FUNDAÇÃO
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EPGE
Escola de Pós-Graduação em
Economia

SEMINÁRIOS
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"A NOTION OF SUBGAME PERFECT NASH EQUILIBRIUM UNDER KNIGHTIAN UNCERTAINTY"

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Praia de Botafogo, 190 - 10º andar - Auditório

DATA
09/10/97 (5ª feira)

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16:00h

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A Notion of Subgame Perfect Nash Equilibrium under Knightian Uncertainty

by

Sérgio Ribeiro da Costa Werlang*

ABSTRACT. We define a subgame perfect Nash equilibrium under Knightian uncertainty for two players, by means of a recursive backward induction procedure. We prove an extension of the Zermelo-von Neumann-Kuhn Theorem for games of perfect information, i. e., that the recursive procedure generates a Nash equilibrium under uncertainty (Dow and Werlang(1994)) of the whole game. We apply the notion for two well known games: the chain store and the centipede. On the one hand, we show that subgame perfection under Knightian uncertainty explains the chain store paradox in a one shot version. On the other hand, we show that subgame perfection under uncertainty does not account for the leaving behavior observed in the centipede game. This is in contrast to Dow, Orioli and Werlang(1996) where we explain by means of Nash equilibria under uncertainty (but not subgame perfect) the experiments of McKelvey and Palfrey(1992). Finally, we show that there may be nontrivial subgame perfect equilibria under uncertainty in more complex extensive form games, as in the case of the finitely repeated prisoner's dilemma, which accounts for cooperation in early stages of the game.

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A Notion of Subgame Perfect Nash Equilibrium under Knightian Uncertainty

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1. Introduction

The most important model of Knightian uncertainty (heretofore simply uncertainty) is due to David Schmeidler and Itzhak Gilboa, in a series of papers (Schmeidler(1982, 1986, 1989), Gilboa(1987), Gilboa and Schmeidler(1989, 1993)). Game theoretic models incorporating uncertainty were soon to come. We will concentrate on models which are akin to Dow and Werlang(1994) (for example, the same type of model is used in Dow, Simonsen and Werlang(1993), Eichberger and Kelsey(1994a), Mukerji(1994), Marinacci(1994), Lo(1995a), Boff and Werlang(1996), and Dow, Orioli and Werlang(1996)). Alternative definitions can be found in Hendon, Jacobsen, Sloth and Tranæs(1993), Klibanoff(1995), Lo(1994), Epstein(1995) and Ghirardato and Le Breton(1996).

In extensive form games, though, the story is different. It is well known that Schmeidler and Gilboa's model, with or without the leading updating rule (known as Dempster-Shafer's updating rule) leads to dynamically inconsistent behavior in one person decision problems (Dow and Werlang(1992b), Epstein and Le Breton(1993), Eichberger and Kelsey(1994b)). Thus, the quest for any equilibrium which makes use of the dynamic structure of extensive form games seems to be very hard. Lo(1995b) attempts a first definition. He imposes dynamically consistent behavior on the equilibrium path. However, off the equilibrium path, nothing is assured.

If the backward induction procedure devised by Kuhn(1953) were to work, then one could define a subgame perfect Nash equilibrium under uncertainty by means of that mechanism. There are two difficulties. The first, is the fact that there is no well established definition of independence. We will use Gilboa and Schmeidler's definition (Gilboa and Schmeidler(1993)), which is the most widely accepted. The second is that Fubini's theorem does not apply to non additive probabilities, even in the case one of the measures is additive. Ghirardato(1994) is an excellent reference on Fubini's theorem for non additive probabilities. The next section will show an example where the backward induction procedure does not work even in a very simple game.

We also show in section two that in two person games of perfect information it is possible to embed any Nash equilibrium under uncertainty of a subgame in a Nash equilibrium under uncertainty of the larger game. The trick is that Fubini's theorem works for the case where one of the probabilities is additive and concentrated in a point. In section three we apply the construction to two games: the chain-store paradox and the centipede. While the procedure delivers a very simple explanation for the chain store paradox, the same does

not occur in the centipede. The centipede game is, however, justified by Nash equilibria of the whole game, as Dow, Orioli and Werlang(1996) show.

