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# "Market Clearing Sealed-Bid Auctions for Non-Identical Objects With Single-Unit Demands"



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# MARKET CLEARING SEALED-BID AUCTIONS FOR NON-IDENTICAL OBJECTS WITH SINGLE -UNIT DEMANDS

By MARILDA SOTOMAYOR[1]


## ABSTRACT

We consider a version of the cooperative buyer-seller market game of Shapley and Shubik (1972). For this market we propose a class of sealed- bid auctions where objects are sold simultaneously at a market clearing price rule. We analyze the strategic games induced by these mechanisms under the complete information approach. We show that these non-cooperative games can be regarded as a competitive process for achieving a cooperative outcome: every Nash equilibrium payoff is a core outcome of the cooperative market game. Precise answers can be given to the strategic questions raised.


Key words: optimal matching, competitive price, Nash equilibrium, mechanism.
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## 1- INTRODUCTION

There are a set of buyers and a set of sellers. Each seller owns one indivisible object and each buyer wants to buy, at most, one object. Each participant places a monetary value on each of the objects. We call these monetary values the reservation prices of the agents. Each buyer has cash on hand sufficient to pay any of her reservation prices. The function of the market is to find a *feasible allocation*, i.e. a *feasible matching* and a *feasible price* for each object. A feasible matching is an assignment of the set of objects to the group of buyers, such that each object is assigned to at most one buyer and each buyer is assigned to at most one object. The price of an object is feasible if it is greater than or equal to the seller's reservation price. The rules of this game are that any buyer and any seller can make a transaction and divide among themselves the gain from trade in any way they like. Moreover a buyer is free to buy nothing and a seller may keep his object unsold. If a buyer acquires an object her payoff is given by the difference between her reservation value for the object and the price paid for it. This model is a version of the cooperative buyer-seller market game of Shapley and Shubik (1972).

Examples of such a market are the real estate market[2], or the market for used cars[3]. In these models a buyer has no additional utility for any supplementary object, although she has preferences defined over the set of all objects. One may also think of the quota as an institutional restriction of the market, as is often the case in public sector auctions; in this case, the restriction aims at limiting the market power of the winners. [4]

A relevant question is: *What are the feasible allocations one would expect to observe in those markets?*

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2 When houses are bought for living in rather than as an investment, i.e., each buyer wants to buy one house and has preferences over different types of houses.

3 When each buyer wants to buy exactly one car and has preferences over a car's different attributes.

4 In Brazil, the licenses to use cellular telephone services vary both in geographic coverage and amount of spectrum covered, and are classified in two types. Each type covers a geographic area corresponding to a certain social level of the population. These licenses were auctioned under the restriction that the buyers could buy at most two licenses, one of each type. One can think of it as two markets of non-identical objects each, such that a buyer can buy at most one license in each market.

The answer to this question lies on the presumption that only Pareto efficient and envy-free feasible allocations will occur. The solution concept for this sort of market game, which captures this intuitive idea of fairness, is that of competitive equilibrium.[5] The idea is then to design mechanisms to produce these competitive outcomes.

Let us recall briefly this solution concept. A price vector is a competitive equilibrium price if it is feasible and supports a feasible matching such that (a) no unsold object has a price greater than its reservation price, (b) if a buyer is assigned to some object then this object maximizes the buyer's surplus, the difference between her valuation and the price of the item, assuming this surplus is non-negative and (c) if a buyer is not assigned to any of the objects then she has a non-positive surplus for every object.

It is proved in Shapley and Shubik (1972) that every competitive equilibrium price supports an optimal matching and every optimal matching is supported by a competitive equilibrium price. An optimal matching is roughly defined as follows. If an object $q$ is assigned to some buyer $p$ then the value of the pair $(p,q)$ is the monetary value placed by $p$ on $q$ minus the reservation price of $q$ to the seller, assuming this difference is non-negative. Given a matching, the sum of the values of the assigned pairs is called value of the matching. A matching is optimal if it attains the maximum value among all feasible matchings.[6] The rigorous definition of such a matching will be given in the text.

Several authors proved the existence of competitive equilibria. Gale (1960) and Shapley and Shubik (1972) proved the existence of competitive equilibria by using linear programming. There is also a simple proof of Sotomayor (1999) using combinatorial arguments. Demange, Gale and Sotomayor (1986) present two dynamic mechanisms that

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[5] Let $a_{jk}$ and $s_k$ be the reservation prices of object $k$ for buyer $j$ and seller $k$, respectively. If the object $k$ is sold to buyer $j$ at the price $\pi \geq s_k$, the payoff of the buyer will be $u_j = a_{jk} - \pi$ and the payoff of the seller will be $v_k = \pi - s_k$. The gain from trade will be $a_{jk} - s_k$. Then $u_j = (a_{jk} - s_k) - v_k$ and $v_k = (a_{jk} - s_k) - u_j$. The allocation of the buyer will be the pair $(\{k\}, -v_k)$ and the allocation of the seller will be the pair $(\{j\}, -u_k)$. If a buyer is not assigned to any object her payoff will be zero and her allocation will be $(\phi, 0)$. If an object is unsold the payoff of the seller will be zero and his allocation will be $(\phi, 0)$. A feasible allocation is envy-free if every agent likes her own allocation at least as well as that of anyone else. It is easy to check that if every buyer (resp. seller) weakly prefers her (resp. his) own allocation to that of any other buyer (resp. seller) then the feasible allocation is envy-free. In the literature a feasible allocation has been called fair if it is Pareto efficient and envy-free. (See Alkan, Demange and Gale (1991)).

[6] For the special case in which there is only one object to be sold and the reservation price of the object to the seller is zero, an optimal matching is a matching which assigns the object to the buyer with the highest value, assuming this value is non- negative. For the case where m buyers are competing for n identical objects, with $m \geq n$, and the reservation prices of the objects to the seller are zero, an optimal matching assigns one object to each of the n buyers with the n highest values, assuming these values are non-negative.

lead competitive equilibria. The first one yields the minimum competitive equilibrium price when all variables are discrete in money. The second one determines a price, which converges to the minimum competitive equilibrium price.[7] Crawford and Knoer (1981) introduced this mechanism, which is a generalization of the deferred acceptance algorithm of Gale and Shapley (1962). The Crawford and Knoer's algorithm finds the competitive equilibrium price for a discrete model. These authors prove the existence of competitive equilibrium for the continuous case by arguing that the algorithm's outcome converges to a competitive equilibrium as the unit of money approaches to zero.

Quinzii (1984), Kaneko (1982), Kaneko and Yamamoto (1986), Alkan (1988, 1992) and Alkan and Gale (1988) prove the existence of competitive equilibria in a more general model.

In the present paper we propose a family of simultaneous sealed-bid auctions under complete information about the reservation values of the bidders. In any of these auctions each bidder chooses a set of objects and specifies a bid for each of them. The auctioneer, according to some auction rule, then finds a competitive equilibrium. An auction rule consists of a price rule and a matching rule. The matching rule will be called a tie-breaking rule. The price rule determines a competitive equilibrium price for each set of joint bids.[8] The tie-breaking rule operates according to two steps. At the first one the auctioneer determines the set of optimal matchings corresponding to the set of submitted bids. Among these matchings he selects those that attain the highest value under the true valuations. At the second step one of these matchings is chosen. Thus, for each set of joint bids the tie-breaking rule selects one corresponding optimal matching. When there is a single object to be sold, this criterion is equivalent to requiring the auctioneer to sell the object to a bidder having the highest valuation among the bidders with the highest bid.

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[7] A competitive equilibrium price is called the minimum competitive equilibrium price if it is smaller, in each coordinate, than any competitive equilibrium price vector. The maximum competitive equilibrium price is analogously defined. The existence and uniqueness of these two prices is a consequence of the complete lattice structure of the set of competitive equilibrium prices (see Roth and Sotomayor, 1990).

[8] Examples of such rules are the minimum competitive equilibrium price rule, the maximum competitive equilibrium price rule, as well as the rule that produces any convex combination of the minimum and the maximum competitive equilibrium prices. The fact that the last rule is well defined is due to the convexity of the set of competitive equilibrium prices (see Roth and Sotomayor, 1990).

Each of these mechanisms induces a strategic game under complete information. These games will be called *auction games*.

The idea of using competitive equilibria as a mechanism for making allocations with desirable properties of fairness and efficiency has been widely explored in the literature. In the second price auction, first described by Vickrey (1961), buyers submit sealed bids for a single object which is then sold to the highest bidder at a price equal to the second highest bid. We can think of the second price auction outcome as an ordinary competitive equilibrium. At the given price the object is demanded by only one buyer, namely the highest bidder, and, as there is only one object, this yields balance of supply and demand. The important property of the second highest bid is that it is the minimum competitive equilibrium price, since for any smaller price at least two bidders would demand the object.

Demange (1982) and Leonard (1983) consider an allocation mechanism which generalizes the second price auction of a single item to the multi-item case. In their multi-item auction mechanism each buyer is required to submit a sealed bid, listing her valuation for all the items. The auctioneer then allocates the objects in accordance with the minimum competitive equilibrium price. These papers prove that the important "incentive compatibility" of the Vickrey auction carries over to the multi-item case, meaning that submitting the true valuation is a dominant strategy for the buyers. More generally, no subset of buyers by jointly misrepresenting valuations can improve the outcome for all of its members[9].

In this paper we will restrict the analysis to the class of auctions where the price rule is not the minimum competitive equilibrium price rule. We show that these games have some common features concerning the strategic possibilities for the buyers.

We are interested on the strategic questions faced by the bidders not only in a particular game, but also in more general situations in which the second step of the tie-breaking rule is not fully known. In such cases the buyers only know the price rule and the

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[9] This result is due to Demange and Gale (1985) and it is a special case of Theorem 9.23 presented in Roth and Sotomayor (1990). Theorem 9.23 is due to Sotomayor (1986). For a comprehensive account of these results see Roth and Sotomayor (1990).

first step of the tie-breaking rule. Thus, they do not know how the auctioneer will choose the final matching from the set of optimal matchings, obtained in the first step of the tie-breaking rule. The strategic analysis of such situations is a central issue of this paper. Without a specification of the rules, the game is not well defined. In this case the notion of equilibrium will be made precise (in the text) employing the concepts of weak sense and strong sense Nash equilibrium.[10] (Roughly speaking, under the weak concept a bidder changes her strategy when she is sure to be better off, while in the strong sense she changes it whenever she has a chance to be better off).

For a particular game we show that when there is more than one competitive equilibrium, truth telling is not always the best policy for at least one buyer. Thus, this result suggests that is meaningful to inquire about the existence of a Nash equilibrium in the usual sense: a joint strategy which has the property that once it is adopted no buyer may profit from deviation.

An example shows that, even in auctions with the same price rule, the true payoffs for the buyers may depend on the matching selected by the auctioneer. Thus the same joint strategy may yield different payoffs to the bidders under different matchings. Another example illustrates a situation in which there is a joint strategy that is a Nash equilibrium in the strong sense under some tie-breaking rule and is not a Nash equilibrium in the weak sense under some other rule. Furthermore the corresponding true payoffs of the buyers are not the same. This suggests to ask whether for a given price rule, there exists a joint strategy that, under any tie-breaking rule, it is a Nash equilibrium and yields the same true payoff for the buyers. The answer is affirmative. We show that there is a joint strategy $\beta^*$ with these properties. The corresponding auction outcome is the minimum competitive equilibrium price under the true valuations (mCEP, for short). Furthermore these properties of $\beta^*$ do not depend on the price rule.

When there exists more than one Nash equilibrium payoff, there may be a conflict of interests among the buyers with respect to their strategic choices. Thus, it would be nice if the mCEP was the only Nash equilibrium payoff under any tie-breaking rule. Nevertheless this is not the case. We show, by using examples, that there may be Nash

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[10] These concepts were used in Demange and Gale (1985).

equilibria different from $\beta^*$ and that there may be more than one Nash equilibrium payoff. The solution of this problem is due to a strong result: *In any auction game every Nash equilibrium outcome is a competitive equilibrium under the true valuations.* Hence every Nash equilibrium payoff is in the core[11] of the cooperative market game. In this sense each non-cooperative auction game can be regarded as a competitive process for achieving a cooperative outcome.[12] Since some buyers will have the incentive to misrepresent their valuations, it is surprising that, even in equilibria, the resulting outcomes are competitive equilibria with respect to the true valuations.

