them generates the same expected output per unit of time spent on it, given by the strictly increasing and concave function $f(\cdot)$, so it presents a diminishing marginal output and the player could produce more by working in both activities. The activity r is riskier than activity s, in the sense that the task r volatility per unit of time spent on it is higher than the volatility of task s. Let τ be the amount of time spent on activity k, the output generated by this task follows a Normal distribution with mean $f(\tau)$ and variance $\tau\sigma_k^2$, i.e. $Y_k^i \sim N(f(\tau),\tau\sigma_k^2)$, where $\sigma_r^2 > \sigma_s^2$.

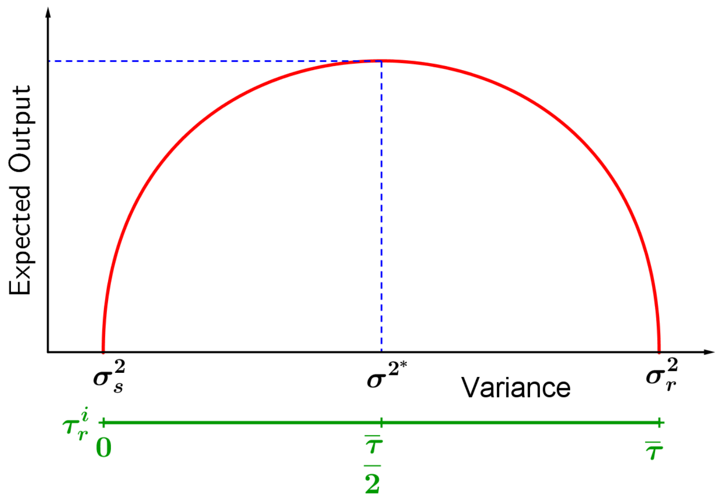
Each player has $\overline{\tau}$ units of time available to allocate in both of activities. Let τ^i be the amount of time player i spends on the riskier activity, r, thus $\overline{\tau} - \tau^i$ is the amount of time spent on the "safe" task, s. After players learn x_1, they choose simultaneously how to spend their time by choosing $\tau^i \in [0,\overline{\tau}]$. Observe that for each possible time allocation by player i, the output Y^i generated by a choice τ^i follows a Normal distribution with mean $g(\tau_t^i) \equiv f(\tau^i) + f(\overline{\tau} - \tau^i)$ and variance $\tau^i\sigma_r^2 + (\overline{\tau} - \tau^i)\,\sigma_s^2$. The outputs are independent, since the random term is independent between players and tasks. Therefore, the total output produce by player i follows a Normal distribution, $Y^i \sim N\left(g(\tau^i), \tau^i\sigma_r^2 + (\overline{\tau} - \tau^i)\,\sigma_s^2\right)$

Note that the variance of the output is basically a convex combination between σ_s^2 and σ_r^2, where the weights are the time allocated in each activity. The function g is even with respect to $\overline{\tau}/2$, i.e. $g(\tau) = g(\overline{\tau} - \tau)$, and thus we have the following two properties: (i) $g'(\tau) = f'(\tau) - f'(\overline{\tau} - \tau) = -g'(\overline{\tau} - \tau)$, so the inclination of g for two different points

equidistant to $\overline{\tau}/2$ has the same magnitude but with different sign, and; (ii) $g(\cdot)$ is strictly increasing in $[0,\overline{\tau}/2)$ and strictly decreasing in $(\overline{\tau}/2,\overline{\tau}]$.

We can see that the maximum expected output is attained when $f'(\tau^i) = f'(\overline{\tau} - \tau^i)$, and since $f(\cdot)$ is strictly increasing, the maximum is achieved when $\tau^i = \overline{\tau}/2 = \overline{\tau} - \tau^i$. The Figure 1.2 shows us the function g, which is equivalent to the set of possible choices a player has.

Figure 1.2: Set of Possible Choices



Since the choice of how the time is allocated allocation is costless, in any payment scheme based on production, one would expected that the player would choose the allocation that maximize the output, i.e., $\tau^i = \overline{\tau}/2$. However, in a tournament payment scheme, when a player is in disadvantage he might be willing to increase variance of his output in order to increase the chance of getting a higher output.

1.3.2 Tournaments Incentives

A tournament is a compensation scheme in which contestant's rewards are based on their ordinal position alone and not on the actual size of their output. This scheme pays $W > 0$ to the player with highest output and zero to the second player. Players decide simultaneously their action, seeking to increase their probability of winning the prize, which depends on player A and B choices and the distribution of the random variables that affects both players output. If we introduce an advantage to player A, denoted by x_1, formally we have,

$$\max_{\tau^A} \; W\mathbb{P}\left[x_1 + Y^A(\tau^A) > Y^B(\tau^B)\right] \qquad (1\text{PP}_A)$$

We can use the properties that the sum of Normal distributions is a Normal distri-

bution, and rewrite the objective function, using the time constraint as

$$\mathbb{P}\left[x_1 + Y^A(\tau^A) > Y^B(\tau^B)\right] = \Phi\left(\frac{x_1 + g(\tau^A) - g(\tau^B)}{\sqrt{(\tau^B + \tau^A)\,\sigma_r^2 + (2\overline{\tau} - \tau^B - \tau^A)\sigma_s^2}}\right),$$

where Φ is the standard Normal cumulative distribution function. In order to ease the notation, denote

$$\eta^2 = \left(\tau^A + \tau^B\right)\sigma_r^2 + (2\overline{\tau} - \tau^A - \tau^B)\sigma_s^2$$

and

$$z = \frac{x_1 + g(\tau^A) - g(\tau^B)}{\eta}.$$

In order to see the incentives behind a tournament scheme, we will study the first order condition of the player A, which is the following

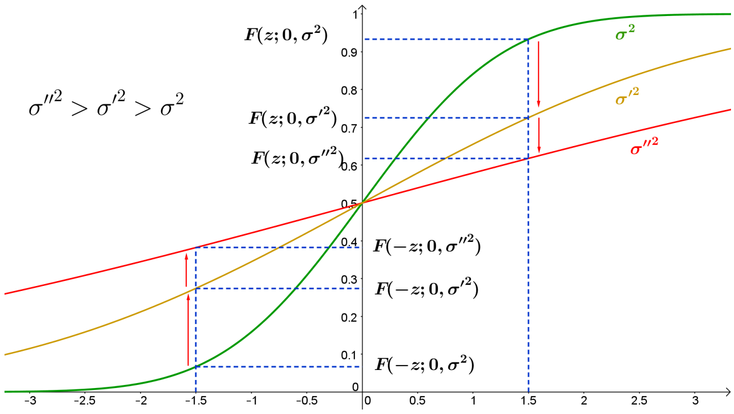
$$\begin{aligned} 0 &= \frac{W}{\eta}\phi(z)\left[f'(\tau^A) - f'(\overline{\tau} - \tau^A) - \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z\right] \\ \iff f'(\tau^A) - f'(\overline{\tau} - \tau^A) &= \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z. \end{aligned} \tag{1.1}$$

As we can see in the above equation, it is not necessarily true that player maximize the expected output, i.e. it's not always true that $f'(\tau^A) - f'(\overline{\tau} - \tau^A) = 0$. This would happen if their contract was based on output production, but since they are working under a relative performance evaluation (RPE) contract, a distortion can happen and the best response for the player will not be to maximize expected output anymore. In a tournament, players want to maximize their probability of winning the prize by allocating their time in two activities. The leader would want to remain with the advantage to win the game, so he is willing to sacrifice some of the expected output in order to reducing the risk. By doing it, the noise effect over leader advantage is reduced. On the other hand, the underdog would be willing to sacrifice some of the expected product he would get by maximizing the output, in order to increase the volatility of his output. Thus, he would be reducing the effect of the disadvantage.

Figure 1.3 illustrates the general idea about incentives in tournaments, when a player has an advantage towards the other. Suppose player A starts with 1.5 units of output in the beginning of the game, while the other player has no output yet. In this simple example, the figure shows the probability player A wins when they both decide to produce at the same expected output, but with different levels of risk. In this example, player A would prefer to reduce his risk, by choosing the variance equals to σ^2. On the other hand, the underdog who is losing by 1.5, we can see his probability of winning on the left side of the figure, where the vertical line intercepts -1.5. In his case, he would be better choosing

higher risk, so the initial disadvantage has lower influence over the outcome.

Figure 1.3: Probability of Player A win the Tournament



In the next subsection, we will provide different ways to read the first order condition from (1.1).

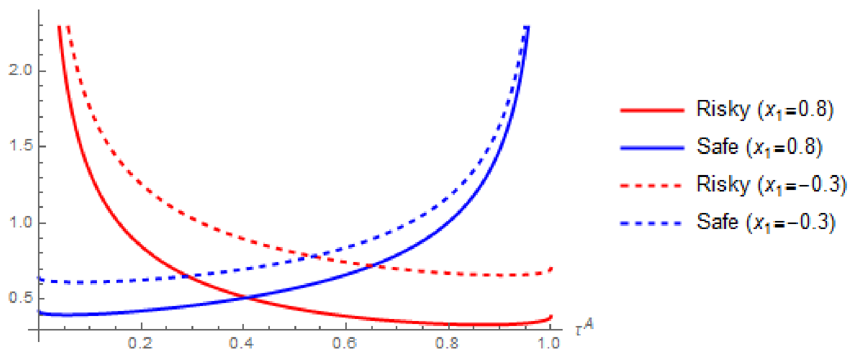
Return on Time Allocation

We can rewrite the Equation (1.1) in terms of marginal return of the time allocate to each activity, and we can see the each player equates the marginal return of the risky and safe activity, as shown by the following equation:

$$f'(\tau^A) - \frac{\sigma_r^2}{2\eta}z = f'(\overline{\tau} - \tau^A) - \frac{\sigma_s^2}{2\eta}z. \tag{1.2}$$

This equation shows on l.h.s. the marginal return of allocating an additional fraction of the time on the risky activity, while the r.h.s. the same for safe activity. Note that since $\sigma_r^2 > \sigma_s^2$, the (dis)advantage affects more the risky choice. The Figure 1.4 illustrates this equation.

Figure 1.4: Player A first order condition



Notes: To plot this figure, it was used $\tau^B = \frac{1}{2}$, $\sigma_s^2 = 1$, $\sigma_r^2 = 2$ and $f(\tau) = \sqrt{\tau}$.

Note that Figure 1.4 is based on the risky activity time allocation, i.e. τ^A. Therefore, this plot can be interpreted as the marginal benefit of the spending time at the risky activity (red line) versus the marginal cost of opportunity to allocate time in this activity (blue line), which is given by the marginal benefit of spending time at the safe activity. First note that they are not symmetric, since $\sigma_r^2 > \sigma_s^2$, hence one player will be willing to deviate from the maximal expected output time allocation, whenever there is a disadvantage. If player A starts with an advantage $x_1 > 0$, we will see that $z > 0$, so working an additional time on the risky task will negatively affect his probability of winning, because $\sigma_r^2 > \sigma_s^2$. We plotted two possible values for x_1, one positive (i.e. player A has advantage) and one negative (i.e. player B has advantage). We can see that when the player A has an advantage, the best response for him is to allocate more time in the safe activity $\tau^A < \bar{\tau}/2$, while when he is losing he prefers to work more on risky activities. We can also see that the risky activity is more sensitive with variations on x_1, since $\sigma_r^2 > \sigma_s^2$.

1.3.3 Equilibrium

In the previous section, we analyzed the best response of a player, and it was shown the effects of the advantage and the variance in this model. Now, we will characterize the Nash equilibrium of this static tournament. From Equation (1.1) we have the best response of player A. For player B, the only change is that the probability of winning is $1 - \Phi(x)$, and thus, the system to be solved has the following equations:

$$f'(\tau^A) - f'(\bar{\tau} - \tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta} z \quad \text{and} \quad f'(\tau^B) - f'(\bar{\tau} - \tau^B) = -\frac{\sigma_r^2 - \sigma_s^2}{2\eta} z \tag{1.3}$$

Proposition 1 *A pure-strategy equilibrium exists, if the following condition holds:*

$$\inf_{\tau^i} \left| \frac{g''(\tau^i)}{g'(\tau^i)} \right| > \frac{\sigma_r^2 - \sigma_s^2}{2\bar{\tau}\sigma_s^2}. \tag{1.4}$$

The players' equilibrium choice will satisfy the following equations:

$$g'(\tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2} \frac{x_1}{\bar{\tau}(\sigma_r^2 + \sigma_s^2)} \quad \textit{and} \quad g'(\tau^B) = -\frac{\sigma_r^2 - \sigma_s^2}{2} \frac{x_1}{\bar{\tau}(\sigma_r^2 + \sigma_s^2)} \tag{1.5}$$

Proof. Note that Φ is a strictly increasing function, so we are interested in maximizing it's argument, defined by

$$z = \frac{x_1 + g(\tau^A) - g(\tau^B)}{\eta}$$

A sufficient condition for a pure-strategy Nash equilibrium to exist is that z strictly concave in player A choice given any choice of his opponent. The existence is guarantee if

$$g''(\tau^A) - \eta_{\tau^A\tau^A}\frac{g'(\tau^A)}{\eta_{\tau^A}} < 0, \text{ for all } \tau^A \text{ and } \tau^B,$$

where η_{τ^A} is the partial derivative of η with respect to τ^A and $\eta_{\tau^A\tau^A}$ is the second order partial derivative of η with respect to τ^A. This condition holds if $g'(\tau^A) = 0$, since g is a concave function. If $g'(\tau^A) < 0$, this condition also holds, because η is a strictly increasing and concave function in η. When $g'(\tau^A) > 0$, a sufficient condition to guarantee the existence is

$$\inf_{\tau^A}\left|\frac{g''(\tau^A)}{g'(\tau^A)}\right| > \sup_{\tau^A,\tau^B}\left|\frac{\eta_{\tau^A\tau^A}}{\eta_{\tau^A}}\right|.$$

Since $\tau^i \in [0,\overline{\tau}]$, the right-hand side is equal to ${(\sigma_r^2-\sigma_s^2)}/{2\overline{\tau}\sigma_s^2}$, thus we have the condition (1.4). The condition for player B is the same, but $-z$ needs to be concave in τ^B. Assuming that (1.4) holds, we can characterize the equilibrium by using the first order condition.

We can rewrite the Equation (1.3) using the function $g(\cdot)$ and equalize them, so we find $g'(\tau^A) = -g'(\tau^B)$. But using that $g'(\tau) = -g'(\overline{\tau} - \tau)$, we rewrite the equilibrium condition as $g'(\tau^A) = g'(\overline{\tau} - \tau^B)$, and since g is strictly concave, we have $\tau^A = \overline{\tau} - \tau^B$, or $\tau^A + \tau^B = \overline{\tau}$.[1] We now rewrite $\eta^2 = \overline{\tau}(\sigma_r^2 + \sigma_s^2)$ and thus at the equilibrium we have

$$z = \frac{x_1}{\sqrt{\overline{\tau}(\sigma_r^2 + \sigma_s^2)}},$$

thus we have the equations (1.5). ■

This proposition shows how players behave depending on the initial advantage x_1. If x_1 is positive, it means that player A is ahead of player B, and that $g'(\tau^A) > 0$, that is, player A will choose $\tau^A < {\overline{\tau}}/{2}$, so he is allocating more time on the safe activity, in order to reduce the volatility of his output. On the other hand, if x_1 is negative, so player A is behind player B, we will have the opposite behavior, that is, he will seek risky activity in order to increase his output volatility, so he can increase the probability of winning the game.

The idea is that whenever a player has some advantage, ‘his probability of winning the tournament is greater than $1/2$. In order to maintain his advantage, he avoids allocating time to risky activity, since it would increase the variance of his output and thus reduce the probability of winning the tournament. On the other side, the player with disadvantage has the opposite incentive, since he could increase his probability of winning by increasing the output variance. However, if no one has any advantage, then both players

[1] Just a reminder, g is strictly concave, thus g' is strictly monotone.

will maximize their expected output.

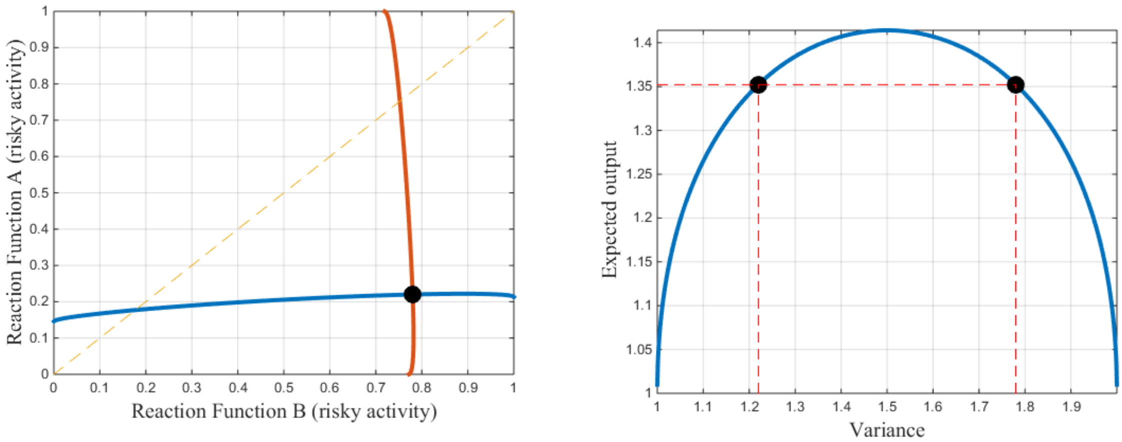
Figure 1.5 shows the reaction function from both players and the equilibrium when $x_1 > 0$, that is, player A starts with an advantage. The figure is plotted in the following way: The blue line is the reaction function for player A, the abscissa contains all possible choices for player B, and the line shows player A best reaction. The red line is the reaction function for player B, and the ordinate has all possible choices for player A. The intersection of these two lines is the Nash equilibrium for this static tournament. The yellow dashed line represents the symmetric choices, that is $\tau^A = \tau^B$.

In order to maintain his advantage, player A will allocate more time to safe activities, reducing the volatility of his output. On the other hand, player B seeks risky activities so that he reduces the probability of player A wins the tournament, since he will be increasing the variance of his output and of the output difference.

Figure 1.5: Equilibrium with $x_1 = 3$

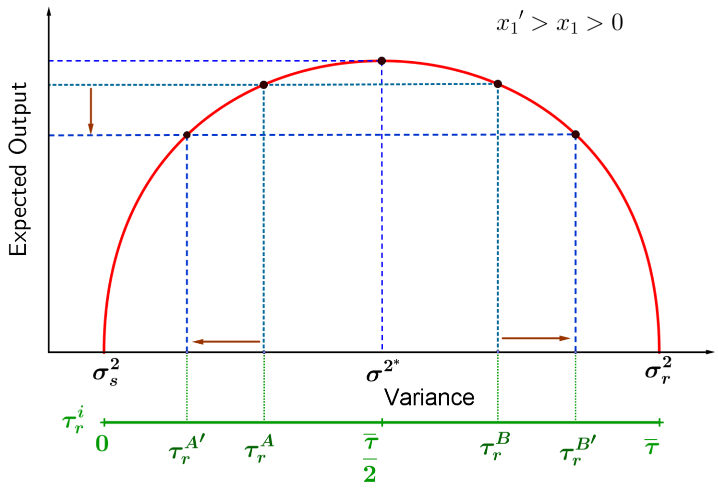
(a) Reaction functions

(b) Equilibrium choices on Mean-Variance Frontier



Equation (1.5) shows the behavior of the players towards changes in the parameters of the model. Formally, we can use the implicit differentiation this equation to view the effect of changes in the parameters. First, note that function g is strictly concave, so the derivative of the inverse function of g' is decreasing. Let us explore this properties beginning with the advantage effects. As $x_1 > 0$ increases, the player A, who has the advantage, reduces his volatility by allocating more time on safe task, while player B, the underdog, seeks for more risk, as we have mentioned before. The Figure 1.6 show us the players' choice.

Figure 1.6: The Effect of x_1 over players' Strategy



Formally, we have that

$$\frac{d\tau^A}{dx_1} = \frac{\sigma_r^2 - \sigma_s^2}{2\overline{\tau}(\sigma_r^2 + \sigma_s^2)g''(\tau^A)} < 0 \quad \text{and} \quad \frac{d\tau^B}{dx_1} = -\frac{\sigma_r^2 - \sigma_s^2}{2\overline{\tau}(\sigma_r^2 + \sigma_s^2)g''(\tau^B)} > 0. \tag{1.6}$$

The magnitude of the change clearly depends on advantage, but also depends on the time endowment, concavity of the expected output and variance parameters. When the variance associated to the risky activity is relatively large to the safe risk, the player who has an advantage will seek safer tasks even more, since he tries to reduce his output volatility, while the underdog do the opposite. The g'' term represents how the marginal expected output changes at each unit of time allocated on the risk activity. The more negative is g'', means that a small change on the risk affects more severely the expected output, so the player who has advantage will avoid reducing too much the risk compared to the optimal choice of $\overline{\tau}/2$, since it would have a large negative impact over his expected output. Finally, the higher is the endowment of time, $\overline{\tau}$, since it also affects g'' in the opposite way, thus the overall effect is ambiguous.

The variance associated with the risky activity also affects players' choice, as expected. The higher is the variance of such activity, the less the leader will allocate time on it, since it will reduce his probability of winning. The marginal effect will depend on the advantage, endowment and on the second derivative of g. Again, the higher is the advantage, the more willing the leader will be to reduce the variance of his output, even though losing in terms of expected output. The g'' term has the same intuition as in the previous paragraph. The trade-off between allocating time to risk and safe activity now appears in this marginal effect:

$$\frac{d\tau^A}{d\sigma_r^2} = \frac{x_1}{g''(\tau^A)}\frac{\sigma_s^2}{\overline{\tau}(\sigma_r^2 + \sigma_s^2)^2} < 0 \quad \text{and} \quad \frac{d\tau^B}{d\sigma_r^2} = -\frac{x_1}{g''(\tau^B)}\frac{\sigma_s^2}{\overline{\tau}(\sigma_r^2 + \sigma_s^2)^2} > 0, \tag{1.7}$$

so, as the variance of the safe activity increases, so the term $\frac{\sigma_s^2}{\overline{\tau}(\sigma_r^2+\sigma_s^2)^2}$ increases, then the leader will not reduce the time allocation on risk activity too much compared to the optimal choice $H/2$. This happens due to the lower trade-off between allocating time to the risk and safe activity, measured by the difference $\sigma_r^2 - \sigma_s^2$.

Finally, we can study how the endowment affects the equilibrium choices. Now, the effect over the leader is clear, increasing the time allocated on risk activity, since the optimal level also increases. However, for the underdog, the effect is not clear. It will depend on his disadvantage and on the g''.

$$\frac{d\tau^A}{d\overline{\tau}} = \frac{f''(\overline{\tau}-\tau^A)}{g''(\tau^A)} - \frac{\sigma_r^2-\sigma_s^2}{2\overline{\tau}^2(\sigma_r^2+\sigma_s^2)}\frac{x_1}{g''(\tau^A)} > 0 \quad \text{and} \quad \frac{d\tau^B}{d\overline{\tau}} = \frac{f''(\overline{\tau}-\tau^B)}{g''(\tau^B)} + \frac{\sigma_r^2-\sigma_s^2}{2\overline{\tau}^2(\sigma_r^2+\sigma_s^2)}\frac{x_1}{g''(\tau^B)}. \tag{1.8}$$

Therefore, we can see that under a tournament scheme, the asymmetry between players and the possibility to take risk will reduce the total expected output. We can also see that the larger is the advantage of one player, the other will seek more risk.

1.3.4 Two Periods Tournament Model

In the previous section, we studied the equilibrium of a static tournament, when one of the player can have an exogenous initial advantage. In this section, we will consider the two period model, in which all players start at the same level, that is, there will not be any player with an initial advantage, i.e. $x_0 = 0$. The objective of this section is to make the advantage endogenous in the model, in the sense that the advantage at the second period is the realization of the stochastic output of each agent after each player chose their time allocation on both tasks.

In the tournament we study in this section, the principal grants the prize $W > 0$ to the player who produce the most, while the other player does not receive anything. This is a typical case of promotion, where W can be seen as the increase of the salary for the promoted player. The output that the principal compares to reward the winner is now the sum of the each stage output. The first period, both players start with no initial output, and then choose how to allocate their time endowment $\overline{\tau}$ of the period. The output is realized at the end of the first period, and players learn their (dis)advantage at the beginning of the second period, so they can choose the risk of their output strategically, just as it was solved in the previous section.