Another way to view the result above is that there are Nash equilibria under uncertainty of the original game that are subgame perfect under the updating rule of Dempster-Shafer (and which are nontrivial, i. e., with non zero uncertainty aversion). In fact, given that the construction uses explicitly the definition of independence, the result is stronger: for any updating rule which is compatible with the definition of independence, one would obtain subgame perfection. Many updating rules are compatible with Schmeidler-Gilboa's independence definition, including Bayes' rule (Bayes' rule coincides with Dempster-Shafer's rule only for additive probabilities).

Finally, although one cannot assure existence in general, in certain games there are Nash equilibria under uncertainty which are subgame perfect when one uses Dempster-Shafer's rule. This is the case of the twice repeated prisoner's dilemma of Dow and Werlang(1994).

2. The Theorem

We assume the reader is aware of the definition of Nash equilibrium under uncertainty. See Dow and Werlang(1994). The backward induction procedure was first used by Zermelo(1913) to prove that chess has a pure strategy Nash equilibrium, and formalized by Kuhn(1953) in the context of a general extensive form game. (For the sake of completeness, one should note that von Neumann and Morgenstern(1953, 24.1.2) defined a general extensive form game with perfect information, and proved the existence of a pure strategy equilibrium, by means of backward induction, even for n-players. However, they dismiss their result for non zero sum games of two or more players as "unimportant", because players could form coalitions. In fact, Kuhn(1953) refers to the perfect information application of the backward induction algorithm as Zermelo-von Neumann.) Selten(1965) defines subgame perfection, whose existence comes from Kuhn's result. We will refer to the backward induction procedure as Zermelo-von Neumann-Kuhn.

Why should not the same algorithm work for games with uncertainty aversion? Let us focus in the case of the game Γ in Figure 1. There are two strict subgames. One starting at the second node of player 1. Call it $\Gamma(1)$. The other starting at the node where player 2 moves. Call it $\Gamma(2)$. To solve $\Gamma(1)$, notice that player 1 strictly prefers R2 to L2. To solve $\Gamma(2)$, player 2's beliefs about player 1 ($P(\cdot)$) have to be such that $P(L2)=0$, because R is a strict best response of player 1. This is where the iterated optimality comes into play. Assume $P(R2)=0.9$. In this case, the expected utility of player 2 in case r is chosen is $1+P(R2)(3-1) = 2.8$. On the other hand, if player 2 plays l, the payoff is 2. Hence, r is a strict best response. If $Q(\cdot)$ represents the beliefs of player 1 about the play of 2, then $Q(l)=0$. In this game, in particular, for the expected utility of player 2 it is irrelevant what is the value of $Q(r)$. We set it to be 0.9. Given these beliefs, the expected utility of player 1 in $\Gamma(2)$ coincides with the

expected utility of $\Gamma(1)$, that is, if R2 is played 2, if L2 is played 0. Since player 1 plays R2, the expected utility is 2. This need not be so in a more general set up. Now, let us focus on the game Γ. We solved for the two smaller subgames. It is now player 1's turn. If the backward induction procedure were to work, then if player 1 plays R1, the expected utility should be 2, and if player 1 played L1 the expected utility should be -1. This means that R1 is a best response. Therefore, any probability p such that $p(R1)=p\leq 1$, and $p(L1)=0$ would do as an equilibrium. Unfortunately, this is not so. Although it is easy to check the optimality of player 1, the response of player 2 may be different from r.

In order to see that, we have to compute the beliefs in the whole game Γ. The strategy space of player 2 does not alter. However, player 1's strategy set now is {L1L2, L1R2, R1R2, R1L2}. To extend the beliefs $P(\cdot)$ and $p(\cdot)$ to this set, we use Gilboa-Schmeidler's independence definition: $p\otimes P$ is defined by its core, $C(p\otimes P)$ = Convex hull of the set of probabilities π defined on {L1,R1}X{L2,R2}, such that $\pi=\pi_1\otimes\pi_2$, where $\pi_1\in C(p)$ and $\pi_2\in C(P)$. Here, C(p) and C(P) are the core of p and P respectilvely. We now have to compute the expected value of player 2's response. It is easy to check that:

$Eu_2(r) = 1 + 0.9\cdot(2-1) + 0.9\cdot p\cdot(3-2) = 1.9 + 0.9p$, and $Eu_2(l) = 2$. It follows that as long as p is small enough so that $0.9p<0.1$, then r is not a best response anymore. For example, $p=0.01$ would do.