To see how the result above solves the conflict of interests problem suppose that some bidder prefers some auction outcome to the minimum competitive equilibrium. Then this outcome will not be a competitive equilibrium under the true valuations, so it will not be produced by a Nash equilibrium strategy, so some buyer will have an incentive to deviate from it. Consequently there is no Nash equilibrium payoff, which is preferred to some bidder than the mCEP. Furthermore the allocation of the objects is an optimal matching under the true valuations. That is, every auction of this class is efficient.[13] Therefore $\beta^*$ can be regarded as the best way of play for the bidders in the sense that a) it is a Nash equilibrium for every auction game and b) under this joint strategy the objects are allocated efficiently and the prices constitute the mCEP, so it gives the bidders the highest possible payoff.

Thus, if buyers play $\beta^*$ when the price rule is not the minimum competitive price rule, the price obtained in any of the auctions will be the mCEP. This also is the price obtained under the minimum competitive price rule when the bidders play their dominant strategies. Consequently, the sellers can expect to sell their objects by the same price under every auction rule.

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[11] Shapley and Subik (1972) proved that the core of this market game coincides with the set of vector payoffs corresponding to the competitive equilibria.

[12] The idea of obtaining competitive allocations via a non-cooperative game was also explored in Wilson (1977). This author considers an exchange economy with indivisible goods, in which the trades are organized as a bidding game. Under several regularity assumptions (which do not apply to our model) it is shown that this game has an equilibrium which yields a core allocation of the corresponding cooperative game of exchange. The buyers are replicated and this core allocation is a competitive equilibrium in the limit.

[13] When the objects are identical an efficient auction is one which puts goods into possession of the buyers who value them most. Among the many authors who have searched for efficient auctions we can mention P. Dasgupta and E. Maskin (1998).

It would be nice if we could state that every competitive equilibrium, under the true valuations, is a Nash equilibrium outcome, because we would then have an implementation result. Nevertheless an example shows that this is not always the case.

Finally, it would be interesting to know under what circumstances can one guarantee that a Nash equilibrium in the strong or in the weak sense does not depend on the tie-breaking rule. We prove that if the buyers only choose acceptable[14] objects, then the Nash equilibrium strategies in the weak or in the strong sense, the prices of the objects and the corresponding true payoffs of the buyers do not depend on the auctioneer's particular choice.

In section 2 we describe the cooperative model. In section 3 we present the sealed - bid auctions. Section 4 is devoted to the analysis of the strategic possibilities for the buyers under any competitive sealed-bid auction. Final remarks are contained in section 5. Proofs are given in an Appendix.

## 2-THE COOPERATIVE MODEL AND SOME KNOWN RESULTS

There is a set $P$ with $m$ buyers and a set $Q$ with $n$ indivisible objects. $P=\{p_1,p_2,...,p_m\}$ and $Q=\{q_1,q_2,...,q_n\}$. Letters $j$ and $k$ will be reserved to index buyers and objects, respectively. The reservation price of object $q_k$ to the seller is $s_k$. Each buyer $p_j$ values object $q_k$ in $\alpha_{jk}\geq 0$ and wants to buy, at most, one object. Thus, if buyer $p_j$ purchases object $q_k$ at a price $\pi \geq s_k$, her payoff will be $u_j=\alpha_{jk}-\pi$ and the payoff of seller $q_k$ will be $v_k=\pi-s_k$. The potential gains from trade between $j$ and $k$ will be $u_j+v_k=\alpha_{jk}-s_k$. When $s_k=0$ for all $q_k\in Q$, this model is the well known Assignment Game presented in the book of Roth and Sotomayor (1990), due to Shapley and Shubik. For our purposes it will be more convenient to work with the values $a_{jk}=\max\{0, \alpha_{jk}-s_k\}$. Since the definition of $a_{jk}$ does not specifies when $\alpha_{jk}-s_k=0$ and when $\alpha_{jk}-s_k<0$ we should inform, for each buyer $j$, the set of objects $k$ such that $\alpha_{jk}-s_k\geq 0$. The introduction

[14] We say that an object is acceptable to a buyer if her reservation value is a feasible price for this object.

of this information in the description of the model yields a change in the definition of feasible matching (assignment) presented in Roth and Sotomayor (1990). The other definitions are kept unchanged. The proofs of the results presented in this section can be found in the book mentioned above. The vector of numbers $(a_{j1},...,a_{jn})$ will be denoted by $a_j$. The matrix $(a_{jk})$ will be denoted by $a$.

**Definition 1**. We say that object $q_k$ is *acceptable* to buyer $p_j$ if $a_{jk}\geq s_k$, and it is not acceptable to buyer $p_j$, otherwise.

Thus, an object is ***not acceptable*** to a buyer if there is no feasible price at which the buyer wishes to buy the object.

The set of acceptable objects to $p_j$ will be denoted by $A_j$. The set of $A_j$'s will be denoted by $A$.

*The market is specified by* $M\equiv(P,Q,(a,A),s)$. We will use the notation $\Sigma_j$ to denote the sum over all $p_j$ in $P$, $\Sigma_k$ to denote the sum of all $q_k$ in $Q$ and $\Sigma_{j,k}$ to denote the sum over all $p_j$ in P and $q_k$ in Q.

**Definition 2**. *A **matching** for $M=(P,Q,(a,A),s)$ is a matrix $x = (x_{jk})$ of zeros and ones. A matching $x$ is **feasible** if it satisfies a) $\Sigma_j x_{jk} \leq 1$ $\forall q_k\in Q$, b) $\Sigma_k x_{jk} \leq 1$ $\forall p_j \in P$ and c) if $x_{jk}=1$ then $q_k\in A_j$.*

Condition a) states that a feasible matching assigns an object to at most one buyer; condition b) states that a feasible matching assigns a buyer to at most one object. As for condition c) it means that the object matched to a buyer is acceptable to her. If $x_{jk}=0$ for all $q_k\in Q$ (resp. $p_j\in P$) we say that $p_j$ (resp. $q_k$) is *unmatched*. If $x_{jk}=1$ we say that $p_j$ is matched to $q_k$ or $q_k$ is matched to $p_j$.

**Definition 3**. *The feasible matching $x$ is **optimal** for $M$ if, for all feasible matchings $x'$, $\Sigma_{j,k}\ a_{jk}x_{jk} \geq \Sigma_{j,k}\ a_{jk}\ x'_{jk}$.*

We call $\sum_{j,k} a_{jk} x_{jk}$ the value of matching $x$. Thus $x$ is optimal if it attains the maximum value among all feasible matchings.

The existence of an optimal matching is guaranteed by the fact that there are only a finite number of matchings.

To see which changes are caused by the introduction of the set $A$ in the description of the model, see Example 1 below.

**Example 1.** Consider the market given by $P=\{p_1,p_2\}$, $Q=\{q_1,q_2\}$, $\alpha_1=(2,3)$, $\alpha_2=(1,2)$ and $s=(2,0)$. Then both objects are acceptable to $p_1$ but $q_1$ is not acceptable to $p_2$, because $\alpha_2<s_1$. Therefore $q_1$ cannot be matched to $p_2$ under a feasible matching. The matrix $a$ is given by $a_1=(0,3)$ and $a_2=(0,2)$. According to the approach of Shapley and Shubik the matching $x$, given by $x_{12}=x_{21}=1$, is optimal because $a_{12}+a_{21}=3>2=a_{11}+a_{22}$. However, according to our approach, $x$ is not optimal for $M=(P,Q,(a,A),s)$, since it is not feasible. The optimal matching for $M$ matches $q_2$ to $p_1$ and leave $q_1$ and $p_2$ unmatched.∎

**Remark 1**. Let $x$ be some optimal matching for a given market. If some $p_j$ and $q_k$ are unmatched, then either $q_k$ is not acceptable for $p_j$ in this market or the pair $(p_j,q_k)$ contributes with a payoff zero. The last assertion is so because otherwise $p_j$ and $q_k$ could be matched to each other contributing with a positive payoff, which contradicts the optimality of $x$. Along this paper we will be considering that $p_j$ and $q_k$ are both unmatched at some optimal matching for a given market if and only if $q_k$ is not acceptable for $p_j$ in this market. There is no loss by doing this. ∎

**Definition 4.** *The pair of vectors $(u,v)$, with $u$ in $R^m$ and $v$ in $R^n$, is called a **feasible payoff** for $M$ if there is a feasible matching $x$ such that $\sum_j u_j + \sum_k v_k=\sum_{j,k} a_{jk} \cdot x_{jk}$.*

In this case we say $(u,v)$ and $x$ are ***compatible*** with each other, and we call $(u,v;x)$ a ***feasible outcome***. Note that a feasible payoff vector may involve monetary transfers

between agents who are not matched to one another.

**Definition 5.** *The outcome* $(u,v;x)$ *is* ***stable for*** $M$ *(or the payoff* $(u,v)$ *with a matching* $x$ *is stable for* $M$*) if it is feasible and*

i) $u_j \geq 0$, $v_k \geq 0$ *(individual rationality) and*

*ii)* $u_j + v_k \geq a_{jk}$

*for all* $(p_j, q_k) \in P x Q$.

The interpretation of condition ii) is the natural one: if it is not satisfied for some $p_j$ and $q_k$, then buyer $p_j$ and the seller of $q_k$ could break up their present partnership and form a new one together, because this could give them each a higher payoff. The pair $(p_j, q_k)$ is called a *blocking pair.*

**Proposition 1*** (Shapley and Shubik, 1972) *Let* $(u,v;x)$ *be a stable outcome for* $M$. *Then*

*a)* $u_j + v_k = a_{jk}$ *if* $x_{jk} = 1$,

*b)* $u_j = 0$ *for all unmatched* $p_j$, *and* $v_k = 0$ *for all unmatched* $q_k$ *at* $x$.

Thus if $(u,v;x)$ is a stable outcome for $M$, and $x_{jk} = 1$, buyer $p_j$ pays $v_k + s_k$ for object $q_k$ and has a payoff equal to $u_j = a_{jk} - v_k$. The net price of object $q_k$ is $v_k$. Proposition 1* implies that at a stable outcome, the only monetary transfers that occur are between buyers and sellers who are matched to each other.

**Remark 2.** Observe that every outcome which satisfies a) and b) of Proposition 1* is feasible for $M$. Hence an outcome is stable if and only if it satisfies i) and ii) of Definition 5 and a) and b) of Proposition 1*.∎

**Theorem 1*.** (Shapley and Shubik, 1972)

*a) The set of stable payoffs and the core of* $M$ *are the same.*

*b) The core of* $M$ *is the (nonempty) set of solutions of the dual linear programming problem corresponding to the matrix* $\boldsymbol{a}$.

**Corollary 1*.** (Shapley and Shubik, 1972) *If* $x$ *is an optimal matching, then it is compatible with any stable payoff* $(u,v)$.

**Corollary 2*.** (Shapley and Shubik, 1972) *If* $(u,v;x)$ *is a stable outcome then* $x$ *is an optimal matching.*

**Proposition 2*** (Demange and Gale, 1985). *Let* $(u,v)$ *be a stable payoff for* $M$. *Then if* $u_j>0$ *(resp.* $v_k>0$*),* $p_j$ *(resp.* $q_k$*) is matched by any optimal matching.*

Shapley and Shubik proved that the set of stable payoffs reflects a polarization of interests between the two sides of the market. Associated with each side of the market, there is an *optimal* stable payoff that is the stable payoff most preferred by every agent on that side of the market and least preferred by every agent on the other side of the market. That is, there is a P-optimal stable payoff such that all buyers (weakly) prefer it to every other stable payoff, and all sellers in Q (weakly) prefer any other stable payoff to it, and there is a Q-optimal stable payoff with the symmetric properties. Formally,

**Definition 6.** *A stable payoff* $(u^*,v_*)$ *is the* ***P-optimal stable payoff*** *for the market* $M$ *if for any stable payoff* $(u,v)$, $u^*\geq u$ *and* $v_*\leq v$. *The* ***Q-optimal stable payoff*** $(u_*,v^*)$ *is symmetrically defined.*

*If* $x$ *is an optimal matching then* $(u^*,v_*;x)$ *is a P-optimal stable outcome and* $(u_*,v^*;x)$ *is a Q-optimal stable outcome.*

**Theorem 2*.** (Shapley and Shubick, 1972) *There always exist the P-optimal stable payoff and the Q-optimal stable payoff for the market* $M$.

