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"Portfolio Selection with Random Transaction Costs"

Professor: Marcelo Nazareth (The University of Chicago)

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Coordenação: Prof. Pedro Cavalcanti Gomes Ferreira
Email: ferreira@fgv.br - ☎ (021) 559-5834

Portfolio Selection with Random Transaction Costs

Marcelo Nazareth *
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Abstract

Transaction costs have a random component in the bid-ask spread. Facing a high bid-ask spread, the consumer has the option to wait for better terms of trade, but only by carrying an undesirable portfolio balance. We present the best policy in this case. We pose the control problem and show that the value function is the unique viscosity solution of the relevant variational inequality. Next, a numerical procedure for the problem is presented.

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1 Introduction

The literature of portfolio selection in a continuous time setting starts with Merton [26]. Merton shows that in the absence of transaction costs, the best strategy for the investor is to keep his holdings in the stock and in the money market account at a constant proportion. To implement this strategy the investor should trade continuously. The first objection to this setup, is that this strategy breaks down in the presence of an arbitrarily small transaction cost. The wild nature of the Brownian motion that drives the stock price movements implies that the strategy has infinite cost. The first model with fixed transaction costs was developed by Constantinides [5]. He shows that the investor couple with transaction costs by refraining to transact too often. The optimal strategy is to transact only when the proportion of stock relative to bonds in the portfolio gets too high or too low. Constantinides also reports that the loss in consumption relative to the Merton paradigm is small. With a 2% transaction cost consumption decreases by 0.4%; with 5% consumption decreases by 1%. Formal proofs of the results in the Constantinides paper can be found in Davis and Norman [7] . For a rather different proof of the same results see Fleming and Soner [12] pg. 342. Were not for the development of the notion of viscosity solutions (see [6] and [12]), anything slightly more complex than the Constantinides model would become impossible to analyze. After the breakthrough, more complex models appeared. Fleming and Zariphopoulou [13] analyzed a model with different borrowing and lending rates, Zariphopoulou [32] studies a model with transaction costs and borrowing constraints. For a model of option pricing with transaction costs, see Davis, Panas and Zariphopoulou [8]. A common characteristic of all the models above is the nature of the optimal policies. The optimal police implies that there is a region where it is optimal not to transact. As soon as the stock holdings try to escape out of this region, the controller is to purchase or sell stock, so as to keep the holdings just within the prescribed region. This paper is a first attempt to look at these policies in the presence of random transaction costs.

Transaction costs faced by consumers usually have a fixed component like brokerage fees, as well as a random component in the form of the bid-ask spread. Furthermore , the bid-ask spread can be viewed as a measure of market liquidity. Suppose large shocks hit the market making the controllers portfolio unbalanced. Suppose also that the market lost liquidity so that the bid-ask spread is temporarily high. Should the controller pay high transaction fees and rebalance the portfolio at once, or exercise his/her option and wait for better terms of trade? This paper is able to give a definite answer to this question: Do wait.

We claim that the chain of events described in the last paragraph is the

rule rather than the exception. Many branches of the market microstructure literature links high volatility in the market with high bid-ask spreads. We have attempted to build this feature in our model. This attempt was not fully successful. The readers can judge by themselves by reading the next section.

The paper is organized as follows: Section 2 develops the dynamics of the transaction costs. Section 3 poses the control problem and proves some elementary properties of the value function. Section 4 establishes the value function as the unique viscosity solution of the relevant Hamilton-Jacobi-Bellman equation. Section 5 deals with utility function in the HARA class. Section 6 deals with the numerical solution of the model. Section 7 presents the numerical results and conclude. The elementary proofs are in the appendix.

2 Modeling Transaction Costs

In the model, the agent faces a decision of purchasing and selling stock. He faces proportional transaction costs which are random. When he purchases y dollars worth of stock, his money market account is debited $y(1+K_t)$ dollars. When he sells y dollars worth of stock, his money market account is credited $y(1+K_t)^{-1}$ dollars. I will be looking for a process K_t which is nonnegative. In this section I develop the dynamics for the stock price S_t and transaction cost K_t. I approach this problem by first developing a discrete time model, where intuition is easier to grasp. The discrete time model is only a device to capture some stylized facts. In the second subsection I proceed to take it's continuous time limit. In later sections, I will only work with the continuous time process.