What is wrong? The answer is that we are always implicitly assuming Fubini's theorem in our reasoning. But Fubini's theorem is not valid for non additive probabilities. In fact, Fubini's theorem does not apply even in the case one of the measures is additive. Ghirardato(1994) is an excellent reference on Fubini's theorem for non additive probabilities. Notice, however, that if p is large (close to 1) then r would still be a best response, and the backward induction algorithm would work. Before proving the main theorem, we need two lemmas.

Lemma 1 Assume that Q is a convex non additive probability defined on A and P is an additive probability concentrated in a point b in B. Then Fubini's theorem is valid, that is, if f is a random variable in AXB, then $E_{P\otimes Q}[f] = E_P[E_Q[f]] = E_Q[E_P[f]]$.

<u>Proof</u>. Simple.

QED.

Lemma 2 Let P and Q be two non additive convex probabilities. Then for any pair[2] suppP and suppQ, there is a support of $P\otimes Q$, such that:

$\text{supp}P\otimes Q \subset \text{supp}P \times \text{supp}Q$.

<u>Proof</u>. If A is suppP, then $P(A^c) = 0$, which means that there is p in the core of P with $p(A^c) = 0$. Also, if B is suppQ, then there is q in the core of Q with $q(B^c) = 0$. Hence, by the definition of independence, $P\otimes Q((A\times B)^c) = 0$. In any set S such that $P\otimes Q(S^c) = 0$ there is a support. The result follows.

QED.

Theorem (Extension of Zermelo-von Neumann-Kuhn) Let G be a two person game of perfect information and G1 a subgame of G. If G1 has a Nash equilibrium under uncertainty (P,Q), then it is possible to construct a Nash equilibrium under uncertainty of G such that its restriction[1] to G1 coincides with (P,Q).

Proof. The proof is by induction on the length of the tree. It suffices to restrict attention to the case where player 1 is choosing among n different alternatives (call them, as in Figure 2, $c_1, ..., c_n$), and there are n different subgames, G1, ..., Gn, each of them with a Nash equilibrium under uncertainty (P_i, Q_i), $i=1,...,n$. Also, assume the strategy spaces are A_i for player 1 after choice c_i, and B_i for player 2 after choice c_i. Then, G is a game such that the strategy space of player 1 is $A = \{c_1, ..., c_n\} X A_1 X ... X A_n$, and the strategy space of player 2 is $B = B_1 X ... X B_n$. Choose an optimal choice c_i of player 1 such that the expected utility of player 1 in subgame Gi is the largest among all the other subgames. Note that there can be several such choices, but we will choose only one. Let p be the additive probability defined on $\{c_1, ..., c_n\}$ which puts mass 1 in $\{c_i\}$. We will show that the pair $P=p \otimes P_1 \otimes ... \otimes P_n$ and $Q=Q_1 \otimes ... \otimes Q_n$ is a Nash equilibrium under uncertainty of the large game G. By Lemma 2, we can choose $suppP \subset supp(p) X suppP_1 X ... X suppP_n$, and $suppQ \subset suppQ_1 X ... X suppQ_n$. To check the optimality of the player 1's choice, note that for $a=(c_j, a_1, ..., a_n) \in A$, $E_Q[u_1(a)]$ is by definition $E_{Qj}[u_1(a_j)]$. Let $\hat{a} \in suppP$. Then, by the definition of c_i and P, $\hat{a}=(c_i, \hat{a}_1, ..., \hat{a}_n)$ and $E_Q[u_1(\hat{a})] \geq E_{Qj}[u_1(\hat{a}_j)]$ for all $j=1, ..., n$. But for any j it is also true that, because $\hat{a}_j \in suppQ_j$, $E_{Qj}[u_1(\hat{a}_j)] \geq E_{Qj}[u_1(a_j)]$ for all $a_j \in A_j$. Also, given any $a=(c_j, a_1, ..., a_n) \in A$, $E_Q[u_1(a)] = E_{Qj}[u_1(a_j)]$. Therefore, by joining all the inequalities and equalities, we have that for all $a \in A$, $E_Q[u_1(a)] \leq E_Q[u_1(\hat{a})]$. Hence, player 1's choices are optimal. To check player 2's choices optimality, we apply Lemma 1 to p and $P_1 \otimes ... \otimes P_n$. As a result, we get that, for all $b=(b_1, ..., b_n) \in B$: $E_P[u_2(b)] = E_{Pi}[u_2(b_i)]$, which implies that player 2's choices are best responses.

QED.