In order to solve this problem, we will find the subgame perfect equilibrium by backward induction. First, we solve the second stage problem for a given initial advantage at the second period, x_1. We starting solving the second stage, considering that player learned x_1, that is, the (dis)advantage that player A has towards player B. This problem

was solved in the Section 1.3.3, so now let us focus on the first stage problem. Player A will solve the following problem

$$\max_{\tau_1^A} \; W\mathbb{P}\left[Y_1^A(\tau_1^A) + Y_2^{A^*} \geqslant Y_1^B(\tau_1^B) + Y_2^{B^*}\right] \tag{2PP$_A$}$$

where $Y_2^{i^*}$ is the player i output when he plays the equilibrium strategy for the second period. First, we use the equilibrium conditions to rewrite the output difference in the second stage:

$$\begin{aligned} Y_2^{A^*} - Y_2^{B^*} &= g(\tau_2^{A^*}) - g(\tau_2^{B^*}) + \sigma_r\left(\sqrt{\tau_2^{A^*}}\varepsilon_{2r}^A - \sqrt{\tau_2^{B^*}}\varepsilon_{2r}^B\right) + \sigma_s\left(\sqrt{\overline{\tau} - \tau_2^{A^*}}\varepsilon_{2s}^A - \sqrt{\overline{\tau} - \tau_2^{B^*}}\varepsilon_{2s}^B\right) \\ &= \sigma_r\left(\sqrt{\tau_2^{A^*}}\varepsilon_{2r}^A - \sqrt{\tau_2^{B^*}}\varepsilon_{2r}^B\right) + \sigma_s\left(\sqrt{\overline{\tau} - \tau_2^{A^*}}\varepsilon_{2s}^A - \sqrt{\overline{\tau} - \tau_2^{B^*}}\varepsilon_{2s}^B\right), \end{aligned}$$

since we know that $g(\tau_2^{A^*}) = g(\tau_2^{B^*})$ by the symmetry of the model, thus the only important part to be considered is the realization of random variables. The difference for the first period is similar to the first line of the above equation:

$$Y_1^A - Y_1^B = g(\tau_1^A) - g(\tau_1^B) + \sigma_r\left(\sqrt{\tau_1^A}\varepsilon_{1r}^A - \sqrt{\tau_1^B}\varepsilon_{1r}^B\right) + \sigma_s\left(\sqrt{\overline{\tau} - \tau_1^A}\varepsilon_{1s}^A - \sqrt{\overline{\tau} - \tau_1^B}\varepsilon_{1s}^B\right),$$

using the equations above and that $\tau_2^{A^*} + \tau_2^{B^*} = \overline{\tau}$ from the equilibrium condition shown in the proof of the Proposition 1, we can rewrite the probability in the following way

$$\mathbb{P}\left[Y_1^A(\tau_1^A) + Y_2^{A^*} \geqslant Y_1^B(\tau_1^B) + Y_2^{B^*}\right] = \Phi\left(\frac{g(\tau_1^A) - g(\tau_1^B)}{\sqrt{\sigma_r^2\left(\overline{\tau} + \tau_1^B + \tau_1^A\right) + \sigma_s^2\left(3\overline{\tau} - \tau_1^B - \overline{\tau} - \tau_1^A\right)}}\right).$$

Now, let us define z_1 and η_1^2:

$$\eta_1^2 = \sigma_r^2\left(\overline{\tau} + \tau_1^B + \tau_1^A\right) + \sigma_s^2\left(3\overline{\tau} - \tau_1^B - \tau_1^A\right)$$

and

$$z_1 = \frac{g(\tau_1^A) - g(\tau_1^B)}{\eta_1}$$

Clearly, substituting these equation in the player A's problem, it is easy to see that it is similar to the static problem, when $x_0 = 0$. So, using the proposition 1 we know that players will maximize their output, by allocating their time equally on both activities, i.e., each player will spend $\tau_{1r}^i = \overline{\tau}/2$ units of time on each task. Therefore, we can see that the (dis)advantage creates incentives for players deviate from the optimal time allocation, and since part of relative position is due to random variables, players will almost surely be deviating from the allocation that maximizes the expected output.

1.4 The Dynamic Model

A natural extension to the previous section is when the tournament has T periods. Now, we will consider a dynamic tournament, in which both players start with the same output level as before. However, as the tournament develops, a player finds out he is ahead of the other and take strategical actions to increases his chances of winning the tournament at each period. The setting is quite similar as the previous one, but now players will be evaluated after T periods, and at the end of each period players will learn their position. Here, the principal compares the sum of the output over the T periods, and the prize will be given to the player with the highest output level at the end of period T.

We consider a $T-$period dynamic stochastic game with two players ($i = A,B$), where each of them chooses how to spend their time at each period in order to produce the output Y_t^i. The sum of the output over time will be considered by the principal to determine the winner, who will be granted a prize $W > 0$, while the loser gets nothing. Let $\Delta Y_t = Y_t^A - Y_t^B$ be the output difference with respect to player A at period t, then the principal will grant the prize to player A if $x_T \equiv \sum_{t=0}^{T} \Delta Y_t > 0$, otherwise player B will receive the prize. Formally, the last period payoff is a function $u_T^i : X \longrightarrow \mathbb{R}$, where $X \equiv \mathbb{R}$ is the state space of the game and

$$u_T^A(x_T) = \begin{cases} W & \text{, if } x_T > 0; \\ W/2 & \text{, if } x_T = 0; \\ 0 & \text{, otherwise,} \end{cases} \quad \text{and} \quad u_T^B(x_T) = \begin{cases} W & \text{, if } x_T < 0; \\ W/2 & \text{, if } x_T = 0; \\ 0 & \text{, otherwise,} \end{cases} \tag{1.9}$$

or, as it will be more convenient for computation later, $u_T^A(x_T) = W\mathbb{1}_{\{x_T>0\}}$, where $\mathbb{1}_{\{x_T>0\}}$ is the indicator function that equals 1 when $x_T > 0$. The utility for player B is similar, except that the prize is given when $x_T < 0$. The tournament studied here does not consider a midterm reward, so the utility function per period $t < T$ equals zero.

Players make their choice after learning the accumulated output difference $x_t \equiv \sum_{l=0}^{t} \Delta Y_l$ from the end of previous period, that is, the accumulated output difference up to the beginning of period t. We assume that at the initial period both players begin the tournament with no output, thus $x_0 = 0$. We can rewrite the law of motion as

$$x_{t+1} = x_t + Y_t^A - Y_t^B, \tag{1.10}$$

where the state of the next period x_{t+1} depends on the current initial state x_t and the players output Y_t^i realization at the end of the period, after players made their decision.

At each period, players can engage in two activities, denoted by $k \in \{r,s\}$, to produce the output Y_t^i, which is a stochastic function of the time spent performing each task. As

in the model from Section 1.3, for each unit of time τ spent on an activity, the expected output will be $f(\tau)$, where $f(\cdot)$ is a strictly increasing and concave function, thus having the same property as before, diminishing marginal productivity. Also, by working in a task k, the player will be subject to output volatility associated with that task, which is given by $\sigma_k^2\tau$, where $\sigma_r^2 > \sigma_s^2$. Therefore, given that the player chooses to spend τ units of time on activity k, the output generated by it follows a Normal distribution with mean $f(\tau)$ and variance $\tau\sigma_k^2$, that is, $Y_{kt}^i \sim N(f(\tau),\tau\sigma_k^2)$.

Both players have $\overline{\tau}$ units of time available to work in both of tasks. Denote $\tau_t^i \in [0,\overline{\tau}] \equiv D^i$ be the amount of time player i spend on the riskier activity, r, thus $\overline{\tau} - \tau_t^i$ is the amount of time spent on the "safe" task, s. Due to the diminishing marginal productivity, players will probably work on both activities, and thus the player i production Y_t^i at the period t follows a Normal distribution with mean $g(\tau_t^i) \equiv f(\tau_t^i) + f(\overline{\tau} - \tau_t^i)$ and variance $\tau_t^i\sigma_r^2 + (\overline{\tau} - \tau_t^i)\,\sigma_s^2$, i.e. $Y_t^i \sim N(g(\tau_t^i),\tau_t^i\sigma_r^2 + (\overline{\tau} - \tau_t^i)\,\sigma_s^2)$, since it is the sum of independent Normal distributions. Again, we assume the players' output is independently distributed, hence we can find the transition probability $q : X \times D \longrightarrow \mathcal{P}(X)$, that is the probability density of next period be the state x' given the current players' choice and state x, i.e. $q(x';\tau^A,\tau^B,x)$, where $\mathcal{P}(X)$ is the set of probability distributions on the set X. The transition function can be found by using the properties from Normal distributions, hence we have

$$x_{t+1} \;\sim\; N\left(x_t + g(\tau_t^A) - g(\tau_t^B), \sigma_r^2\left(\tau_t^A + \tau_t^B\right) + \sigma_s^2\left(2\overline{\tau} - \tau_t^A - \tau_t^B\right)\right), \tag{1.11}$$

therefore, q is the probability density function (pdf) of a Normal distribution given by (1.11) and we denote Q as its cumulative density function (cdf).

Formally, the game is a tuple $[X,(D^i,u_T^i)_{i=A,B},Q]_{t=0}^T$, where $X = \mathbb{R}$ for all t is the state space, $D^i = [0,\overline{\tau}]$ for $i = A,B$ is the action space, that is, the players' endowment of time and $Q : X \times D^A \times D^B \rightarrow [0,1]$ is the cumulative distribution of the state transition function from x_t to x_{t+1} given a choice (τ^A,τ^B).

Therefore, the game proceeds as follows. At period $t = 0$, both players learn that they are starting the game with no initial output, that is, the initial state is $x_0 = 0$. After observing the initial state, players choose how to spend their time endowment of $\overline{\tau}$ over the two tasks, $\tau_0 = \left(\tau_0^A,\tau_0^B\right)$, simultaneously and independently from each other. After the realization of both outputs, the state x_0 transits to state x_1 according to the distribution mentioned above, that follows the Normal distribution $N(x_0 + g(\tau_0^A) - g(\tau_0^B),\sigma_r^2(\tau_0^A + \tau_0^B) + \sigma_s^2(2\overline{\tau} - \tau_0^A - \tau_0^B))$. In the next round, at period $t = 1$, after observing the state x_1, players choose how to allocate their time τ_1, and after the realization of the outputs, the state changes again. The game goes on this way for $T - 1$ periods. At the beginning of period T, when the outputs are realized, the principal learns who produced the most by

observing x_T, and grant the prize W to player A if $x_T > 0$, otherwise player B wins the prize W.

The nature of the tournament, that is the compensation scheme depending only on the x_T, makes different histories of the game lead to a same states, which is the variable observed by the players. This property allow us to focus our attention to *Markov strategies.* Let $\xi_t^i : X \to D^i$ denote the Markov strategy for period t, and $\xi^{i^t} \equiv \{\xi_l^i\}_{l=0}^t$ the strategy up to period t. The complete strategy for the tournament for player i is denoted by $\xi^i \equiv \xi^{i^T}$. Now, we can write the expected payoff function based on the strategy profile $\xi \equiv \left(\xi^A, \xi^B\right)$, given by

$$U(\xi) \equiv \mathbb{E}^{\xi}\left[u_T^i\left(x_T\right)\right], \tag{1.12}$$

where $\mathbb{E}^{\xi}$ is the expectation taken with respect to the measure induced by transition probability $q(\cdot)$ and the strategy profile ξ. The equilibrium concept we use to solve the game is the Markov perfect equilibrium (MPE), which considers that players condition their actions only on the current state of the game ξ_t. A strategy profile is a Markov perfect equilibrium, if the strategy profile is a subgame perfect Nash equilibrium and the strategies is Markovian. Since we consider a finite horizon game, using the one-shot deviation principle, we can find the equilibrium by backward induction, so we have the following definition.

Definition 1 *A (nonstationary) Markov perfect equilibrium (MPE) in this setting is a pair of a finite sequence of value function $\{V_t^A, V_t^B\}_{t=0}^T$ and a Markov strategy profile ξ such that*

- *The system of Bellman equation*

$$V_{T-1}^i(x_{T-1}) \equiv \max_{\tau_{T-1}^i} \int u_T^i(x_T) q(x_T | x_{T-1}, \tau_{T-1}^i, \xi_{T-1}^{-i}) dx_T \tag{1.13}$$

$$V_t^i(x_t) \equiv \max_{\tau_t^i} \int V_{t+1}^i(x_{t+1}) q(x_{t+1} | x_t, \tau_t^i, \xi_t^{-i}) dx_{t+1} \tag{1.14}$$

  is satisfied for $i = A, B$, with the expectation induced by the probability distribution q.

- *The strategy of players are optimal each period t:*

$$\xi_{T-1}^i \in \arg\max_{\tau_{T-1}^i} \int u_T^i(x_T) q(x_T | x_{T-1}, \tau_{T-1}^i, \xi_{T-1}^{-i}) dx_T$$

$$\xi_t^i \in \arg\max_{\tau_t^i} \int V_t^i(x_{t+1}) q(x_{t+1} | x_t, \tau_t^i, \xi_t^{-i}) dx_{t+1}$$

In other words, to carry out backward induction, we start at the last period of

players' action, $T-1$, and for each possible state x_{T-1} we calculate the time $T-1$ value functions and strategies. Next, we move backward one period to time $T-2$, the expected value function considers that players will behave according to the strategy found at the period $T-1$ in the future, and solve the players' problem considering the probability induced by the next period strategies and the decision at the period $T-2$. We continue the backward induction recursively for the periods $T-3$,$T-4$,... until we reach time period $t=0$, and thus we have the equilibrium strategy profile ξ.

Proposition 2 *There is a unique interior Markov perfect equilibrium, in which the strategy profile ξ is such that ξ_t^i satisfies at each period t the following equation*

$$g'(\xi_t^A) = \frac{\sigma_r^2 - \sigma_s^2}{2} \frac{x_t}{(T-t)\,\overline{\tau}(\sigma_r^2+\sigma_s^2)} \quad \textit{and} \quad g'(\xi_t^B) = -g'(\xi_t^A), \tag{1.15}$$

and the value function for each period t is given by

$$V_t^A(x_t) = W\Phi\left(\frac{x_t}{(T-t)\,\overline{\tau}(\sigma_r^2+\sigma_s^2)}\right) \quad \textit{and} \quad V_t^B(x_t) = W\left[1-\Phi\left(\frac{x_t}{(T-t)\,\overline{\tau}(\sigma_r^2+\sigma_s^2)}\right)\right], \tag{1.16}$$

if the following condition holds:

$$\inf_{\tau^i}\left|\frac{g''(\tau^i)}{g'(\tau^i)}\right| > \frac{\sigma_r^2-\sigma_s^2}{2\overline{\tau}\sigma_s^2}. \tag{1.17}$$

Proof. At the last period of players' decision, $T-1$, the problem is similar to the one presented in Section 1.3.1. Players learn x_{T-1} and must choose how to spend their time in order to maximize their probability of winning the tournament, which depends on the law of motion and the transition distribution. Using the law of motion given by $x_T = x_{T-1} + Y_{T-1}^A - Y_{T-1}^B$, and the Normal distribution of the outputs, the distribution of x_T will be a Normal distribution with mean $x_{T-1} + g(\tau_{T-1}^A) - g(\tau_{T-1}^B)$ and variance $\eta_{T-1}^2 = \sigma_r^2\left(\tau_{T-1}^A + \tau_{T-1}^B\right) + \sigma_s^2\left(2\overline{\tau} - \tau_{T-1}^A - \tau_{T-1}^B\right)$. The objective function from Equation (1.13) for player A can be rewritten as

$$\begin{aligned}\int u_T^A(x_T) q(x_T|x_{T-1},\tau_{T-1}^A,\tau_{T-1}^B)dx_T &= \int W\mathbb{1}_{\{x_T>0\}}\frac{1}{\eta_{T-1}}\phi\left(\frac{x_T - x_{T-1} - g(\tau_{T-1}^A) + g(\tau_{T-1}^B)}{\eta_{T-1}}\right)dx_T \\ &= W\int_0^{+\infty}\frac{1}{\eta_{T-1}}\phi\left(\frac{x_T - x_{T-1} - g(\tau_{T-1}^A) + g(\tau_{T-1}^B)}{\eta_{T-1}}\right)dx_T \\ &= W\Phi\left(\frac{x_{T-1} + g(\tau_{T-1}^A) - g(\tau_{T-1}^B)}{\eta_{T-1}}\right),\end{aligned}$$

since the integrand is the density of a standardize normal distribution in the second line and we used the property from Normal distribution that $1-\Phi(x) = \Phi(-x)$. Now it is clear that both problem from period T and the static one presented in Section 1.3.1 is

the same. Therefore, the Proposition 1 solves the problem to find the strategy profile for period $T-1$:

$$g'(\xi_{T-1}^{A}) = \frac{\sigma_r^2 - \sigma_s^2}{2} \frac{x_{T-1}}{\overline{\tau}(\sigma_r^2 + \sigma_s^2)} \quad \text{and} \quad g'(\xi_{T-1}^{B}) = -g'(\xi_{T-1}^{A}). \tag{1.18}$$

Due to symmetry of the production function of the activities, r and s, $g'(\xi_{T-1}^{B}) = -g'(\xi_{T-1}^{A})$ implies that both players will choose an allocation that generates the same expected output, $g(\xi_{T-1}^{B}) = g(\xi_{T-1}^{A})$, but different levels of risk such that $\xi_{T-1}^{A} + \xi_{T-1}^{B} = \overline{\tau}$, so $\eta_{T-1}^2 = \overline{\tau}(\sigma_r^2 + \sigma_s^2)$. Using this information, we now can find a closed-form solution for the last period expected value function,

$$V_{T-1}^{A}(x_{T-1}) = W\Phi\left(\frac{x_{T-1}}{\overline{\tau}(\sigma_r^2 + \sigma_s^2)}\right) \quad \text{and} \quad V_{T-1}^{B}(x_{T-1}) = W\left[1 - \Phi\left(\frac{x_{T-1}}{\overline{\tau}(\sigma_r^2 + \sigma_s^2)}\right)\right].$$

We use the guess-and-verify method to find the value function for the period t. In order to find an educated guess, we can compute the probability of winning by each player, when they play the same strategy of the period $T-1$, but taking into account that there will be $T-t$ random variables realizing until the end of the game. So, let the following function be our guess for the player A's value function for period t,

$$V_t^{A}(x_t) = W\Phi\left(\frac{x_t}{\sqrt{(\sigma_r^2 + \sigma_s^2)\,\overline{\tau}(T-t)}}\right) \quad \text{and} \quad V_t^{B}(x_t) = W\left[1 - \Phi\left(\frac{x_t}{\sqrt{(\sigma_r^2 + \sigma_s^2)\,\overline{\tau}(T-t)}}\right)\right]. \tag{1.19}$$

We will solve the problem for player A, since the player B problem can be done by the same way. In order to verify the guess for the value function, we move forward in the time, so we have $V_{t+1}^{A}(x_{t+1})$ and substituting it in the Equation (1.14), we can solve player A's problem. First, let $\eta_t^{*^2} = (\sigma_r^2 + \sigma_s^2)\,\overline{\tau}(T-t-1)$, we can rewrite the integral using our guess as

$$\int \Phi\left(\frac{x_{t+1}}{\eta_{t+1}^{*}}\right) \frac{1}{\eta_t} \phi\left(\frac{x_{t+1} - x_t - g(\tau_t^{A}) + g(\tau_t^{B})}{\eta_t}\right) dx_{t+1} = \int F_{W_1}(w_1) f_{W_2}(w_1 - m) dw_1,$$

where F_{W_1} is a cdf of a Normal distribution with mean 0 and variance $\eta_t^{*^2}$, the function f_{W_2} is a pdf of a Normal distribution with mean 0 and variance η_t^2, and $m = x_t + g(\tau_t^{A}) - g(\tau_t^{B})$. Integrating by parts, we can rewrite it as

$$\Lambda(m) = \int f_{W_1}(w_1)\left[1 - F_{W_2}(w_1 - m)\right] dw_1.$$

Differentiating it with respect to x_t, we find $\Lambda'(m) = \int f_{W_1}(w_1) f_{W_2}(w_1 - m) dw_1$, a convolution of two normal density function. Therefore, we know that $\Lambda'(m)$ will be a pdf of

a Normal distribution with mean zero and variance given by the sum of both variances, and hence

$$\Lambda\left(x_t + g(\tau_t^A) - g(\tau_t^B)\right) = \Phi\left(\frac{x_t + g(\tau_t^A) - g(\tau_t^B)}{\sqrt{\sigma_r^2\left[\tau_t^A + \tau_t^B + \overline{\tau}\,(T-t-1)\right] + \sigma_s^2\left[2\overline{\tau} - \tau_t^A - \tau_t^B + \overline{\tau}\,(T-t-1)\right]}}\right).$$

Now we can compute the Nash equilibrium for the period t using our guess, where players solve the following problems

$$\max_{\tau_t^A} W\Phi\left(\frac{x_t + g(\tau_t^A) - g(\tau_t^B)}{\sqrt{\sigma_r^2\left[\tau_t^A + \tau_t^B + \overline{\tau}\,(T-t-1)\right] + \sigma_s^2\left[2\overline{\tau} - \tau_t^A - \tau_t^B + \overline{\tau}\,(T-t-1)\right]}}\right)$$

$$\max_{\tau_t^B} W\left[1 - \Phi\left(\frac{x_t + g(\tau_t^A) - g(\tau_t^B)}{\sqrt{\sigma_r^2\left[\tau_t^A + \tau_t^B + \overline{\tau}\,(T-t-1)\right] + \sigma_s^2\left[2\overline{\tau} - \tau_t^A - \tau_t^B + \overline{\tau}\,(T-t-1)\right]}}\right)\right].$$

The solution of these problem can be found by the same way we use in the Proposition 1, and thus must satisfies the following conditions

$$g'(\tau_t^A) = \frac{\sigma_r^2 - \sigma_s^2}{2}\frac{x_t}{(T-t)\,\overline{\tau}(\sigma_r^2 + \sigma_s^2)} \quad \text{and} \quad g'(\tau_t^B) = -g'(\tau_t^A), \tag{1.20}$$

where again we have $g(\tau_t^A) = g(\tau_t^B)$ and $\tau_t^A + \tau_t^B = \overline{\tau}$. Finally, using these conditions, we can check that our guess (1.19) for the value function is indeed the correct value function:

$$V_t^A(x_t) = W\Phi\left(\frac{x_t}{\sqrt{(\sigma_r^2 + \sigma_s^2)\,\overline{\tau}(T-t)}}\right) \quad \text{and} \quad V_t^B(x_t) = W\left[1 - \Phi\left(\frac{x_t}{\sqrt{(\sigma_r^2 + \sigma_s^2)\,\overline{\tau}(T-t)}}\right)\right].$$

Thus, by the procedure of guess-and-verify, we found the value function and the strategy profile for period t. The sufficient condition for uniqueness can be found by the same way as in Proposition 1, which concludes the demonstration. ■

The strategy profile for period t is similar to the static equilibrium, except that now it depends on the accumulated output difference up to period t, and there is a new term, $T - t$, which is how many periods have left until the end of the game. This term allow us to formalize the idea about how player behave over a multi-stage tournament.

This idea is formalized in the following corollary.

Corollary 2 *Players with (dis)advantage will seek (more) less risk as the game comes to the end.*

Proof. Suppose player A has some advantage, i.e. $x_t > 0$. By the implicit differentiation, we have

$$\frac{d\tau_t^A}{d(T-t+1)} = -\frac{\sigma_r^2 - \sigma_s^2}{2\overline{\tau}^2(T-t)^2(\sigma_r^2 + \sigma_s^2)}\frac{x_t}{g''(\tau_t^A)} > 0,$$

and

$$\frac{d\tau_t^B}{d(T-t)} = \frac{\sigma_r^2 - \sigma_s^2}{2\overline{\tau}^2(T-t)^2(\sigma_r^2+\sigma_s^2)} \frac{x_t}{g''(\tau_t^B)} < 0,$$

where the inequality signs come from the fact that $g(\cdot)$ is a strictly concave function. ■

This corollary shows us how the player behave along the tournament. In words, given an advantage for player A, if the tournament is just beginning and it will endure for a long time, the term on denominator $T-t$ will be higher, reducing the value of x_t. The intuition is that, if the tournament will continue for a long time, there will be higher "*uncertainty*" about the outcome at the end of the tournament, and thus trying to reduce the variance of the period t outcome won't have much impact at the final outcome. Therefore, players prefer to maximize their output on the early stages of the game. As the game come to the end, the player with advantage will try to secure his position by reducing the time allocate on the risky activity, as now it will have greater impact on the final output. On the other hand, the underdog will seek risk.