2.1 Discrete Time Model

I start with a discrete time, standard log-normal process for the stock price. For $m = 0, 1, 2, \ldots$ define

$$S_{m+1} = S_m + \mu\, S_m + \sigma\, S_m\, Z_m \tag{1}$$

$$S_0 \text{ given },\, Z_m \text{ i.i.d. } N(0,1)$$

Next, I postulate the behavior of the random transaction costs. The idea is to capture some stylized facts hinted by the market microstructure literature. In the inventory branch of the literature I cite Stoll [30], Ho and Stoll [19] and O'Hara and Oldfield [28]. These all imply that the market maker responds to a more volatile market by widening the bid-ask spread. The same qualitative result appear in some information based models. The story behind the well known papers by Glosten and Milgrom [14] and Easley and O'Hara [9] goes like: Suppose a group of traders learn privately that the market price of a certain stock is below its true value. Acting in self interest they will send to the market

maker an abnormal flow of buy orders. He, in turn, will respond by increasing the bid and ask prices, as well as by widening the spread. The overall effect is a positive association between the size of the spread and the absolute value of the stock price change. The effect also tend to be temporary, in the sense that, once the market maker learns the information conveyed by the flow of orders, the spread is reduced. Empirical tests of the results above have been carried out by Bollerslev and Melvin [3], Hasbrouk [15] and Hausman, Lo and MacKinlay [16]. Hasbrouk finds evidence linking large trades with wide spreads, what is consistent with the Easley and O'Hara model. Hausman, Lo and MacKinlay find positive links between stock price volatility and bid-ask spreads in the U.S.. The same relationship was uncovered by Bollerslev and Melvin in foreign exchange data. The idea, hence, is that transaction costs are higher when the market is volatile. A sequence of abnormally large shocks hitting the market are supposed to make transaction costs grow temporarily. A calm period, in the other hand, should make transaction costs smaller. The situation is reminiscent of a ARCH like modeling. I assume that current transaction costs K_{m+1} depends only on the past shocks $(Z_m, Z_{m-1}, \ldots)$ and that (K_{m+1}, Z_{m+1}) is markovian.

$$K_{m+1} = f(Z_m, Z_{m-1}, \ldots) = g(Z_m, K_m)$$

What are the features I want from the function g? Well, if $|Z_m|$ is large, I want K_{m+1} to increase, at least temporarily. Whereas, if $|Z_m|$ is small, I want K_{m+1} to decrease. On top of that, I also want the flexibility to let transaction costs respond asymmetrically with respect to positive and negative shocks. Like the EGARCH model, I assume that a large negative shock increases transaction costs more than a shock that is equally large but positive. The rational in the EGARCH model is to capture the so called "leverage effect". Here, it just means that it is particularly expensive to transact during crashes.[1] Summing up:

$$\begin{cases} \text{If } Z_m \text{ is large and positive} & \Rightarrow \quad K_{m+1} \uparrow \\ \text{If } Z_m \text{ is small} & \Rightarrow \quad K_{m+1} \downarrow \\ \text{If } Z_m \text{ is large and negative} & \Rightarrow \quad K_{m+1} \uparrow\uparrow \end{cases}$$

Before I proceed, I should warn the reader that the situation is only analogous to ARCH modeling. In my model the volatility of the stock prices is always constant. I completely abstract from stochastic volatility considerations, trying to isolate the effect of random transaction costs.[2] First, I specialize the function $g(Z_m, K_m)$. With a view toward the continuous time limit I pick the "square root" process.

$$K_{m+1} \quad = \quad K_m + \beta(\alpha - K_m) + \lambda \sqrt{K_m} \ \times \tag{2}$$

[1] No empirical evidence to support this particular feature will be presented. This just gives me extra flexibility and can be easily undone by setting the parameter ρ, below, appropriately.

[2] See the end of this section for a stochastic volatility model.

$$\times [\rho Z_m + \gamma (|Z_m| - (2/\pi)^{1/2})]$$

K_0 given

the constants satisfy,

$$\alpha, \beta, \lambda > 0$$
$$-1 < \rho < 1$$
$$2 \beta \alpha > \lambda^2$$
$$\gamma = (\frac{1 - \rho^2}{1 - 2/\pi})^{1/2}$$

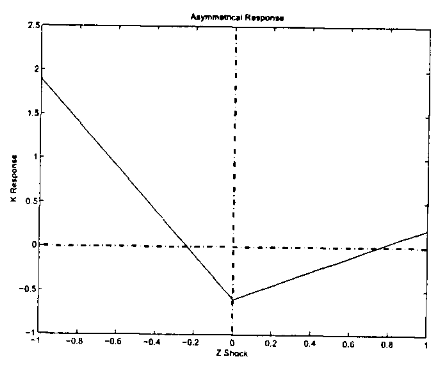

Figure 1: Shock Response

The response of $K_{m+1}/\sqrt{K_{m+1}}$ to shocks Z_m are plotted in figure 1 for $\rho < 0$. I favor the case $\rho < 0$. The choice of the process K_m is had-oc. It is dictated, however, by the desired response to shocks described above as well as by the properties of the continuous time limit of the system (1–2).