**Theorem 3*** (Demange and Gale, 1985). *For the market* $M$, *let* $(u^*,v_*)$ *and* $(u_*,v^*)$ *be the P-optimal stable payoff and the Q-optimal stable payoff. Let* $P'\subseteq P$. *Let* $M'=(P',Q,(a',A'),s)$, *where* $(a',A')$ *is the restriction of* $(a,A)$ *to* $P'$. *For the market* $M'$,

*let* $(u^*(M'),v_*(M'))$ *and* $(u_*(M'),v^*(M'))$ *be the P-optimal stable payoff and the Q-optimal stale payoff, respectively. Then* $u^*_j \leq u^*_j(M')$, $u_{*j} \leq u_{*j}(M')$, $v_{*k} \geq v_{*k}(M')$ *and* $v^*_k \geq v^*_k(M')$ *for all* $p_j \in P'$ *and all* $q_k \in Q$.

Theorem 3* states that if a set of buyers leave the market, then $u^*$ and $u_*$ do not decrease and $v^*$ and $v_*$ do not increase.

**Definition 7.** *A* ***feasible price vector*** $v$ *for the market* $M$ *is a function from* $Q$ *to* $R$ *such that* $v_k \equiv v(q_k)$ *is greater than or equal to zero.*

**Definition 8.** *A* ***feasible allocation*** *is a pair* $(v,x)$, *where* $v$ *is a feasible price and* $x$ *is a feasible matching.*

The *demand set* of a buyer $p_j$ at a feasible price $v$ is defined by

$$D_j\{v\}=\{q_k \in A_j;\ a_{jk}-v_k \geq 0 \text{ and } a_{jk}-v_k \geq a_{jt}-v_t \text{ for all } q_t \in Q\}.$$

That is, among all the acceptable objects that $p_j$ can buy at price $v$, $p_j$ chooses those ones which maximize her payoff at the given prices.

**Definition 9**. *The feasible allocation* $(v, x)$ *is a* ***competitive equilibrium*** *for* $M=(P,Q,(a,A),s)$ *if a) for all pair* $(p_j,q_k)$ *with* $x_{jk}=1$ *we have that* $q_k$ *is in* $D_j\{v\}$, *b) if* $p_j$ *is unmatched then* $a_{jk}-v_k \leq 0$ $\forall q_k \in Q$ *and c) if* $q_k$ *is unmatched then* $v_k=0$.

If $(v,x)$ is a competitive equilibrium, $v$ will be called a competitive equilibrium price vector or simply a competitive equilibrium price. In this case we say that $x$ is supported by $v$ or it is compatible with $v$. It is easy to verify that if $(v,x)$ is a competitive equilibrium, then the outcome $(u,v;x)$ is stable for $M$, where $u_j=a_{jk}-v_k$ if $x_{jk}=1$ and $u_j=0$ if $p_j$ is unmatched at $x$. We say that $(u,v;x)$ *corresponds* to $(v,x)$ and vice-versa. Thus the P-optimal stable payoff $(u^*,v_*)$ corresponds to the minimum competitive equilibrium price $v_*$ and the Q-optimal stable payoff $(u_*,v^*)$ corresponds to the maximum competitive equilibrium price $v^*$. Corollary 2* implies that $x$ is an optimal

matching. If $x$ is an optimal matching and $v$ is a competitive equilibrium price vector, Corollary 1* implies that $(v,x)$ is a competitive equilibrium.

For all $R \subseteq P$ and $S \subseteq Q$ we define $V(R,S)=\max \Sigma_{RxS}\, a_{jk}\, x_{jk}$, with the maximum to be taken over all feasible matchings $x$. $V(R,S)$ is a measure of the worth of the coalition $R \cup S$ in the market $M$.

**Theorem 4*** (Demange (1982), Leonard (1983)). *Let $v^*$ and $v_*$ be the maximum and minimum competitive equilibrium prices, respectively, for M. For all $q_k$ in $Q$*

*a)$v_{*k} = V(P-\{p_j\},Q) - V(P-\{p_j\},Q-\{q_k\})$ if $x_{jk}=1$ and $v_{*k}=0$ if $q_k$ is not matched by $x$.*

*b)$v^*_k = V(P,Q)-V(P,Q-\{q_k\})$*

**Theorem 5*** (Demange (1982), Leonard (1983)). *In a mechanism that produces the minimum competitive equilibrium price, truth telling is a dominant strategy for each buyer.*

## 3. SIMULTANEOUS SEALED BID MULTI-ITEM AUCTIONS

In this section we will interpret $P$ as a set of bidders. We will describe a class of simultaneous sealed bid auctions for the market $M=(P,Q,(a,A),s)$. The agents can communicate to each other so that the market $M$ becomes common knowledge among them. In any of the auctions each buyer $p_j$ selects a set $B_j$ of objects. Next she deposits the amount corresponding to the reservation price of that object, in $B_j$, which has the highest reservation price, among all objects in $B_j$. Then she submits a bid $b_{jk} \geq 0$ for each object $q_k \in Q$. The vector $b_j=(b_{j1},...,b_{jn})$ is enclosed in a sealed envelope and given to the auctioneer. If $q_k \notin B_j$ then $b_{jk}=0$ and $q_k$ cannot be matched to $p_j$ in the auction. We will use the notation $\beta=(b,B)$ to denote the collection of $(b_j,B_j)$' s.

Having $\beta$ the auctioneer finds a competitive equilibrium for the market $M(\beta)=(P,Q,\beta,s)$ according to the *auction rule* $(v(.),x(.))$. The component $v(.)$ is called *price rule*. For each $\beta$ it determines a competitive equilibrium price $v(\beta)$ for the market $M(\beta)=(P,Q,\beta,s)$. The component $x(.)$ is called *tie-breaking rule*. This rule operates in two

steps. In the first step the auctioneer finds $\Sigma(\beta)$, the set of all optimal matchings for $M(\beta)$. That is, $\Sigma(\beta)$ is the set of feasible matchings which attain the highest value in $M(\beta)$. Then if, say, $\Sigma(\beta)=\{x^1,x^2,\dots,x^t\}$ is the set of optimal matchings for $M(\beta)$, the auctioneer determines the set of matchings $x^*\in\Sigma(\beta)$ such that $\Sigma_{j,k}\, a_{jk}\, x^*_{jk}=\max\{\Sigma_{j,k}\, a_{jk}\, x^1_{jk}, \Sigma_{j,k}\, a_{jk}\, x^2_{jk}, \dots, \Sigma_{j,k}\, a_{jk}\, x^t_{jk}\}$. This set will be denoted by $\Sigma^*(\beta)$. That is, $\Sigma^*(\beta)$ is the set of matchings of $\Sigma(\beta)$ which attain the highest true value. At the second step of the tie-breaking rule the auctioneer chooses $x(\beta)\in\Sigma^*(\beta)$. Therefore if the buyers bid $\beta$ *the auction outcome will be* $(v(\beta),x(\beta))$. (The fact that $x(\beta)$ and $v(\beta)$ are compatible follows from Corollary 1*). If $q_k$ is matched to $p_j$ under $x(\beta)$, buyer $p_j$ will pay $v_k(\beta)$ to the auctioneer and will receive back the difference between the value of her deposit and $s_k$. Thus the auctioneer will receive the total of $(v_k(\beta)+s_k)$ from $p_j$. If $p_j$ is not matched to any object then $p_j$ will receive back the full amount deposited.

Two special types of auction rules are the minimum competitive equilibrium price sealed bid *(mCPSB)* auction rules and the maximum competitive equilibrium price sealed bid *(MCPSB)* auction rules. Under any *mCPSB* (resp. *MCPSB*) auction rule the auctioneer allocates the objects in accordance with some minimum competitive equilibrium (resp. maximum competitive equilibrium) for $M(\beta)$. That is, if the buyers bid $\beta$, the outcome of the *mCPSB* (resp. *MCPSB*) auction with the tie-breaking rule $x(.)$ is $(v_*(\beta), x(\beta))$ (resp. $(v^*(\beta),x(\beta))$. The prices $v_*(\beta)$ and $v^*(\beta)$ can be computed by using the rule given in Theorem 4*. For all $R\subseteq P$ and $S\subseteq Q$, $V_\beta(R,S)=\max \Sigma_{RxS}\, b_{jk}.x_{jk}$, with the maximum to be taken over all feasible matchings $x$ for the market $M(\beta)$. Thus, for each object $q_k$,

$$v_{*k}(\beta)= V_\beta(P-\{p_j\},Q) - V_\beta(P-\{p_j\},Q-\{q_k\}) \text{ if } x^*_{jk}=1$$

$$v_{*k}(\beta)=0 \text{ if } q_k \text{ is not matched by } x^* \text{ and}$$

$$v^*_k(\beta)= V_\beta(P,Q)-V_\beta(P,Q-\{q_k\}).$$

We will assume that the price rule and the first step of the tie-breaking rule are known by every agent before starting the auction.

If $\varphi$ is an auction rule and $\varphi$ is not any of the mCPSB auction rules, we say that $\varphi$ is a CPSB auction rule.

Example 2 below illustrates the auction mechanisms.

**Example 2.** Let $M=(P,Q,(a,A),s)$ where $P=\{p_1,p_2,p_3\}$, $Q=\{q_1,q_2\}$, $a_1=(8,6)$, $a_2=(7,5)$, $a_3=(4,2)$, $A_j=Q$ for all $j=1,2,3$ and $s=(0,0)$. Let the buyers select the joint strategy $\beta=(b,B)$, where $B_j=Q$ for all $j=1,2,3$, $b_1=(4,2)$, $b_2=(4,2)$ and $b_3=(1,2)$. The auction rule is given by some price rule $v(.)$ and some tie-breaking rule $x(.)$.

The optimal matchings for the market $M(\beta)$ are

$$x^1=\begin{matrix}1 & 0\\ 0 & 1\\ 0 & 0\end{matrix}\qquad x^2=\begin{matrix}1 & 0\\ 0 & 0\\ 0 & 1\end{matrix}\qquad x^3=\begin{matrix}0 & 0\\ 1 & 0\\ 0 & 1\end{matrix}\qquad x^4=\begin{matrix}0 & 1\\ 1 & 0\\ 0 & 0\end{matrix}$$

We can compute $V_\beta(P-\{p_j\},Q)=6$ for all $j=1,2,3$; $V_\beta(P-\{p_1\},Q-\{q_1\})=2$; $V_\beta(P-\{p_2\},Q-\{q_2\})=4$; $V_\beta(P,Q-\{q_1\})=2$; $V_\beta(P,Q-\{q_2\})=4$ and $V_\beta(P,Q)=6$. We can use any of the above matchings to compute $v_*(\beta)$. Then, $v^*_1(\beta)=v_{*1}(\beta)=6-2=4$ and $v^*_2(\beta)=v_{*2}(\beta)=6-4=2$. Thus $(4,2)$ is the only competitive equilibrium price in $M(\beta)$, so $v(\beta)=(4,2)$.

The tie-breaking rule considers the values of the above matchings in the market $M$: $\Sigma_{j,k}\, a_{jk}x^1_{jk}=13$, $\Sigma_{j,k}\, a_{jk}x^2_{jk}=10$, $\Sigma_{j,k}\, a_{jk}x^3_{jk}=9$ and $\Sigma_{j,k}\, a_{jk}x^4_{jk}=13$.