In order to provide an examples and see how player behave over time in a dynamic tournament, we made a simulation of a twenty-period tournament which is shown in Figure 1.7. There are five different realizations of the random variables involved in the tournament, each represented by different color. Each color is a new tournament starting from zero. We show three different figures, where the first is the accumulated output difference, that is, x_t, which is the state of the game at the beginning of period t. When the lines are above zero, it means player A is ahead, and player B is losing. For example, reading figure 1.7a, the blue line represents a tournament realization. Both players starts at 0, so there is no advantage at the first period. By the period 5, player A has 4 units of output more than player B. At the end of the tournament, represented by period $t = 21$, we can see who was the winner. The dark blue tournament, player A won approximately by 6.8 output units more than player B. The light blue line, player B won the tournament by 10 units more than player A.

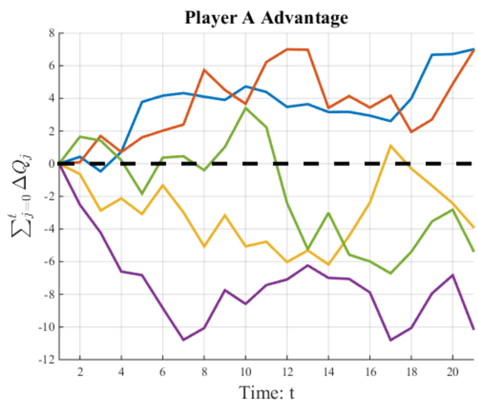
In the Figure 1.7b is depicted how player allocate their time on risk activity. As it was shown by proposition 1, since players start without any advantage, both players chose $\overline{\tau}/2$, and thus maximize their expected output. As the tournament evolves, some player will obtain some advantage, and they start to deviate from the optimal choice, $\overline{\tau}/2$. We can see the proposition 1 working here. Take the dark blue line for example. From the Figure 1.7a, we know that player A is always ahead of player B, so he reduces the volatility of his output by allocating more time to safe activities, i.e. $\tau_t^A < \overline{\tau}/2$.

The Figure 1.7c represents the expected output for player A. We can see that he avoids to deviate from the optimal choice $\overline{\tau}/2$ during most part of the tournament, which generates an expected output of 1.4142 in this setting.

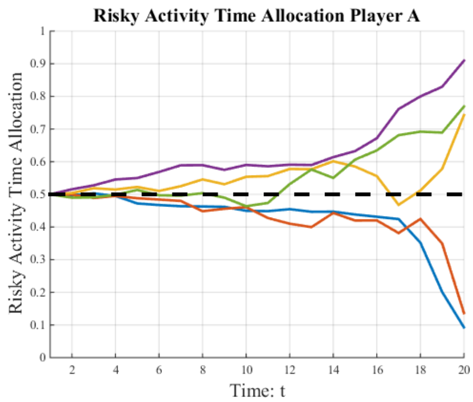
We can also see the proposition 2 working here, using this three figures. Consider the tournament represented by the light blue line. We can see player A with a disadvantage equal to 10 in 5 different moments ($t \approx$ 7,8,15,18,20). In Figure 1.7b we can see player A time allocation on risky activity. By looking at the periods mentioned above, we can see that as the tournament nears the end, he seeks more risk, although he has the same disadvantage at those moments. By the end of the game, he allocates more than 90% of his time on the risky activity. Figure 1.7c shows that players avoid distortions on expected output during most part of the tournament.

Figure 1.7: Tournament with 20 periods

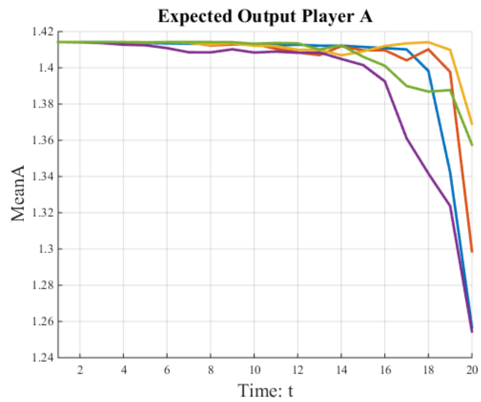
(a) Player A Advantage



(b) Player A Time Allocation on Risky Activity



(c) Player A Expected Output



1.5 Other Extensions for the Static Tournament

In this section we discuss some possible modifications and extensions on the static version of the model presented in Section 1.3.1, and see how it affects players behavior.

1.5.1 Asymmetric expected output

We shall discuss the implications of the symmetry of the expected output function $f(\cdot)$ and what would happen if we relax this assumption. The symmetry of the expected output function helped us to find a closed-form solution for both the static and, especially, the dynamic tournament. The symmetry implies that the function g is even with respect to $\overline{\tau}/2$, i.e. $g(\tau) = g(\overline{\tau} - \tau)$, and so the inclination of g for two different points equidistant to $\overline{\tau}/2$ has the same magnitude but with different sign, i.e. $g'(\tau) = g'(\overline{\tau} - \tau)$. Therefore, the equilibrium condition that $g'(\tau^A) = -g'(\tau^B)$ implies that $g(\tau^A) = g(\tau^B)$. Another implication that follows from the function g is even with respect to $\overline{\tau}/2$ is that we can write $g'(\tau) = -g'(\overline{\tau} - \tau)$, and using the assumption that g is a concave function, then g' will be monotone, and so the equilibrium condition implies that $\tau^A = \overline{\tau} - \tau^B$. By doing it, we were able to find a closed form for the variance η^2 and solve the players' problem.

We can relax this assumption by considering that each activity has a different expected output, say $f_r(\cdot)$ and $f_s(\cdot)$. Now, for each τ_k unit time spent on task k by player i, the output generated by it follows a Normal distribution, i.e. $Y_k^i \sim N(f_k(\tau_k),\tau_k\sigma_k^2)$. Denote now the expected total output by player i as $g(\tau_r^i) = f_r(\tau_r^i) + f_s(\overline{\tau} - \tau_k^i)$.

The first order condition would be similar to the Equation (1.3), but now considering two different function for the expected output,

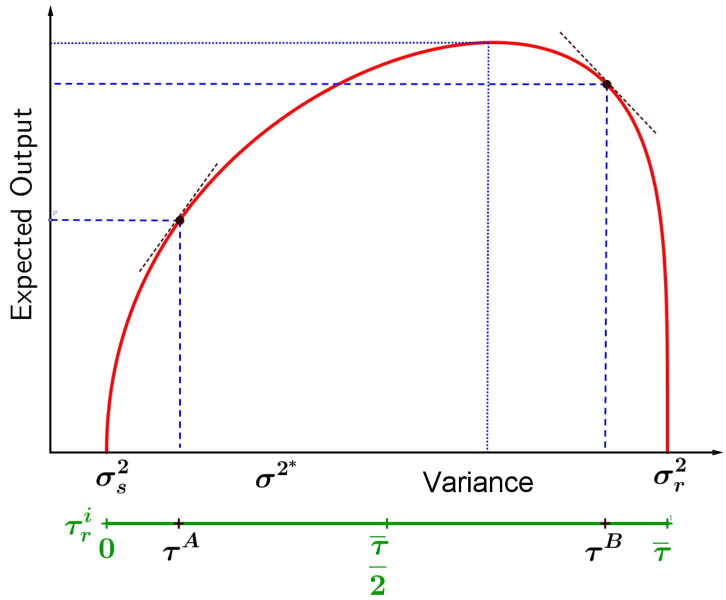
$$f_r'(\tau^A) - f_s'(\overline{\tau} - \tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z \quad \text{and} \quad f_r'(\tau^B) - f_s'(\overline{\tau} - \tau^B) = -\frac{\sigma_r^2 - \sigma_s^2}{2\eta}z, \tag{1.21}$$

and again the condition for the equilibrium is

$$g'(\tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z = -g'(\tau^B).$$

However, since the expected output functions are not symmetric, we cannot find a closed-form solution to the problem as before, and thus we would not be able to use the guess-and-verify method to find the equilibrium in the dynamic model. The Figure 1.8 below illustrates a situation in which f_r'/f_s' is an increasing function of τ, that is, the marginal rate of technical substitution is increasing. So, in order to maintain the same level of expected output, reducing one unit of time allocated to risky activity, a player needs to increase in less than one unit of time allocated to the safe activity. We illustrate numerically the case where $x_1 > 0$, that is, player A has an advantage. In this situation, it becomes harder to determine analytically the signal of z, that is, if $x_1 + g(\tau^A) - g(\tau^B)$ is positive or negative.

Figure 1.8: Set of possible choices of the asymmetric model



1.5.2 Tie-Break Rule

Now, we consider the static model presented in section 1.3.1 again, but we will introduce a tie-break rule, in the sense that, if the output of a player are not sufficiently greater than his opponent' output, the principal will declare a tie. The motivation of this section is to provide further insights about tournaments when the precision of the measure to decide who will be the winner is not good enough. For example, in the swimming competition and car racing, it is possible to determine the winner no matter how close they are, using precision timing of milliseconds. However, in the fund industry, a small difference over performance may not be enough for a fund receive more deposits than the other one. Maybe the investor prefer to move their money to other funds only if they are sufficiently better than the one where the money is deposited due to transaction costs.

In order to do that, let us introduce the tie-break as $\delta > 0$ and now we will need three prizes. To ease the computation, we will consider a symmetric prizes: W,0 and $-W$, that is, the winner receives W, if there is a tie both players receive nothing, and the loser must pay W. So, for example, player A wins the tournament, if his output overcomes player B output plus δ, i.e. $Y^A > Y^B + \delta$. This setting could be thought as the case of investment funds inflow. The winner fund is the one with sufficiently higher performance and will receive an inflow of W deposit, while the loser will notice an outflow of W from their deposits. However, if the performance of one fund was not sufficiently better, there will be no transfers.

The problem player A faces now is

$$\max_{\tau^A} \; W\left\{\mathbb{P}\left[x_1 + Y^A(\tau^A) > Y^B(\tau^B) + \delta\right] - \mathbb{P}\left[x_1 + Y^A(\tau^A) + \delta < Y^B(\tau^B)\right]\right\} \quad (\text{TBP}_A)$$

Rewriting the probabilities using the standard Normal distributions as before

$$\mathbb{P}\left[x_1 + Y^A(\tau^A) > Y^B(\tau^B) + \delta\right] = \Phi\left(\frac{x_1 - \delta + g(\tau^A) - g(\tau^B)}{\sqrt{\left(\tau^A + \tau^B\right)\sigma_r^2 + (2\overline{\tau} - \tau^A - \tau^B)\sigma_s^2}}\right)$$

and

$$\mathbb{P}\left[x_1 + Y^A(\tau^A) + \delta < Y^B(\tau^B)\right] = 1 - \Phi\left(\frac{x_1 + \delta + g(\tau^A) - g(\tau^B)}{\sqrt{\left(\tau^A + \tau^B\right)\sigma_r^2 + (2\overline{\tau} - \tau^A - \tau^B)\sigma_s^2}}\right).$$

Thus, the problem can be rewritten as

$$\max_{\tau^A} \quad W\left[\Phi(z_A) + \Phi(z_B) - 1\right], \qquad (\text{TBP}_A)$$

where $\eta^2 = \left(\tau^A + \tau^B\right)\sigma_r^2 + (2\overline{\tau} - \tau^A - \tau^B)\sigma_s^2$ is the variance of the random variables difference of the output, $z_A = \frac{x_1 - \delta + g(\tau^A) - g(\tau^B)}{\eta}$ is the argument of the winning probability of player A and $z_B = \frac{x_1 + \delta + g(\tau^A) - g(\tau^B)}{\eta}$ is the argument of player B probability of winning the tournament.

The FOC is

$$\begin{aligned}
0 = W&\left\{\frac{\phi(z_A)}{\eta}\left[g'(\tau^A) - \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z_A\right] + \frac{\phi(z_B)}{\eta}\left[g'(\tau^A) - \frac{\sigma_r^2 - \sigma_s^2}{2\eta}z_B\right]\right\} \\
\iff &\left[\phi(z_A) + \phi(z_B)\right]g'(\tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta}\left[z_A\phi(z_A) + z_B\phi(z_B)\right]. \\
\iff &\left[\phi(z_A) + \phi(z_B)\right]g'(\tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta}\left\{\left[x_1 + g(\tau^A) - g(\tau^B)\right]\left[\phi(z_A) + \phi(z_B)\right] + \delta\left[\phi(z_B) - \phi(z_A)\right]\right\}.
\end{aligned} \qquad (1.22)$$

The equilibrium condition is

$$\left[\phi(z_A) + \phi(z_B)\right]\left[g'(\tau^A) + g'(\tau^B)\right] = 0,$$

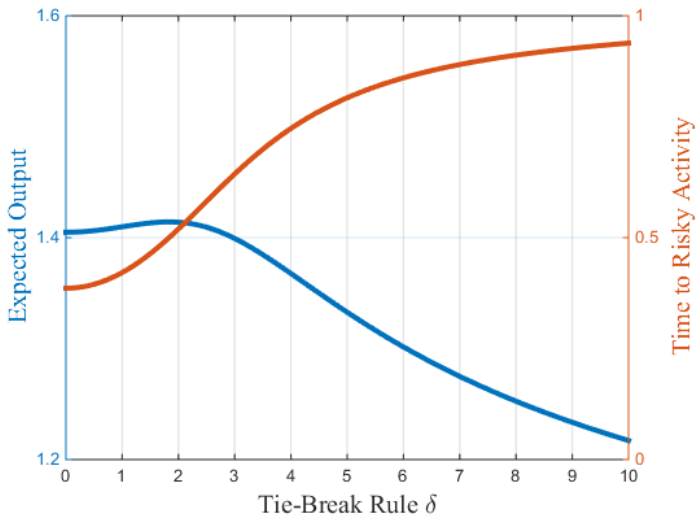
which implies that $g(\tau^A) = g(\tau^B)$, and thus we can simplify as before

$$g'(\tau^A) = \frac{\sigma_r^2 - \sigma_s^2}{2\eta}\left\{\Delta Q_1 + \delta\frac{\phi(z_B) - \phi(z_A)}{\phi(z_B) + \phi(z_A)}\right\} \qquad (1.23)$$

Figure 1.9 shows how player A behaves, when he has 1 unit of output in advantage

with respect to player B. The vertical axis, on the right side is how player A spends his time on the risky activity, and is represented by the red line. On the other side, it shows the expected output generated by this player, and it is plotted as a blue line. The horizontal axis contains all tie-break rules considered for this example, that goes from zero up to 10.

Figure 1.9: Player A Choice of Expected Output and Volatility.



In this Figure we can see that when the principal sets the tie-break rule equals to 2, player A will choose the maximal expected output, but for greater tie-break rule, he becomes the underdog and will seek more risk, even though he has started with an advantage.

Figure 1.10: Player B Choice of Expected Output and Volatility.

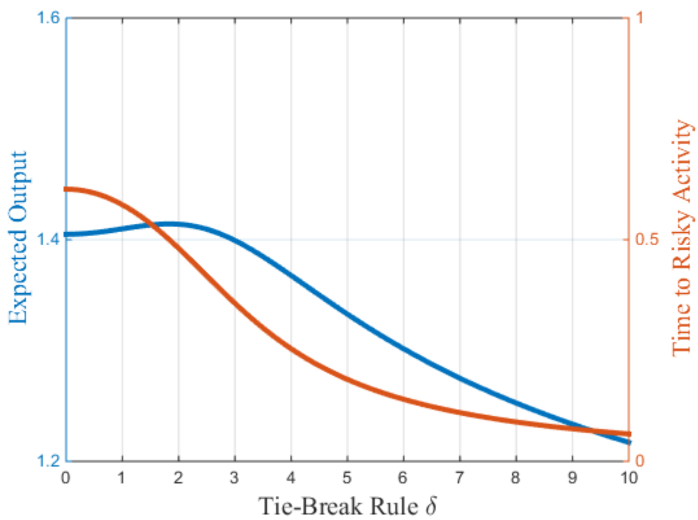


Figure 1.10 shows the same plot, but for player B. We can see that the tie-break

rule also induce the maximal expected output when it is equal to 2. In this case, and for values larger than it, this player start acting as the leader, and reduces its risk.

1.6 Conclusion

Tournaments are a pervasive circumstance, as they serve as an important tool for providing incentives. Most work on tournament theory has used the labor economics predominantly as its background. However, it is quite general, since we can apply it for investment funds, research and development, elections and political campaigns, students' competition for better recommendation letter and so on.

The initial literature showed that tournaments can be a good instrument for encouraging the optimal effort level under a situation where players are subject to a common risk, affecting the output of all contestants. However, in a dynamic setting, players can deviate from what is the best choice from the principal point of view in order to increase their chance of winning the prize, due to the asymmetric position they will almost surely be. It also can induce risk-seeking behavior by the player in disadvantage, since overcoming his opponent becomes harder through the effort, and volatility will be more likely the way to increase his chance of winning the prize.

This paper has studied how players behave in a tournament towards the risk taking in a static and dynamic setting. Our environment considered different tasks that a player can work on, and each of them is associated with different risk levels. The standard model in the tournament literature involves no trade-off between the expected output and variance, and the choice of risk does not involve any cost. However, in order to take more risk, sometimes agents need to give up some expected return. The trade-off between expected output and variance allowed us to find an interior solution for the problem, while in most of the literature that do not consider a direct cost over the variance choice end up with extreme variance choices. In order to deal with it, most of the studies allow players to choose between two possible values, that is high risk and low risk. The problem is that in a dynamic setting, the equilibrium in the two-stage model increases the difficult of find a unique solution for the game. In our setting, the simultaneous choice of risk and expected return allow us to consider the dynamic problem in a more tractable way. We were able to find the Markov perfect equilibrium for the dynamic tournament, and thus formalizing the common wisdom that contestant seeks more risk by the end of the tournament.

Nonetheless, many questions kept unanswered, and should be a matter of concern for the development of this research. We assumed that activities have a symmetric expected output, since it can easily characterize the equilibrium. However, we might lose some

interesting properties from the equilibrium by doing this assumption. It could be possible to understand better, how player perceive the trade-off between the mean and variance, by relaxing the symmetry assumption. In addition, a growing part of literature are concerned about the optimal design of the prizes and the property of convexity of it, so it could be a future extension for this model to compute the optimal prize. Another extension could be to study what happen when more players are in the game. Finally, it would be interesting to test empirically the risk behavior the players have along a tournament.

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Chapter 2

Unitização de Jazidas de Petróleo: Uma Aplicação do Modelo de Green e Porter

Abstract

Neste trabalho abordamos a unitização como uma reinterpretação de cartel, partindo do modelo clássico de Green e Porter. A incerteza geológica é representada por um componente estocástico no custo marginal. Caracterizamos o contrato ótimo e, a partir da estática comparativa, avaliamos a eficiência e a viabilidade da cooperação. O preço e o grau da externalidade afetam positivamente o nível de eficiência do contrato ótimo. Mas enquanto preços elevados viabilizam os acordos, o grau de externalidade elevado pode conduzir a equilíbrios ineficientes ou mesmo inviabilizar a produção. O mesmo resultado ocorre com os custos fixos. Adicionalmente, quanto maior for o número de firmas envolvidas no acordo, menor será a chance de existir um contrato mais eficiente que a regra da captura.

2.1 Introdução

Um setor de petróleo descentralizado cria condição para um problema potencial inerente aos recursos de propriedade comum, a tragédia dos comuns. Este problema surge quando duas ou mais firmas têm direito de exploração um mesmo reservatório. Devido à natureza migratória, a extração de uma firma irá afetar a produção e o custo das outras firmas. Caso não haja algum tipo de esforço para cooperação, o resultado será a exploração em níveis elevados. O custo da extração será demasiadamente elevado e o retorno da produção do reservatório reduzirá. Logo o resultado da competição será a ineficiência.

Para evitar a perda associada à extração competitiva, as firmas podem tentar coop-

erar, definindo consensualmente um plano de produção ótimo. A unitização é um acordo de cooperação para a extração conjunta, de forma mais racional do reservatório. Porém, a incerteza geológica dificulta a negociação destes acordos, pois ela induz à assimetria de informação entre as firmas. Alguns trabalhos empíricos, como Libecap and Wiggins (1985), reportam que nem sempre estes acordos são concluídos com sucesso. Diante deste problema, a questão que surge naturalmente é: como o ambiente influencia a viabilidade de contratos que possam reduzir a ineficiência da competição sob a regra da captura.

Neste trabalho foi abordada a cooperação, como uma reinterpretação de cartel, pois as firmas têm incentivo natural a aumentar a produção, em detrimento da redução do lucro da indústria. O acordo de cartel deve ser escrito de forma que cada uma das firmas envolvidas aceite o acordo de forma descentralizada. Partindo do modelo clássico de Green e Porter, o contrato ótimo é caracterizado e a estática comparativa traz algumas respostas para a questão. Tanto o preço quanto o grau da externalidade afetam positivamente o nível de eficiência possível do contrato ótimo. Porém, enquanto o aumento nos preços viabiliza acordos de cooperação, elevado grau de externalidade podem conduzir a indústria para equilíbrios ineficientes ou mesmo inviabilizar a produção. O mesmo ocorre quando os custos fixos são elevados, de forma que é ótimo não produzir. Adicionalmente, quanto maior for o número de firmas envolvidas no acordo, menor será a chance de existir um contrato mais eficiente que a regra da captura.

Este trabalho está dividido em duas seções. Na seção 2.2 apresentamos o problema que a indústria de hidrocarboneto está submetida. Em seguida, é destacada a unitização como uma alternativa para evitar ineficiência na produção. Alguns resultados empíricos e teóricos sobre a viabilidade dos acordos de unitização são citados. Ao final da seção é apresentado o modelo de Green e Porter, o qual foi usado como base para este modelo. Na seção 2.3 é apresentado o modelo formalmente e os resultados obtidos, como a caracterização do contrato ótimo e a estática comparativa para avaliar a eficiência e a viabilidade do contrato. Ao final do trabalho, na seção 2.4, comparamos os resultados obtidos com os modelos iniciais de unitização.

2.2 O problema dos recursos comuns e a unitização

A indústria de hidrocarbonetos exibe as características clássicas de recursos de propriedade comum. A dificuldade de mapear os reservatórios antes do processo de licitação cria condições para que blocos contíguos estejam delimitados sobre uma mesma reserva. Neste caso, as firmas podem se valer da regra da captura para decidir o nível de produção. Segundo esta regra, a propriedade do petróleo é concedida a firma que extraiu o recurso; porém, devido à natureza migratória do hidrocarboneto, esta regra provoca uma extração predatória das reservas, elevando os custos da produção. Em particular, a extração exager-

ada reduzirá demasiadamente a pressão interna do reservatório, podendo comprometer o fator de recuperação do reservatório com uma política predatória de produção. À medida que a pressão se reduz, são necessários métodos mais custosos para a extração do hidrocarboneto. Assim, todas as firmas que produzem sobre um mesmo reservatório estão sujeitas a esta externalidade.

Uma alternativa para minimizar os efeitos da exploração exagerada é o acordo de unitização. Basicamente, este acordo é celebrado entre firmas com o objetivo da produção conjunta do reservatório. A coordenação permite que a produção seja feita de forma mais eficiente, se aproximando do nível de produção de monopólio. Uma forma de interpretar este acordo é como a formação de um cartel. Neste caso, o cartel permite alcançar resultados mais eficientes em relação à regra da captura, pois o acordo estabelecerá um nível de produção do que considera a presença da externalidade. Os ganhos de um acordo de cooperação são retratados por alguns autores, como Libecap and Wiggins (1984) apresentam dados do *Federal Oil Conservation Board* de 1926, que estima que a taxa de recuperação de petróleo sob competição é de apenas 20 a 25%, enquanto a taxa de recuperação quando apenas uma firma operasse poderia alcançar 85 a 90%. Já Libecap and Wiggins (1985) apresentam estimativas ainda mais otimistas, a partir da publicação *Oil Weekly* de 1942, no qual a recuperação em áreas unitizadas poderia aumentar a produção em duas a cinco vezes.

