# On the construction of explicit exponential-based schemes for stiff SDEs

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$$dx(t) = f(t, x(t))dt + \sum_{j=1}^{m} g_j(t, x(t))dw_t^j$$

$x \in \mathbb{R}^d$

$w_t = (w_t^1, \ldots, w_t^m)$ $m-$dimensional Brownian Motion,

$f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$g_j : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

## semi-linear SDE

$$dx(t) = (Ax(t) + f(x(t)))\, dt + \sum_{j=1}^{m} g_j(x(t)) dw_t^j$$


Y. Komori, K. Burrage, A stochastic **Exponential Euler scheme** for simulation of stiff biochemical reaction system, BIT, 10.1007/s10543-014-0485-1 (2014)

C. Shi, Y. Xiao, C. Zhang, The convergence and MS stability of **Exponential Euler method** for semilinear SDEs, Abstract and Applied Analysis, 10.1155/2012/350407 (2012)

By Ito formula to $f, L^0 f, L^j f$:

$$f(t,x(t)) = f(t_n, x(t_n)) + \int_{t_n}^{t_{n+1}} L^0 f(s,x(s))ds + \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} L^j f(s,x(s))dw_t^j$$

By Ito formula to $f, L^0 f, L^j f$:

$$
\begin{aligned}
f(t,x(t)) &= f(t_n,x(t_n)) + \int_{t_n}^{t_{n+1}} L^0 f(s,x(s))ds + \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} L^j f(s,x(s))dw_t^j \\
&= f(t_n,x(t_n)) + L^0 f(t_n,x(t_n)) \int_{t_n}^{t_{n+1}} ds + \sum_{j=1}^{m} L^j f(t_n,x(t_n)) \int_{t_n}^{t_{n+1}} dw_t^j + R
\end{aligned}
$$

By Ito formula to $f, L^0 f, L^j f$:

$$
\begin{aligned}
f(t,x(t)) &= f(t_n,x(t_n)) + \int_{t_n}^{t_{n+1}} L^0 f(s,x(s))ds + \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} L^j f(s,x(s))dw_t^j \\
&= f(t_n,x(t_n)) + L^0 f(t_n,x(t_n)) \int_{t_n}^{t_{n+1}} ds + \sum_{j=1}^{m} L^j f(t_n,x(t_n)) \int_{t_n}^{t_{n+1}} dw_t^j + R
\end{aligned}
$$

where

$$L^0 f = \frac{\partial f}{\partial t} + J_f \; f + \frac{1}{2}\sum_{j=1}^{m} (\mathbf{I} \otimes g_j^{\intercal}) \; f_{xx} \; g_j$$

$$L^j f = J_f \; g_j,$$

**Then**:

$$f(t,x(t)) = f(t_n,x(t_n)) + \left[\frac{\partial f}{\partial t} + J_f\ f + \frac{1}{2}\sum_{j=1}^{m}(\mathbf{I}\otimes g_j^{\intercal})\ f_{xx}\ g_j\right]_{(t_n,x(t_n))}(t-t_n)$$
$$+ \sum_{j=1}^{m}[J_f\ g_j]_{(t_n,x(t_n))}(w_t^j - w_{t_n}^j) + R_1$$

**Then**:

$$f(t,x(t)) \quad = \quad f(t_n,x(t_n)) + \left[\frac{\partial f}{\partial t} + J_f \ f + \frac{1}{2}\sum_{j=1}^{m}(\mathbf{I}\otimes g_j^{\intercal}) \ f_{xx} \ g_j\right]_{(t_n,x(t_n))}(t-t_n)$$
$$+\sum_{j=1}^{m}[J_f \ g_j]_{(t_n,x(t_n))}(w_t^j - w_{t_n}^j) + R_1$$

Also

$$x(t) = x(t_n) + f(t_n,x(t_n))(t-t_n) + \sum_{j=1}^{m} g_j(t_n,x(t_n))(w_t^j - w_{t_n}^j) + R_2$$