The maximum is attained in $x^1$ and $x^4$. The second step of the tie-breaking rule chooses one of these two matchings. If say, $x(\beta)=x^1$ then $q_1$ will be matched to $p_1$ at price 4, so $p_1$'s true payoff will be 4; $q_2$ will be matched to $p_2$ at price 2, so $p_2$'s true payoff will be 3. Since any optimal matching is compatible with $v(\beta)$, the prices obtained in the auction do not depend on the particular matching chosen by the auctioneer. It is a matter of verification that, in this example, the true payoffs of the buyers do not change if the tie-breaking rule chooses the matching $x^4$.

Now suppose that the auction rule is $\varphi(.)=(1/2(v^*(.)+v_*(.)),x(.))$. Let the buyers bid $b'_1=(0,2)$, $b'_2=(6,8)$ and $b'_3=(1,3)$. After the computations we get that $v^*_1(\beta')=9-8=1$

and $v^*_2(\beta')=9\text{-}6=3$; $v_{*1}(\beta')=3\text{-}3=0$ and $v_{*2}(\beta')=8\text{-}6=2$. Then $v(\beta')=(0.5,\ 2.5)$. The optimal matchings for the market $M(\beta')$ are

$$y^1 = \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \quad \text{and} \quad y^2 = \begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}$$

The true values of these matchings are $\Sigma_{j,k}\, a_{jk}y^1{}_{jk}=9$ and $\Sigma_{j,k}\, a_{jk}y^2{}_{jk}=9$. If the second step of the tie-breaking rule chooses $y^1$ then $q_1$ will be matched to $p_2$ at the price ½, so $p_2$'s true payoff will be 7-1/2=6.5; $q_2$ will be matched to $p_3$ at the price 2.5, so $p_3$'s true payoff will be 2-2.5=-0.5. However if $y^2$ is chosen, the true payoff of $p_2$ will decrease to 5-2,5=2,5 and the true payoff of $p_3$ will increase to 4-1/2=3.5. ▮

Example 2 illustrates that the bidders' true payoffs may depend on the matching selected at the second step of the tie-breaking rule. The reason by which the buyers are indifferent between the choices of the auctioneer under strategy $\beta$, and are not indifferent under strategy $\beta'$, will be understood after Theorem 5.

## 4. INCENTIVES FACING THE BUYERS

Let the auction rule be given by $\varphi=(v(.),x(.))$. When $x(.)$ is known before buyers are called to bid, the mechanism induces a well-defined strategic game. The set of players is the set of buyers; the true value of object $q_k$ for buyer $j$ is $a_{jk}=max\{0,\ \alpha_{jk}\text{-}s_k\}$. The set of acceptable objects for buyer $p_j$ is $A_j$. A strategy of player $p_j$ is a pair $\beta_j=(b_j,B_j)$, where $B_j\subseteq Q$ and $b_j=(b_{j1},\dots,b_{jn})\in R^n{}_+$, with $b_{jk}=0$ for all $q_k\notin B_j$. Thus when a buyer is called to play she must select some group of objects and a bid for each object in this group. Having the buyers selected a joint strategy $\beta$, the outcome function, given by $\varphi$, finds $(v(\beta);x(\beta))$. The allocation $(v(\beta);x(\beta))$ is a competitive equilibrium for the market $M(\beta)=(P,Q,\ \beta,\ s)$. We will denote this game by $\Gamma(\varphi)$ and we will refer to it as **the auction game induced by $\varphi$**. Some times we will call $\beta=(b,B)$ a joint strategy or a joint bid,

without reference to a specific auction game. Let $\varphi=(v(.),x(.))$ be some auction rule. We will use the following notation:

$M=(P,Q,(a,A),s)$ is the cooperative market game associated to $(a,A)$.

$v^*$ is the maximum competitive equilibrium price for $M$.

$(u_*,v^*)$ is the Q-optimal stable payoff for $M$.

$v_*$ is the minimum competitive equilibrium price for $M$.

$(u^*,v_*)$ is the P-optimal stable payoff for $M$.

$(u,v)$ is the payoff vector obtained under $\varphi$ when the bidders tell the truth and $\varphi$ is any CPSB auction.

$\Sigma(M)$ is the set of all optimal matchings for $M$.

$E(M)$ is the set of competitive equilibrium of $M$.

$C(M)$ is the core of $M$.

For all $R\subseteq P$ and $S\subseteq Q$, $V(R,S)=\max\Sigma_{RxS}\, a_{jk}\, x_{jk}$ with the maximum to be taken over all feasible matchings $x$ of $M$.

If $\beta=(b,B)$ is a joint strategy for the bidders:

$M(\beta)=(P,Q,\ \beta=(b,B),s)$ is the cooperative market game associated to $\beta$.

$v^*(\beta)$ is the maximum competitive equilibrium price for $M(\beta)$. It is the price vector yielded by $\varphi$ when the price rule is the maximum competitive equilibrium price rule and the bidders choose $\beta$.

$(u_*(\beta),v^*(\beta))$ is the Q-optimal stable payoff for $M(\beta)$.

$v_*(\beta)$ is the minimum competitive equilibrium price for $M(\beta)$. It is the price vector yielded by $\varphi$ when the price rule is the minimum competitive equilibrium price rule and the bidders choose $\beta$.

$(u^*(\beta),v_*(\beta))$ is the P-optimal stable payoff for $M(\beta)$.

$(u(\beta),v(\beta))$is the payoff obtained under $\varphi$ when the bidders choose the joint strategy $\beta$ and $v(.)\neq v_*(.)$.

$\Sigma(\beta)$ is the set of all optimal matchings for $M(\beta)$.

$\Sigma^*(\beta)=\{x\in\Sigma(\beta);\ \Sigma_{j,k}\, a_{jk}\, x_{jk} \geq \Sigma_{j,k}\, a_{jk}\, x'_{jk},$ for all $x'\in\Sigma(\beta)\}$.

$x(\beta)$ is the matching in $\Sigma^*(\beta)$ determined by $x(.)$.

$E(\beta)$ is the set of competitive equilibria of $M(\beta)$.

$C(\beta)$ is the core of $M(\beta)$.

For all $R\subseteq P$ and $S\subseteq Q$, $V_\beta(R,S)=$ max $\Sigma_{RxS}\, b_{jk}.x_{jk}$, with the maximum to be taken over all feasible matchings $x$ for the market $M(\beta)$.

We need the following definition:

**Definition 10.** *Let $x(\beta)\equiv x$. The true payoff of bidder $p_j$ under the outcome $(u(\beta),v(\beta);x)$ is:*

$Uj(\beta,x)= (\alpha_{jk}-s_k) -v_k(\beta)$ if $x_{jk}=1$ and

$Uj(\beta,x)=0$ if $p_j$ is unmatched at $x$.

Of course, if $x_{jk}=1$ and $q_k\in A_j$ then $\alpha_{jk}-s_k=a_{jk}$, so $Uj(\beta,x)=(a_{jk}-v_k(\beta))$. Then $(Uj(\beta,x),,v(\beta))$ is feasible by Remark 2. The vector $U(\beta,x)=(U_1(\beta,x),...,U_m(\beta,x))$ is called true payoff under $(u(\beta),v(\beta);x)$.

It is noteworthy that when there are several optimal matchings in $\Sigma^*(\beta)$, the true payoff of a bidder $p_j$ may depend on the matching determined in the second step of the tie-breaking rule, but it is not the case when she tells the truth, i.e., $b_j=a_j$.

Theorems 4* and 5* show the way in which the Vickrey second price auction for a single object is generalized by the multi-item auction mechanism that gives buyers their optimal stable payoff. Both mechanisms give buyers their marginal contributions to coalitional values. The price paid by buyer $p_j$ in any mCPSB auction does not depend on any valuations $\alpha_{jk}$'s. Thus, as in the Vickrey second-price auction for a single object, the price a buyer pays is not determined by her bids. This is the critical observation for the proof of Theorem 5*, which can be restated as follows:

*Let $\varphi$ be some mCPSB auction. Then truth telling is a dominant strategy for each bidder in the game $\Gamma(\varphi)$.*

Therefore, under any *mCPSB* auction, it is optimal for the buyers to bid their true value, independently of the tie-breaking rule. In any of these auctions the auctioneer expects to sell each object $q_k$ by $v*_k+s_k$. In the special case in which there is only one competitive equilibrium price in $M$ truth telling is a dominant strategy for each bidder $p_j$ under any game. This conclusion is implied by the result below, which is a particular case of the non-manipulability Theorem of Demange and Gale (1985), which in its turn is a special case of a result of Sotomayor (1986)[15]. Theorem 5* is clearly a corollary of Theorem 6*.

**Theorem 6*** (Demange and Gale (1985), Sotomayor (1986)). *Let $p_j$ be a buyer who misrepresents her preferences. Let $(u',v';x)$ be any stable outcome for the market $M'=(P,Q,(a',A'),s)$, where $a'_k=a_k$ and $A'_k=A_k$ for all $p_k \in P$ with $k \neq j$. Let $(u,v)$ be the true payoff under $(u',v';x)$. Then $u*_j \geq u_j$.*

In this section we consider the incentives facing the bidders when the auctioneer uses some price rule other than the minimum competitive equilibrium price rule. Our first result implies that, when there is more than one stable payoff for the market $M$ and the price rule is not the minimum competitive equilibrium price rule, stating true valuations is not the best policy for, at least, one bidder, in the sense that there will always be advantage to her in misrepresenting her valuations.

**Theorem 1.** *Suppose that there is more than one competitive equilibrium price for the market $M$. Then it is always possible for some bidder to improve her payoff under any CPSB auction, by misrepresenting her valuations, assuming the others tell the truth.*

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15 See Roth and Sotomayor (1990).

This result can be illustrated by the following simple example.

**Example 3**. Let $M=(P,Q,A,s)$. Let $P=\{p_1,p_2,p_3\}$, $Q=\{q_1\}$, $a=(8,6,4)$, $A_j=Q$ $\forall$ $j=1,2,3$, $s=(0,0)$. Let the price rule be given by $v(.)=\lambda v_*(.) + (1-\lambda)v^*(.)$, with $\lambda\in[0,1)$. The maximum competitive equilibrium price for $M$ *is* $v^*=8$ and the minimum competitive equilibrium price for $M$ is $v_*=6$. If the bidders bid their true values then $p_1$ will win the auction and will pay $v=6\lambda + 8(1-\lambda)$. However if the bidders bid $\beta=(b_1=7,$ $b_2=6,\ b_3=4;\ B_1=B_2=B_3=Q)$ then $v_*(\beta)=6$ and $v^*(\beta)=7$. Then $p_1$ will win the auction and will pay $v(\beta)=6\lambda + 7(1-\lambda) < 6\lambda + 8(1-\lambda)=v,$ $\forall\lambda\in[0,1)$.

**Definition 11**: *Let* $\Gamma((v(.),x(.))$ *an auction game. The pair* $\beta=(b,B)$ *is a* ***Nash*** *equilibrium (in the usual sense) for* $\Gamma((v(.),x(.))$ *if for no* $p_j\in P$, *there is* $\beta'=(b',B')$ *with* $b'_t=b_t$ *and* $B'_t=B_t$ *for all* $p_t\neq p_j$, *such that* $U_j(\beta', x(\beta'))> U_j(\beta,x(\beta))$.

**In what follows we will be considering that the price rule is some fixed** $v(.)$.

**Definition 12**: *Let* $\beta=(b,B)$ *and* $x\in\Sigma^*(\beta)$. *The pair* $\beta=(b,B)$ *is a* ***Nash equilibrium in the weak*** *(resp.* ***strong****)* ***sense for x*** *if for no* $p_j\in P$, *there is* $\beta'=(b',B')$ *with* $b'_t=b_t$ *and* $B'_t=B_t$ *for all* $p_t\neq p_j$, *such that for every (resp. for some)* $y\in\Sigma^*(\beta')$, $U_j(\beta', y)>U_j(\beta,x)$.

*The pair* $\beta=(b,B)$ *is a* ***Nash equilibrium in the weak*** *(resp.* ***strong****)* ***sense*** *if* $\beta=(b,B)$ *is a Nash equilibrium in the weak (resp. strong) sense for every* $x\in\Sigma^*(\beta)$.