Apesar existir ganhos potenciais ao cooperar, a experiência americana demonstrou que há dificuldades para conclusão destes acordos quando não há intervenção do governo. Gary Libecap e co-autores escreveram trabalhos nos anos 1980 e 1990, explorando empiricamente esta dificuldade.[1] Wiggins and Libecap (1985) afirmam que as principais dificuldades para conclusão dos acordos está relacionado com a assimetria de informação e incerteza geológica. Segundo os autores, a unitização pode atrasar ou não ocorrer em função da divergência sobre as estimativas do valor da jazida, ou seja, a incerteza está induzindo a assimetria de informação.[2] Outro motivo para atrasar uma unitização pode ser estratégico. Quando uma firma não entra na unitização, gerará perdas agregadas que afeta as firmas que participam da unitização. Isto dará maior poder de barganha para as firmas que não participam da unitização, aumentando a partilha recebida por ela. Contudo, há poucos trabalhos teóricos abordando este tema e como o ambiente afeta a viabilidade de um acordo de cooperação.

A unitização de jazidas de petróleo pode ser uma reinterpretação dos acordos para cartelização. No caso do cartel, cada firma tem incentivo a produzir acima no nível

[1]Ver Libecap and Smith (1999), Libecap and Wiggins (1985) e Wiggins and Libecap (1985). Para uma revisão um pouco mais ampla, ver Kim and Mahoney (2002).

[2]O estudo geológico e sísmico de reservatório pode ser interpretado de formas diferentes entre os especialistas, mesmo que estejam usando os mesmos dados.

de produção que maximiza o lucro da indústria, pois assim teria maior participação no mercado e na receita. Deste modo, as firmas não estão internalizando que a produção exagerada reduzirá o preço de mercado. Analogamente, temos este resultado para o caso dos recursos de propriedade comum, como mostra a tragédia dos comuns. A unitização neste caso é representada pela formação de um cartel; ou seja, de uma única empresa para internalizar o custo da externalidade. Neste caso, o preço pode ser considerado fixo, mas a externalidade é representada no custo de extração do recurso, afetando assim o valor presente do lucro esperado. Desta forma, o nível de extração do recurso seria Pareto superior ao nível de extração de Cournot[3].

Partindo da ideia da unitização como formação de cartel, Deilami (1991) é um dos primeiros trabalhos teóricos sobre a unitização. O autor analisa como o ambiente afeta a viabilidade de existir um mecanismo descentralizado de cooperação factível, ou seja, mecanismos que utilizam apenas o recurso gerado para compensar as firmas não produtoras. As firmas são heterogêneas e podem reportar falsamente seus custos marginais de produção. Neste modelo, é considerado o problema de definir o operador e partilhar o recurso entre as firmas na unitização. Os incentivos deste ambiente fazem com que as firmas de custo elevado prefiram reportar um custo inferior para sinalizar que no caso de produção competitiva, elas teriam retorno superior. Enquanto as firmas de custo baixo preferem reportar um custo superior para barganhar maior participação na partilha do lucro. No caso da unitização não ocorrer, as firmas competirão pelo recurso seguindo a regra da captura. O autor mostra que os preços aumentam o benefício da cooperação, facilitando a existência de acordos. Note que preços elevados induzem a receitas maiores, que por sua vez aumenta o recurso disponível para compensar as firmas não produtoras. Este fator não foi considerado pelo Wiggins and Libecap (1985).[4] A magnitude da externalidade do reservatório e os custos fixos também influenciam positivamente a factibilidade do acordo, pois neste caso quando duas ou mais firmas produzem competitivamente, o custo de extração aumentará em função da externalidade que cada um delas gera para as demais; e quanto maior os custos fixos, devido à duplicidade de custos fixos, menor será o lucro gerado pela jazida.

Contudo, os resultados analíticos deste modelo dependem do número de firmas ser grande o suficiente. Note que quando o número de firmas é grande, o benefício de uma unitização será maior e mais provável neste modelo, pois maior será a economia gerada por não competir com custos fixos e externalidade. Esta hipótese não é uma boa aproximação da realidade, pois quanto maior o número de firmas envolvidas num acordo, mais difícil será a conclusão dele. Mailath and Postlewaite (1990) apresentam um modelo de barganha

[3]Neste trabalho, a alocação de Cournot se refere à extração quando as firmas não fazem acordo para cooperar.

[4]Os dados utilizados por Wiggins and Libecap (1985) eram da década de 1960. Os preços se elevaram após os dois choque de petróleo da década de 1970.

com assimetria de informação, no qual mesmo que haja ganhos sociais de um acordo ser concluído, quando o número de agentes aumenta se torna difícil de chegar a um acordo. Adicionalmente, o ambiente considerado por Deilami (1991) é estático e não há incerteza sobre o reservatório. Ao considerar interações repetidas podem surgir incentivos distintos dos incentivos dos jogos estáticos. Interações repetidas permitem aos jogadores estabelecerem formas de punição no caso de deserção da cooperação. Green and Porter (1984) e Porter (1983b) desenvolveram um modelo clássico que estuda uma indústria que enfrenta o problema de detectar e impedir desvios do nível de produção do acordo de cartel. Porter (1983a) observou que ocorriam episódios recorrentes de alterações no preço e lucro da indústria e tais alterações eram estatisticamente significativas. No modelo de Porter (1983b), a indústria incorre em custos para coletar e disseminar dados de preços e quantidades entre os membros, bem como custos para verificar a acuidade da informação. Desta forma, as firmas observam apenas sua própria produção e o preço[5] de mercado, mas não observam a quantidade produzida por cada uma das outras firmas. O autor considera um jogo de Cournot repetido infinitamente e representa esta incerteza usando um componente estocástico na curva de demanda. Desta forma, uma queda inesperada nos preços pode sinalizar tanto um desvio da produção do cartel como um choque negativo sobre a demanda.

No ambiente considerado por Green e Porter, as firmas podem evitar desvios do nível de produção de cooperação usando uma estratégia de gatilho. Quando o preço de mercado fica abaixo do preço gatilho, as firmas produzirão no nível de Cournot por um tempo fixado previamente, denominado de período de punição. Uma firma que considere expandir sua produção secretamente, deve ponderar o *trade-off* entre o ganho imediato de lucro com a elevação da probabilidade dos preços de mercado ficarem abaixo do gatilho de preços. Isto aumentará a chance de haver lucros inferiores nos períodos seguintes, uma vez que a indústria reverte o nível de produção para o equilíbrio de Cournot. Green and Porter (1984) mostram que, mesmo no equilíbrio não cooperativo, haverá episódios recorrentes de "Guerra de Preços", ou seja, períodos que a indústria produz competitivamente. Estes episódios são causados pelo choque negativo sobre a demanda, mas nunca por um desvio voluntário de alguma firma.

Na próxima seção será apresentada formalmente esta uma modificação do modelo de Porter (1983b) para aplicá-lo ao caso da unitização. A motivação desta modificação consiste em avaliar a eficiência e a viabilidade da cooperação entre as firmas quando não há regulação. Para isto, é caracterizado o equilíbrio ótimo e a partir da estática comparativa será avaliado como o preço, a incerteza e o grau de externalidade afetam a possibilidade de haver cooperação.

[5] Por hipótese, a produção das firmas é homogênea, logo o preço de mercado é o mesmo para todas as firmas.

2.3 Aplicação do Modelo de Green e Porter

Considere um jogo dinâmico em tempo discreto, no qual $n \in \mathbb{N}$ firmas possuem direito de extração de uma mesma jazida. A cada período $t \geqslant 0$, cada firma i escolhe o nível de extração[6] $q_{it} \geqslant 0$ simultaneamente. Denote $\overline{q}_t = (q_{1t},...,q_{nt})$ o vetor de extração no período t e a extração total por $Q_t = \sum_{i=1}^{n} q_{it}$. A extração das firmas é de qualidade homogênea e o preço de mercado é constante, dado por p. Suponha que as firmas possuem a mesma tecnologia de extração. Logo, o custo marginal de extração *ex post* é comum entre as firmas. Suponha que o custo marginal de extração tem um componente estocástico, de forma que as firmas observam somente $\hat{c}_t$, dado por

$$\hat{c}_t = \theta_t c_{mg}\left(Q_t\right),$$

onde $\{\theta_t\}$ é uma sequência de variáveis aleatórias i.i.d., não observadas diretamente pelas firmas, com média μ, função de densidade f e distribuição F. O componente estocástico do custo marginal representa a incerteza geológica que as firmas estão sujeitas. Mais precisamente este componente altera o grau de externalidade do reservatório. Suponha que $F(0) = 0$ e que $F(\infty) = 1$, e que tanto F como f são continuamente diferenciáveis. Adicionalmente, $c_{mg} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ é dada por

$$c_{mg}\left(Q_t\right) = c + bQ_t. \tag{2.1}$$

Observe que o termo bQ_t representa a externalidade da extração no custo marginal c_{mg}, onde $b > 0$ é o coeficiente de externalidade. O custo total da firma i é dado por

$$C\left(q_{it},Q_t\right) = \theta_t c_{mg}\left(Q_t\right) q_{it} + K, \tag{2.2}$$

onde K é o custo fixo comum de cada firma. Podemos escrever a função lucro esperado[7] da firma i de um período como

$$\pi_i\left(\overline{q}\right) \equiv \pi\left(q_i,Q_i\right) = \left[A - B\left(Q_i + q_i\right)\right] q_i - K, \tag{2.3}$$

onde $Q_i = Q - q_i = \sum_{j \neq i} q_j$ é a soma da extração das outras firmas, $A = p - \mu c$ e $B = \mu b$. Suponha que $0 < \mu c < p$, e que $b > 0$, logo A e B são constantes positivas. No ambiente dinâmico, cada firma i escolhe um plano de produção $\{q_{it}\}_{t=0}^{\infty}$ que maximiza o lucro esperado em valor presente, com fator de desconto comum $\beta \in (0,1)$, i.e.

[6] Extração e produção serão usados como sinônimos neste trabalho.
[7] O subscrito do tempo foi ocultado para simplificar a notação.

$$\max_{\{q_{it}\}_{t=0}^{\infty}} \mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\left[pq_{it}-\theta_t c_{mg}(Q_t)q_{it}-K\right]\right\}.$$

Antes de avaliarmos o jogo dinâmico será apresentado o equilíbrio do jogo estático, os quais serão *benchmarks* para avaliar a eficiência da estratégia no caso dinâmico. O equilíbrio de Cournot do *stage game* será representado por $\overline{s}=(s_i)_{i=1}^n$, onde $s_i\equiv s$, para todo $i=1,...,n$, e é dado por

$$s=\frac{A}{B\left(n+1\right)}.$$

com lucro esperado positivo[8] para cada firma i e dado por

$$\pi_i\left(\overline{s}\right)=\frac{A^2}{B\left(n+1\right)^2}-K. \tag{2.4}$$

Para evitarmos equilíbrios com produção nula, suponha que o número de firmas n é pequeno o suficiente para que

$$0<K<\frac{\left(p-\mu c\right)^2}{\mu b\left(n+1\right)^2}=\frac{A^2}{B\left(n+1\right)^2}. \tag{2.5}$$

No caso de um monopólio produzindo com n plantas idênticas, onde cada planta contribui com $1/n$ da produção, seria equivalente ao caso de um cartel perfeito. Assim, cada firma extrairá o recurso à um nível que maximize o retorno esperado da indústria. Denote o vetor de extração de monopólio como $\overline{r}=(r_i)_{i=1}^n$, onde é possível mostrar pela simetria que $r_i\equiv r$ para todas as firmas $i=1,...,n$ e é dado por[9]

$$r=\frac{A}{2Bn},$$

com lucro esperado de cada firma dado por

$$\pi_i\left(\overline{r}\right)=\frac{A^2}{4Bn}-K. \tag{2.6}$$

É possível mostrar que a alocação eficiente deste modelo será dada pelo nível de produção r. Note que a demanda é totalmente elástica, de modo que o excedente de um consumidor é nulo. Assim, a alocação que maximiza o bem-estar da economia será equivalente à solução de monopólio. Neste modelo também ocorre a tragédia dos comuns; ou seja, o nível de extração no equilíbrio de Cournot é superior ao nível de extração

[8] Usando a equação (2.5).

[9] Ao longo do trabalho, o nível de produção de monopólio se refere a extração igual a r de cada uma das plantas.

eficiente, i.e., $s > r$, o que implica que o custo esperado de equilíbrio será superior ao nível eficiente. O lucro esperado no equilíbrio de Cournot será estritamente inferior ao lucro esperado da alocação eficiente. Da mesma forma como ocorre no dilema dos prisioneiros, o aumento da extração unilateral implica retorno superior para firma i em relação ao nível de extração da unitização.

No restante do trabalho, verificamos que é possível encontrar um acordo com nível de produção Pareto superior à produção do equilíbrio de Cournot quando as firmas seguem a estratégia de Green e Porter. Neste ambiente, não existe um regulador e os acordos devem ser implementados a partir de um equilíbrio de Nash. Ou seja, mesmo atuando de forma competitiva, as firmas irão preferir fazer parte do acordo de unitização para auferir lucros superiores, e não irão desviar do acordo. No caso de desvio, a probabilidade de incorrer em punição será maior. Logo o lucro esperado do desvio será inferior ao da unitização. Considere a estratégia na qual as firmas inicialmente extraem no nível determinado pelo acordo, e mantêm este nível de extração até que o custo marginal observado ultrapasse um gatilho de custo $\tilde{c}$, previamente determinado. Quando o custo marginal exceder o gatilho, as firmas produzirão no nível de Cournot por $T - 1$ períodos (episódios reversionários), independentemente do que aconteça com o custo marginal nestes períodos. A partir do período T após o acionamento do gatilho, as firmas voltarão a produzir no nível determinado pelo acordo. O período de punição desta estratégia equivale ao caso em que voltam a competir de acordo com a regra da captura, produzindo exageradamente e com custos elevados. Os períodos cooperativos e não cooperativos podem ser definidos recursivamente por

(a) $t = 0$ é um período cooperativo;

(b) se t é um período cooperativo e $\hat{c}_t \leqslant \tilde{c}$, então $t + 1$ é um período cooperativo;

(c) se t é um período cooperativo e $\hat{c}_t > \tilde{c}$, então $t + 1, ..., t + T - 1$ são períodos não cooperativos, com nível de produção $\overline{q}_t = \overline{s}$, e $t + T$ é um período cooperativo.

Note que não há desvio lucrativo do nível de extração de Cournot para uma firma, quando todas as outras estão produzindo no nível de Cournot. Desta forma, a estratégia que as firmas irão seguir é definida por[10]

$$q_{it} = \begin{cases} q, & \text{se } t \text{ for um período cooperativo,} \\ s, & \text{se } t \text{ for um período não-cooperativo.} \end{cases}$$

Uma hipótese implícita do modelo se refere à estacionaridade do problema dinâmico.

[10] Na subseção 2.3.1 será mostrado que, em geral, não é possível alcançar a produção de monopólio r nos períodos cooperativos.

Cada firma está sujeita à um processo de Markov estacionário de dois estados (cooperativo e não cooperativo) com T períodos.[11] Portanto, quando as firmas seguem esta estratégia, a função valor esperado descontado da produção satisfaz uma relação recursiva. Então, dado que as firmas irão produzir ao nível q predeterminado no acordo, o valor esperado descontado para a firma i é definido por

$$V_i\left(\overline{q}\right) = \pi_i\left(\overline{q}\right) + \Pr\left[\hat{c}_t \leqslant \tilde{c}\right]\beta V_i\left(\overline{q}\right) + \Pr\left[\hat{c}_t > \tilde{c}\right]\left[\sum_{\tau=1}^{T-1}\beta^{\tau}\pi_i\left(\overline{s}\right) + \beta^{T}V_i\left(\overline{q}\right)\right]. \tag{2.7}$$

Ou seja, o valor é dado pelo lucro do período presente somado ao valor esperado do próximo período. Caso o custo marginal observado seja inferior ao custo de gatilho, o valor do próximo período será o valor do período presente descontado por β. Caso contrário, as firmas receberão o lucro de Cournot por $T-1$ períodos e então voltam a cooperar. Podemos reescrever a equação (2.7) como

$$V_i\left(\overline{q}\right) = \frac{\pi_i\left(\overline{s}\right)}{1-\beta} + \frac{\pi_i\left(\overline{q}\right) - \pi_i\left(\overline{s}\right)}{1-\beta^T - \left(\beta-\beta^T\right)F\left(\frac{\tilde{c}}{c_{mg}(Q)}\right)}. \tag{2.8}$$

Portanto, o valor para a firma i é o lucro esperado descontado de Cournot somado ao adicional de lucro dos períodos de cooperação relativo ao lucro de Cournot, descontado de forma apropriada. Note que o fator de desconto do lucro adicional em relação ao lucro de Cournot é um fator de desconto esperado, dado por $(1-F)\beta^T + F\beta$, onde o argumento de F é $\tilde{c}/c_{mg}(Q)$.

Definition 3 *Uma estratégia de gatilho de custos* $(\overline{q}^{\star},T,\tilde{c})$ *é um equilíbrio de Nash, se para todo* $i=1,...,n$*,*

$$V_i\left(\overline{q}^{\star}\right) \geqslant V_i\left(q_i,q_{-i}^{\star}\right), \text{ para todo } q_i \geqslant 0.$$

A condição de primeira ordem do problema da firma i implica

$$\begin{aligned}\pi_i^i\left(\overline{q}^{\star}\right)\left[1-\beta^T-\left(\beta-\beta^T\right)F\left(\frac{\tilde{c}}{c_{mg}\left(Q^{\star}\right)}\right)\right] &= \\ \left(\beta-\beta^T\right)f\left(\frac{\tilde{c}}{c_{mg}\left(Q^{\star}\right)}\right)\frac{\tilde{c}c'_{mg}\left(Q^{\star}\right)}{c_{mg}\left(Q^{\star}\right)^2}\left[\pi_i\left(\overline{q}^{\star}\right)-\pi_i\left(\overline{s}\right)\right]&, \end{aligned} \tag{2.9}$$

para todo $i=1,...,n$, onde $\pi_i^i\left(\overline{q}^{\star}\right) = \partial\pi_i\left(\overline{q}^{\star}\right)/\partial q_i$, $c'_{mg}\left(Q^{\star}\right) = dc_{mg}\left(Q^{\star}\right)/dQ$, e $Q^{\star} = \sum_{i=1}^{n} q_i^{\star}$. A equação (2.9) representa o benefício e o custo marginal da produção para a firma i, sob a estratégia proposta. No equilíbrio de Cournot, a firma i maximizaria quando $\pi_i^i\left(\overline{q}^{\star}\right)=0$. Contudo, o lado direito da equação não será zero em geral. Logo, a

[11] Para o caso de cartel com choque no preço, o processo Markoviano é caracterizado por Green and Porter (1984).

firma irá escolher uma produção inferior ao nível de Cournot, indicando uma melhora no sentido de Pareto.

Como será mostrado adiante, $\overline{r}$ não será uma alocação de equilíbrio. Analogamente ao ambiente de Porter (1983b), o problema deste modelo não é de existência de equilíbrio, mas caracterizar o equilíbrio que é Pareto superior em relação aos demais. A proposição 3 a seguir mostra que a produção de Cournot é sempre um equilíbrio, para qualquer par $(\tilde{c},T)$ dado.

Proposition 3 *Para qualquer função lucro estritamente côncava, o vetor de produção $\overline{s}$ de equilíbrio de Cournot do jogo estático, é um nível de produção de equilíbrio nos períodos cooperativos, para quaisquer valores de $\tilde{c}$ e T.*

Proof. Para verificar que $\overline{s}$ é uma alocação de equilíbrio, note que lado direito de (2.9) é trivialmente nulo. É fácil verificar que $\pi_i^i(\overline{s}) = 0$. Portanto $\overline{s}$ satisfaz a condição de primeira ordem. Adicionalmente, $\overline{s}$ satisfaz a condição de segunda ordem, dada por

$$\frac{\partial^2 V_i(\overline{s})}{\partial q_i^2} = \frac{\frac{\partial^2 \pi_i(\overline{s})}{\partial q_i^2}}{1 - \beta^T - (\beta - \beta^T) F\left(\frac{\tilde{c}}{c_{mg}(Q)}\right)} < 0,$$

usando a concavidade da função lucro. Portanto $\overline{s}$ é um nível de produção de equilíbrio nos períodos cooperativos. ■

Podemos restringir o conjunto das alocações possíveis de equilíbrio. Como o lucro das firmas é simétrico, a extração de equilíbrio será igual para todas as firmas, i.e. $q_j^\star = q_k^\star \equiv q^\star$ para todo $j,k = 1,...,n$. Desta forma evitaremos o problema de conciliar interesses distintos, e permite avaliar como a incerteza atua sobre a solução de equilíbrio. Adicionalmente, a proposição 4 restringe o conjunto dos níveis de produção de equilíbrio possíveis, de modo que $q^\star \in (s/n,s]$.

Proposition 4 *Em qualquer equilíbrio de Nash $(\overline{q}^\star,T,\tilde{c})$, todas as firmas produzem no mesmo nível, i.e. $q_j^\star = q_k^\star \equiv q^\star$ para todo $j,k = 1,...,n$, e a produção de equilíbrio $q^\star$ pertence ao intervalo $(s/n, s]$.*

Proof. Suponha que $q_j^\star$ e $q_k^\star$ são positivos para algum j e k. Usando as equações (2.3) e (2.4), podemos reescrever a condição de primeira ordem (2.9) como

$$\alpha\left[A - BQ^\star - Bq_i^\star\right] = \gamma\left[(A - BQ^\star) q_i^\star - \frac{A^2}{B(n+1)^2}\right], \text{ para } i = j,k, \tag{2.10}$$

onde $\alpha = 1 - \beta^T - \left(\beta - \beta^T\right) F > 0$ e $\gamma = \left(\beta - \beta^T\right) f\tilde{c}c'_{mg}(Q^\star) / c_{mg}(Q^\star)^2 > 0$, pois $\beta \in (0,1)$, e são comuns para ambas as firmas.

Subtraindo a equação (2.10) da firma j pela equação da firma j, temos

$$\alpha B\left(q_k^\star - q_j^\star\right) = \gamma\left(A - BQ^\star\right)\left(q_k^\star - q_j^\star\right), \tag{2.11}$$

que é satisfeita se e somente se $q_k^\star = q_j^\star$

Portanto, as firmas que produzem positivamente no equilíbrio, produzirão no mesmo nível. Denotando $Q^\star = nq^\star$, podemos reescrever a equação (2.9) como

$$\left[1 - \beta^T - \left(\beta - \beta^T\right) F\left(\frac{\tilde{c}}{c + bQ^\star}\right)\right]\left[A - B\left(n+1\right)q^\star\right] = \\ \left(\beta - \beta^T\right) f\left(\frac{\tilde{c}}{c + bQ^\star}\right)\frac{\tilde{c}b}{\left(c + bQ^\star\right)^2}\left[\left(A - BQ^\star\right)q^\star - \frac{A^2}{B\left(n+1\right)^2}\right]. \tag{2.12}$$

Como o nível de extração de Cournot é um equilíbrio, podemos restringir a atenção para os valores de $q^\star$ no qual as firmas não ficam pior do que extrair no nível s. Note que pela equação (2.8), $V_i\left(\overline{q}\right) \geqslant V_i\left(\overline{s}\right)$ vale se e somente se $\pi_i\left(\overline{q}\right) \geqslant \pi_i\left(\overline{s}\right)$, ou ainda,

$$q\left(A - Bnq\right) \geqslant \frac{A^2}{B\left(n+1\right)^2}, \tag{2.13}$$

que é satisfeita se e somente se $q \in [s/n,s]$. Porém, s/n não satisfaz a condição de primeira ordem (2.9), pois $\pi_i^i\left(\overline{s}/n\right) = A(n-1)/n > 0$ enquanto $\pi_i\left(\overline{s}/n\right) - \pi_i\left(\overline{s}\right) = 0$.

Assim, $\overline{s}/n$ não é uma alocação de equilíbrio. Portanto, em qualquer equilíbrio devemos ter que $q^\star \in (s/n, s]$. ■

Vimos pela proposição 3 que a função valor $V_i\left(\overline{q}\right)$ é côncava em q_i quando é avaliada no ponto $\overline{s}$ para qualquer par $(\tilde{c},T)$, desde que a função lucro esperado seja côncava. Porém, para garantimos que exista um máximo para a função valor avaliada nos equilíbrios, precisamos restringir o formato da função de distribuição F. A função valor deve ser côncava em q_i quando todas as outras firmas estão no equilíbrio de Nash. Como todas as firmas agem igualmente, a extração que maximiza o valor de uma firma, é o mesmo valor que irá maximizar o valor do recurso comum.