**thus**,

$$J_f \ (x(t)-x(t_n)) = J_f \ f(t_n,x(t_n))(t-t_n) + \sum_{j=1}^{m}[J_f \ g_j]_{(t_n,x(t_n))}(w_t^j - w_{t_n}^j) + \bar{R}_2$$

**Then**:

$$f(t, x(t)) \approx f(t_n, x(t_n)) + J_f \; (x(t) - x(t_n)) + \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{j=1}^{m} (\mathbf{I} \otimes g_j^{\intercal}) \; f_{xx} \; g_j \right]_{(t_n, x(t_n))} (t - t_n)$$

$$
\begin{aligned}
dx(t) \quad = \quad & \left( J_f \; (x(t) - x_n) + \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{j=1}^{m} (\mathbf{I} \otimes g_j^{\intercal}) \; f_{xx} \; g_j \right]_{(t_n, x_n)} (t - t_n) + f(t_n, x_n) \right) dt \\
& + \sum_{j=1}^{m} g_j(t, x_n) dw_t^j
\end{aligned}
$$

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dx(t) &= \left( J_f \ (x(t) - x_n) + \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sum_{j=1}^{m} (\mathbf{I} \otimes g_j^{\intercal}) \ f_{xx} \ g_j \right]_{(t_n, x_n)} (t - t_n) + f(t_n, x_n) \right) dt \\
&\quad + \sum_{j=1}^{m} g_j(t, x_n) dw_t^j \\
&= \left( \mathbf{A}_n \ (x(t) - x_n) + \mathbf{b}_n (t - t_n) + f(t_n, x_n) \right) dt + G(t, x_n) dW_t
\end{aligned}
$$

Let $\mathbf{Z}(t) = (x(t) - x_n, t - t_n, f(t_n, x_n), 1)^\top \in \mathbb{R}^{2d+2}$ then

$$d\mathbf{Z}(t) = \mathbf{M}\ \mathbf{Z}(t)dt + \begin{pmatrix} G(t, x_n) \\ 0 \\ 0 \end{pmatrix} dW_t$$

$$\mathbf{Z}(t_n) = [\mathbf{0}_{1\times d}\ 0\ f(t_n, x_n)\ 1]^\top,$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_n & \mathbf{b}_n & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 0 \end{pmatrix}.$$

Solution:

$$\mathbf{Z}(t) = e^{\mathbf{M}(t-t_n)}\ \mathbf{Z}(t_n) + \int_{t_n}^{t} e^{\mathbf{M}(t-s)} \begin{pmatrix} G(t, x_n) \\ 0 \\ 0 \end{pmatrix} dW_s$$

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-Looking at the first component of $\mathbf{Z}(t)$, it follows that $x(t)$, can be computed by

$$\begin{aligned} x(t) = x_n &+ \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} e^{\mathbf{M}(t-t_n)} \left[\mathbf{0}_{d\times d}\ 0\ f(t_n,x_n)\ 1\right]^{\top} \\ &+ \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} \int_{t_n}^{t} e^{\mathbf{M}(t-s)} \begin{pmatrix} G(t,x_n) \\ 0 \\ 0 \end{pmatrix} dW_s \\ = x_n &+ \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} e^{\mathbf{M}(t-t_n)} \left[\mathbf{0}_{d\times d}\ 0\ f(t_n,x_n)\ 1\right]^{\top} + \int_{t_n}^{t} e^{\mathbf{A}_n(t-s)} G(t,x_n) dW_s \end{aligned}$$

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Thus, in particular

$$x(t_{n+1}) = x_n + \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} e^{\mathbf{M}h} \left[\mathbf{0}_{d\times d}\ 0\ f(t_n, x_n)\ 1\right]^{\top} + \int_{t_n}^{t_{n+1}} e^{\mathbf{A}_n(t_{n+1}-s)} G(t, x_n) dW_s$$