If $\beta=(b, B)$ is a Nash equilibrium for $\Gamma((v(.),x(.))$, we say that the allocation $(v(\beta),x(\beta))$ is the *Nash equilibrium outcome* corresponding to $\beta$. The payoff $(U(\beta,x),v(\beta))$ is called the *Nash equilibrium payoff* corresponding to $\beta$. Thus, a Nash payoff is the payoff in $M$ corresponding to the transfers obtained from a Nash equilibrium.

If $\beta=(b, B)$ is a Nash equilibrium in the strong (respectively weak) sense for

$x \in \Sigma^*(\beta)$, then the Nash equilibrium outcome and the Nash equilibrium payoff in the strong (resp. weak) sense under $x$ corresponding to $\beta$ is analogously defined.

Under the weak concept a bidder changes her strategy when she is sure to be better off, while in the strong one she changes it whenever she has a chance to be better off. Consequently every Nash equilibrium in the strong sense is a Nash equilibrium in the weak sense.

Definition 12 is equivalent to:

**Definition 12'**: *Let $\beta=(b,B)$ and $x \in \Sigma^*(\beta)$. The pair $\beta=(b,B)$ is a **Nash equilibrium in the weak** (resp. **strong**) **sense for $x$** if for no $p_j \in P$, there is $\beta'=(b',B')$ with $b'_t=b_t$ and $B'_t=B_t$ for all $p_t \neq p_j$, such that for every (resp. for some) $x(.)$ such that $x(\beta)=x$, we have that $U_j(\beta', x(\beta')) > U_j(\beta,x)$.*

Observe that $\beta$ is a Nash equilibrium in the strong sense for $x \in \Sigma^*(\beta)$ if and only if it is a Nash equilibrium in the usual sense of the game $\Gamma(v(.),x(.))$, for every $x(.)$ such that $x(\beta)=x$. Thus $\beta$ is a Nash equilibrium in the strong sense if and only if it is a Nash equilibrium in the usual sense of every auction game with price rule $v(.)$. If $\beta$ is a Nash equilibrium in the usual sense of the game $\Gamma(v(.),x(.))$, then $\beta$ is a Nash equilibrium in the weak sense for $x=x(\beta)$.

**Remark 3.** If $\beta$ is a Nash equilibrium in the weak sense for $x \in \Sigma^*(\beta)$ then $U_j(\beta,x) \geq 0$ for all $p_j$. In fact, if $U_j(\beta,x)<0$ for some $p_j$ then $p_j$ can be better off by choosing $B'_j=\phi$ and $b'_{jk}=0$ for all $q_k$, because then her true payoff will be zero at any optimal matching for $M(b,B)$. But this contradicts the assumption that $\beta$ is a Nash equilibrium in the weak sense for $x \in \Sigma^*(\beta)$. Hence $U_j(\beta,x) \geq 0$. Consequently if $x_{jk}=1$ we have that $(\alpha_{jk}-s_k)-v_k(\beta) \geq 0$ and so $\alpha_{jk}-s_k=a_{jk}$. Therefore $U_j(\beta,x)=a_{jk}-v_k(\beta)$ if $x_{jk}=1$ and $U_j(\beta,x)=0$ if $p_j$ is unmatched at $x$, from which follows that $(U(\beta,x),v(\beta))$ is feasible and individually rational.∎

The existence of a Nash equilibrium in the strong sense is given by Theorem 2. We need the following results.

**Lemma 1.** *For each bidder* $p_j$ *let* $\beta_j=(b_j, B_j)$ *be a strategy where* $B_j=\{q_k \in A_j;\ a_{jk}-\lambda_j \geq 0\}$, *with* $0 \leq \lambda_j \leq u^*_j$, *and* $b_{jk}=max\{0, (a_{jk}-\lambda_j)\}$ *for all* $q_k \in Q$. *Then* $\Sigma(M)=\Sigma^*(\beta)$.

**Lemma 2.** *For each bidder* $p_j$ *let* $\beta_j=(b_j,B_j)$ *be a strategy, where* $B_j=\{q_k \in A_j;\ a_{jk}-\lambda_j \geq 0\}$, *with* $0 \leq \lambda_j$, *and* $b_{jk}=max\{0, (a_{jk}-\lambda_j)\}$ *for all* $q_k \in Q$. *Let* $\lambda=(\lambda_1,\ldots,\lambda_m)$.

*a)Let* $x \in \Sigma(M)$. *If* $(u^1,v^1;x)$ *is stable for* $M(\beta)$ *and* $0 \leq \lambda \leq u^*$ *then* $(u^1+\lambda, v^1;x)$ *is stable for* $M$.

*b) If* $(u^2,v^2;x)$ *is stable for* $M$ *and* $0 \leq \lambda \leq u^2$ *then* $(u^2-\lambda, v^2;x)$ *is stable for* $M(\beta)$.

**Proposition 1.** *For each bidder* $p_j$ *let* $\beta_j=(b_j,B_j)$ *be a strategy, where* $B_j=\{q_k \in A_j;\ a_{jk}-\lambda_j \geq 0\}$, *with* $0 \leq \lambda_j$, *and* $b_{jk}=max\{0, (a_{jk}-\lambda_j)\}$ *for all* $q_k \in Q$.

*If* $0 \leq \lambda \leq u^*$ *and* $|C(\beta)|=1$ *then* $\beta=(b,B)$ *is a Nash equilibrium in the strong sense (and consequently in the weak sense). Furthermore* $(u^*,v^*)$ *is the corresponding Nash equilibrium payoff in the strong sense.*

The condition $0 \leq \lambda \leq u^*$ is necessary to avoid cases like the one described in the example below, in which $\beta$ is not a Nash equilibrium in the strong sense and $|C(\beta)|=1$.

**Example 4.** $P=\{p_1,p_2\}$, $Q=\{q_1\}$, $a_{11}=9$, $a_{21}=7$. Then the P-optimal stable payoff is given by $u^*_1=2$, $u^*_2=0$, $v^*_1=7$ and the only optimal matching matches the object to buyer $p_1$. Let the buyers choose $\lambda_1=-1$ and $\lambda_2=-3$, which gives $b_{11}=b_{21}=10$ and $\beta_1=\beta_2=Q$. The payoff $u_1=u_2=0$ and $v_1=10$ is the only stable in $M(\beta)$. The object is matched to $p_1$ who gets a negative true payoff. In this case $|C(\beta)|=1$ but $\beta$ is not a Nash equilibrium, since $p_1$ has an incentive to deviate by telling the truth. Similarly, if $\lambda_1=3$ and $\lambda_2=1$ the only stable payoff allocates the object to $p_1$ at the price 6. In this case $p_2$

can deviate by bidding 6.1, for example, which makes her better off. ∎

**Proposition 2.** *Let $\beta=(b,B)$ be a Nash equilibrium in the weak sense for $x \in \Sigma^*(\beta)$. Then $|C(\beta)|=1$.*

**Theorem 2 .** *Let $(u',v')$ be a stable payoff for $M$. For each bidder $p_j$ let $\beta_j=(b_j,B_j)$ be the strategy where $B_j=\{q_k \in A_j;\ (a_{jk}-u'_j)\geq 0\}$ and $b_{jk}=\max\{0,\ (a_{jk}-u'_j)\}$ for all $q_k \in Q$. Then (a) if $\beta=(b,B)$ is a Nash equilibrium in the weak sense for some $x \in \Sigma^*(\beta)$ then $u'=u^*$ and (b) if $u'=u^*$ then $\beta=(b,B)$ is a Nash equilibrium in the strong sense and $(u^*,v^*)$ is the Nash equilibrium payoff in the strong sense.*

The following corollary is immediate from Theorem 2-b).

**Corollary 1-** *For each bidder $p_j$ let $\beta_j=(b_j,B_j)$ be the strategy where $B_j=\{q_k \in A_j;\ (a_{jk}-u^*_j)\geq 0\}$ and $b_{jk}=\max\{0,\ (a_{jk}-u^*_j)\}$ for all $q_k \in Q$. Then $\beta=(b,B)$ is a Nash equilibrium ( in both the weak and strong senses). Furthermore the payoff $(u^*,v^*)$ is the Nash equilibrium payoff ( in both the weak and strong senses).*

We may have Nash equilibrium strategies different from the strategy $\beta$ defined in Corollary 1. See the following example.

**Example 5.** $P=\{p_1,p_2\}$, $Q=\{q_1,q_2\}$, $a_j=(4,3)$ for all $j=1,2$. The P-optimal stable payoff is given by $u^*_1=u^*_2=3$, $v^*_1=1$, $v^*_2=0$. Now consider $\lambda_1=2$, $\lambda_2=3$. Let $\beta=(b,B)$ be the strategy defined in Lemma 1. Then $B_1=B_2=Q$, $b_1=(2,1)$, $b_2=(1,0)$. By Proposition 1, $\beta=(b,B)$ is a Nash equilibrium in the strong sense with payoff $(u^*,v^*)$, since $0\leq\lambda\leq u^*$ and $|C(\beta)|=1$. However $\lambda\neq u^*$. ∎

On the other hand we may have Nash equilibrium payoff different from $(u^*,v^*)$, as we show below.

**Example 6.** $P=\{p_1,p_2,p_3\}$, $Q=\{q_1,q_2\}$, $a_1=(8,7)$, $a_2=(5,6)$, $a_3=(4,5)$. The minimum and the maximum competitive equilibrium prices for this market are given by $v_*=(4,5)$ and $v^*=(7,6)$, respectively. Let the buyers choose $\beta=(b,B)$, where $b_1=(5,5)$, $b_2=(5,5)$, $b_3=(4,5)$, $B_1=B_2=B_3=Q$. Then $v_*(\beta)=v^*(\beta)=(5,5)$. Therefore there is only one competitive equilibrium for $M(\beta)$. Hence if $(v(.),x(.))$ is some auction rule with $v(.)\neq v_*(.)$, $v(\beta)=(5,5)$. The only optimal matching in $\Sigma^*(\beta)$ gives $q_1$ to $p_1$ and $q_2$ to $p_2$. The true payoff vector is $U(\beta,x(\beta))=(3,1,0)$. Clearly $(v(\beta),x(\beta))$ is a competitive equilibrium price under the true valuations. We claim that $\beta$ is a Nash equilibrium in the strong sense. In fact, suppose that $p_1$ deviates by choosing $\beta'_1$. Let $\beta'=(\beta'_1,\beta^1_2,\beta^1_3)$. If $p_1$ is matched under $x(\beta')$, then $p_3$ is unmatched under $x(\beta')$, so $v_2(\beta')\geq a_{32}=5$. If $p_1$ is matched to $q_1$ then $p_2$ is matched to $q_2$, so $v_2(\beta')\leq 5$. Then $v_2(\beta')=5$ and so $v_1(\beta')\geq 5$, for if not $p_2$ will demand $q_1$ instead of $q_2$. If $p_1$ is matched to $q_2$ she will pay $v_2(\beta')\geq 5$. In any case $p_1$ is not able to pay less than 5. Hence her true payoff will not be greater than 3, so she does not have any incentive to deviate from $\beta$.

With an analogous argument we can see that buyer $p_2$ does not have any incentive to deviate from $\beta$. For $p_3$ observe that in order to be matched she must bid $b'_{31}>5$ and $b'_{32}>5$, which gives to her a negative payoff. Hence $p_3$ does not have any incentive to deviate from $\beta$. Hence $\beta$ is a Nash equilibrium in the strong sense and the corresponding Nash equilibrium payoff is given by $U(\beta,x(\beta))=(3,1,0)$, $v(\beta)=(5,5)$.

It is a matter of verification that the joint strategy given by $b_1=b_2=(4.5, 5)$, $b_3=(4, 5)$, $B_1=B_2=B_3=Q$ is also a Nash equilibrium in the strong sense with payoff $U(\beta, x(\beta))=(3.5, 1, 0)$, $v(\beta)=(4.5, 5)$.∎

Example 6 poses the following question: How will buyers make their strategic choices in those cases where there is more than one Nash equilibrium payoff? We argue that a solution exists and it is implied by Theorem 3 below.

**Theorem 3.** *Let* $x \in \Sigma^*(\beta)$. *Let* $\beta=(b,B)$ *be a Nash equilibrium in the weak sense for* $x$. *Then the corresponding Nash payoff in the weak sense,* $(U(\beta,x),v(\beta))$, *is stable for* $M$.