2.2 Continuous Time Model

I now turn to the continuous time limit of the system (1–2). All the missing details can be found in Nelson [27], which I follow closely. I will be considering 3 kinds of processes (where X stands for S, K and Z).

- Sequences of discrete time processes $\{X^h_{mh}\}$ that depend on h and the discrete time index mh, $m = 0, 1, 2, \ldots$. I set $X^h_0 = X_0 \; \forall h$.
- Sequences of continuous time processes $\{X^h_t\}$ defined as

$$P^h[X^h_t = X^h_{mh} \; , \; mh \leq t < (m+1)h] = 1$$

That is $\{X^h_t\}$ is formed as step functions from the processes $\{X^h_{mh}\}$ with jumps at $h, 2h, \ldots$.

- Limit diffusion processes $\{X_t\}$ to which $\{X_t^h\}$ will converge weakly as $h \downarrow 0$.

Now rewrite the system (1–2) as

$$\begin{aligned}
S^h_{(m+1)h} &= S^h_{mh} + \mu\, S^h_{mh}\, h + \sigma\, S^h_{mh}\, Z^h_{mh} && (3)\\
K^h_{(m+1)h} &= K^h_{mh} + \beta(\alpha - K^h_{mh})h + \lambda \sqrt{K^h_{mh}}\ \times && (4)\\
&\quad \times [\, \rho\, Z^h_{mh} + \gamma\, (\, |Z^h_{mh}| - (2\, h/\pi)^{1/2})]
\end{aligned}$$

$$\begin{aligned}
(S^h_0, K^h_0) &= (S_0, K_0) \text{ given}\\
Z^h_{mh} &\quad \text{i.i.d.} \quad N(0,h)
\end{aligned}$$

The following proposition is an easy consequence of theorem 3.1 in Nelson. It's proof is in the appendix

Proposition 1 $(S_t^h, K_t^h) \Longrightarrow (S_t, K_t)$ *(weakly) as* $h \downarrow 0$. *Where* (S_t, K_t) *satisfy:*

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_{1,t} \tag{5}$$

$$dK_t = \beta(\alpha - K_t)\, dt + \lambda\, \sqrt{K_t}\, dW_{2,t} \tag{6}$$

$$S_0, K_0 \text{ given}$$

and $[W_{1,t}\ W_{2,t}]$ is a 2-dimensional standard Brownian motion satisfying

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \begin{bmatrix} dW_{1,t} & dW_{2,t} \end{bmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt.$$

Intuitively, $(S^h_{(m+1)h} - S^h_{mh})$, $(K^h_{(m+1)h} - K^h_{mh})$ and h , are the discrete time counterparts to dS, dK and dt, respectively. Perhaps it is not so easy to see that Z^h_{mh} and $\rho\, Z^h_{mh} + \gamma\, (\, |Z^h_{mh}| - (2\, h/\pi)^{1/2})$ are also the discrete time counterparts of $[dW_{1,t}\ dW_{2,t}]$.

The reason this holds, is because if I define for each $m = 1, 2, \ldots$ and each $t \in [0, 1]$

$$Q_{m,t} \equiv (1 - 2/\pi)^{-1/2} \sum_{j=1}^{[mt]} \{|W_{(j+1)/m} - W_{j/m}| - (2/\pi m)^{1/2}\}$$

$[mt]$ is the integer part of mt and W_t is a standard Brownian motion on $[0, 1]$.