Let

$$\int_{t_n}^{t} e^{\mathbf{A}_n(t-s)} G(s, x_n) dW_s = e^{\mathbf{A}_n t} \boldsymbol{\eta}(t)$$

where

$$\boldsymbol{\eta}(t) = \sum_{i=1}^{m} \int_{t_n}^{t} e^{-\mathbf{A}_n u} g_i(u, x_n) dw_u^i$$

is the solution of the SDE

$$d\boldsymbol{\eta}(t) = \sum_{i=1}^{m} e^{-\mathbf{A}_n t} g_i(t, x_n) dw_t^i$$

$$\boldsymbol{\eta}(t_n) = \mathbf{0}$$

- We need to compute $e^{\mathbf{A}_n t_{n+1}} \boldsymbol{\eta}(t_{n+1})$

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- The application of the Ito formula to $\mathbf{U}(t, \boldsymbol{\eta}) = \boldsymbol{\rho}(t) = e^{\mathbf{A}_n t_{n+1}} \boldsymbol{\eta}(t)$ yields

$$
\begin{aligned}
d\boldsymbol{\rho}(t) &= \sum_{i=1}^{m} e^{\mathbf{A}_n (t_{n+1} - t)} g_i(t, x_n) dw^i(t) \\
\boldsymbol{\rho}(t_n) &= \mathbf{0}
\end{aligned}
$$

- We need to compute $e^{\mathbf{A}_n t_{n+1}} \boldsymbol{\eta}(t_{n+1})$
- The application of the Ito formula to $\mathbf{U}(t, \boldsymbol{\eta}) = \boldsymbol{\rho}(t) = e^{\mathbf{A}_n t_{n+1}} \boldsymbol{\eta}(t)$ yields

$$d\boldsymbol{\rho}(t) = \sum_{i=1}^{m} e^{\mathbf{A}_n (t_{n+1} - t)} g_i(t, x_n) dw^i(t)$$
$$\boldsymbol{\rho}(t_n) = \mathbf{0}$$

- By the order-1.5 strong Taylor scheme we have the approximation

$$\boldsymbol{\rho}(t_{n+1}) = e^{\mathbf{A}_n h} \sum_{i=1}^{m} \left( g_i(t_n, x_n) \Delta w_n^i + (-\mathbf{A}_n g_i(t_n, x_n) \Delta z_n^i + \frac{dg_i(t_n, x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i) \right)$$

where $\Delta w_n^i = w_{t_{n+1}}^i - w_{t_n}^i = \sqrt{h} u_1^i$ and $\Delta z_n^i = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dw_{s_1}^i ds_2 = \frac{1}{2} h^{\frac{2}{3}} (u_1^i + \frac{1}{\sqrt{3}} u_2^i)$

$$
\begin{aligned}
x_{n+1} &= x_n + \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} e^{\mathbf{M}h} \left[\mathbf{0}_{d\times d}\ 0\ f(t_n,x_n)\ 1\right]^{\top} \\
&\quad + e^{\mathbf{A}_n h} \sum_{i=1}^{m} \left( g_i(t_n,x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n,x_n)\Delta z_n^i + \frac{dg_i(t_n,x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i) \right),
\end{aligned}
$$

# Implementation

The *Padé algorithm with scaling-squaring strategy* for computing $e^{\mathbf{C}}$ [Golub-Van Loan]:

1. Determine the minimum integer $k$ such that $\left\| \frac{\mathbf{C}}{2^k} \right\| < \frac{1}{2}$

2. Compute

$$
\begin{aligned}
\mathbf{N}_q\left(\frac{\mathbf{C}}{2^k}\right) &= \sum_{j=0}^{q} \frac{(2q-j)!q!}{(2q)!j!(q-j)!} \mathbf{C}^j \\
\mathbf{D}_q\left(\frac{\mathbf{C}}{2^k}\right) &= \sum_{j=0}^{q} \frac{(2q-j)!q!}{(2q)!j!(q-j)!} (-\mathbf{C})^j .
\end{aligned}
$$

3. Compute $\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = [\mathbf{D}_q(\frac{\mathbf{C}}{2^k})]^{-1}\mathbf{N}_q(\frac{\mathbf{C}}{2^k})$, (solving the system $\mathbf{D}_q(\frac{\mathbf{C}}{2^k})\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = \mathbf{N}_q(\frac{\mathbf{C}}{2^k})$)