Uma condição suficiente para concavidade da função valor é apresentada pela proposição 5 a seguir. A restrição sobre a elasticidade da função de distribuição F implica que esta função é estritamente côncava sobre o suporte.[12]

Proposition 5 *Suponha que* $q_j = q \in (s/n,s]$ *e satisfaz a condição de primeira ordem* (2.9) *para todas as firmas* $j \neq i$ *e que a função distribuição* F *satisfaz* $\left(2 + \frac{f'(\theta)}{f(\theta)}\theta\right) \leqslant 0$

[12]De forma mais geral, pela equação (2.14) temos que, se $2 - \frac{b}{c_{mg}(Q)B}\left[A - B\left(n+1\right)q\right]\left(2 + \frac{f'}{f}\frac{\tilde{c}}{c_{mg}(Q)}\right) \geqslant 0$, onde o argumento das funções f' e f é dado por $\tilde{c}/c_{mg}(Q)$, então V_i é côncava em q_i.

para todo θ no suporte da distribuição. Então $V_i(\overline{q})$ é côncava em q_i.

Proof. Usando a simetria e a condição de primeira ordem (2.9) das firmas $j \neq i$, chegamos que a condição de segunda ordem é dada por

$$\frac{\partial^2 V_i(\overline{q})}{\partial q_i^2} = \frac{-B}{1-\beta^T-(\beta-\beta^T)F}\left\{2-\frac{b}{c_{mg}(Q)\,B}\left[A-B(n+1)\,q\right]\left(2+\frac{f'}{f}\frac{\tilde{c}}{c_{mg}(Q)}\right)\right\}<0, \tag{2.14}$$

pois $q \in (s/n,s]$ implica que $A - B(n+1)\,q \geqslant 0$, onde o argumento das funções f' e f é $\tilde{c}/c_{mg}(Q)$. ■

Um resultado importante de Green and Porter (1984) deve ser observado. A estrutura de incentivos do contrato — a estratégia de punição do Green e Porter —, garante que as firmas não desviem do nível de extração $q^\star$ nos períodos cooperativos. Portanto, o custo irá exceder o gatilho apenas nas realizações suficientemente altas do componente estocástico θ, e nunca por desvios unilaterais. Da mesma forma, não é racional desviar do nível de extração de Cournot nos períodos de reversão. Observe que realizações altas de θ implica elevados custos de externalidade. Sendo assim, a tragédia dos comuns será precedida por um período cujo coeficiente de externalidade *ex post*, θb, é elevado e não por desvios unilaterais das firmas.

2.3.1 Desenho do contrato ótimo

Nesta subseção será caracterizado o gatilho e o período de punição ótimo, que induzem níveis de produção mais eficiente em relação ao nível de Cournot. Será caracterizado apenas as soluções interiores do problema, ou seja, quando $\tilde{c} > 0$ e $1 < T < \infty$ para utilizar a condição necessária de máximo do problema.[13] Trataremos T como uma variável contínua, pois veremos que existe um grau de liberdade na escolha ótima. Então basta escolher um valor inteiro superior e próximo a T.

Uma quantidade de equilíbrio de Nash $q^\star$ deste modelo representa também a quantidade de equilíbrio não cooperativo no espaço das estratégias $(\tilde{c},T,q,\overline{s})$. Desta forma, $q^\star$ é a solução da condição de primeira ordem (2.12), que é uma função do gatilho de custos $\tilde{c}$ e de T, onde $T-1$ é o período de punição. Consequentemente, a função valor esperado da firma i, no equilíbrio é uma função do par $(\tilde{c},T)$, i.e. $V_i(q^\star(\tilde{c},T);\tilde{c},T)$, para $i=1,\ldots,n$. Os valores ótimos de $\tilde{c}$ e do T, sob o ponto de vista da firma i, são aqueles que maximizam $V_i(q^\star(\tilde{c},T);\tilde{c},T)$ sujeito a $\tilde{c} \geqslant 0$ e do $T \geqslant 1$. Mas note que a simetria das firmas implica que maximizando para a firma i, também estaremos maximizando para a indústria. Mas o nível de produção que maximiza o lucro da indústria é equivalente ao nível que minimiza a ineficiência da produção competitiva. Portanto, a escolha ótima de $\tilde{c}$ e do T implicam

[13] Mais precisamente, as soluções interiores de $\tilde{c}$ são tais que $\tilde{c}/c_{mg}(Q^\star)$ pertençam ao suporte da distribuição F.

em nível de produção fracamente Pareto superiores ao equilíbrio de Cournot.

Resolvendo a condição necessária de primeira ordem para T do problema da escolha do par $(\tilde{c},T)$ ótimo[14], obtemos

$$-\frac{f}{1-F}\frac{\tilde{c}^\star}{c^\star} = \eta^\star \Longleftrightarrow -\frac{f}{1-F}\frac{\tilde{c}^\star}{c^\star} - \frac{f'}{f}\frac{\tilde{c}^\star}{c^\star} = 1, \tag{2.15}$$

onde $c^\star = c_{mg}(nq^\star)$, $\tilde{c}^\star$ é o gatilho ótimo e o argumento das funções F, f e f' é dado por $\tilde{c}^\star/c^\star$. Esta condição implica que $\eta^\star \leqslant 0$, logo a distribuição F deve ser côncava.[15] Como veremos adiante, podemos interpretar η como uma medida de ruído sobre o custo marginal, de forma que quanto menor for $|\eta|$, maior será a variabilidade do choque sobre o custo. Na subseção 2.3.2 apresentamos um exemplo de distribuição que satisfaz esta propriedade, conhecida como distribuição de Pareto. Denote $\theta^\star = \tilde{c}^\star/c^\star$, logo podemos reescrever esta condição como

$$-\frac{f(\theta^\star)}{1-F(\theta^\star)}\theta^\star - \frac{f'(\theta^\star)}{f(\theta^\star)}\theta^\star = 1. \tag{2.16}$$

Se a solução da equação (2.16) para $\theta^\star$ for única, então $\tilde{c}^\star$ será ajustado exatamente na mesma proporção que $c_{mg}(nq^\star)$, em resposta a mudanças nos parâmetros c, b, n, A e B, pois $\theta^\star = \tilde{c}^\star/c^\star$. Esta condição sobre F ressalta a importância do choque sobre o modelo. A concavidade da distribuição implica que as realizações dos choques sobre o custo devem tem maior massa de densidade de probabilidade na parte inferior do suporte.

A condição necessária de primeira ordem para $\tilde{c}$ implica que

$$q^\star = \frac{A}{2Bn}\left[\frac{n+\eta^\star-(n+1)\,\frac{c}{A}\frac{b}{B}}{n+\eta^\star+1}\right] = r\left[\frac{n+\eta^\star-(n+1)\,\frac{c}{A}\frac{b}{B}}{n+\eta^\star+1}\right]. \tag{2.17}$$

A incerteza neste modelo impede que as firmas alcancem a produção eficiente em geral, como será mostrado na proposição 8. No caso sem incerteza, é possível verificar que usando a produção de Cournot como punição, a alocação eficiente é alcançada quando o fator de desconto for suficientemente alto; porém, a incerteza sobre a jazida possui um papel importante na eficiência do modelo, como mostra a proposição 7. Primeiro, será mostrado que um aumento no número de firmas pode inviabilizar a cooperação. É possível mostrar que

$$\frac{dq^\star}{dn} = -\frac{q^\star}{n} + \frac{A}{2Bn}\left[\frac{1-\eta^\star\frac{c}{A}\frac{b}{B}}{(n+\eta^\star+1)^2}\right] < 0$$

[14] O apêndice 2.4 apresenta o problema e a resolução do modelo.

[15] Podemos reescrever esta condição como a diferença da elasticidade de $1-F$ em relação a $\theta^\star$ pela elasticidade de $-f$ em relação a $\theta^\star$, onde $\theta^\star = \tilde{c}^\star/c^\star$.

e que

$$\frac{d\left(nq^{\star}\right)}{dn}=\frac{A-c\left(\eta^{\star}+2\right)}{2B\left(n+\eta^{\star}+1\right)^{2}}>0.$$

Logo, quando o número de firmas aumenta, o nível de extração individual decresce. Contudo, o nível de extração agregado aumentará. Quanto maior o número de firmas explorando a jazida, maior será a dificuldade de haver um acordo que induza a equilíbrios mais eficientes em relação ao equilíbrio de Cournot. Note que quando n se torna infinitamente grande, $q^{\star}$,r e s tendem a zero. Porém,

$$\lim_{n\to\infty} nq^{\star}=\lim_{n\to\infty} ns=\frac{A}{B}>\frac{A}{2B}=\lim_{n\to\infty} nr,$$

pois para um n suficientemente grande, $q^{\star}$ será maior que s, logo as firmas irão extrair no nível de equilíbrio de Cournot s. Formalmente, temos que o seguinte resultado:

Proposition 6 *Se o número de firmas for grande o suficiente, o nível de extração ótimo nos períodos cooperativos será igual ao nível de extração de equilíbrio de Cournot.*

Proof. Suponha, sem perda de generalidade, que o custo fixo K seja nulo.[16] Se $n\leqslant n^{o}$, então temos que $q^{\star}\leqslant s$, onde n^{o} é dado implicitamente por

$$0=-\eta^{\star}+\frac{c}{A}\frac{b}{B}(n^{o}+1)^{2}+n^{o}(n^{o}+\eta^{\star}+1).$$

Se $n>n^{o}$, então $q^{\star}>s$. Mas como as firmas otimizam suas decisões, o nível de extração escolhido pelas firmas será s, pois s é sempre um equilíbrio possível e neste caso $V_i(\overline{s})>V_i(\overline{q}^{\star})$. ■

Uma forma alternativa de obter este resultado mostra a importância da concavidade da função de distribuição F sobre o nível de extração de equilíbrio ótimo.

Proposition 7 *O nível de extração de equilíbrio $q^{\star}$ é uma função crescente de $\eta^{\star}$, com taxa de crescimento dado por*

$$\frac{dq^{\star}}{d\eta^{\star}}=\begin{cases}\frac{A+(n+1)c\frac{b}{B}}{2Bn(n+\eta^{\star}+1)^{2}}>0, & se\ \eta^{\star}\leqslant\eta^{o}\\ 0, & c.c.,\end{cases}\tag{2.18}$$

onde

$$\eta^{o}=-\frac{n+1}{n-1}\left[n+(n+1)\frac{c}{A}\frac{b}{B}\right],$$

que é uma função decrescente de n. Quando $\eta^{\star}$ tende para $-\infty$, $q^{\star}$ converge para r, o nível de extração de monopólio.

[16] Esta não é uma condição necessária para este resultado. Caso $K>0$, seria possível encontrar $\hat{n}$ tal que $\pi_i\left(q^{\star}\left(\hat{n}\right)\right)\geqslant 0\geqslant\pi_i\left(q^{\star}\left(\hat{n}+1\right)\right)$, e obteríamos o mesmo resultado.

Proof. Como $\eta^\star < 0$, então $q^\star > s/n$, satisfazendo a proposição 4. Se $\eta^\star \leqslant \eta^o$, então $q^\star \leqslant s$. Quando $\eta^\star > \eta^o$, temos que $q^\star > s$. Neste caso $V_i(\overline{s}) > V_i(\overline{q}^\star)$, e como s é sempre um equilíbrio de Nash, então as firmas produziriam no nível de extração de Cournot s. A equação (2.18) é obtida derivando (2.17) em relação $\eta^\star$, e $\lim_{\eta^\star \to -\infty} q^\star = r$ é trivialmente obtido a partir da equação (2.17). ■

A medida que número de firmas cresce, menor será η^o. Desta forma, mais difícil será encontrar um acordo tal que $q^\star < s$. Este resultado é análogo a proposição 6. Quando o número de firmas for grande, é necessário que a distribuição seja "suficientemente côncava" para suportar um equilíbrio mais eficiente que o equilíbrio de Cournot.[17] Outra forma de interpretar o resultado, está relacionado ao ruído do componente estocástico. No exemplo da subseção 2.3.2, fica claro que quando η não for suficientemente negativo, estamos no caso em que há muito ruído, ou ainda, a variância da distribuição é elevada. Se a variância é elevada, se torna mais difícil definir um gatilho, pois o custo irá variar muito mesmo que mantenha o nível de extração constante. Isto dará espaço para comportamento oportunístico das firmas. A probabilidade das firmas entrarem no período de punição será alta, logo será inviável manter um equilíbrio Pareto superior ao equilíbrio de Cournot.

Podemos considerar este modelo como uma extensão do caso determinístico. Quando o ruído do componente estocástico for baixo o suficiente retornamos ao caso determinístico, e o equilíbrio Pareto ótimo é alcançado. Note que mantendo a média μ constante, quando $\eta^\star$ tende para $-\infty$, a massa de densidade de probabilidade se acumula no ponto médio μ, de forma que $\theta^\star$ converge para μ. Quando $\mu = 1$, voltamos para o caso sem incerteza, e teremos que $\tilde{c}^\star = c_{mg}(nq^\star) = c_{mg}(nr)$. Logo é possível induzir um equilíbrio Pareto ótimo usando a estratégia de gatilho de Green e Porter. Quando μ for livre, a medida que F se torna mais côncava, maior será a densidade de probabilidade na parte inferior do suporte. Ou seja, os choques baixos ocorrerão com maior probabilidade. Tomando uma distribuição com suporte (θ_0,θ_1), com $\theta_1 > \theta_0$, quando $\eta^\star$ tende para $-\infty$, teremos que no limite $\Pr[\theta = \theta_0] = 1$, ou seja, a externalidade ocorre no menor grau possível e novamente estamos num caso sem incerteza. Isto pode ser observado no exemplo da subseção 2.3.2.

Vimos acima os dois casos extremos, quando há muito ruído e o equilíbrio será de Cournot e quando a pouco ruído e é possível induzir um equilíbrio Pareto ótimo. Considere agora o caso intermediário, quando $\eta^o \geqslant \eta^\star > -\infty$, no qual não será possível alcançar resultados eficientes. A incerteza sobre a jazida dificulta definir um gatilho que induza as firmas extraírem no nível r. A produção ótima $q^\star$ será maior do que r, pois caso contrário a punição para induzir $q^\star < r$ não seria factível, uma vez que o desvio unilateral será lucrativo. Logo o resultado seria o nível de extração de Cournot. Com $q^\star > r$, é possível encontrar uma punição, ou mais precisamente, um par $(\tilde{c}^\star,T^\star)$, de modo que a punição

[17] De forma mais precisa, é necessário que $\eta^\star$ seja suficientemente baixo.

seja factível. Factibilidade da punição significa que existe um par ($\tilde{c}^{\star}$,$T^{\star}$) tal que as firmas aceitam participar e que preferem respeitar o acordo, pois o retorno do desvio será inferior ao retorno de respeitar o acordo. Temos então o seguinte resultado:

Proposition 8 *Suponha que $\tilde{c}$ e T foram escolhidos de modo a maximizar o valor esperado da exploração da jazida e a solução ótima é interior, i.e. $T > 1$ e $\tilde{c}/c_{mg}(nq^{\star})$ pertence ao suporte da distribuição F. Suponha ainda que $\eta^{o} \geqslant \eta^{\star} > -\infty$. Então o nível de extração ótimo $q^{\star}$ será maior do que o nível de extração de monopólio r, i.e. $q^{\star} > r$. Como corolário, custo marginal de extração será maior que no nível eficiente, i.e. $c_{mg}(nq^{\star}) > c_{mg}(nr)$.*

Proof. Basta verificar que, quanto $\eta^{\star}$ for suficientemente baixo, da equação (2.17) temos

$$\frac{n+\eta^{\star}-(n+1)\,\frac{c}{A}\frac{b}{B}}{n+\eta^{\star}+1} > 1,$$

pois tanto o denominador quanto o numerador serão negativos e

$$\left| n+\eta^{\star}-(n+1)\,\frac{c}{A}\frac{b}{B} \right| > \left| n+\eta^{\star}+1 \right|.$$

■

Vimos que o número de firmas afeta a viabilidade de acordos de cooperação. Quanto maior o número de firmas, um nível de extração mais eficiente se torna menos provável. Adicionalmente, o preço p e o coeficiente esperado da externalidade, $B = \mu b$, também influenciam o nível ótimo de extração. Lembre que $A = p - \mu c$, onde p é o preço e c é o custo marginal individual e que $\tilde{c}^{\star} = \theta^{\star} c_{mg}(nq^{\star})$. Primeiro note que quanto mais elevado for preço, maior será η^{o}, o que torna mais provável haver cooperação. Ao mesmo tempo, temos que

$$\frac{dq^{\star}}{dp} = \frac{dq^{\star}}{dA}\frac{dA}{dp} = \frac{dq^{\star}}{dA} = \frac{n+\eta^{\star}}{2Bn\,(n+\eta^{\star}+1)} > 0$$

logo,

$$\frac{d\tilde{c}^{\star}}{dp} = \frac{d\tilde{c}^{\star}}{dA} = \frac{\theta^{\star}\,(n+\eta^{\star})}{2B\,(n+\eta^{\star}+1)} > 0.$$

Quando o preço é elevado ou a diferença entre o preço e o custo marginal individual for elevado, a receita gerada pela jazida será maior. O nível de produção ótimo aumentará como também o nível de produção de monopólio. Mas a taxa de crescimento de r em relação a aumentos marginais de A ou p será inferior ao aumento de $q^{\star}$ em relação a aumentos marginais de A ou p. Ou seja, o nível de extração ótimo aumentará mais do que o nível de extração de monopólio, e assim gerando resultados mais ineficientes. A razão disto ocorrer é que a punição necessária para manter o nível de extração próximo do nível de monopólio deveria aumentar. Mas aumentando a punição, seja pelo aumento

do período de punição ou pela redução do gatilho, aumentaria a ineficiência da jazida, pois as firmas extrairiam no nível de Cournot com maior probabilidade. Quando se escolhe um nível de produção mais alto, permite que a punição seja mais branda, e assim o retorno esperado da jazida será maior.

A externalidade reduz o lucro de desviar do acordo e pode melhorar o nível de produção do acordo. Note que variações no coeficiente b não afeta η^o, e que

$$\frac{dq^\star}{dB} = -\frac{q^\star}{B} < 0$$

logo,

$$\frac{d\tilde{c}}{dB} = -\theta^\star n q^\star < 0.$$

Se a externalidade é alta entre as firmas, o nível de extração ótimo se aproxima do nível de extração de monopólio. No caso do monopólio haverá apenas aumento de custo, pois não há outras firmas explorando a jazida. Assim, a redução da produção não precisa ser tão severa. No caso do acordo ótimo, a externalidade reduz o ganho de desviar do acordo, funcionando como uma punição. Isto permite punições mais brandas e que o nível de extração se aproxime no nível de monopólio.

Vale observar que embora o custo fixo K não tenha influência explícita sobre acordo ótimo, implicitamente afeta todo o modelo. A medida que K aumenta, será necessário aumentar o nível de extração da jazida, gerando mais externalidade sobre as firmas. Isto diminui o intervalo dos níveis de extração Pareto superiores ao equilíbrio de Cournot. Se o custo fixo for excessivamente alto, o único equilíbrio trivialmente será não produzir.

Portanto, a solução interior do acordo ótimo é caracterizada pela seguinte proposição:

Proposition 9 *Seja* $(\tilde{c}^\star, T^\star)$ *o par da escolha ótima que maximiza o retorno da jazida, e* $q^\star$ *o nível de extração individual de equilíbrio. Suponha que* $1 < T^\star < \infty$ *e que* $\theta^\star \equiv \tilde{c}^\star / c_{mg}(nq^\star)$ *pertence ao suporte da distribuição* F*. Então a tripla* $(q^\star, \tilde{c}^\star, T^\star)$ *é determinada por*

$$q^\star = \begin{cases} \frac{A}{2Bn}\left[\frac{n+\eta^\star-(n+1)\frac{c}{A}\frac{b}{B}}{n+\eta^\star+1}\right], & se\ \eta^\star \leqslant \eta^o, \\ s, & c.c.; \end{cases} \tag{2.19}$$

$$-\frac{f(\theta^\star)}{1-F(\theta^\star)}\theta^\star - \frac{f'(\theta^\star)}{f(\theta^\star)}\theta^\star = 1; \tag{2.20}$$

e

$$T^\star = \frac{1}{\ln\beta}\ln\left\{\beta - \frac{(1-\beta)(1-F(\theta^\star))\left[A - B(n+1)q^\star\right]}{f(\theta^\star)\frac{b\theta^\star}{c^\star}\Delta - (1-F(\theta^\star))\left[A-B(n+1)q^\star\right]}\right\}, \tag{2.21}$$

onde $c^\star = c_{mg}(nq^\star)$, $\Delta \equiv \pi_i\left(\overline{q}^\star\right) - \pi_i\left(\overline{s}\right)$ *e* $\eta^o \equiv -\frac{n+1}{n-1}\left[n + (n+1)\frac{c}{A}\frac{b}{B}\right]$.

Ao longo desta subseção foram examinadas as soluções interiores do acordo de cooperação ótimo. Contudo, os valores ótimos para $\tilde{c}$ ou T ou ambos podem não ser interiores. De acordo com Porter (1983b), é possível mostrar que $\tilde{c}$ interior é uma condição necessária para que exista equilíbrios simétricos mais eficientes em relação ao nível de produção de Cournot. Isto significa que, nesta modificação, o gatilho de custo $\tilde{c}$ nulo ou infinito implicaria que somente equilíbrios de Cournot seriam observados. Note que um gatilho nulo significa que qualquer nível de produção positivo resultaria em períodos de punição, logo não é possível suportar equilíbrios mais eficientes neste caso. Um gatilho infinito significa que para qualquer nível de produção finita, o gatilho não seria atingido. Sendo assim, as firmas voltam para o caso sem punição e portanto não haveria incentivo para as firmas cooperarem.

Por fim, Porter (1983b) examina as soluções interiores para T. Primeiro note que $T > 1$, pois caso $T = 1$ significa que as firmas não seriam punidas, uma vez que o período de punição é dado por $T - 1$. Logo a solução do modelo seria o nível de produção de Cournot. Desta forma, dentro das soluções não interiores, $T^\star$ infinito e $\theta^\star = \tilde{c}^\star/c^\star$ pertencer ao suporte de F são condições necessárias para existirem equilíbrios simétricos mais eficientes que o equilíbrio Cournot. Isto significa que o cartel deixa de existir a partir do momento em que o gatilho for excedido e que, de acordo com Porter (1983b), a expectativa de vida do cartel é dada por $F(\theta^\star)^{-1}$, função que depende apenas do número de firmas e dos parâmetros da distribuição. Contudo, o autor comenta que algumas funções de distribuição não suportam equilíbrios simétricos.

Na subseção a seguir, apresentamos um exemplo para o caso das soluções serem interiores.

2.3.2 Exemplo com a distribuição de Pareto

Uma distribuição que satisfaz a propriedade $\eta + 1 \leqslant 0$ é a distribuição de Pareto. Esta distribuição é caracterizada por dois parâmetros: $\alpha > 0$ que está relacionada com a curvatura da função distribuição e θ_m que é a moda da distribuição. A função de distribuição é dada por

$$F(\theta;\alpha,\theta_m) = \begin{cases} 1 - \left(\frac{\theta_m}{\theta}\right)^\alpha, & \text{se } \theta \geqslant \theta_m, \\ 0, & \text{caso contrário.} \end{cases} \tag{2.22}$$

Assim, temos que

$$f(\theta) = -\frac{\alpha}{\theta}\left(\frac{\theta_m}{\theta}\right)^\alpha,\ f'(\theta) = -\frac{\alpha(\alpha+1)}{\theta^2}\left(\frac{\theta_m}{\theta}\right)^\alpha,\ \mathbb{E}[\theta] = \frac{\alpha\theta_m}{\alpha-1} \text{ e } \mathrm{Var}[\theta] = \frac{\alpha\theta_m^2}{(\alpha-2)(\alpha-1)^2}.$$

A escolha desta distribuição decorre da facilidade para resolver o sistema de equações da proposição 9, pois η é constante, no suporte da distribuição:

$$\eta = \frac{f'(\theta)\,\theta}{f(\theta)} + 1 = \frac{-f(\theta)\,\theta}{1 - F(\theta)} = -\alpha$$

Usando a proposição 9 podemos encontrar o nível de extração ótimo,

$$q^\star = \begin{cases} \frac{A}{2Bn}\left[\frac{n-\alpha-(n+1)\frac{c}{A}\frac{b}{B}}{n-\alpha+1}\right], & \text{se } \alpha > \alpha^o, \\ \frac{A}{B(n+1)}, & \text{se } \alpha \leqslant \alpha^o, \end{cases}$$

onde $\alpha^o = \frac{n+1}{n-1}\left[n + (n+1)\frac{c}{A}\frac{b}{B}\right]$.