3. Two Examples

The first example, in Figure 3, is the chain-store paradox. The game was introduced by Selten(1978). As it is well known, the only subgame perfect equilibrium of the game consists of the entrant entering the market and the monopolist acquiescing, even if the game is repeated (finitely) many times. Kreps and Wilson(1982) and Milgrom and Roberts(1982) proposed the most

[1] We did not define the meaning of restriction, because we did mention what is the updating rule that is to be used. However, as we are building equilibria where there is independence, we may use almost any rule. In particular Dempster-Shafer rule will do.

[2] The definition of support of a nonadditive measure can be found in Dow and Werlang(1994).

widely accepted solution to the paradox - they showed that the entrant may stay out if the game is repeated many times, and there is a small chance that the monopolist is irrational, in the sense that the monopolist might like to fight entry. The payoffs are self-explanatory: P_m the monopolistic profit, P_C the Cournot duopoly profit per firm and L the absolute value of the loss, in case the monopolist decides to fight entry. Clearly, the monopolist will acquiesce. This means that the entrant will set a zero probability of the monopolist fighting. Let p be the probability that the entrant assigns to the monopolist conceding. Then, as long as p is greater than $L/(L+P_C)$, the entrant enters. However, if p is less than this value, the entrant stays out, despite the fact that the monopolist would acquiesce (i. e., upon entry it would be infinitely more likely for the monopolist to concede than to fight) if there were entry. The interpretation of p being less than some value is the same as saying that the uncertainty aversion[3] of the entrant is larger than $P_C/(L+P_C)$. Thus, the intuition is clear: if the profits after entry are not high with respect to the potential losses in case there is a fight, then the entrant stays out. This seems to be a much more reasonable behavior on the part of the entrant. Of course, the same would occur in a repetition of this game. Here we see a situation where the backward induction algorithm works and delivers a very interesting result.

The next example is the centipede game, in Figure 4. First described by Rosenthal(1981), the centipede game has been tested experimentally (McKelvey and Palfrey(1992)), and an "irrational" behavior explanation was found. It is fair to note that many other similar games have been analyzed, as the TOL (take-it-or-leave-it) game of Reny(1986). Dow, Orioli and Werlang(1996) explain the majority of the observed phenomena in McKelvey and Palfrey's experiment by using the Nash equilibrium under uncertainty of the whole game. It is easy to prove that the backward induction procedure that we devised here will select as a best response for player 1 taking in the first round. Therefore, it does not account for the experimental results.

4. Conclusion

Our results are very partial, and show just that a class of subgame perfect equilibria under uncertainty exists, despite the fact that the one person decision problem under Knightian uncertainty is known to be (in general) inconsistent. it is also somewhat surprising that a backward induction algorithm akin to Zermelo-von Neumann-Kuhn works for perfect information games. Our results also point to the fact that this class may be small, mainly because it fails to explain some observed phenomena.

Dow and Werlang(1994) show that cooperation arises in the twice repeated prisoner's dilemma. In their equilibrium in the second stage the players never cooperate. Therefore, it is clear that there is a larger class of

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[3] The uncertainty aversion of a probability P at an event A is $c(P,A)=1-P(A)-P(A^c)$. It is usually interpreted as the degree or the dislike of uncertainty - see Dow and Werlang(1992a).

games for which the subgame perfect equilibria under uncertainty exist. We believe that there are several ways to proceed. The most important is to prove a broader result that will allow embedding any equilibrium of a subgame into the larger game (obviously, with a different way to join the beliefs). We also think that the problem of existence of subgame perfect equilibria with arbitrary degree of uncertainty (measured by the uncertainty aversion) in any subgame is interesting, and may even be false in all generality. The clear understanding of the reasons that subgame perfection cannot explain some observed phenomena is also a challenge for the theory. Finally, another way to pursue is to follow along the lines of Lo(1995b).

References

Boff, Hugo Pedro and Sérgio Ribeiro da Costa Werlang (1995), "Oligopolistic Competition under Knightian Uncertainty". Mimeo, EPGE/FGV.

Dow, James, Sérgio Orioli and Sérgio Ribeiro da Costa Werlang(1996), "The Centipede Game under Knightian Uncertainty". Mimeo, EPGE/Getulio Vargas Foundation.

Dow, James, Mario Henrique Simonsen and Sérgio Ribeiro da Costa Werlang(1993), "Knightian Rational Expectations, Inflationary Inertia and Money Neutrality". Mimeo, EPGE/Getulio Vargas Foundation.