4. Compute $[\mathbf{P}_q(\frac{\mathbf{C}}{2^k})]^{2^k}$ by squaring $\mathbf{P}_q(\frac{\mathbf{C}}{2^k})$ $k$ times

$\mathbf{C} = \mathbf{M}h$. Denote $\mathbf{A} = \mathbf{A}_n h$ and $\mathbf{b} = \mathbf{b}_n h$, then

$$\left(\frac{\mathbf{C}}{2^k}\right) = \begin{pmatrix} \mathbf{A} & \mathbf{b} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 0 \end{pmatrix} \frac{h}{2^k}$$

and, for any $j \geq 2$

$$\left(\frac{\mathbf{C}}{2^k}\right)^j = \begin{pmatrix} \mathbf{A}^j & \mathbf{A}^{j-1}\mathbf{b} & \mathbf{A}^{j-1} & \mathbf{A}^{j-2}\mathbf{b} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 0 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 0 \end{pmatrix} \left(\frac{h}{2^k}\right)^j.$$

$$\mathbf{N}_q\left(\frac{\mathbf{C}}{2^k}\right) = \begin{pmatrix} \mathbf{I} + \mathbf{A}\left(\alpha_1\mathbf{I} + \mathbf{AS}\right) & \left(\alpha_1\mathbf{I} + \mathbf{AS}\right)\mathbf{b} & \left(\alpha_1\mathbf{I} + \mathbf{AS}\right) & \mathbf{Sb} \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & \alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix},$$

and

$$\mathbf{D}_q\left(\frac{\mathbf{C}}{2^k}\right) = \begin{pmatrix} \mathbf{I} + \mathbf{A}\left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right) & \left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right)\mathbf{b} & \left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right) & \hat{\mathbf{S}}\mathbf{b} \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & -\alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{S} &= \left(\alpha_2\mathbf{I} + \alpha_3\mathbf{A} + \ldots + \alpha_q\left(\mathbf{A}\right)^{q-2}\right), \\ \hat{\mathbf{S}} &= \left(\alpha_2\mathbf{I} - \alpha_3\mathbf{A} + \ldots + (-1)^{q-2}\alpha_q\left(\mathbf{A}\right)^{q-2}\right), \end{aligned}$$

and $\alpha_j = c_j\left(\frac{\bar{h}}{2^k}\right)^j$, $\quad \left(c_j = \frac{(2q-j)!q!}{(2q)!j!(q-j)!}\right)$

# The Adapted Padé algorithm for computing exp(Mh)

## Padé Approximation

$$\mathbf{D}_q(\frac{\mathbf{C}}{2^k})\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = \mathbf{N}_q(\frac{\mathbf{C}}{2^k})$$

$\mathbf{P}_q(\frac{\mathbf{C}}{2^k})$ has the form:

$$\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = \begin{pmatrix} \mathbf{U1} & \mathbf{U2} & \mathbf{U3} & \mathbf{U4} \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & \alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix}$$

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$\mathbf{P}_q(\frac{\mathbf{C}}{2^k})$ has the form:

$$\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = \begin{pmatrix} \mathbf{U1} & \mathbf{U}2 & \mathbf{U3} & \mathbf{U}4 \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & \alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix}$$

and $\mathbf{U}1$, $\mathbf{U}2$, $\mathbf{U}3$, $\mathbf{U}4$ satisfy:

$$\begin{aligned}
\left[\mathbf{I} + \mathbf{A}\left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right)\right]\mathbf{U}1 &= \left[\mathbf{I} + \mathbf{A}\left(\alpha_1\mathbf{I} + \mathbf{AS}\right)\right] \\
\left[\mathbf{I} + \mathbf{A}\left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right)\right]\mathbf{U}3 &= \left(\alpha_1\mathbf{I} + \mathbf{AS}\right) - \left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right) \\
\left[\mathbf{I} + \mathbf{A}\left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right)\right]\mathbf{U}4 &= \left[\mathbf{S} - 2\alpha_1\left(-\alpha_1\mathbf{I} + \mathbf{A}\hat{\mathbf{S}}\right) - \hat{\mathbf{S}}\right]\mathbf{b} \\
\mathbf{U}2 &= (\mathbf{U}3)\mathbf{b}
\end{aligned}$$