Since any Nash equilibrium in the strong sense is a Nash equilibrium in the weak sense, it is immediate from Theorem 3 that

**Corollary 2.** *Let* $x \in \Sigma^*(\beta)$. *Let* $\beta=(b,B)$ *be a Nash equilibrium in the strong sense for* $x \in \Sigma^*(\beta)$. *Then the corresponding Nash payoff in the strong sense,* $(U(\beta,x),v(\beta))$, *is stable for* $M$.

Consequently there is no Nash equilibrium payoff, which is preferred by some bidder to the minimum competitive equilibrium under the true valuations. Hence, as it is explained in session 1, the strategy described in Theorem 2 can be regarded as the best way of play for the bidders.

It would be nice if we could say that every competitive equilibrium is a Nash equilibrium outcome. Nevertheless the converse of Corollary 2 is not true. There may be stable payoffs that are not Nash payoffs. See Example 7.

**Example 7.** Let the market $M$ be given by $P=\{p_1\}$, $Q=\{q_1\}$, $a_{11}=2$. Then the payoff $(u_1=1,v_1=1)$ is stable for $M$. Nevertheless there is no $\beta=(b,B)$ such that $\beta$ is a Nash equilibrium with payoff $(u_1,v_1)$. In fact, given $\beta=(b,B)$, the payoff $(u'_1=b_{11},v'_1=0)$ is always in $C(\beta)$. Thus if $v(\beta)=1$ we must have that $(u(\beta)=b_{11}-1,v(\beta)=1)$ is in $C(\beta)$. Then $|C(\beta)|>1$ and $\beta$ is not a Nash equilibrium by Proposition 2.∎

Assuming that the bidders play the Nash equilibrium strategies described in Corollary 1, the price obtained in any CPSB auction is the minimum competitive equilibrium price for $M$. Theorem 5* says that the minimum competitive equilibrium price is the price obtained in the mPSB auctions if the bidders play their dominant strategies. Hence

**Theorem 4.** *(REVENUE EQUIVALENCE)- If the bidders play the Nash equilibrium strategies described in Corollary 1, under any CPSB auction, and play their dominant strategies under the mCPSB auction, then the auctioneer gets the same vector of prices in all the auctions.*

The fact that the outcome produced by a Nash equilibrium in the weak sense is stable for $M$ implies that under such an equilibrium the buyers are allocated to objects which are acceptable to them. Thus a quite reasonable assumption is that the buyers always choose a set of objects which are acceptable to them. Under this hypothesis, the following theorem asserts that when buyers select Nash equilibrium strategies (in the weak or in the strong senses) the knowledge of the tie-breaking rule is irrelevant.

**Theorem 5.** *Let $\beta=(b,B)$ be a Nash equilibrium in the strong (resp. weak) sense for some $x\in\Sigma^*(\beta)$. Let $B_j\subseteq A_j$ for all $p_j$. Then $\beta$ is a Nash equilibrium in the strong (resp. weak) sense for every $x\in\Sigma^*(\beta)$.*

Consequently, for $\beta$ to be a Nash equilibrium in the usual sense, under every tie-breaking rule, it is sufficient that it be a Nash equilibrium in the strong sense for some $x\in\Sigma^*(\beta)$.

Corollary 3 asserts that the Nash payoff in the weak or in the strong senses does not depend on the particular matching chosen at the second step of the tie-breaking rule.

**Corollary 3.** *Let $\beta=(b,B)$ be a joint strategy with $B_j\subseteq A_j$ for all $p_j$. Suppose that $\beta$ is Nash equilibrium in the strong (resp. weak) sense for some $x\in\Sigma^*(\beta)$. Then $\beta$ is Nash equilibrium in the strong (resp. weak) sense. Furthermore if the buyers choose $\beta$ then $U_j(\beta,x)=U_j(\beta,x')$, $\forall x'\in\Sigma^*(\beta)$ and $p_j\in P$.*

These results imply the following. Consider a situation where buyers are playing

some auction game with price rule $v(.)$. Suppose they do not know the second step of the tie-breaking rule. Let $\beta$ be a Nash equilibrium for some game $\Gamma(v(.),x(.))$ where $x(\beta)=x$. Then the buyers can force the payoff $(U(\beta,x),v(\beta))$ by playing $\beta$. Now let $\beta$ be a Nash equilibrium of every game $\Gamma(v(.),x(.))$ where $x(\beta)=x$. Then, without knowing the second step of the tie-breaking rule, they will play in equilibrium by choosing $\beta$.

Theorem 5 and Corollary 3 are not necessarily true when $B_j$ contains some object that is unacceptable to $p_j$. In fact, see the example below.

**Example 8.** Consider $P=\{p_1,p_2\}$, $Q=\{q_1\}$, $\alpha_{11}=\alpha_{12}=8$, $s_1=9$. Then $a_{11}=a_{12}=0$ and $A_1=A_2=\phi$. Let $\beta=(b,B)$, where $b_{11}=b_{21}=0$, $B_1=\phi$ and $B_2=\{q_1\}$. Under any price rule the object will be sold at price zero. The tie-breaking rule can choose one of the two matchings: $x_{11}=x_{21}=0$ and $x'_{11}=0$, $x'_{21}=1$. The true payoffs of the buyers under these matchings are: $U_1(\beta,x)=U_2(\beta,x)=0$ and $U_1(\beta,x')=0$, $U_2(\beta,x')=8\text{-}9=\text{-}1$. In this example the best for the buyers is to be unmatched. Thus $\beta$ is a Nash equilibrium in the strong sense for $x$ and is not a Nash equilibrium in the weak sense for $x'$. If the tie-breaking rule chooses $x'$, $p_2$ will be better off by choosing $b'_{21}=b_{21}=0$, $B'_2=\{\phi\}$. ∎

## 5. FINAL REMARKS

Our model is a version of the Assignment Game of Shapley and Shubik (1972), which in its turn is a special case of Demange and Gale's model. Demange and Gale (1985) present a general market for buyers and sellers where the preferences' structure is represented by utility functions which are continuous, and not necessarily linear, in the money variable. That paper investigates the strategic behavior of buyers and sellers in the game induced by the mechanism which yields the maximum competitive equilibrium. For a given joint bid the mechanism chooses the corresponding maximum competitive equilibrium price and some compatible matching. For this game it is natural to assume that the sellers always tell the truth, because this strategy is dominant for each seller. The authors show the existence of a strong equilibrium payoff and then focus attention to the

case where buyers can only manipulate their reservation utilities. In the context of our model this assumption is equivalent to restricting the bids of any buyer $p_j$ to the type (a): $b_{jk}= (a_{jk}-\lambda_j)\geq 0$ for any object $q_k$, with $\lambda_j\geq 0$ or to the type (b): $b_{jk}=(a_{jk}+\lambda_j)$ for any object $q_k$ with $\lambda_j\geq 0$. They then prove that when the bidders only use strategies of the type (a), the minimum competitive equilibrium is the only Nash equilibrium outcome.

In the present paper we presented a market game for buyers and sellers which was modeled and analyzed using the tools, core and strategic equilibrium, drawn from what has traditionally been called cooperative and non-cooperative game theory. Their use here together should clarify that these two approaches to game theory are complementary rather than substitutes. In order to obtain the competitive equilibria (which correspond to the core outcomes) the buyers were stimulated to act strategically in the games induced by a class of sealed bid auctions. No restriction on the strategic space of the buyers was assumed. Each auction rule was determined by a given price rule and some matching rule, called tie-breaking rule. The outcomes of such mechanisms were competitive equilibria for the market corresponding to the submitted bids. All the results proved in this paper hold, not only for the maximum competitive equilibrium price rule, but for every competitive equilibrium price rule other than the minimum competitive equilibrium price rule.

We distinguished some common features in these games. Once proved that, in markets with more than one competitive equilibrium price, telling the truth is not a dominant strategy for every buyer in any auction game, we were able to conclude that at least one buyer has incentive to misrepresent her valuations. Then it follows that Nash equilibria involve misrepresentation of the valuations by at least one buyer. By playing a Nash equilibrium, a competitive equilibrium under the true valuations is obtained. Example 7 shows that not all competitive equilibria under the true valuations is a Nash equilibrium outcome. Example 6 shows that we may have a Nash equilibrium outcome different from the minimum competitive equilibrium, even when buyers do not bid over their true values.

We studied the strategic questions faced by the bidders, not only in a particular game, but also in the situations in which the second step of the tie-breaking rule is not fully known. Under the assumption that buyers only choose objects which are acceptable to them, Theorem 5 and Corollary 3 imply that the Nash equilibria in the weak sense or in the

strong sense, as well as the Nash payoffs of a given game, do not depend on the particular matching obtained at the second step of the tie-breaking rule. The existence theorem provides a joint strategy $\beta^*$ which is a Nash equilibrium in any of the auction games. Moreover the corresponding Nash equilibrium payoff is the P-optimal stable payoff of the cooperative market game. Therefore, it is reasonable to expect that the buyers will play $\beta^*$ when the price rule is different from the minimum competitive equilibrium price rule, and will play the sincere strategies, otherwise. Thus, the sellers will sell their objects by the same prices under any auction rule, including the minimum competitive equilibrium price rule.

The first step of the tie-breaking rule plays a crucial role in our mechanism. In the model of Demange and Gale, because of the absence of a convenient matching rule, an equilibrium outcome only results if buyers select the correct bids and if the mechanism selects the correct matching. Being a Nash equilibrium or a Nash payoff in that model depends on the particular matching rule used by the mechanism. The following example illustrates this.

**Example** . There are two buyers and one object with reservation price 0. The true values are $a_{11}=8$ and $a_{21}=6$. The auction rule is some MCPSB auction rule. Let the buyers bid $B_1=B_2=Q$ and $b_{11}=b_{21}=6$. The optimal matchings are $x$ and $x'$, where $x$ matches the object to $p_1$ and $x'$ matches the object to $p_2$. In any case the price of the object is 6.

In our model the auctioneer will never choose $x'$, because of the first step of the tie-breaking rule. The matching selected will be $x$ and no buyer will profit from deviation. *Thus $(B,b)$ is a Nash equilibrium in the usual sense. By playing $(B,b)$ the buyers force the outcome to be the minimum competitive equilibrium price under the true valuations.*

According to the approach of Demange and Gale the mechanism is allowed to choose $x'$. In this game $p_1$ will not be playing her best response: by playing, say, $b'_{11}=7$, she can obtain the object for 7, with a true payoff of $1$ instead of $0$. Therefore *$(B,b)$ is not a Nash equilibrium under the usual sense.*

We must point out that if in the event of more than one optimal matching for $M(\beta)$ the ties are broken randomly, and if any matching in $\Sigma(\beta)$ has equal probability to be

chosen by the auctioneer, then the game may have no Nash equilibrium. In fact in the previous example let $v(.)\neq v*(.)$. It is a matter of verification that $\beta=(b,B)$, with $b_1\neq b_2$ or $B_j=\phi$ for some $p_j$, is not a Nash equilibrium. If $b_1=b_2=x$ and $B_1=B_2=Q$, $p_1$'s true payoff will be $(8-x)/2$ and $p_2$'s true payoff will be $(6-x)/2$. Then $x\leq 6$, for otherwise, $p_2$ will be better off by choosing $b'_2=0$. Thus, if $p_1$ bids $b'_1=x+\varepsilon$, with $\varepsilon<(8-x)/2$, she will win the auction and her true payoff will be $8-x-\varepsilon > (8-x)/2$. Therefore, $p_1$ always profits from deviation and there is no Nash equilibrium.

Although the approach of incomplete information is the most realistic, the games treated here were modeled under the assumption of complete information on the true reservation values of the objects to the buyers. The reason for that lies on the fact that the structure of auction games with non identical objects is not well-known in the case of incomplete information. On the other hand, we are convinced that a full understanding of the operation of the market mechanisms with incomplete information must be built on an analysis of the certainty case. Thus the complete information set-up constitutes a starting point. We believe that the modeling and analysis developed in this paper can be useful to the study of models with more complex preferences or where buyers may want to purchase more than one object.