Neste exemplo podemos reproduzir os resultados obtidos. Note que α^o é uma função crescente em n, logo η^o é decrescente em n. Quando n tende para infinito, teremos que α^o vai para infinito também, e η^o vai para menos infinito. Logo, $\lim_{n\to\infty} nq^* = \lim_{n\to\infty} ns = \frac{A}{B}$. Quando α tende para o infinito, $q^\star$ se aproxima de r. Note que neste caso $\mathrm{Var}(\theta) = 0$, e retornamos para o ambiente sem incerteza. Adicionalmente, quanto menor for α, maior o ruído sobre o modelo, e como consequência $q^\star$ se aproxima de s. Logo, a elasticidade da função de distribuição F em relação a θ tem um papel importante sobre a eficiência do modelo.

Para ilustração, considere o caso em que $p = 5$, $c = 1$, $b = 1/2$, $K = 0$, e $n = 3$. Os parâmetros da distribuição são dados por $\alpha = 15$ e $\theta_m = 14/15$, logo $\mu = 1$. Logo, temos que $A = 4$, $B = 1/2$, $\alpha^o = 8$ e os níveis de extração e custo marginais serão $r = 4/3$, $c_{mg}(nr) = 3$, $s = 2$ e $c_{mg}(ns) = 4$. O nível de extração ótimo será dado por

$$q^\star = \begin{cases} \frac{4}{3}\frac{(2-\alpha)}{(4-\alpha)}, & \text{se } \alpha > \alpha^o, \\ 2, & \text{se } \alpha \leqslant \alpha^o, \end{cases}$$

e o custo marginal de extração ótimo será dado por

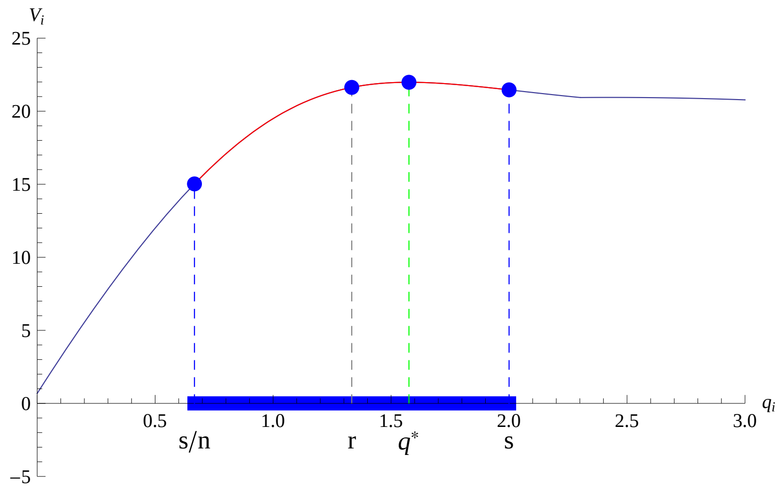
$$c_{mg}(\overline{q}^\star) = \begin{cases} 1 + 2\frac{(2-\alpha)}{(4-\alpha)}, & \text{se } \alpha > \alpha^o, \\ 4, & \text{se } \alpha \leqslant \alpha^o. \end{cases}$$

O gatilho ótimo terá um grau de liberdade, pois se ajustará de acordo com a escolha de T, e será dado por

$$\tilde{c}^\star = \theta_m \left[1 + 2\left(\frac{2-\alpha}{4-\alpha}\right)\right]\left(\frac{\beta - \beta^T}{1-\beta}\frac{5\alpha\,(4-\alpha)}{8\,(8-3\alpha)}\right)^{\frac{1}{\alpha}}.$$

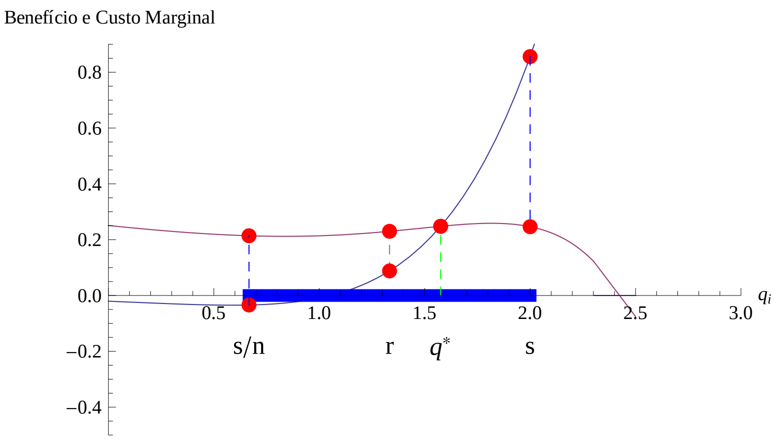
Neste exemplo é possível observar que a função valor será côncava no intervalo de interesse $(s/n, s]$, como é mostrado na figura 2.1. Isto garante que é possível encontrar um máximo, quando todas as outras firmas estão extraindo no equilíbrio de Nash.

Figure 2.1: Função Valor V_i



A figura 2.1 é a representação gráfica da função V_i quando as firmas $j \neq i$ estão extraindo no nível estabelecido pelo acordo ótimo, $q^\star$. O intervalo que é possível suportar equilíbrio, $(s/n, s]$, está representado pela região azul no eixo da abscissa. Fica claro que se a firma i produzir no nível do acordo ótimo, $q^\star$, estará maximizando seu retorno. Neste exemplo, o retorno do monopólio, dividido entre n firmas seria igual a 26,6667, com extração agregada igual a 4 a cada período, e o retorno no equilíbrio de Cournot seria igual a 20, com extração agregada igual a 6 a cada período. Quando as firmas usam o acordo ótimo, o retorno é igual a 21,9798 e a extração agregada de cada período é 4,72727. Pela simetria das firmas, o nível de extração $q^\star$ também maximiza o retorno da jazida no ambiente de Green e Porter.

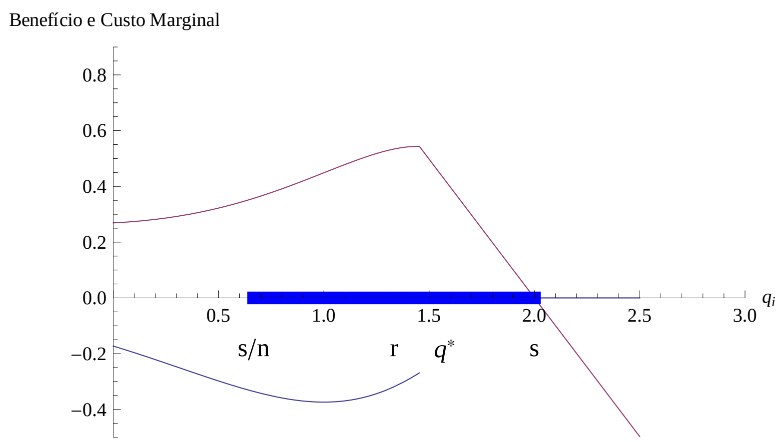
Figure 2.2: Condição de Primeira Ordem da firma i quando $j \neq i$ jogam $q^\star$



O comportamento da firma i pode ser avaliado na figura 2.2. A figura mostra o

benefício marginal e o custo marginal da firma i, quando todas as demais firmas estão extraindo no nível estabelecido pelo acordo ótimo, $q^\star$. A linha azul representa o custo marginal dado pelo lado direito da condição de primeira ordem da firma i, na equação (2.9), e a linha roxa representa o benefício marginal dado pelo lado esquerdo da mesma equação. A linha azul não é contínua, pois a medida que q_i aumenta, a razão $\tilde{c}^\star/c_{mg}(nq^\star)$ diminui, chegando a um ponto em que fica fora do suporte da distribuição de Pareto. A partir deste ponto, a a função será igual a zero. A área delimitada entre as curvas do ponto $q^\star$ até s é menor que a área entre as curvas do ponto s até o ponto em que ambas se cruzam. Mais precisamente, a integral da diferença do benefício marginal e custo marginal com limite superior onde as curvas se cruzam e limite inferior igual a $q^\star$, é igual a $-0{,}413033$. Ou seja, é ótimo para firma i produzir no nível $q^\star$, quanto todas as demais firmas produzem no nível $q^\star$. Logo $\overline{q}^\star$ é um equilíbrio de Nash.

Figure 2.3: Condição de Primeira Ordem da firma i quando $j \neq i$ jogam s



Da mesma forma, se todas as firmas $j \neq i$ produzem no nível s, é ótimo a firma produzir no nível s, como mostra a figura 2.3, indicando o outro equilíbrio de Nash, $\overline{s}$, embora este seja Pareto inferior ao equilíbrio do acordo ótimo.

2.4 Conclusão

Este trabalho examinou como o ambiente influencia a viabilidade de um acordo de unitização sem a intervenção governamental. O ganho agregado potencial da cooperação é superior ao retorno da extração pela regra da captura. Contudo Wiggins and Libecap (1985) e Libecap and Wiggins (1985) mostram a dificuldade do contrato ocorrer com sucesso. Dentre os motivos desta dificuldade, figuram a incerteza geológica e a assimetria de informação inerente à indústria de hidrocarbonetos.

A unitização pode ser tratada como formação de um cartel, no qual a cooperação das firmas envolvidas pode gerar resultados mais eficientes. Neste sentido, Deilami (1991)

desenvolveu um dos primeiros modelos teóricos aplicado a unitização. Porém, este modelo é tratado num ambiente estático, que reduz o espaço dos contratos compatíveis com incentivo. Quando consideramos um modelo dinâmico temos maior espaço para incentivar o conluio usando punições ao longo do tempo. Esta foi a motivação desta aplicação do modelo de Green e Porter. Um modelo clássico de formação de cartel, que considera a incerteza e a assimetria de informação foi desenvolvido por Porter (1983b). Ele analisa da viabilidade da cooperação neste ambiente. Desta forma, a reinterpretação deste modelo para o caso da unitização é oportuna.

Este trabalho apresentou a modificação do modelo de Green e Porter para verificar a viabilidade de cooperação na exploração de recursos de propriedade comum. A incerteza geológica foi representada por um componente estocástico do custo de produção. O custo de produção varia de acordo com a produção das outras firmas. Neste sentido, a incerteza geológica está representada pelo coeficiente estocástico de externalidade do reservatório. Mesmo que o custo seja comum entre as firmas, uma elevação no custo pode ocorrer tanto por uma realização alta como por um desvio unilateral de alguma firma. Assim, a incerteza está induzindo à incompletude de informação entre as firmas.

O acordo ótimo entre as firmas foi caracterizado neste ambiente usando o espaço das estratégias de Green e Porter. Esta estratégia é definida por um gatilho, representado pelo custo marginal comum entre as firmas, e por um período de punição. Quando o custo exceder um valor predeterminado, iniciará o período de punição. Durante este período, as firmas voltam a competir sob a regra da captura, reduzindo assim o retorno da exploração.

O contrato ótimo permite que a exploração do reservatório seja feita de forma mais eficiente do que a regra da captura, representado pelo equilíbrio de Cournot. A partir da estática comparativa, foi avaliado como a incerteza, o número de firmas, o preço e o coeficiente de externalidade influenciam a viabilidade do acordo e a eficiência da produção. A incerteza dificulta a cooperação entre as firmas, como Wiggins and Libecap (1985) descrevem. Quanto maior o ruído sobre o custo, menos informação uma firma terá sobre as demais firmas. Desta forma, estabelecer o gatilho correto se torna inviável e o resultado será a competição pela regra da captura. Pode-se observar o mesmo resultado quando o número de firmas é elevado. O ganho da cooperação não será suficientemente alto para que punição tenha eficácia. Este resultado difere de Deilami (1991), que depende de um grande número de firmas para obter os resultados analíticos.

Preços elevados facilitam o acordo entre as firmas, pois o retorno esperado da cooperação será maior, assim como ocorre em Deilami (1991). Preços elevados facilitam o acordo entre as firmas, pois o retorno esperado da cooperação será maior, assim como ocorre em Deilami (1991). Porém, o nível de produção será menos eficiente, uma vez que aumentos no preço elevam o retorno da regra da captura. Para evitar uma punição

severa, que implicaria em aumento na duração da punição e menor eficiência, o nível de produção ótimo é elevado. Assim, o retorno de um desvio será menor e a punição mais branda. Embora o coeficiente de externalidade não influencie diretamente a viabilidade de um acordo, coeficientes elevados funcionam como uma punição pela não cooperação. Desta forma, coeficientes elevados permitem níveis de produção mais eficientes, devido ao ganho da cooperação ser superior. Mas se o coeficiente de externalidade for muito elevado, embora a viabilidade do contrato não varie explicitamente com este termo, pode ser que não haja equilíbrio Pareto superior ao equilíbrio de Cournot, ou mesmo à um equilíbrio trivial com produção nula. O mesmo ocorre com o custo fixo neste modelo.

Por fim, cabe ressaltar que o modelo usa hipóteses pouco realistas. A primeira está relacionada com o custo de produção e o tempo de produção. Neste modelo, a firmas produzem por um tempo infinito, pois o custo de extração não aumenta com o tempo. Mas esta hipótese surge da necessidade do modelo usar a estacionaridade problema e da solução, como foi ressaltado na seção 2.3. Um extensão possível seria o uso de uma função de custo que cresce a medida que o estoque se reduz. Porém, esta extensão não é trivial, uma vez que há poucos modelos com esta propriedade mesmo dentro da literatura de organização industrial. Outra hipótese é a simetria entre as firmas, que foi usada para isolar o efeito da incerteza sobre a viabilidade do contrato. Uma outra extensão possível seria um modelo com duas firmas com tecnologia distintas. Assim, haveria tanto o efeito da incerteza quando a heterogeneidade entre as firmas. Porém, este modelo também não seria trivial, pois possivelmente — dependendo de como o choque ocorre a cada período — seria necessário usar métodos de programação dinâmica estocástica, o que torna a resolução do modelo tecnicamente mais difícil. Por fim, este modelo trata de ambientes sem regulação. Uma extensão natural possível seria colocar o papel de um regulador com poder coercitivo, capaz de interromper a produção na eventualidade de uma deserção. Este modelo poderia reproduzir com um pouco de mais realidade a unitização no caso brasileiro.

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Apêndice A - Condições de primeira ordem do contrato ótimo

Considere o problema de escolha do gatilho ótimo e período de punição, i.e.

$$\max_{\tilde{c}>0,\ T>1} V_i \left(q^{\star}\left(\tilde{c},T\right), \tilde{c}, T\right). \tag{2.23}$$

A condição necessária de primeira ordem do problema da escolha do par $(\tilde{c},T)$ ótimo, quando a solução é interior, é dada por

$$0 = \frac{dV_i\left(q^{\star}\left(\tilde{c},T\right);\tilde{c},T\right)}{d\tilde{c}}, \tag{2.24}$$

e

$$0 = \frac{dV_i\left(q^{\star}\left(\tilde{c},T\right);\tilde{c},T\right)}{dT}, \tag{2.25}$$

que é a mesma para cada firma $i = 1,...,n$, devido a simetria. Usando a simetria e a condição de primeira ordem (2.9) da firma i, podemos reescrever estas equações como

$$0 = (n-1)\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^{\star}}{\partial \tilde{c}} + \frac{\partial V_i}{\partial \tilde{c}}, \text{ para } j \neq i, \tag{2.26}$$

e

$$0 = (n-1)\frac{\partial V_i}{\partial q_j}\frac{\partial q_j^{\star}}{\partial T} + \frac{\partial V_i}{\partial T}, \text{ para } j \neq i. \tag{2.27}$$

Para resolver a condição de primeira ordem (2.27), usando a equação (2.8) da função valor V_i e a condição de primeira ordem (2.12) da firma i, e obtemos

$$\frac{\partial V_i}{\partial q_j} = \frac{-\left(A - Bnq^{\star}\right)}{1-\beta^T - \left(\beta - \beta^T\right)F}, \tag{2.28}$$

e

$$\frac{\partial V_i}{\partial T} = \frac{c^{\star 2}\left[A - B\left(n+1\right)q^{\star}\right]\beta^T \ln\beta\left(1-F\right)}{\left(\beta-\beta^T\right)f\left[1-\beta^T-\left(\beta-\beta^T\right)F\right]\tilde{c}b}, \tag{2.29}$$

Derivando implicitamente a condição de primeira ordem (2.12) da firma i, temos

$$\frac{\partial q^{\star}}{\partial T} = \frac{\beta^T \ln\beta c^{\star}\left\{f\tilde{c}b\Delta - \left(1-F\right)c^{\star 2}\left[A - B\left(n+1\right)q^{\star}\right]\right\}}{\tilde{c}fb\left(\beta-\beta^T\right)K}, \tag{2.30}$$

onde $c^{\star} = c_{mg}(nq^{\star})$,o argumento das funções F, f e f' é dado por $\frac{\tilde{c}}{c^{\star}}$ e

$$\Delta = \pi_i\left(\overline{q}^{\star}\right) - \pi_i\left(\overline{s}\right),$$

$$\eta = \frac{f'}{f}\frac{\tilde{c}}{c^{\star}} + 1,$$

e

$$K \equiv \frac{B(n+1)c^\star\Delta}{A - B(n+1)q^\star} - bn\Delta(\eta+1) - c^\star(n-1)(A - Bnq^\star).$$

Substituindo as equações (2.28), (2.29) e (2.30) na condição de ótimo (2.27) para T obtemos

$$-\frac{f}{1-F}\frac{\tilde{c}^\star}{c^\star} = \eta^\star = \frac{f'}{f}\frac{\tilde{c}^\star}{c^\star} + 1. \tag{2.31}$$

Da mesma forma, podemos resolver a condição de primeira ordem (2.26) usando a equação (2.8) da função valor V_i e a condição de primeira ordem (2.12) da firma i, para obter

$$\frac{\partial V_i}{\partial \tilde{c}} = \frac{c^\star[A - B(n+1)q^\star]}{\tilde{c}b[1 - \beta^T - (\beta - \beta^T)F]} \tag{2.32}$$

e derivando implicitamente a condição de primeira ordem (2.12) da firma i, temos

$$\frac{\partial q^\star}{\partial \tilde{c}} = \frac{-c^\star\{[A - B(n+1)q^\star]c^\star + \Delta b\eta\}}{\tilde{c}bK}. \tag{2.33}$$

Substituindo as equações (2.28), (2.32) e (2.33) na condição de ótimo (2.26) para $\tilde{c}^\star$ temos que

$$q^\star = \frac{A}{2Bn}\left[\frac{n + \eta^\star - (n+1)\frac{c}{A}\frac{b}{B}}{n + \eta^\star + 1}\right] = r\left[\frac{n + \eta^\star - (n+1)\frac{c}{A}\frac{b}{B}}{n + \eta^\star + 1}\right]. \tag{2.34}$$

Por último, usando a condição de primeira ordem (2.12) da firma i, podemos encontrar o período de punição ótimo, que é dado por

$$T^\star = \frac{1}{\ln\beta}\ln\left\{\beta - \frac{(1-\beta)(1-F(\theta^\star))[A - B(n+1)q^\star]}{f(\theta^\star)\frac{b\theta^\star}{c^\star}\Delta - (1-F(\theta^\star))[A - B(n+1)q^\star]}\right\}. \tag{2.35}$$

Chapter 3

Oilfield Unitization Under Dual Fiscal Regime: The Regulator Role over the Bargaining

Abstract

We study how the existence of dual fiscal regimes affects the unitization process. We consider identical firms operating under two different regulatory regimes. The negotiation of share and production plan is modeled as a bargaining problem and we solve it using the axiomatic solution proposed by Nash. We show that firms operating the leases that must share larger portion of its profit with the government end up being benefited in the negotiation. Furthermore, the regulator can reduce the distortion generated by the dual fiscal regime if the regulator is impartial with respect to government revenue and firms profits or worries more about firms than the government.

3.1 Introduction

In the oil industry, the government grant licenses for companies to search for commercially feasible deposits of petroleum. The geographical area defined by the government for the tender process is based on roughly estimates of possible oil leases. Due to the lack of precise information, sometimes two or more oil companies end up having access to the same reservoir. Without any regulation, whenever two or more companies have the access to the same resource, the overexploitation of it will occur as predicted by the common-pool problem literature. Under this so-called Rule of Capture, each company will try to extract as much oil as it can, before the other companies do the same. It also increases the production cost, since both companies have to make capital investments, creating two

structures to extract the resource. This is not optimal for the society due the duplication of costs.

A solution for this problem used by the industry is called unitization. This is an agreement in which it determines who will be responsible for extracting the resource, named operator, the production plan, and the profits will be shared. The agreement sometimes can reduce economic losses from the common-pool problem by 50%, according to Kim and Mahoney (2002). The negotiation between the parties can take a long time in general, due to the complexities involved on this agreement. However, sometimes this bargaining may breaks down and companies' decision will take into account the rule of capture to produce, which would be wasteful. Aware of it, a government could demand firms reach an agreement within a time limit and then offer a take-it-or-leave-it agreement. Under this scheme, firms would incorporate it in the game to threaten the other parties during the bargaining process.

Since companies take into account a possible sharing rule offered by the regulator, it is important to consider the role of legal framework under which companies produce. Johnston (1994) surveys fiscal regimes around the world and shows that some countries are adopting the production sharing agreements (PSA) in which the government remains the owner of the resources and only reimburses the cost of the oil companies and share a portion of the profit. The two special features of this contract are: (i) The government can interfere the production plan; (ii) The oil company share the profit with the government based on auction that the company won. The interference can occur, for example, when the bargaining process breaks down and the government offers a take-it-or-leave-it proposal.

In 2007, Brazil poised to become one of the top oil producers in the world after discovery a huge reserves in the pre-salt layer. The International Energy Agency (IEA, 2013) estimates the country's oil production will triple by 2035, turning Brazil into the world's sixth-largest oil producer, and the pre-salt layer will account for 52% of total domestic oil production by 2018, according to the report by Petrobras, Brazil's semi-public multinational energy corporation. It is estimated that the total recoverable oil and natural gas to be about 100 billion barrels of oil equivalent in the pre-salt area, which would triple the national reserves.

In response, the federal government considered necessary to amend the regulatory framework in order to create specific rules for the exploration and production of Petroleum in these areas. The new regulatory framework was proposed by the federal government in 2009 and approved by the Brazilian Congress in 2010, which considered it would afford the greatest benefit to the country and its people. The system adopted for the pre-salt reserves is the production sharing agreement. Now, Brazil have at least two fiscal regime for the

oil industry, and some of the reserves might be under these two regimes. According to the Brazilian law, firms operating contiguous leases with access to the same reserve must unitize their production, and the heterogeneous fiscal regime will affect the negotiation of the unit agreement.

Using the axiomatic solution for bargaining, we show that even if the firms are identical in terms of technology and costs, but operating under different fiscal regime, one of the firm will be benefited in the unitization process. This benefit is associated to a larger share that a firm receives compared to the profit that the firm is helping to generate. We consider two possible resolution for the bargaining, that is, two possible scenarios if the negotiation breaks-down. The first one is the rule of capture, where firms will compete for the resource, while the other one is the regulator proposal according to the Brazilian law.

In the next section, we will introduce the Brazilian Fiscal regimes. In Section 3.3 we survey a short literature about unitization. In Section 3.4 we present our environment of the model and the common-pool problem. In Section 3.5, we introduce the unitization as a bargaining problem and we solve it using the axiomatic solution propose by Nash. In Section 3.6 we study the two resolutions considered by this paper, and we conclude in Section 3.7.

3.2 Brazilian Fiscal Regimes

The fiscal regime of a country is a regulation that rules the exploration and production of hydrocarbons, hence establishing the rights, obligations and the division of revenues between the government and companies. Throughout the world, we can find many fiscal regimes relative to the oil exploration and production (E&P), but the two most common fiscal regimes are concession and production sharing agreement (PSA).

According to the Brazilian Federal Constitution, enacted in 1988, all national oil and natural gas reserves were public property and belonged to the Federal Government. The Union held a natural monopoly over research, exploration, production, refinement, transportation, importation, and exportation of oil and its by-products. Petrobras, the Brazilian National Oil Company, was the sole agent to explore the Union's monopoly.