# Squaring the scaled Padé Approx.

Once we have **U**1, **U**2, **U**3, **U**4, we can compute $\left(\mathbf{P}_q\left(\frac{\mathbf{C}}{2^k}\right)\right)^{2^k}$ :

$$\left(\mathbf{P}_q\left(\frac{\mathbf{C}}{2^k}\right)\right)^{2^k} = \begin{pmatrix} (\mathbf{U1})^{2^k} & \left(\sum_{i=0}^{2^k-1} (\mathbf{U1})^i\right)\mathbf{U2} & \left(\sum_{i=0}^{2^k-1} (\mathbf{U1})^i\right)\mathbf{U3} & \left(\sum_{i=0}^{2^k-1} (\mathbf{U1})^i\right)\mathbf{U4} + \alpha_1 \left(\sum_{i=0}^{2^k-2} (m-1-i)\,(\mathbf{U1})^i\right)\mathbf{U2} \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & 2^k\alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix}.$$

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**Thus,**

$$\begin{aligned} x_{n+1} &= x_n + \left[\mathbf{I}_{d\times d} \;\; \mathbf{0}_{d\times(d+2)}\right]\left(\mathbf{P}_q(\frac{\mathbf{M}h}{2^k})\right)^{2^k}\left[\mathbf{0}_{d\times d} \;\; 0 \;\; f(t_n,x_n) \;\; 1\right]^\top \\ &+ \left(\mathbf{P}_q(\frac{\mathbf{A}_n h}{2^k})\right)^{2^k}\sum_{i=1}^{m}\left(g_i(t_n,x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n,x_n)\Delta z_n^i + \frac{dg_i(t_n,x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i)\right) \end{aligned}$$

# Squaring the scaled Padé Approx.

Once we have $\mathbf{U}1$, $\mathbf{U}2$, $\mathbf{U}3$, $\mathbf{U}4$, we can compute $\left(\mathbf{P}_q(\frac{\mathbf{C}}{2^k})\right)^{2^k}$:

$$\left(\mathbf{P}_q(\frac{\mathbf{C}}{2^k})\right)^{2^k} = \begin{pmatrix} (\mathbf{U1})^{2^k} & \left(\sum_{i=0}^{2^k-1}(\mathbf{U1})^i\right)\mathbf{U2} & \left(\sum_{i=0}^{2^k-1}(\mathbf{U1})^i\right)\mathbf{U3} & \left(\sum_{i=0}^{2^k-1}(\mathbf{U1})^i\right)\mathbf{U}4+\alpha_1\left(\sum_{i=0}^{2^k-2}(m-1-i)(\mathbf{U1})^i\right)\mathbf{U}2 \\ \mathbf{0}_{1\times d} & 1 & \mathbf{0}_{1\times d} & 2^k\alpha_1 \\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times 1} & \mathbf{I}_{d\times d} & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times d} & 0 & \mathbf{0}_{1\times d} & 1 \end{pmatrix}.$$

**Thus,**

$$\begin{aligned} x_{n+1} &= x_n + \left[\mathbf{I}_{d\times d} \;\; \mathbf{0}_{d\times(d+2)}\right]\left(\mathbf{P}_q(\frac{\mathbf{M}h}{2^k})\right)^{2^k}\left[\mathbf{0}_{d\times d}\;0\;f(t_n,x_n)\;1\right]^\top \\ &\quad + \left(\mathbf{P}_q(\frac{\mathbf{A}_n h}{2^k})\right)^{2^k}\sum_{i=1}^{m}\left(g_i(t_n,x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n,x_n)\Delta z_n^i + \frac{dg_i(t_n,x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i)\right) \\ &= x_n + \mathbf{L}f(t_n,x_n) + \mathbf{Q} \\ &\quad + \mathbf{R}\sum_{i=1}^{m}\left(g_i(t_n,x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n,x_n)\Delta z_n^i + \frac{dg_i(t_n,x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i)\right) \end{aligned}$$