Dow, James and Sérgio Ribeiro da Costa Werlang (1992a), "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio", Econometrica 60: 197-204.

Dow, James and Sérgio Ribeiro da Costa Werlang (1992b), "The Ex-ante Non-optimality of the Dempster-Shafer Updating Rule". In Moses, Yoram ed., Proceedings of the Fourth Conference on Theoretical Aspects of Reasoning About Knowledge. San Mateo, CA: Morgan Kaufman Publishers.

Dow, James and Sérgio Ribeiro da Costa Werlang (1994), "Nash Equilibrium under Knightian Uncertainty: Braking Down Backward Induction". Journal of Economic Theory 64: 305-324.

Eichberger, Jürgen and David Kelsey(1994a), "Non-additive Beliefs and Game Theory". University of Birmingham discussion paper no. 94-01.

Eichberger, Jürgen and David Kelsey(1994b), "Uncertainty Aversion and Dynamic Consistency". Mimeo, Department of Economics, University of Birmingham.

Epstein, Larry (1995), "Preference, Rationality and Rationalizability". Decision under Uncertainty: Theory, Applications, Experimental Evidence, Conference of the International School of Economic Research, University of Siena, July, pp. 131-149.

Epstein, Larry and Michel Le Breton (1993), "Dynamically Consistent Beliefs Must be Bayesian", Journal of Economic Theory 61:1-22.

Ghirardato, Paolo (1994), "Some Remarks on Product Spaces and a Fubini Theorem". Mimeo, Department of Economics, University of California at Berkeley.

Ghirardato, Paolo and Michel Le Breton (1996), "Choquet Rationaizability". Mimeo, California Institute of Technology.

Gilboa, Itzhak (1987), "Expected Utility with Purely Subjective Non-Additive Priors", Journal of Mathematical Economics 16: 279 - 304.

Gilboa, Itzhak and David Schmeidler (1989), "Maxmin Expected Utility with a Non-Unique Prior", Journal of Mathematical Economics 18: 141- 153.

Gilboa, Itzhak and David Schmeidler (1993), "Updating Ambiguous Beliefs", Journal of Economic Theory 59: 33-49.

Hendon, Ebbe, Hans Jørgen Jacobsen, Birgitte Sloth and Torben Tranæs(1993), "Nash Equilibrium with Lower Probabilities", mimeo, Instituite of Economics, University of Copenhagen.

Klibanoff, Peter (1995), "Uncertainty, Decision, and Normal Form Games". Journal of Economic Theory forthcoming.

Kreps, David and Robert Wison (1982), "Reputation and Imperfect Information", Journal of Economic Theory 27:253-279.

Kuhn, Harold (1953), "Extensive Games and the Problem of Information", Annals of Mathematics Studies no. 28. Princeton University Press.

Lo, Kin Chung (1994), "Equilibrium Beliefs under Uncertainty". Mimeo, Department of Economics, University of Toronto.

Lo, Kin Chung (1995a), "Nash Equlibrium without Mutual Knowldge of Rationality". Mimeo, Department of Economics, University of Toronto.

Lo, Kin Chung (1995b), "Extensive Form Games with Uncertainty Averse Players". Mimeo, Department of Economics, University of Toronto.

Marinacci, Massimo (1994), "Equilibrium in Ambiguous Beliefs". Mimeo, Northwestern University.

McKelvey Richard and Thomas Palfrey (1992), "An Experimental Study of the Centipede Game". Econometrica 60: 803-836.

Milgrom, Paul and John Roberts (1982), "Predation, Reputation and Entry Deterrence", Journal of Economic Theory 27:280-312.

Mukerji, S. (1994), "A Theory of Play for Games in Strategic Form when Rationality is not Common Knowledge", mimeo, University of Pennsylvania.

von Neumann, John and Oskar Morgenstern (1953), Theory of Games and Economic Behavior, 3rd edition. Princeton: Princeton University Press.

Reny, Philip (1986), "Rationality, Common Knowledge, and the Theory of Games". Ph.D. Dissertation, Princeton University.

Rosenthal, Robert (1981), "Games of Perfect Information, Predatory Pricing and the Chain-store Paradox". Journal of Economic Theory 25: 92-100.

Schmeidler, David (1982), "Subjective Probability and Expected Utility Without Additivity". Foerder Institute for Economic Research, Tel-Aviv University, working paper.