Then, by Donsker's theorem, as $m \to \infty$

$$[Q_{m,t}\ W_t] \Longrightarrow W_t^*,$$

W_t^* a 2-dimensional Brownian motion on $[0,1]$. $Q_{m,t}$ depends on the whole path of W_s for $s \leq t$. But as $m \to \infty$, $Q_{m,t}$ converges in distribution to a Brownian motion independent of W_t . The way to get a sequence of processes that converges to a Brownian motion that is correlated with W_t is by taking a linear combination of W_t and $Q_{m,t}$. I claim that the choice of a square root process for the transaction costs is reasonable. With this choice, I get a process K_t that

- Is stationary.
- Is positive with probability 1.
- It mean reverts, and when properly initialized, the mean of K_t is equal to the mean reversing point α.

The last property leads to a very natural way to access the effects of random transaction costs as opposed to fixed transaction costs. Namely, I will be comparing the effects of fixed costs $[(1+\alpha),(1+\alpha)^{-1}]$ as opposed to random costs $[(1+K_t),(1+K_t)^{-1}]$ initialized at $K_0 = \alpha$. Things are not as nice as they look, however. This setup has a drawback and is to be regarded as a first attempt to modeling random transaction costs. The reason being that all the nice intuition of the discrete time model 1–2 have vanished out of thin air. The continuous time model 5–6 has only a flavor of the association between transaction costs and volatility. In the limit all that remained was a negative association between prices and transaction costs (if $\rho < 0$).[3] The natural way to recover the negative correlation would be to regard K_t as having the double role of stochastic volatility and transaction costs as in

$$dS_t = \mu S_t \, dt + \frac{\sigma}{\alpha} K_t \, S_t \, dW_{1,t} \tag{7}$$

$$dK_t = \beta(\alpha - K_t) \, dt + \lambda \sqrt{K_t} \, dW_{2,t} \tag{8}$$

$$S_0, K_0 \text{ given}$$

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \begin{bmatrix} dW_{1,t} & dW_{2,t} \end{bmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt.$$

The drawback of this alternative setup is two-fold. The first is the increase in complexity, which can be severe. The second is the difficulty of interpretation. Whatever results I get with the setup 7–8, cannot be attributed separately to stochastic volatility or random transaction costs. I will stick to the setup 5–6, which is much simpler and can be more easily interpreted. In the end of section 4, I comment on what remains true with the alternative setup 7–8.

[3]The easiest way to see this is the case, is by considering a standard Euler approximation for the system 5–6. Approximating $(W_{1,t}, W_{2,t})$ by correlated normals (Z_1, Z_2) one sees that $E[Z_2 \mid |Z_1|] = 0$. The effect, thus, is absent in all stages of the approximation and therefore absent in the continuous time limit.

3 Investment-Consumption Model

In the first sub-section pose the control problem for the consumer and establishes some properties of the value function. In the second sub-section, I write down the Hamilton-Jacobi-Bellman equation, the value is supposed to satisfy.

3.1 The Control Formulation

The market has 2 securities. The first is a risk free money market account, which pays a constant interest rate r

$$B(t) = e^{rt} \tag{9}$$

The second security is a stock $S(t)$. It's dynamics is given together with the dynamics of the transaction costs $K(t)$ by

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_{1,t} \tag{10}$$

$$dK_t = \beta(\alpha - K_t) \, dt + \lambda\sqrt{K_t} \, dW_{2,t} \tag{11}$$

$$S_0, K_0 \text{ given}$$

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \begin{bmatrix} dW_{1,t} & dW_{2,t} \end{bmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt. \tag{12}$$

$[\, W_{1,t} \; W_{2,t} \,]$ is a two-dimensional Brownian Motion on a complete probability space (Ω, F_t, F, P). F_t is the completion of $\sigma(W_{1,t}, W_{2,t}; s \leq t)$.

The amounts held in the money market and stock accounts are denoted $X(t)$ and $Y(t)$ respectively. The consumer chooses processes $(C(t), L(t), M(t))$.

- $C(t)$ is the dollar consumption process.
- $L(t)$ is the cumulative dollar transfer into the stock account.
- $M(t)$ is the cumulative dollar transfer into the money market account.

Transfers between the stock account $Y(t)$ and the money market account $X(t)$ incur transaction costs $(1 + K(t))$. More specifically, a instantaneous transfer into the stock account of size $dL(t)$ reduces the bond account by $(1 + K(t))dL(t)$. A instantaneous transfer out of the stock account of size $dM(t)$ increases the bond account by $(1 + K(t))^{-1}dM(t)$. My bookkeeping always penalizes the money market account.

The consumption process $C(t)$ drains directly the bond account. Transfers out of the money market account to consumption incur no transaction costs.