## APPENDIX

**Proof of Theorem 1.** Let $\varphi=(v(.),x(.))$ be some auction rule where $v(.)$ is not the minimum competitive equilibrium price rule. Let $(u,v)$ be the payoff vector produced by $\varphi$ when bidders tell the truth. If there is more than one competitive equilibrium in the market $M$ then $v\neq v_*$ and consequently $u^*\neq u$. Thus there is at least one bidder $p_j$ such that

$u^*_j>u_j\geq 0.$ (1)

Then for some positive $\lambda$,

$u^*_j>u_j+\lambda$ (2)

Let $\beta=(b,B)$ be some joint strategy where

$B_j=\{q_k\in A_j\ ;\ a_{jk}-(u_j+\lambda)\geq 0\}$, $b_{jk}=a_{jk}-(u_j+\lambda)$ *for all* $q_k\in B_j$ and $b_{jk}=0$ *otherwise;* $b_t=a_t$ and $B_t=A_t$ if $t\neq j$.

We will show that $p_j$ can improve her payoff by declaring $(b_j,B_j)$ instead of $(a_j,A_j)$. In fact, (1) and Proposition 2* imply that bidder $p_j$ is matched under any optimal matching for $M$. Then let $x\in\Sigma(M)$. Let $q_k\in Q$ such that $x_{jk}=1$. Thus $a_{jk}-u^*_j=v_{*k}$. It follows from (2) that $a_{jk}-(u_j+\lambda)>a_{jk}-u^*_j$. Therefore

$a_{jk}-(u_j+\lambda)>v_{*k}\geq 0,$ (3)

so $a_{jk}-(u_j+\lambda)\geq 0$, so $q_k\in B_j$ and consequently

$b_{jk}=a_{jk}-(u_j+\lambda).$ (4)

We claim that $p_j$ is matched under any optimal matching for $M(\beta)=(P,Q,\beta=(b,B),s)$. In fact, arguing by contradiction, suppose $p_j$ is unmatched under some $x'\in\Sigma(\beta)$. Then $V(P-\{p_j\},Q)=V_\beta(P,Q)\geq b_{jk}+V_\beta(P-\{p_j\},Q-\{q_k\})=a_{jk}-(u_j+\lambda)+V_\beta(P-\{p_j\},Q-\{q_k\})>v_{*k}+V(P-\{p_j\},Q-\{q_k\})$, where the second equality follows from (4) and the second inequality follows from (3) and the definition of $V$. Therefore $v_{*k}<V(P-\{p_j\},Q)-V(P-\{p_j\},Q-\{q_k\})$, which contradicts Theorem 4*-a).

Thus let $q_t$ be the object matched to $p_j$ by $x(\beta)$. Then $q_t\in B_j$ and so $q_t\in A_j$. Therefore the true payoff for $p_j$ under $(u(\beta),v(\beta);x(\beta))$ is $U_j(\beta,x(\beta))=a_{jt}-v_t(\beta)$. From the individual rationality of $u_j(\beta)$ it follows that $v_t(\beta)\leq b_{jt}$, so $U_j(\beta,x(\beta))\geq a_{jt}-b_{jt}$. Using that

$b_{jt}=a_{jt}-(u_j+\lambda)$ we get that $U_j(\beta,x(\beta)) \geq u_j+\lambda>u_j$. Hence $p_j$ can improve her payoff by misrepresenting her valuations and the proof is complete.∎

**Proof of Lemma 1.** We will first show that $\Sigma(M)\subseteq\Sigma^*(\beta)$. Let $x\in\Sigma(M)$ and let $x'$ be any feasible matching for $M(\beta)$. To see that $x$ is an optimal matching for $M(\beta)$ we must show that $\Sigma_{j,k}\, b_{jk}x_{jk} \geq \Sigma_{j,k}\, b_{jk}x'_{jk}$. If $x'_{jk}=1$, the feasibility of $x'$ in $M(\beta)$ implies that $q_k\in B_j$, so $q_k\in A_j$. Thus $x'$ is also feasible for $M$. From the optimality of $x$ in $M$ it follows that

$$\Sigma_{j,k}\, a_{jk}x_{jk} \geq \Sigma_{j,k}\, a_{jk}x'_{jk} \qquad (1)$$

The definition of $\lambda_j$ implies that $u^*_j=\lambda_j=0$ if $p_j$ is unmatched at $x$. On the other hand $u^*_j-\lambda_j\geq 0$ by hypothesis. Then

$$\Sigma_{j,k}\ (u^*_j-\lambda_j)\, x_{jk} =\Sigma_j\ (u^*_j-\lambda_j)\geq\Sigma_{j,k}\ (u^*_j-\lambda_j)x'_{jk} \qquad (2)$$

Now observe that by Proposition 1* applied to $(u^*,v^*;x)$, $a_{jk}-u^*_j=v^*_k\geq 0$ if $x_{jk}=1$. Therefore if $x_{jk}=1$ we can write (recalling that $u^*_j-\lambda_j\geq 0$): $a_{jk}-\lambda_j=(a_{jk}-u^*_j)+(u^*_j-\lambda_j)\geq 0$, so

$$b_{jk}=a_{jk}-\lambda_j. \qquad (3)$$

On the other hand, the feasibility of $x'$ in $M(\beta)$ implies that $q_k\in B_j$ if $x'_{jk}=1$, so

$$b_{jk}=a_{jk}-\lambda_j \ \text{ if } x'_{jk}=1. \qquad (4)$$

Now use that if $x_{jk}=1$ then the stability of $(u^*,v^*;x)$ implies that $a_{jk}-u^*_j=v^*_j\geq a_{tk}-u^*_t$, for all $p_t$. Hence

$$\Sigma_j\ (a_{jk}-u^*_j)x_{jk}\geq\Sigma_j\ (a_{jk}-u^*_j)x'_{jk}, \text{ for all } q_k. \qquad (5).$$

Then, for al $q_k$ we have that $\Sigma_j\ (a_{jk}-\lambda_j)x_{jk}=\Sigma_j(a_{jk}-u^*_j)x_{jk}+\Sigma_j(u^*_j-\lambda_j)x_{jk}\geq\Sigma_j(a_{jk}-u^*_j)x'_{jk}+\Sigma_j(u^*_j-\lambda_j)x_{jk}$.

Adding up yields:

$\Sigma_{j,k}\ (a_{jk}-\lambda_j)x_{jk}\ \geq\ \Sigma_{j,k}(a_{jk}-u^*_j)x'_{jk}+\Sigma_{j,k}(u'_j-\lambda_j)\ x_{jk}\ \geq\ \Sigma_{j,k}\ (a_{jk}-u^*_j)\ x'_{jk}+\Sigma_{j,k}(u^*_j-\lambda_j)\cdot x'_{jk}=\Sigma_{j,k}(a_{jk}-\lambda_j)x'_{jk}$, where the second inequality follows from (2). Hence

$$\Sigma_{j,k}\ (a_{jk}-\lambda_j)x_{jk}\ \geq \Sigma_{j,k}\ (a_{jk}-\lambda_j)x'_{jk} \qquad (6)$$

Now use (3), (4) and (6) to get that $\Sigma_{j,k}\, b_{jk}x_{jk} \geq \Sigma_{j,k}\, b_{jk}x'_{jk}$.

To see that $x \in \Sigma^*(\beta)$ suppose $x' \in \Sigma^*(\beta)$ and use $(1)$. Then

$$\Sigma(M) \subseteq \Sigma^*(\beta). \tag{7}$$

To prove the other inclusion let $x'' \in \Sigma^*(\beta)$. Let x be any optimal matching for M. By (7), $x \in \Sigma^*(\beta)$. Then $\Sigma_{j,k} a_{jk} x''_{jk} = \Sigma_{j,k} a_{jk} x_{jk}$ from which follows that $x'' \in \Sigma(M)$. Hence

$$\Sigma(M) \supseteq \Sigma^*(\beta). \tag{8}$$

The desired result follows from (7) and (8). ∎

**Proof of Lemma 2.** a) If $x_{jk}=1$ then $a_{jk} \geq u^*_j \geq \lambda_j$, so $a_{jk}-\lambda_j \geq 0$ and so $b_{jk}=a_{jk}-\lambda_j$. By Lemma 1, $x \in \Sigma^*(\beta)$, so $u^1_j+\lambda_j+v^1_k=a_{jk}$. If $p_j$ is unmatched at $x$ then $u^1=0$. On the other hand $u^*_j=0$ by Corollary 1*, so $0=u^*_j \geq \lambda_j \geq 0$, so $\lambda_j=0$ and so $u^1_j+\lambda_j=0$; if $q_k$ is unmatched at $x$ then $v^1_k=0$. Thus $(u^1+\lambda_j, v^1;x)$ is feasible for $M(\beta)$ by Remark 2. The individual rationality of $(u^1,v^1;x)$ in $M(\beta)$ implies the individual rationality of $(u^1+\lambda, v^1;x)$ in $M$. To see that there is no blocking pair, consider $(p_j,q_k) \in PxQ$. Then $(u^1_j+\lambda_j)+(v^1_k) \geq b_{jk}+\lambda_j$ from the stability of $(u^1,v^1;x)$ in $M(\beta)$. Now use that $b_{jk}+\lambda_j = a_{jk}-\lambda_j+\lambda_j=a_{jk}$ if $q_k \in B_j$ and $b_{jk}+\lambda_j=\lambda_j > a_{jk}$ if $q_k \notin B_j$, to see that $u^1_j+\lambda_j+v^1_k \geq a_{jk}$. Hence $(u^1+\lambda, v^1;x)$ is stable for $M$.

b) By hypothesis $u^2_j-\lambda_j \geq 0$ for all $p_j$. If $x_{jk}=1$ we have that $a_{jk}-\lambda_j=(u^2_j-\lambda_j)+v^2_k \geq 0$, so $(u^2_j-\lambda_j)+v^2_k=b_{jk}$. If $p_j$ is unmatched under $x$ then $0 \leq \lambda_j \leq u^2_j=0$, so $\lambda_j=0$, and so $u^2_j-\lambda_j=0$. If $q_k$ is unmatched under $x$ then $v^2_k=0$. Thus $(u^2-\lambda, v^2;x)$ is feasible for $M(\beta)$, by Remark 2. The individual rationality of $(u^2-\lambda, v^2;x)$ follows from the individual rationality of $(u^2,v^2)$ and the fact that $u^2_j-\lambda_j \geq 0$ for all $p_j$. That there is no blocking pair is immediate from the stability of $(u^2,v^2)$ in $M$. In fact for every $(p_j,q_k) \in PxQ$, $(u^2_j-\lambda_j)+v^2_k=(u^2_j+v^2_k)-\lambda_j \geq a_{jk}-\lambda_j=b_{jk}$ if $q_k \in B_j$ and $(u^2_j-\lambda_j)+v^2_k \geq 0=b_{jk}$ if $q_k \notin B_j$. Hence $(u^2-\lambda, v^2;x)$ is stable for $M(\beta)$. ∎

**Proof of Proposition 1.** Arguing by contradiction, suppose $\beta$ is not a Nash equilibrium in the strong sense for some $x \in \Sigma^*(\beta)$. That is, there exists some bidder $p_j$, some joint strategy $\beta'=(b',B')$ with $B'_t=B_t$ and $b'_t=b_t$ if $p_t \neq p_j$, such that

$$U_j(\beta',x')>U_j(\beta,x) \text{ for some } x'\in\Sigma^*(\beta'). \qquad (1)$$

Lemma 1 implies that $x\in\Sigma(M)$. Lemma 2-a) implies that $(u(\beta)+\lambda, v(\beta);x)$ is stable for $M$, so

$$U(\beta,x)=u(\beta)+\lambda. \qquad (2)$$

By (1) and (2),

$$U_j(\beta',x')> u_j(\beta)+\lambda_j\geq\lambda_j\geq 0 \qquad (3)$$

from which follows that $U_j(\beta',x')>0$. Then $p_j$ is matched to some $q_k$ under $x'$, by Definition 10. Furthermore, from the facts that $(a_{jk}-s_k)-v_k(\beta')=U_j(\beta',x')>0$ and $v_k(\beta')\geq 0$ we can conclude that $a_{jk}-s_k\geq 0$, so $q_k\in A_j$ and so

$$U_j(\beta',x')=a_{jk}-v_k(\beta'). \qquad (4)$$

By (3) and (4), $(a_{jk}-v_k(\beta')>\lambda_j$, so $(a_{jk}-\lambda_j)>v_k(\beta')\geq 0$, from which follows that $b_{jk}=a_{jk}-\lambda_j$. Now let $U'_j(\beta',x')$ be the transfer associated to $x'$ in $M(\beta)$. That is, $U'_j(\beta',x')=b_{jk}-v_k(\beta')$. Then $U'_j(\beta',x')=(a_{jk}-\lambda_j)-v_k(\beta')= U_j(\beta',x')-\lambda_j>u_j(\beta)$, where the last equality and the last inequality follows from (4) and (3), respectively. Now use that $|C(\beta)|=1$. Then $u^*(\beta)=u(\beta)$ and so $U'_j(\beta',x') > u^*_j(\beta)$, contradicting Theorem 6* applied to $M(\beta)$. Thus $\beta=(b,B)$ is a Nash equilibrium in the strong sense for every $x\in\Sigma^*(\beta)$. Hence $\beta$ is a Nash equilibrium in the strong sense.