Schmeidler, David (1989), "Subjective Probability and Expected Utility Without Additivity", Econometrica 57: 571 - 587.

Selten, Reinhard (1965), "Spieltheoretische Behandlung eines Oligoplmodells mit Nachfrageträgheit", Zeitschrift für die Gesamte Staatswiessenchaft 12: 301-324.

Selten, Reinhard (1978), "The Chain-store Paradox", Theory and Decision 9: 127-159.

Simonsen, Mario H. and Sérgio R. C. Werlang (1991), "Subadditive Probabilities and Portfolio Inertia", Revista de Econometria 11: 1 - 20.

Zermelo, E. (1913), "Über eine Anwendung der Mengenlehre auf der Theorie des Schachspiels", in Annals of the Fifth International Congress on Mathematics.

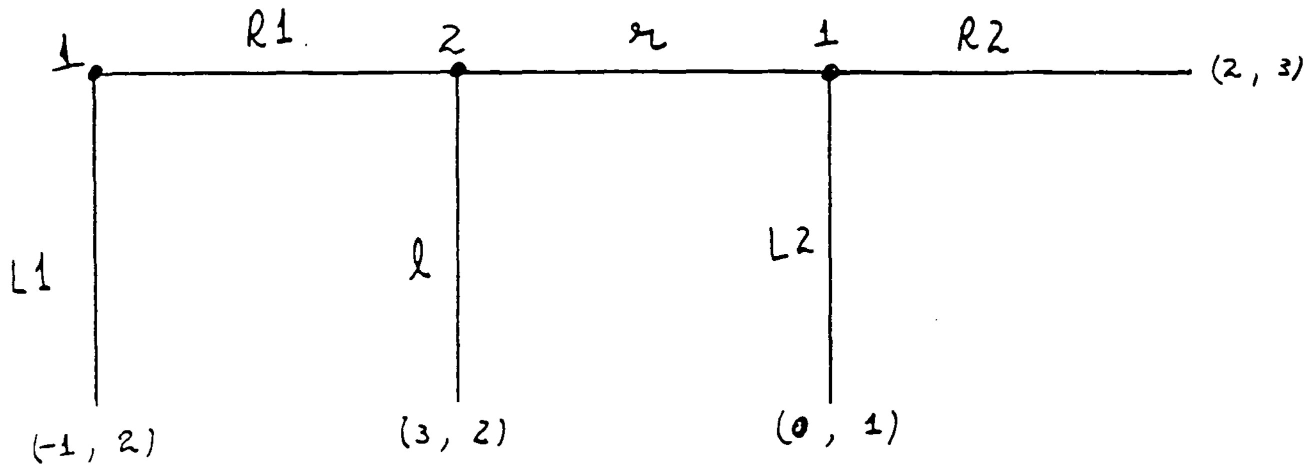

FIGURE 1

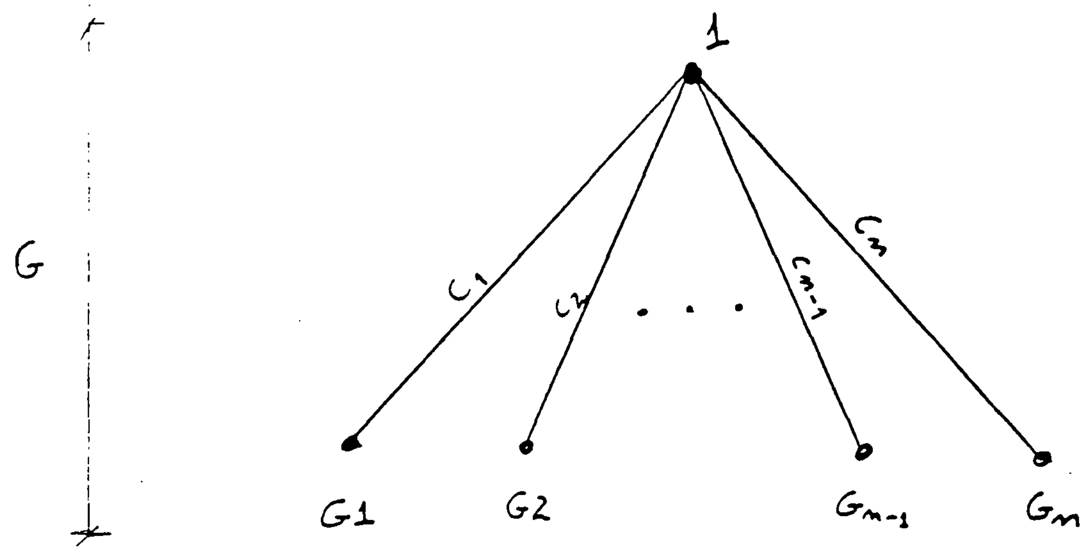

FIGURE 2

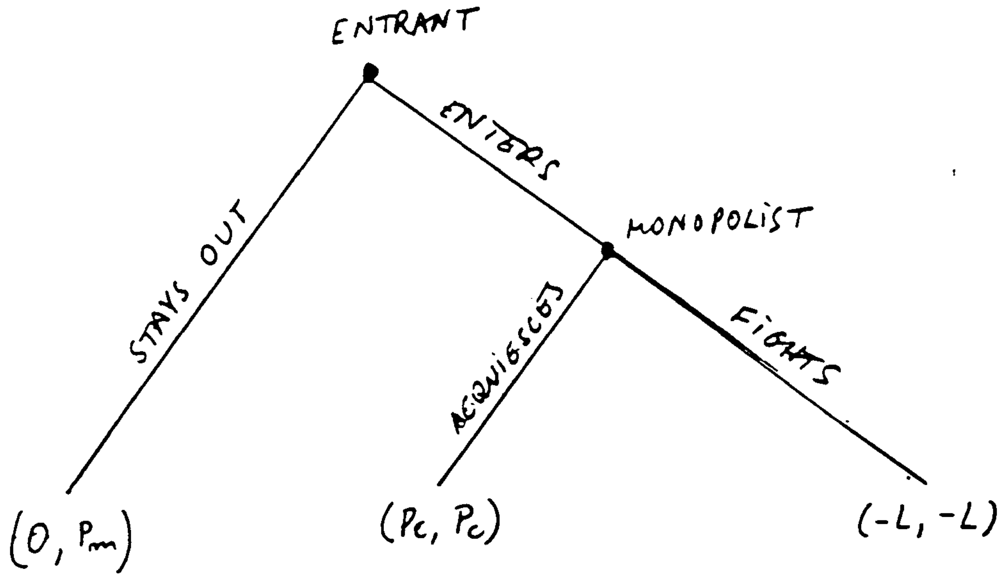
ENTRANT
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FIGURE 3

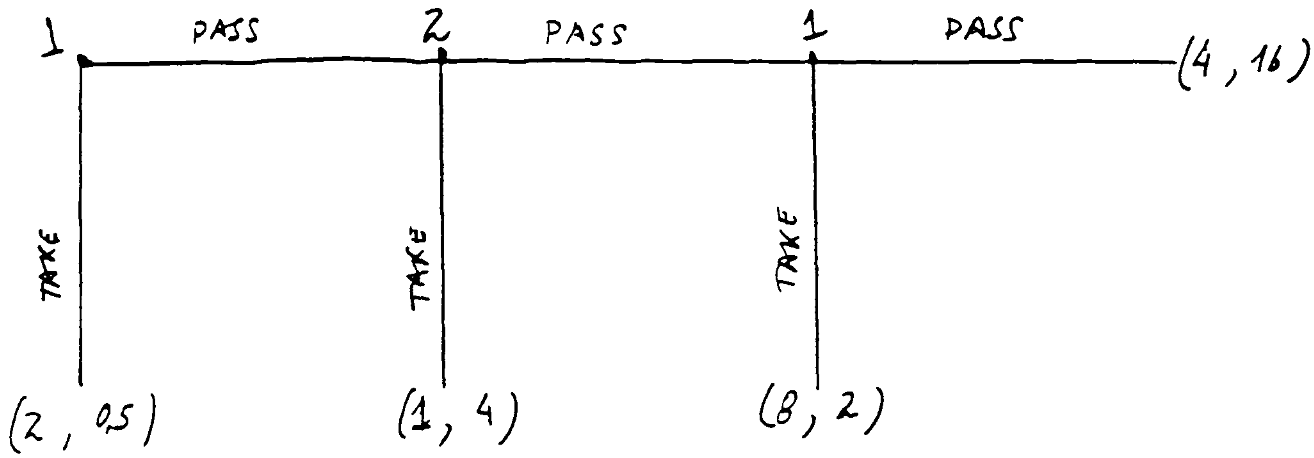

FIGURE 4