# The Proposed Integrator

$$
\begin{aligned}
x_{n+1} &= x_n + \mathbf{L} f(t_n, x_n) + \mathbf{Q} \\
&+ \mathbf{R} \sum_{i=1}^{m} \left( g_i(t_n, x_n) \Delta w_n^i + (-\mathbf{A}_n g_i(t_n, x_n) \Delta z_n^i + \frac{d g_i(t_n, x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i) \right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{L} &= \left( \sum_{i=0}^{2^k - 1} (\mathbf{U1})^i \right) \mathbf{U3}, \\
\mathbf{Q} &= \left( \sum_{i=0}^{2^k - 1} (\mathbf{U1})^i \right) \mathbf{U}4 + \alpha_1 \left( \sum_{i=0}^{2^k - 2} (m - 1 - i) (\mathbf{U1})^i \right) (\mathbf{U}3)\, \mathbf{b} \\
\mathbf{R} &= (\mathbf{U1})^{2^k}
\end{aligned}
$$

**Convergence and Stability**

Let

$$x_{n+1} = x_n + \boldsymbol{\phi}(t_n, x_n; h) + \boldsymbol{\xi}(t_n, x_n; h)$$

the order-$\gamma$ method, and

$$\widetilde{x}_{n+1} = \widetilde{x}_n + \widetilde{\boldsymbol{\phi}}(t_n, \widetilde{x}_n; h) + \widetilde{\boldsymbol{\xi}}(t_n, \widetilde{x}_n; h)$$

a **numerical implementation** of $x_{n+1}$, where $\widetilde{\boldsymbol{\phi}}$ and $\widetilde{\boldsymbol{\xi}}$ denote **numerical algorithms to compute $\boldsymbol{\phi}$** and $\boldsymbol{\xi}$. And suppose that

$$\left| E\left( \boldsymbol{\phi}(t_n, \widetilde{x}_n; h) - \widetilde{\boldsymbol{\phi}}(t_n, \widetilde{x}_n; h) \,\middle|\, \mathcal{F}_{t_n} \right) \right| \leq K_1 (1 + |\widetilde{x}_n|^2)^{1/2} \; h^{k_1}$$

$$(E\left( \left| \boldsymbol{\phi}(t_n, \widetilde{x}_n; h) - \widetilde{\boldsymbol{\phi}}(t_n, \widetilde{x}_n; h) \right|^2 \,\middle|\, \mathcal{F}_{t_n} \right))^{1/2} \leq K_1 (1 + |\widetilde{x}_n|^2)^{1/2} \; h^{k_2 + 1/2}$$

$$\left| E\left( \boldsymbol{\xi}(t_n, \widetilde{x}_n; h) - \widetilde{\boldsymbol{\xi}}(t_n, \widetilde{x}_n; h) \,\middle|\, \mathcal{F}_{t_n} \right) \right| \leq K_2 (1 + |\widetilde{x}_n|^2)^{1/2} \; h^{p_1}$$

$$(E\left( \left| \boldsymbol{\xi}(t_n, \widetilde{x}_n; h) - \widetilde{\boldsymbol{\xi}}(t_n, \widetilde{x}_n; h) \right|^2 \,\middle|\, \mathcal{F}_{t_n} \right))^{1/2} \leq K_2 (1 + |\widetilde{x}_n|^2)^{1/2} \; h^{p_2 + 1/2}$$

for some positive numbers $k_2, p_2 \geq 1/2$, $k_1 \geq k_2 + 1$, $p_1 \geq p_2 + 1$, then

$$\left( E\left( |x(t_n) - \widetilde{x}_n|^2 \,\middle|\, \mathcal{F}_{t_0} \right) \right)^{\frac{1}{2}} = O(h^{\min\{\gamma, k_2, p_2\}})$$