In 1995, the Brazilian government reformed the oil and gas regulatory framework, and authorized the Union to contract any company incorporated and headquartered in Brazil to conduct the activities previously reserved only to Petrobras. It ended the oil and gas public monopoly in Brazil. The Oil Act, enacted in 1997 by the Congress, allowed competition in all segments of the oil and gas industry, and established that exploration and production of oil and gas in Brazil should be carried out under the concession regime.

In the concession regime, the exploration, development and production are conducted at the oil company's sole cost and risk. In return, the concessionaire is granted a contract that last up to 35 years, becoming the owner of the hydrocarbons extracted and having exclusive control of operations and holding both commercialization and exportation rights. The contracts are granted based on signature bonus and local content auction organized by the National Petroleum Agency (NPA). A signature bonus is the amount of money offered by the bidder to explore and produce the field, paid when the concession agreement is signed, and the local content represents the bidder's commitment to contract a minimum percentage of goods and services from Brazilian companies.

Therefore, according to The Oil Act and concession agreements, the concessionaires must pay (i) signature bonuses, which consist of a lump sum payment by the winning company(s) of the bidding procedure upon the signing of the concession agreement; (ii) royalties, which are defined by law as a financial compensation payable as a result of production of Petroleum; (iii) annual rental fees, which consist of the value paid annually, from the execution of the concession agreement and fixed per square kilometer or fraction of the block area; and (iv) special participation, which is defined as an extraordinary financial compensation payable only in cases of high volume of production or high profitability. The special participation is a tax over the net revenue of the concessionaire based on its location, onshore or offshore, and the production level. The two most important government take in this regime, along with signature bonus, is the royalties and special participation, in terms of total revenue. According to the Statistical Yearbook of NPA of 2010, the royalties and special participation accounts for 50.6% and 47.8%, on average from 2000 to 2009, of the revenue not taking into account the signature bonus.

In late 2006, Petrobras announced it had found evidence of ultra-deep-water oil reserves beneath the underground salt layer, the so-called pre-salt. This layer extends along the Brazilian outer continental shelf at about 300 kilometers from the shoreline. The prospect of huge reserves of hydrocarbons in the pre-salt with low exploratory risk[1] raised discussion of whether concessions are the appropriate legal regime to govern the development of oil and gas in these locations. In response to that, a new regulatory framework was designed to govern the exploration and production of the oil discovered in the pre-salt area, and in 2010, Brazil adopted the production sharing agreement regime.

In the PSA, differently from concession contracts, the ownership of the hydrocarbons is inverted, so instead of it being owned by the oil company, the extracted oil still belongs to the government. The contractor will, at its own risk, explore, develop and produce the oil, and in the case of any commercial discovery, such contractor is entitled to receive a share of the produced oil to reimburse the exploration and production cost, the so-called

[1]Although the exploratory risk is reduced by the findings from Petrobras, the companies still bears the development risk, which is considered very high in these areas.

cost oil. The remaining oil, the so-called profit oil, is shared between the government and the contractor according to the terms of the production sharing agreement, which is based on bid proposed by the winning contractor for the lease auction. Thus, under the PSA, the government take is paid through its share of profit oil as well the payment of a signature bonus and royalties.

The 2010 Federal law also established the Petróleo e Gás Natural S.A. (PPSA), a 100% government-owned company, to be responsible for managing the production sharing agreements and the sale of the government's share of oil output. The company represented the Union in the PSA and in the unitization process, but it has no responsibility over exploration and production activities. PPSA does not bear exploratory and development risk, but it has a mandatory participation in all consortia formed as well as the right to nominate the majority of the operational committee, including its president who has a tie-breaking vote. The Brazilian PSA also established the Petrobras as the sole operator of the oilfields, and it must have at least 30% of each field tendered. Petrobras may increase its participation in the oil field by participating with the other parties. Finally, when the reserve is under two different leases, possibly under two different fiscal regimes, it is mandatory to unitize the production in order to reduce the cost of production of the reserve, and as soon as one consortia finds its reserves goes beyond its area, it must notice NPA. If the parties involved in the unitization process do not reach an agreement before a deadline specified in the PSA, the NPA will propose an agreement, and in case it is not accept by the parties, their contract can be terminated.

As an example, the Table 3.1 presents the companies and government take under two different regimes. Under concession, it considers the royalty, denoted by r as 10%, while in PSA we consider 15%, which is following the concession and PSA laws. Under concession, we consider the special participation, denoted by SP, of 20%, and under PSA the profit share is 50%. We consider a tax of 25% and the CSLL of 9%, for this example.[2] It is clear that under PSA the government take is greater than under concession, which should be related to the lower exploratory risk that consortia would bear.

Consequently, Brazil now has a dual fiscal regime for hydrocarbons exploration and production: while the PSA governs exploration and production of the oil fields located in the pre-salt area as well as those considered strategic by a presidential decree, the concession regime governs most of the other fields. However, since it is not possible to delimit the areas according to the reserve, there will be situation in which companies have to unitize their production, even operating under different regimes. This will be the focus of this paper, to study how the dual fiscal regime will affect the negotiation of the unitization.

[2] CSLL is the abbreviation for Contribuição Social sobre o Lucro Líquido (Social Contribution on Net Profits). It is one of the contributions destined to finance the Brazilian social security system.

Table 3.1: Company and government take example

Regime	Concession (r = 10% and SP = 20%)	PSA (r = 15% and Share = 50%)
Gross Revenue	100	100
Royalties	10	15
Costs	30	30
Net Revenue	60	55
SP	12	0
Profit Oil	0	27.5
Firm Gross Profit	48	27.5
Tax	12	6.875
CSLL	4.32	2.47
Firms Net Profit	31.68	18.15
FNP/(Rev-Cost)	45.3%	25.9%
Gov. Take	38.32	51.85
GT/(Rev-Cost)	54,74%	74.1%

3.3 Literature

The US oil industry flourished after 1890s and started an oil rush in many states in the country. In most of the states, the ownership of the underground resources was initially given to the company that extracted the oil, the so-called rule of capture. A landowner neighbor of a land in which the oil was found had incentives to rent his land to an oil company, which would extract the oil probably from the same oil reserve. This competition and the decreasing crude oil prices experienced until 1970 became a concern for the industry, and a topic for economist to discuss.

Gary Libecap and Steven Wiggins studied the competition in the industry and its rent dissipation, among others, during the 1980s. Libecap and Wiggins (1984) show an estimate from Federal Oil Conservation Board that the estimated recovery rate was only $20 - 25\%$ under competitive extraction, while under controlled withdrawal could be $85 - 90\%$. As firm compete for migratory oil and gas, they dissipate reservoir rents with excessive capital, too rapid production, and lost total recovery. However, with a complete unitization agreement among the producers, a single firm is designated as the unit operator to develop the entire reservoir. The gains from an agreement can be huge both from savings in capital costs and from increasing the overall production that can be from two to five times unregulated output.

Even though there is a huge rent dissipation due to the rule of capture, studies

showed that by 1947, only 12 in 3,000 fields in the US were fully unitized. Libecap and Wiggins (1984, 1985a) considered three possible causes for this failure of reaching an agreement. The first one is the concentration in the field, as the more firms have access to the same tract, the harder will be to reach a mutual interest agreement. Another reason, related to the previous one, is the heterogeneity of firms' interest and information. As most of the unitization process started during development phase after the commercial petroleum deposits have been found, the information about firms' tracts and their estimated about the value of it diverges, making it harder for parties reach an agreement.

In a comparative analysis, Libecap and Wiggins (1985b) evaluated how different regulatory policies affects the unitization process in the US. They chose three US states that have different regulatory policies with high production level. In Wyoming was observed high level of success in achieving unitization agreements. In this state, most of the oil field are federal land, where the policy encourages early exploratory units, before the commercial petroleum deposits have been found. It allows for large potential gains from unitization, and ease the bargaining process since the information about the deposit is more homogeneous. The Brazilian regulatory framework adopted the same principle, where firms must notice NPA as soon as it is found that the tract extends beyond their lease. Another incentives provided by the federal policy in the US is that firms are granted the area for up to 20 years, but if a lease become unitized, the leases are automatically extended for the life of the unit. In the other states, the unitization typically starts in late field development, and it was observed lower number of successful unitization agreements.[3]

More recently, Libecap and Smith (1999, 2001) shows that if the field contains two (or more) substances that differ in kind (like oil and gas), then it is possible that different forms of non-unitized ownership and operation (with conflicted production incentives) may dominate unitized development of the resource.[4] According to them, unitization may not be a Pareto improving in situations where the tract contains oil and gas, due to different interests about timing and forms of extraction needed. Mohan and Goorha (2008) showed that the nature of contractual incompleteness of the unitization could also provide wrong incentives to firms, resulting in an under-investment during the exploratory phase.

In a setting more closely related to ours, Smith (1987) solves an axiomatic bargaining, and shows that small-interests holders are benefited in the unitization when the alternative solution is the rule of capture. This idea follows Libecap and Wiggins (1984, 1985a) hypothesis, where it was shown that small firms are reluctant to enter agreements, expecting that large firms offer concession for the small firms. Smith argues that, if arbi-

[3]For an extensive comparative analysis of country regulation about unitization, see Weaver and Asmus (2006).

[4]For an extensive review about what involve the unitization contracts, see Libecap and Smith (1999).

tration were possible, and neutral with respect to distribution of the profit, large-interests firms would issue a threat to invoke arbitration, if the cost for it is not sufficiently high. Differently from our model, the firms' difference is exogenous; he does not consider the fiscal regime the firms are operating. Thus, our model shows explicitly the effect of the fiscal regime over the bargaining position of the firms involved in a unitization process. Also, we show the role of a regulator that consider both government take and firms profit.[5]

3.4 The Environment

We consider two firms in this model, which are operating over their granted land to explore and develop oil, but the oil extracted by the firms comes from the a single reservoir underneath their areas. In our model, the average production cost is represented by the function $AC_i(q_1,q_2)$, and it satisfies the following assumption of externality, leading to the common-pool problem:

Assumption 1 $AC_i(q_1,q_2) = cq_i + b(q_1 + q_2)$ *and* $\frac{\partial AC_i}{\partial q_{-i}} = b > 0$.

Thus, the extraction by one of the firms leads to an increase in the production cost of the other firm. We suppose the price of the produced oil is given by p, thus the gross profit function for a firm i is

$$\tilde{\pi}_i(q_1,q_2) = \left[p - AC_i(q_1,q_2)\right] q_i. \tag{3.1}$$

Each firm may be operating under different fiscal regime. We consider two possible types of contract that can be used by the government to lease areas for exploration and development of oil. The fiscal regime determines the economic relation between firms and government, and in particular, what is the government take on the total output revenue. The first type of contract we describe is the PSA, described in Section 3.2. Under this contract, the profit from the production is shared with the government according to the parameter λ_i, established during the auction, and the firm still needs to pay royalties to the government. The (net) profit function is

$$\pi_i(q_1,q_2) = (1 - \lambda_i)\left[\tilde{\pi}_i(q_1,q_2) - r_i q_i\right], \tag{3.2}$$

and, as we assume there is no tax, thus the government revenue is given by

[5] Regarding the PSA contracts, Aghion and Quesada (2010) study the moral hazard and risk sharing problem that affects the production under different contracts, where they consider that government may change the agreement to increase its take, like what happened in Venezuela and Bolivia, and thus companies anticipate this possibility of holdup and under-invest. For a survey about the PSA fiscal regime, see Johnston (1994), and for a comparison of the legal framework between Brazil and the US, see Pinto (2013).

$$\psi(q_1,q_2) = \lambda_i \left[\tilde{\pi}_i(q_1,q_2) - rq_i\right] + rq_i. \tag{3.3}$$

The other type of fiscal regime is a concession contract, which grants the lease to a firm in exchange for a fixed payment in advance and requires the firm to pay the royalties based on the production. The advanced payment is sunk for the firm at the exploratory and development phase, so the production level takes into account only the royalty, along with price and production cost. We consider royalties as a function of the production level. Therefore, the (net) profit function and government revenue can be represented by Equations (3.2) and (3.3) when λ_i is zero.

In order to easy the computation, we will also consider that firms are identical, so $c_1 = c_2 = c$, and the royalty will be the same under both regimes, based on the production of the firms, thus $r_1 = r_2 = r$. By doing it, we are isolating the problem of the fiscal regime, excluding the heterogeneity of the firms and evaluating only the impact of the existence of two fiscal regimes over the unitization process. In addition, throughout this paper, we will consider only interior solution. In the next subsection we will show the common-pool problem under these fiscal regimes.

3.4.1 The common-pool problem

In this section we introduce the common-pool problem existing in this setting. In order to do so, we examine two benchmark cases: (i) when firms act non-cooperatively by strategically choosing a production level seeking to maximize his own profit; and (ii) firms act as one and seeks to maximize the joint profit.

First, let us consider what would happen if both firms were acting according to the rule of capture. It establishes that the ownership of the resource extracted is given to firm responsible for extracting it. This allocation rule induces firms to act strategically, seeking to extract as much as possible without considering its negative effects over the other firm. A simple solution concept that captures this behavior is the Nash equilibrium, which will represent the case when firms are competing against each other when the allocation rule is determined by the rule of capture. Henceforth, this solution concept will be called as rule of capture solution.

Firms seeking to maximize their profit function (3.2) do not take into account the fiscal regime λ_i, as it is just a fixed proportion of its profit. On the other hand, the royalty r acts as an additional marginal cost for the firm, thus reducing the output level of both firms. As we consider both firms identical in terms of cost structure, the rule of capture solution involves both firms choosing the same output level denoted by q^N, where N indicates the Nash equilibrium concept. The externality introduced in the previous

section reduces the total profit that firms could make if they behave as a single firm operating in two different fiscal regimes, as they would be internalizing the externality of the setting, as we can see in the following proposition.

Proposition 10 *The total profit under the rule of capture is strictly less than the profit made by a single firm operating two distinct areas.*

Proof. Consider the problem of a single firm operating two areas under different fiscal regime, which is given by

$$\max_{q_1,q_2} \sum_{i=1,2} (1-\lambda_i)\left[\tilde{\pi}_i(q_1,q_2) - rq_i\right]. \tag{3.4}$$

We denote q_i^* the solution for this problem and $Q^* = q_1^* + q_2^*$. The first order condition for this problem implies that the total marginal profit is greater than zero, while summing the first order condition of each firm from the rule of capture solution, using the same notation $Q^N = q_1^N + q_2^N$, it implies

$$p - cQ^N - b\frac{Q^N}{2} - bQ^N = 0 < p - cQ^* - b\frac{Q^*}{2} - bQ^*. \tag{3.5}$$

The inequality above implies that the total production under a single firm operation is less than under a competitive solution, i.e. $Q^N > Q^*$. Since the cost function is strictly convex, and prices are fixed, thus the total output from competition is strictly less than the profit made by a single firm operating both areas. ■

The common-pool problem occurs when a firm does not take into account the negative externality the others firms will bear due to its production. In the oil industry, this would mean that firms should reduce its production, since the costs become higher due to competition, as they would have to introduce more costly ways of extraction. In our static model, we are not considering the stock level and the cost associated to it, but the competition induces firms to produce more at higher costs, and thus the profit reduces.

The total profit loss due to competition could be internalize if firms cooperate and then split the profit between them. However, it is not a trivial problem to decide the proportion of the profit that each player will receive. In the next section we solve this problem using the Nash's Bargaining solution.

3.5 The Bargaining Problem

Consider a model in which two firms found that they have access to the same oil reservoir. Due to the common-pool problem, competition would reduce the total profit that would be possible to achieve if they cooperate. However, negotiating a cooperation agreement

involves attrition about sharing rule for the profit. According to Osborne and Rubinstein (1990), this is a typical bargaining situation, because cooperating is mutually beneficial and there is a conflict of interest about which agreement to conclude.

We are interested in studying how the dual regimes affects the negotiation between firms. Thus, we model the unitization process as a Nash solution to a bargaining game between both firms over the agreement that includes the production plan (q_1,q_2), and a sharing rule $\alpha \in [0,1]$, where α represents the joint-profit portion given to firm 1. Suppose the profit function of each firm is denoted by $\pi_i(q_1,q_2)$, so the joint-profit is $\Pi(q_1,q_2) = \sum_i \pi_i(q_1,q_2)$. As it was discussed above, sometimes the bargaining process breaks down, so the firms would compete for the resource or the government could offer a take-it-or-leave-it agreement. We will analyze both resolution on Section 3.6. For the moment, these threat points are denoted by $\overline{\pi}_i$. Nash (1950) showed that we can solve this problem by the following maximization

$$\max_{(\alpha,q_1,q_2)\in S} \left\{\alpha\Pi(q_1,q_2) - \overline{\pi}_1\right\} \cdot \left\{(1-\alpha)\Pi(q_1,q_2) - \overline{\pi}_2\right\} \tag{BP}$$

where $S = \{(\alpha,q_1,q_2) : \alpha \in [0,1] \text{ and } q_i \in \mathbb{R}_+ \text{ for } i = 1,2\}$ is the feasible choice set.

Figure 3.1 shows a representation of the stake of the bargaining. Both vertical and horizontal axis represents the profit of each firm, in particular, the vertical axis represents firm 1 profit, and the horizontal axis represents firm 2 profit. The green line represents the profit possibility frontier, which is found by solving the following problem, for each possible π_2,

$$\max_{q_1,q_2} \quad \pi_1(q_1,q_2) \text{ s.t.} \pi_2(q_1,q_2) = \pi_2.$$

The point $D = (\overline{\pi}_1,\overline{\pi}_2)$ represents the threat point of each firm, and it is given by the outside option that firms have. For example, a firm could prefer not to agree to any deal proposed, and decide to compete for the resource according to the rule of capture, if it is possible. Thus, the profit this firm would have is equal to the rule of capture profit level. However, due to the common-pool problem, we can see that there exist deals that can improve both firms payoff, represented in the blue area.

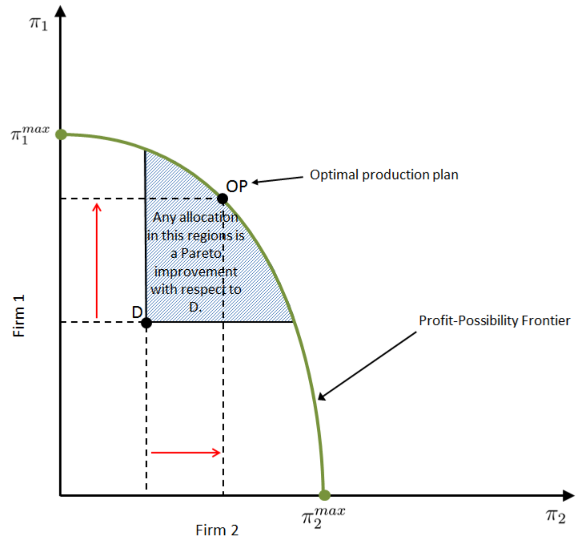

Figure 3.1: Optimal production plan

According to the axiomatic bargaining proposed by Nash, we know that the solution must be Pareto optimal, otherwise there would still be a room for a costless negotiation that could improve both players. Since both firms are splitting the profit generated, it would be optimal for them when they choose the production plan (q_1,q_2) such that maximize the total profit $\Pi(q_1,q_2)$. Let $q^* = (q_1^*,q_2^*)$ be the optimal production plan when players cooperate and produce at the quantity that maximize the total profit $\Pi(q)$. Therefore, under a unitization we can solve the problem of common pool, since only one of the firms will explore the reservoir, and in order to maximize his payoff, he must choose the optimal production, which will generate the highest payoff for both firms.

Let $\Pi^* = \Pi(q_1^*,q_2^*)$, thus the unitization problem can be reduced to the problem of splitting the profit, i.e.

$$\max_{\alpha} \{\alpha\Pi^* - \overline{\pi}_1\} \cdot \{(1-\alpha)\,\Pi^* - \overline{\pi}_2\}\,. \tag{BP'}$$

Now, the problem of how the firms actually share profit shows the importance of the threat points, as we can see it clearly from the first order condition:

$$\overline{\alpha} = \frac{1}{2} + \frac{\overline{\pi}_1 - \overline{\pi}_2}{2\Pi^*} \quad \text{and} \quad 1 - \overline{\alpha} = \frac{1}{2} + \frac{\overline{\pi}_2 - \overline{\pi}_1}{2\Pi^*} \tag{3.6}$$

From these equations, it is clear that whenever the threat point is the same for both firms, they will share the profit equally to each other, otherwise one of them will end up receiving a large share. Clearly, player i will have greater share when he generates more profit at the threat point.

In the next section we present two models of breakdowns we induces the threat point of the firms. Based on them, we study the solution of the bargaining problem introduced above.

3.6 Two Models of Negotiation Breakdown

The unitization process involves a complicated phase where companies work on the deal they want to reach. The firms negotiating the deal are usually aware of what the other company can do if an agreement is not reached, and they use it to get the best possible deal. This is the threat point considered by the Nash solution for a bargaining. As we saw in the previous section, the bargaining solution will depend on the threat point that comes from the solution of possible resolutions the game have. A natural threat point would be a non-cooperative solution, when firms compete for the oil and face the common-pool problem. The other option is when the regulator proposes an agreement and firms can accept it or not, following the idea of the Brazilian regulatory framework. If they do not accept, the government will terminate their contract and they will end up with no profit.

In this section, we will see how the fiscal regimes affect the unitization, first considering the benchmark case, where firms will compete for the resource according to the rule of capture. In this situation, both firms can improve their profit due to the common-pool problem by cooperating, the question is how the fiscal regimes will affect the bargaining of the shares and thus if it will involve any transfer between players.

3.6.1 Rule of capture resolution

The rule of capture states that the ownership of the resource extract from the nature belongs to the one who extracted it. In the past, in some countries like United States, the government did not interfere in the competition in the oil industry. This is our benchmark, as now we will consider the rule of capture as our status quo condition, in the sense that if firms cannot reach an agreement, they will compete according to the rule of capture. As we saw before, it can be viewed as the Nash equilibrium for the common-pool problem, which solution was called rule of capture solution. Thus, the threat point is given by $\overline{\pi}_i = \pi_i^N$. By using it as a threat point, it becomes an individual rationality condition, which is the minimum a firm must receive to accept a deal from the bargaining.

As seen in the previous section, the solution of the bargaining is such that a firm will receive a portion of the joint profit, $\alpha^N \Pi^*$, when the production plan is the one that maximize the joint profit, and α^* is a function of the threat points, given by Equation (3.6).

Proposition 11 *The firm that will receive larger shares is the one with lower government*

share.

Proof. The rule of capture solution can be found by solving the system of equations that induces the Nash equilibrium of the game, which is the following: The output choice for each firm i solves

$$q_i^N = \arg\max_{q_i}(1-\lambda_i)\left[\tilde{\pi}_i(q_1,q_2) - rq_i\right]. \tag{3.7}$$

It is clear that the output choice of each player does not depend on the fiscal regime λ_i, as it is a constant for each firm. However, the profit is a decreasing function of λ_i. So, suppose, without loss of generality, that $\lambda_2 > \lambda_1$, thus the firm 2 has lower profit than firm 1, that is, $\pi_1^N > \pi_2^N$, and finally, according to the Equation (3.6), $\alpha^N > 1/2$. Thus, the firm that shares large portion of the profit receives lower share of the total profit. ■

Although firm 2 received larger portion of the joint profit, it might be the case that he will receive less than his facilities usage generates as profit due to a transfer to firm 1 in terms of profit share. The role of the transfer is to insure the deal will be reach, and the firm with larger benefit from cooperation will be more willing to pay to the other firm.

We can see that the firm with higher threat point gets a larger share of the profit, but does this payoff is equal to the firm contribution for the total profit? Let the transfer be define as the amount of payoff a player receives in excess of his contribution to total profit. So, for example, the transfer received by the player 1 is given by $t_1^N = \alpha^N \Pi^* - \pi_1(q_1^*,q_2^*)$. When $t_1^N > 0$ means that the firm 2 is giving implicit transfer for firm 1 to be able to reach an agreement. This transfer occurs by giving by a larger share of the profit to one player in order to guarantee the deal. At the solution, this transfer is given by

$$t_1^N = \frac{\pi_2(q^*) - \pi_1(q^*)}{2} + \frac{\pi_1^N - \pi_2^N}{2}, \tag{3.8}$$

and $t_2^N = -t_1^N$. We can see the representation of the transfer in the Figure 3.2. In this figure, firm 2 is receiving a transfer from firm 1, after using the threat point as his bargaining position, since player 2 won't be benefited as much as player 1.