To see that $(u^*,v^*)$ is the corresponding Nash equilibrium payoff in the strong sense, use Lemma 2-(b) to get that $(u^*-\lambda,v^*)$ is stable for $M(\beta)$. Then $u^*-\lambda=u^*(\beta)$ and $v^*=v^*(\beta)$, by the fact that $|C(\beta)|=1$.. By (2) $U(\beta,x)=u(\beta)+\lambda=u^*(\beta)+\lambda=u^*$. Since $v(\beta)=v^*(\beta)=v^*$ we get that $(U(\beta,x),v(\beta))=(u^*,v^*)$ and the proof is complete. ∎

**Proof of Proposition 2.** Suppose on the contrary that there is more than one stable payoff in $M(\beta)$. Then $u^*_j(\beta)>u_j(\beta)\geq 0$ for some $p_j\in P$ (recall that the auctioneer is not using the minimum competitive price rule), so $p_j$ must be matched to some $q_k$ under $x$ by Proposition 2*. Let $\lambda\in R^n{}_+$ such that

$$u^*_j(\beta)>\lambda_j>u_j(\beta). \qquad (1)$$

Define $\beta'=(b',B')$ such that $B'_t=B_t$, $b'_t=b_t$ if $p_t \neq p_j$, $B'_j=\{q_k\}$, $b'_{jk}=b_{jk}-\lambda_j$ and $b'_{jt}=0$ if $q_t \neq q_k$. Using (1) we can write that $b_{jk}-\lambda_j > b_{jk}- u^*_j(\beta)$, and using that $x_{jk}=1$ we can write that $v_*(\beta)=b_{jk}-u^*_j(\beta)$, so

$$b'_{jk} > v_{*k}(\beta), \qquad (2)$$

so $b'_{jk} \geq 0$ and so $b'_{jk}$ is well defined. We claim that $p_j$ is matched under any optimal matching in $\Sigma(\beta')$. Arguing by contradiction, suppose that $p_j$ is unmatched under some $x' \in \Sigma(\beta')$. By definition of the function $V_\beta$ we have that $V_\beta(P-\{p_j\},Q)=V_{\beta'}(P,Q) \geq b'_{jk}+V_{\beta'}(P-\{p_j\},Q-\{q_k\})=(b_{jk}-\lambda_j)+V_{\beta'}(P-\{p_j\},Q\{q_k\})>v_{*k}(\beta)+ V_\beta (P-\{p_j\},Q-\{q_k\})$, where in the last inequality was used (2) and the definition of $b'$. Then $v_{*k}(\beta)<V_\beta(P-\{p_j\},Q) - V_\beta(P-\{p_j\},Q-\{q_k\})$, which contradicts Theorem 4*-a.

Thus $p_j$ is matched to $q_k$ under any matching $x'$ in $\Sigma(\beta')$, since any other object is not in $B'_j$. Because of this $v_k(\beta') \leq b'_{jk}= b_{jk}-\lambda_j$. However $b_{jk}-\lambda_j < b_{jk}-u_j(\beta)$ by (1) and by the fact that $x_{jk}=1$. Then $v_k(\beta')< b_{jk}-u_j(\beta) = v_k(\beta)$. Thus $U_j(\beta',x')=\alpha_{jk}-v_k(\beta')-s_k> \alpha_{jk}-v_k(\beta)-s_k=U_j(\beta,x)$, for all $x' \in \Sigma(\beta')$. That is, $U_j(\beta',x')> U_j(\beta,x)$ for all $x' \in \Sigma(\beta')$, which contradicts the fact that $\beta=(b,B)$ is a Nash equilibrium in the weak sense for $x$. Hence $|C(\beta)|=1$.∎

**Proof of Theorem 2.** To prove (a) take $x \in \Sigma^*(\beta)$ and suppose that $\beta=(b,B)$ is a Nash equilibrium in the weak sense for $x$. Since $0 \leq u'_j \leq u^*_j$ for all $p_j$, we can apply Lemma 1, with $\lambda=u'$ and get that $x \in \Sigma(M)$. Proposition 2 implies that $|C(\beta)|=1$. However Lemma 2-b) applied to the payoffs $(u',v';x)$ and $(u^*,v_*;x)$, with $\lambda=u'$, implies that $(0=u'-u',v';x)$ and $(u^*-u',v_*;x)$ are stable for $M(\beta)$. Hence $u^*-u'=0$, and so $u^*=u'$.

For the proof of (b) suppose that $u^*=u'$. Let $(u^1,v^1;x)$ *be some stable outcome for* $M(\beta)$ with $x \in \Sigma^*(\beta)$. Lemma 1 implies that $x \in \Sigma(M)$. Lemma 2-a) applied to the payoff $(u^1,v^1;x)$, with $\lambda=u^*$, implies that $(u^1+u^*,v^1;x)$ is stable for $M$. The maximality of $u^*$ in $M$ and the individual rationality of $u^1$ imply that $u^1_j=0$ for all bidder $p_j$, so $u^1+u^*=u^*$ and $v'^1=v_*$. Therefore $|C(\beta)|=1$. Now use Proposition 1 to conclude that $\beta=(b,B)$ is a Nash equilibrium in the strong sense and $(u^*,v_*)$ is the corresponding Nash

equilibrium payoff in the strong sense.▮

**Proof of Theorem 3.** Suppose that $(U(\beta,x),v(\beta))$ is not stable for $M$. The payoff $(U(\beta,x),v(\beta))$ is feasible and individually rational, by Remark 3. Then there exists a blocking pair, that is a pair $(p_j,q_k)\in PxQ$ such that $U_j(\beta,x)+v_k(\beta)<a_{jk}$ (observe that $a_{jk}>0$ by individual rationality of $(U(\beta,x),v(\beta))$, so $q_k\in A_j$). Hence, for some $\lambda>0$,

$$v_k(\beta)<a_{jk} - (U_j(\beta,x)+\lambda). \qquad (1)$$

*Now define* $\beta'=(b',B')$ *such that* $B'_t=B_t$, $b'_t=b_t$ *if* $p_t\neq p_j$; $B'_j=\{q_k\}$, $b'_{jk}=a_{jk}-(U_j(\beta,x)+\lambda)$ *and* $b'_{jt}=0$ *if* $q_t\neq q_k$. *That* $b'_j$ *is well defined follows from* (1) *and from the fact that* $v_k(\beta)\geq 0$.

Take any $y$ in $\Sigma^*(\beta')$. We will show that $y_{jk}=1$ and so, using that $q_k\in A_j$ and $v_k(\beta')\leq b'_{jk}$, we can get that $U_j(\beta',y)=a_{jk}-v_k(\beta')\geq a_{jk}-b'_{jk}=a_{jk}-a_{jk}+U_j(\beta,x)+\lambda=U_j(\beta,x)+\lambda>U_j(\beta,x)$. But this contradicts the assumption that $\beta=(b,B)$ is a Nash equilibrium for $x$ in the weak sense. In fact, first observe that $p_j$ cannot be matched to an object other than $q_k$ since any other object is not in $B'_j$. Set $M'\equiv(P-\{p_j\}, Q, \beta_{-j}, s)$, where $\beta_{-j}$ is the restriction of $\beta$ to $P-\{p_j\}$. Let $(u*(M'),v^*(M'))$ be the Q-optimal payoff for the market $M'$. If $p_j$ is unmatched under $y$, then, using Theorem 4*-b), $v^*_k(\beta')=V_{\beta'}(P,Q) - V_{\beta'}(P,Q-\{q_k\})= V_{\beta'}(P-\{p_j\},Q)- V_{\beta'}(P-\{p_j\},Q-\{q_k\})=v^*_k(M')$, so $v^*_k(\beta')=v^*_k(M')$. Theorem 3* applied to the markets $M(\beta)$ and $M'$ implies that $v^*_k(\beta)\geq v^*_k(M')$, so

$$v^*_k(\beta) \geq v^*_k(\beta'). \qquad (2)$$

Since $\beta$ is a Nash equilibrium in the weak sense, we can apply Proposition 2 and get that $|C(\beta)|=1$, so

$$v_k(\beta)=v_k^*(\beta). \qquad (3)$$

Thus $v^*_k(\beta') \geq b'_{jk} > v_k(\beta)=v^*_k(\beta)\geq v^*_k(\beta')$ where the first inequality follows from the stability of $(u*(\beta'), v^*(\beta');y)$ and from the assumption that $p_j$ is unmatched at y; in the second inequality we used (1); the equality is given by (3) and the last inequality follows from (2). Hence $v^*_k(\beta') > v^*_k(\beta')$, absurd. Therefore $y_{jk}=1$ and the proof is complete.

▮

**Proof of Theorem 5.** Let $x$ and $x'$ be in $\Sigma^*(\beta)$. Then $\Sigma_j U_j(\beta,x) + \Sigma_k v_k(\beta) = \Sigma_{j,k} a_{jk} x_{jk} = \Sigma_{j,k} a_{jk} x'_{jk} = \Sigma_j U_j(\beta,x') + \Sigma_k v_k(\beta)$, where the first and the third equalities follow from the assumption that $B_j \subseteq A_j$, for all $p_j$, and Remark 2, and in the second equality was used the optimality of $x$ and $x'$. Then

$$\Sigma_j U_j(\beta,x) = \Sigma_j U_j(\beta,x') \qquad (1)$$

Suppose that $\beta$ is a Nash equilibrium in the strong (resp. weak) sense for $x$. Corollary 1 (resp. Theorem 2) implies that $(U(\beta,x), v(\beta); x)$ is stable for M. Consequently $U_j(\beta,x) \geq a_{jk} - v_k(\beta)$ for all $(p_j,q_k) \in PxQ$. Particularly, using the definition of $U_j(\beta,x')$,

$$U_j(\beta,x) \geq U_j(\beta,x') \text{ for all } p_j \in P. \qquad (2)$$

By (1) and (2) it follows that

$$U_j(\beta,x) = U_j(\beta,x') \text{ for all } p_j \in P. \qquad (3)$$

The desired result is immediate from (3) and Definition 11. ∎

**Proof of Corollary 3.** This is immediate from equality (3) obtained in the proof of Theorem 5. ∎

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# MARKET CLEARING SEALED-BID AUCTIONS FOR NON-IDENTICAL OBJECTS WITH SINGLE -UNIT DEMANDS

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**ABSTRACT**

We consider a version of the cooperative buyer-seller market game of Shapley and Shubik (1972). For this market we propose a class of sealed- bid auctions where objects are sold simultaneously at a market clearing price rule. We analyze the strategic games induced by these mechanisms under the complete information approach. We show that these non-cooperative games can be regarded as a competitive process for achieving a cooperative outcome: every Nash equilibrium payoff is a core outcome of the cooperative market game. Precise answers can be given to the strategic questions raised.

KEYWORDS: optimal matching, competitive price, Nash equilibrium, mechanism.

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