Note that we have

$$\begin{aligned}
\widetilde{\boldsymbol{\phi}}(t_n,\widetilde{x}_n;h) &= \begin{bmatrix}\mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)}\end{bmatrix}\left(\mathbf{P}_q(\frac{\mathbf{M}h}{2^k})\right)^{2^k}[\mathbf{0}_{d\times d}\ 0\ f(t_n,x_n)\ 1]^\top \\
\widetilde{\boldsymbol{\xi}}(t_n,\widetilde{\mathbf{y}}_n;h) &= \left(\mathbf{P}_q(\frac{\mathbf{A}_n h}{2^k})\right)^{2^k}\sum_{i=1}^{m}\left(g_i(t_n,x_n)\Delta w_n^i+(-\mathbf{A}_n g_i(t_n,x_n)\Delta z_n^i+\frac{dg_i(t_n,x_n)}{dt})(\Delta w_n^i h-\Delta \mathbf{z}_n^i)\right)
\end{aligned}$$

Note that we have

$$\widetilde{\boldsymbol{\phi}}(t_n, \widetilde{x}_n; h) = \begin{bmatrix} \mathbf{I}_{d\times d} & \mathbf{0}_{d\times(d+2)} \end{bmatrix} \left( \mathbf{P}_q(\frac{\mathbf{M}h}{2^k}) \right)^{2^k} \left[ \mathbf{0}_{d\times d} \ 0 \ f(t_n, x_n) \ 1 \right]^\top$$

$$\widetilde{\boldsymbol{\xi}}(t_n, \widetilde{\mathbf{y}}_n; h) = \left( \mathbf{P}_q(\frac{\mathbf{A}_n h}{2^k}) \right)^{2^k} \sum_{i=1}^{m} \left( g_i(t_n, x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n, x_n)\Delta z_n^i + \frac{dg_i(t_n, x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i) \right)$$

and by using that

$$\left| e^{\mathbf{X}} - (\mathbf{P}_q(2^{-k}\mathbf{X}))^{2^k} \right| \leq c_q(k, |\mathbf{X}|) \, |\mathbf{X}|^{p+q+1} ,$$

where $c_q(k, |\mathbf{X}|) = \alpha 2^{-k(2q)+3} e^{(1+\epsilon_{p,q})|\mathbf{X}|}$ and $\alpha = \frac{(q!)^2}{(2q)!(2q+1)!}$, $\epsilon_q = \alpha(\frac{1}{2})^{2q-3}$

Note that we have

$$
\begin{aligned}
\widetilde{\boldsymbol{\phi}}\left(t_n, \widetilde{x}_n; h\right) &= \left[\mathbf{I}_{d\times d} \;\; \mathbf{0}_{d\times(d+2)}\right] \left(\mathbf{P}_q(\frac{\mathbf{M}h}{2^k})\right)^{2^k} \left[\mathbf{0}_{d\times d} \; 0 \; f(t_n, x_n) \; 1\right]^{\top} \\
\widetilde{\boldsymbol{\xi}}\left(t_n, \widetilde{\mathbf{y}}_n; h\right) &= \left(\mathbf{P}_q(\frac{\mathbf{A}_n h}{2^k})\right)^{2^k} \sum_{i=1}^{m} \left(g_i(t_n, x_n)\Delta w_n^i + (-\mathbf{A}_n g_i(t_n, x_n)\Delta z_n^i + \frac{dg_i(t_n, x_n)}{dt})(\Delta w_n^i h - \Delta \mathbf{z}_n^i)\right)
\end{aligned}
$$

and by using that

$$
\left| e^{\mathbf{X}} - (\mathbf{P}_q(2^{-k}\mathbf{X}))^{2^k} \right| \leq c_q(k, |\mathbf{X}|)\, |\mathbf{X}|^{p+q+1},
$$

where $c_q(k, |\mathbf{X}|) = \alpha 2^{-k(2q)+3} e^{(1+\epsilon_{p,q})|\mathbf{X}|}$ and $\alpha = \frac{(q!)^2}{(2q)!(2q+1)!}$, $\epsilon_q = \alpha(\frac{1}{2})^{2q-3}$