Note that the firm 1 will receive the transfer when $\pi_2(q^*) - \pi_2^N > \pi_1(q^*) - \pi_2^N$, that is, when his gain from cooperation is less than the other player's gain. The idea here is that the player with lower gain from cooperation will ask a transfer in order to agree with the terms of the deal. The cooperation generates more benefits for the other firm, hence he would be willing to transfer some of its benefit from cooperation and still be in a better situation than the status quo, where the regulator would impose them an agreement.

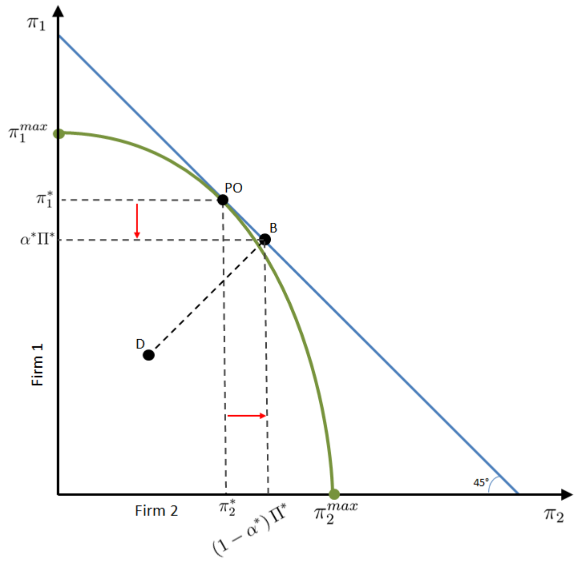

Figure 3.2: Transfer

Before studying who receives transfer, we need the following lemma. The problem for a single firm operating two areas with different fiscal regimes is also called cooperation problem, when two firms do not acts strategically to extract larger profit from the lease.

Lemma 4 *If a single firm were operating under two different fiscal regimes, the area responsible to produce more and to generate more profit for the firm is the one with lower government share.*

Proof. Without loss of generality, assume $\lambda_1 > \lambda_2$. First, let us show that $q_2^* > q_1^*$. The first order condition to solve problem (3.4) implies for each firm i the following equation

$$\frac{\partial \tilde{\pi}_i(q_1^*, q_2^*)}{\partial q_i} = r + \frac{1-\lambda_j}{1-\lambda_i} b q_j^*, \tag{3.9}$$

and using the Equation (3.1) we can compare both first order condition through the following equation

$$p - bQ^* - 2cq_1^* = \frac{1-\lambda_2}{1-\lambda_1}\left(p - bQ^* - 2cq_2^*\right). \tag{3.10}$$

By the assumption $\lambda_1 > \lambda_2$, we have $\frac{1-\lambda_2}{1-\lambda_1} > 1$, and thus it is clear that $q_2 > q_1$ by the Equation (3.10), proving the first part of this lemma. Now, let us prove that the plant on area 2 has greater profit than the facilities located on the area that shares larger portion of the profit with the government. In order to prove it, we assume again that $\lambda_1 > \lambda_2$, which implies that $\frac{1-\lambda_2}{1-\lambda_1} > 1 > \frac{1-\lambda_1}{1-\lambda_2}$ and $q_2 > q_1$. According to the first order condition state by Equation (3.9), we have the following inequalities

$$\frac{\partial \tilde{\pi}_1(q_1^*, q_2^*)}{\partial q_1} = r + \frac{1-\lambda_2}{1-\lambda_1} b q_2^* > r + bq_2^* > r + bq_1^* > r + \frac{1-\lambda_1}{1-\lambda_2} b q_1^* = \frac{\partial \tilde{\pi}_2(q_1^*, q_2^*)}{\partial q_2}, \tag{3.11}$$

and by concavity of the profit function, it implies that $\tilde{\pi}_1(q_1^*,q_2^*) < \tilde{\pi}_2(q_1^*,q_2^*)$, and thus $\pi_1(q_1^*,q_2^*) < \pi_2(q_1^*,q_2^*)$. ■

The production plan as a solution of the cooperation problem is affected by the fiscal regime. The choice is such that the firm that pays lower portion of the profit to the government will produce more, increasing its profit, while the other one will reduce the production, internalizing the externality, which is now a composition of the fiscal regimes and the externality parameter. The regimes makes each firms be affected by the externality with different intensity. The firm with larger λ_i will be less sensitive to the externality, as we see the fraction $(1-\lambda_i)/(1-\lambda_j)$. Note that in this problem, each firm has a weight given by $1-\lambda_i$, so, the firm with lower λ_i will have larger importance in the maximization problem, and so it would be responsible to generate larger profit, while the other firm should reduce its production in order to reduce the negative externality. Therefore, the firm with lower λ can increase the production and thus increase the profit, while having a minor negative effect from the other firm.

Now we can study which firm receives a transfer t_i^N. When the fiscal regime is the same, that is $\lambda_1 = \lambda_2$, at the rule of capture solution we have $\pi_1^N = \pi_2^N$, and at the cooperative solution, firms' profit will be the same, i.e. $\pi_1^* = \pi_2^*$. Thus, there will not be a transfer from one firm to another. However, when the fiscal regime is different, we have the following proposition.

Proposition 12 *The firm receiving transfer is the one with larger government share.*

Proof. Assume, without loss of generality, that $\lambda_1 > \lambda_2$. Since firms have identical cost functions, we have $\tilde{\pi}_1^N = \tilde{\pi}_2^N$, while according to the Lemma 4, we found that $\tilde{\pi}_1^* < \tilde{\pi}_2^*$. Using this two relations, we find the following inequalities

$$\tilde{\pi}_1^* - \tilde{\pi}_1^c < \tilde{\pi}_2^* - \tilde{\pi}_2^c < \frac{1-\lambda_2}{1-\lambda_1}\left(\tilde{\pi}_2^* - \tilde{\pi}_2^c\right),$$

and thus, rewriting it we find $\pi_2^* - \pi_2^N > \pi_1^* - \pi_2^N$, the condition which shows that $t_1^N > 0$. The inequality above holds, since $\tilde{\pi}_2^* - \tilde{\pi}_2^c$ is positive if $\lambda_1 > \lambda_2$. It cannot be negative, because it could not be the solution for the cooperation problem, as the firm 2 weights is greater than firm 1. ■

Therefore, although firm are identical, by operating under different fiscal regimes creates a distortion which one firm can use it strategically to gain more rent from the unit deal, in the sense that firm 1 will earn more that it is generating for the unit. In the next section, we will introduce the role of the regulator, following the Brazilian legal framework. He will be responsible to propose a deal whenever firms can't reach one. We will study how it affects the bargaining position and how firm use it strategically to gain more rent from the unit.

3.6.2 Regulator resolution

According to the Brazilian legal framework, during a unitization process firms will have a period to negotiate and try to reach an agreement, however after a deadline, the regulator will propose an agreement. This agent takes into account the government role and the firms' role. The government is interested in collecting revenue to finance his policies, revenue that is based on profit share and royalties from the firms. According to our model, the government utility is given by

$$\Psi(q) = \sum_i \lambda_i \left[\tilde{\pi}_i(q) - rq_i\right] + r\left(q_1 + q_2\right) \tag{3.12}$$

so the government has two revenue sources: (i) profit share from the PSA; (ii) and royalties. The profit share is aligned with firms' goal, which is to maximize their profit, however the royalty adds an extra burden to the firms. In some cases, the government would prefer firms to operate above the optimal production level in order to collect royalties and increase its utility.

The regulator role in this model is to consider both firms objective and government objective. A simple way to model it is by assigning weights for firms' profit and government's revenue, which we denote $\rho \in [0,1]$, where $\rho > 1/2$ means that the regulator cares more about revenues than profits, and otherwise for $\rho < 1/2$. Therefore, when $\rho = 0$, means that regulator just care about firms' profits, or firms' efficiency, by not considering the government's revenue. On the opposite way, for $\rho = 1$, the regulator is totally aligned to the government and thus it only worry about revenues. The regulator will be impartial when $\rho = 1/2$, since it assigns the same weight for firms' profits and government's revenues. Hence, we can write the regulator utility as

$$W(q) = \rho\Psi(q) + (1-\rho)\Pi(q). \tag{3.13}$$

We can rewrite it as a weighted average of firms profit and the royalty term, using the weight denoted by $\Lambda_i = \rho\lambda_i + (1-\rho)(1-\lambda_i)$, in which the first term is related to the government take given the regulator weight for it, while the second term is referred to the firms' profits. The regulator utility is now given by

$$W(q) = \sum_i \Lambda_i \left[\tilde{\pi}_i(q) - rq_i\right] + \rho r\left(q_1 + q_2\right). \tag{3.14}$$

First note that when $\rho = 1/2$, we have $\Lambda_i = 1/2$ and then the solution for the maximization of $W(q)$ would be the same as efficient solution, where firms operate under no regime, i.e. $\lambda_i = 0$ and $r = 0$. Also, in this equation we can see the efficiency trade-

off the regulator face, similarly to the government, in which increasing the production generates more utility to the regulator through higher level of government take. It is clear that when the regulator only carries about the firms, it would find the same solution as the cooperation's problem. However, increasing the importance of the government, higher level of production can generates more government's take, yielding higher benefit for the regulator. The first order condition for this problem implies

$$\frac{\partial \tilde{\pi}_i(q_1^R,q_2^R)}{\partial q_i} = (1-2\rho)\,\frac{1-\lambda_i}{\Lambda_i} r + \frac{\Lambda_j}{\Lambda_i} b q_j^R. \tag{3.15}$$

When $\rho = 1/2$, this equation would be equal to bq_j^R, which is the same condition for the efficient production, where firms cooperate and do not take into account the fiscal regime. Note that when $\rho = 0$, we have the same first order condition as the cooperation problem. The first term related to the royalties is decreasing in ρ, so the highest value it assume is 1, the same as in the cooperation problem. As the regulator increases the importance of the government revenue, this term decreases indicating he would be willing to increase the production, when we consider only the effect of the royalties. The second term is related to the externality and the fiscal regimes. If $\rho = 0$ or firms are in the same regime, this term is equal to the term in the cooperation first order condition. However, suppose $\lambda_2 > \lambda_1$, that is, firm 2 share a larger portion of its profit with the government, the fraction Λ_2/Λ_1 will be greater than $(1-\lambda_2)/(1-\lambda_1)$ and increasing in ρ. Thus, the optimal production level for firm 1 will be reduced compare to the cooperative solution, and this reduction will be amplified as ρ increases. On the other hand, firm 2 will increase its production.

Now, the firm that will produce the most will be determined by the regulator alignment ρ and the fiscal regime λ_i, which is shown in the following proposition.

Proposition 13 *A firm will produce more than the other if, and only if, it shares lower portion of its profit with the government and the regulator is more aligned with the firms, i.e.* $q_1^R > q_2^R$ *if, and only if* $\lambda_2 > \lambda_1$ *and* $\rho < 1/2$.

Proof. We can check that by subtracting the first order condition for q_1 and q_2, and analyzing the following equality

$$p - bQ^R - 2cq_1^R = \frac{\Lambda_2}{\Lambda_1}\left(p - bQ^R - 2cq_2^R\right), \tag{3.16}$$

as before, but now we are comparing Λ_1 and Λ_2. If $\rho < 1/2$ and $\lambda_2 > \lambda_1$, then $\Lambda_1 > \Lambda_2$, and thus $q_1^R > q_2^R$. ■

According to Equation (3.14), we saw that Λ_i is the weight for firm i in the weighted average that represented the regulator utility. Under the condition above, we know that

firm 1 will produce more and it will have greater weight than firm 2, so we can expect that firm 1 will generate greater profit level than firm 2, if the royalties are equal to zero.

Proposition 14 *Suppose there are no royalties, that is, $r = 0$. A firm will have larger profit under the regulator solution if, and only if, it shares lower portion of its profit with the government and the regulator is more aligned with the firms, i.e. $\pi_1^R > \pi_2^R$ if, and only if $\lambda_2 > \lambda_1$ and $\rho < 1/2$.*

Proof. We can check that using the same idea used before for the cooperation problem. More specifically, we can prove that at the interior solution for the regulator problem, $\tilde{\pi}_1(q^R) > \tilde{\pi}_2(q^R)$ if, and only if, $\rho < 1/2$ and $\lambda_2 > \lambda_1$. In order to check that, we can compare the first order solution, when $r = 0$, in which

$$\frac{\partial \tilde{\pi}_1(q_1^R, q_2^R)}{\partial q_1} = \frac{\Lambda_2}{\Lambda_1} b q_2^R > b q_2^R > b q_1^R > \frac{\Lambda_1}{\Lambda_2} b q_1^R = \frac{\partial \tilde{\pi}_2(q_1^R, q_2^R)}{\partial q_2}, \tag{3.17}$$

and again, using the concavity of $\tilde{\pi}$ we have $\pi_1^R > \pi_2^R$. ■

This proof can fail if $r > 0$ is sufficiently high, because we would not use the fact that the partial derivative with respect to q_i is positive, which certifies us that the solution is at the increasing part of the profit function. Note that here, the regulator are increasing the production of the firm have greater Λ_i, which is the firm i weight in the regulator utility. In addition, since the first order condition implies that the production level is at the increasing part of the profit function, when $r = 0$, that we can say that by increasing the quantity of firm i, the regulator is increasing its profit level.

Numerical analysis

Unfortunately, we cannot use the same idea to prove that $\pi_1(q^R) > \pi_2(q^R)$, because ρ can invert the inequality sign, and it does not allow us to prove the converse of the statement. In order to provide insights for this problem, we solved this regulator problem numerically and it is possible to identify parameter regions in which the profit of one firm will be greater than the other.

An illustration of this solution is plotted in the Figure 3.3 below. This figures are the contour plot of the difference of firm 1 profit relative to firm 2, i.e. $\pi_1(q^R) - \pi_2(q^R)$ when $\lambda_2 = 0.4$. The lighter color means that $\pi_1(q^R) > \pi_2(q^R)$, and on the other hand, the darker color means $\pi_1(q^R) < \pi_2(q^R)$. In the plot on the left, it was assumed $b = 3$, while in the plot on the right, it was assumed $b = 2$. As we can see, the upper right and the lower left corners of the figure are the regions of (λ_1, ρ) implies greater profit level for firm 1 than firm 2. The vertical line at $\lambda_1 = 0.4$ clear represents a condition to identify who earns more profit. However, the ρ will be a function of the parameters of the model,

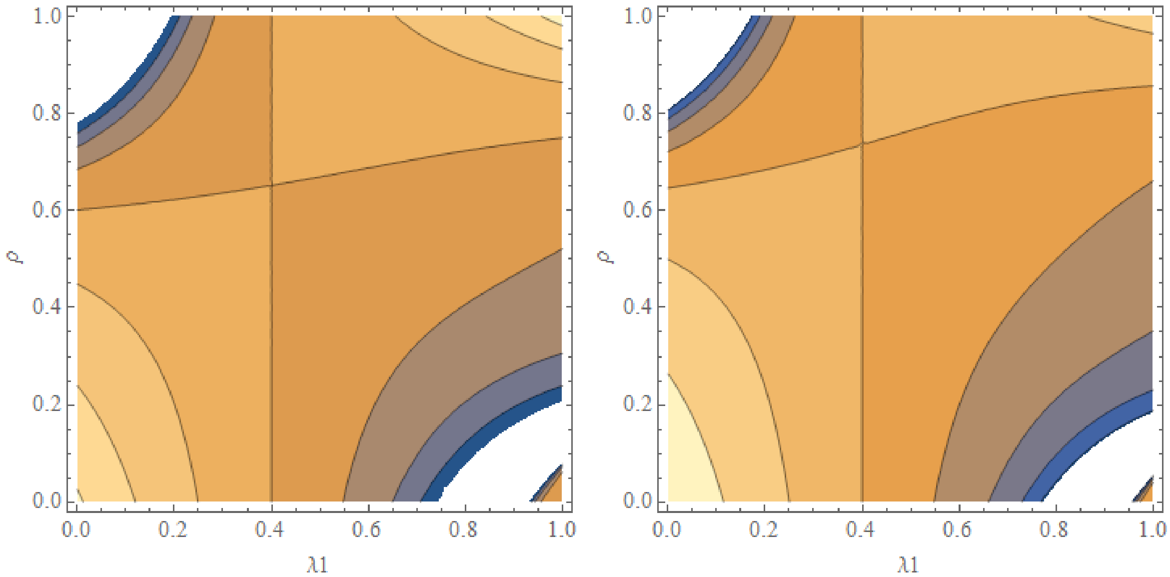

Figure 3.3: $\pi_1(q^R) - \pi_2(q^R)$ when $\lambda_2 = 0.4$, for $b = 3$ and $b = 2$, respectively.

like the regimes, cost and externality level, as we can see by comparing both of the plots. In summary, we can argue that the firm that share less profit will generate more profit relative to the other firm, when the regulator is sufficiently aligned with the firms, that is, worries more about the profit of the firm than the government revenue.

Returning to the bargaining problem, when firms use the solution of the regulator to threat as an alternative for the bargaining problem. Using the Equation (3.6) and the numerical solution explained above, we can say that firm 1 will have greater share than firm 2, i.e. $\alpha^R > 1/2$, if the firms profit share portion is less than firm 2 share and the regulator worries more about firms profit, or when firm 1 shares more of its profits and the regulator gives more importance for government revenue.

Even though firm 1 receives a larger portion of the joint profit under the condition mentioned above, the transfer that a firm will make to the other does not depend on ρ, according to the numerical exercise. Interestingly, the transfer appears to depend only on the fiscal regime, and the transfer will be positive for the firm that shares larger portion of its profit to the government, that is, firm 1 will receive a transfer if $\lambda_1 > \lambda_2$. The illustration of this solution is in the Figure 3.4, where we show a contour plot of the firm 1 transfer, and the lighter color means $t_1^R > 0$ and the darker color means firm 1 pays transfer to firm 2. We assumed again that $\lambda_2 = 0.4$. Note that the transfer do not depend on the regulator parameter ρ, but only on the fiscal regime. Hence, as in the rule of capture situation, the firm that will be benefited is the one that shares a larger portion of its profit to the government.

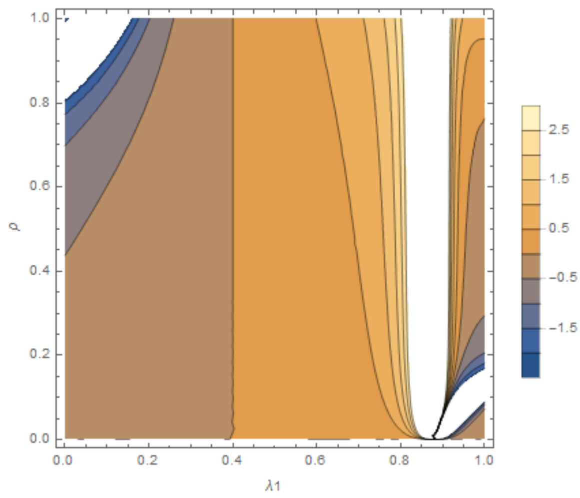

Figure 3.4: t_1^R contour plot

3.6.3 Comparing the solutions by numerical analysis

Another question we explore here is whether the existence of the regulator increases or decreases the transfer needed to reach an agreement. To address this case, we will compare the transfer when players are imposed the regulator deal, where the transfer is denote by t_i^R, with the case when players extract oil based on the rule of capture. Let t_i^c be the transfer when the threat point is given by the Nash equilibrium. The answer to this question can be found by subtracting the absolute value of the transfer in each situation, and if both values are positive, we have

$$t_i^R - t_i^N = \frac{\pi_i^R - \pi_j^R}{2} - \frac{\pi_i^N - \pi_j^N}{2} \Longleftrightarrow t_i - t_i^N = \frac{\pi_i^R - \pi_i^N}{2} - \frac{\pi_j^R - \pi_j^N}{2}, \tag{3.18}$$

hence, all we need is to compare the threat points if the firm receiving the transfer is the same. Looking at this equations, the transfer under the regulator setting would be larger than under the rule of capture if, and only if $\pi_i^R - \pi_j^R > \pi_i^c - \pi_j^c$, that is, the firm receiving the transfer will have larger transfer whenever difference of the profit under the regulator choice is larger than under rule of capture. The larger is the difference between players profit in each threat point situation, the larger will be the transfer needed to reach an agreement. The transfer occurs due to the fiscal regime players are operating, and not by their production and cost function, since we assumed they are identical. Therefore, the role of the regulator should be the one to minimize the misallocation of the rents, reducing the transfer between players.

So far, we identified the player who will be benefited in the unitization, when operating in a dual fiscal regime, whether firms threat according to the rule of capture or according to the regulator solution. Nevertheless, an important question now is whether the regulator are able to reduce the distortion caused by the dual fiscal regime. We should

expected that the role of the regulator in an economy to be such that reduce the distortion caused by a previous policy.

In order to check whether the regulator is reducing the distortion, we can use the Equation (3.18). Note that according to the equation, the distortion will be lower when the difference in the profit evaluated at the threat point solution is lower. The Figure 3.5 below illustrates that the distortion is increasing with respect to ρ, that is, the more importance the regulator gives to the government's revenue, the more distortion it will generate, and higher will be the transfer. However, the transfer will be lower than the transfer using the rule of capture as a threat point, as long as the regulator is impartial between government revenue and firms profit, that is, when $\rho = 1/2$. If the regulator worries more about government's take, than the distortion in the economy will be higher, the transfer needed to reach an agreement will be larger.

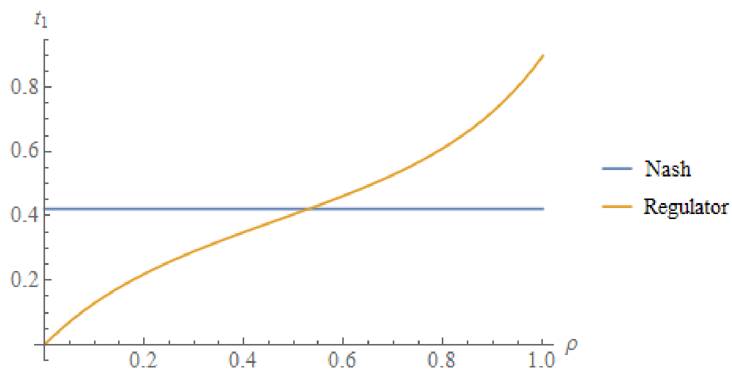

Figure 3.5: Transfer comparison

On the horizontal axis of this plot is all possible alignment between the regulator and government, and on the vertical axis is the transfer that firm 1 receives. We assume that $\lambda_1 > \lambda_2$. The blue line in this figure represents the transfer when firms threat according to the rule of capture, while the orange line represents the transfer when firms threat according to the regulator proposal. As we can see, the transfer associated to the regulator's problem will be lower if $\rho < 1/2$.

3.7 Conclusion

In this paper, we studied the effects of the dual fiscal regime over the unitization negotiation. The new regulatory framework for the pre-salt areas in Brazil introduced the production sharing agreement to lease the oil field for exploration and development. However, the concession contracts will co-exist in different areas, but some of them might be contiguous to the pre-salt area. Thus, the study proposed here shed a light about the distortion the dual regime will impose to the firms and how firms will strategically use the regulator as a threat to be benefited in the negotiation.

Our paper modeled the negotiation of the unitization agreement as a bargaining problem and solved it using the axiomatic solution proposed by Nash (1950). We showed that even if the firms are identical, but operating under different fiscal regime, one of the firm will be benefited. This benefit is associated to a larger share that a firm receives than the profit that the firm are helping to generate. This finding holds for the two possible resolution considered here, that is, two possible scenarios if the negotiation breaks-down. The rule of capture was used as a benchmark, where firms compete for the resource suffering from the common-pool problem, while the other one is the regulator proposal according to the Brazilian law. We also showed that the regulator could reduce this distortion as long as he is impartial between government take and firms profit or it is more worried about firms profit than government revenue.

Finally, the future steps of this research should consider finding analytically the regions that defines the player benefited by the fiscal regime. It can also study the dynamic setting, there price is uncertain and firms operating in different fiscal regime might be interested in producing in a different timing to reduce the risks. In the concession, firms are able to choose how fast they can produce, while in the PSA, since the firm is a partner of the regulator, the board might not accept a faster recovery rate.

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