The scheme

$$
\widetilde{x}_{n+1} = \widetilde{x}_n + \widetilde{\boldsymbol{\phi}}(t_n, \widetilde{x}_n; h) + \widetilde{\boldsymbol{\xi}}(t_n, \widetilde{x}_n; h)
$$

satisfies

$$
\left(E\left(\left|x(t_n) - \widetilde{x}_n\right|^2 \,\middle|\, \mathcal{F}_{t_0}\right)\right)^{\frac{1}{2}} = O(h^{\min\{\frac{1}{2}, q-1\}})
$$

# Numerical Experiments

## Example

**Two connected oscillators**

$$
\begin{aligned}
dx_1 &= x_2 \\
dx_2 &= (\frac{1}{2}x_2(1-x_1^2) - x_1 + 0.1x_3)dt + 10^{-3}x_1 dw^1 \\
dx_3 &= x_4 \\
dx_4 &= (5x_4(1-x_3^2) - x_3 + 0.1x_1)dt + 10^{-3}x_3 dw^2
\end{aligned}
$$

$(x_1^0, x_2^0, y_1^0, y_2^0) = (1, 1, 0, 0).$

# phase space second V. der Pol oscillator

Click me!

# Numerical Experiments

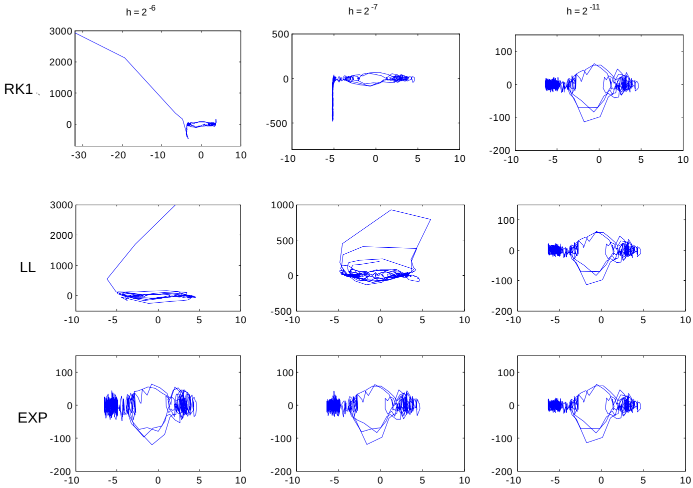
## Example

**Stochastic Van der Pol equat.**

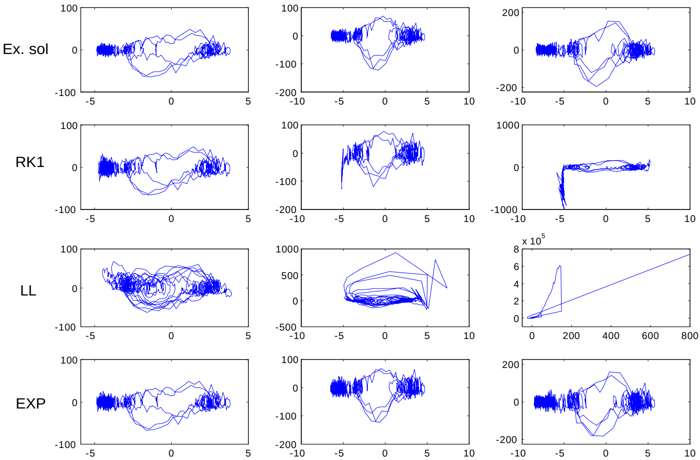
$$
\begin{aligned}
dx^1 &= x^2 dt, \\
dx^2 &= E\left(\left(1-\left(x^1\right)^2\right)x^2 - x^1\right)dt + \sigma d\mathbf{W}_t
\end{aligned}
$$

$E = 10, \quad [t_0, T] = [0, 12], \; \mathbf{x}(t_0) = (2, 0).$

## $\sigma = 200$



$\sigma_1 = 100, \sigma_2 = 200, \sigma_3 = 400$



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