Everything considered, the processes $X(t), Y(t)$ and $K(t)$ are given by:

$$X(t) = x + \int_0^t (rX(s) - C(s))ds \quad - \quad \int_0^t (1 + K(s))dL(s)$$
$$+ \quad \int_0^t (1 + K(s))^{-1} dM(s) \qquad (13)$$

$$Y(t) = y + \int_0^t \mu\, Y(s)\, ds + \int_0^t \sigma\, Y(s)\, dW_{1,s} + L(t) - M(t) \qquad (14)$$

$$K(t) = k + \int_0^t \beta(\alpha - K(s))ds + \int_0^t \lambda\, \sqrt{K(s)}\, dW_{2,s} \qquad (15)$$

The investor has Von Neumann-Morgestern preferences

$$\mathrm{E}\,[\int_0^\infty e^{-\delta t}\, U(C(t))\, dt\,]$$

over the consumption process $\{C(t); t \geq 0\}$. δ is the subjective discount factor and $U(C(t))$ has the following properties:

- $U : R^+ \mapsto R$, $U \in C^2$ with $U(0) = 0$, $U' > 0$, $U'' < 0$.
- $0 \leq U(c) \leq M\,(1 + c)^\gamma$ for some $M > 0$, $\gamma \in (0, 1)$
- U satisfy the Inada conditions $U'(0) = \infty$, $U'(\infty) = 0$

The investor faces the following problem. Given $(X(0) = x, Y(0) = y, K(0) = k)$ maximize

$$V(x, y, k) = \sup_{A_{(x,y,k)}} \mathrm{E}^{(x,y,k)}\,[\int_0^\infty e^{-\delta\, t}\, U(C(t))\, dt\,] \qquad (16)$$

$A_{(x,y,k)}$ is the set of triples $(L(t), M(t), C(t))$ satisfying

A-1 $(L(t), M(t))$ are right continuous, adapted, non-decreasing processes. I set $L(0) = M(0) = 0$.

A-2 $C(t)$ is adapted, continuous, non-negative and $\int_0^\infty C(t)dt < \infty$.

A-3 $X(t), Y(t) \geq 0$.

A–3 is admittedly restrictive. It does not allow leveraged or short positions in the stock. One excuse is that the parameters in the numerical computations will be such that the consumer never let $X(t) \leq 0$ or $Y(t) \leq 0$.

The real reason, however, is that the natural alternative assumption leads to a pathology. It would be natural to assume instead

$$0 \leq Z(t) \equiv \begin{cases} X(t) + Y(t)(1 + K(t))^{-1} & \text{if } Y(t) \geq 0 \\ X(t) + Y(t)(1 + K(t)) & \text{if } Y(t) < 0 \end{cases}$$

this says that, should the investor be called upon liquidating his position in the stock, at current transaction cost, he remains solvent.

But consider two sequences of points $a_n = (x_n, y_n, k_n) = (n, -1/n, n^2 - 1)$ and $b_n = (n, 0, n^2 - 1)$. Along b_n the consumer gets infinitely rich. It is feasible never to transact and consume arbitrarily large amounts directly from the money market account. Along a_n he is always broke. The only feasible strategy is to liquidate his stock position immediately and consume zero forever. Acting otherwise risks making $Z(t) < 0$ which is not allowed.

The pathology is that you may have an arbitrarily small short position in the stock $(-1/n)$ but transaction costs are so high $(n^2 - 1)$ that you are in fact insolvent. This implies that $|a_n - b_n| \to 0$ but $|V(a_n) - V(b_n)| \to U(\infty) - U(0) > 0$. This shows that under $Z \geq 0$ the value function is not uniformly continuous, a property that is essential for my uniqueness result.

I will assume that the problem 16 has a solution. As usual, if the parameters are set arbitrarily, investing everything in the stock, followed by massive consumption achieves an arbitrarily large value. I assume this is not the case. One possibility is to set the parameters as in the Merton problem.

$$\delta > \gamma(r + \frac{(\mu - r)^2}{\sigma^2(1 - \gamma)})$$

I close this section proving some elementary properties of the value function. The proof is in the appendix.

Proposition 2 **(a)** *$V(x, y, k)$ is concave in (x, y).*

(b) *$V(x, y, k)$ is strictly increasing in (x, y), non-increasing in k.*

(c) *$V(x, y, k)$ is uniformly continuous.*

3.2 The H-J-B Equation

In this sub-section I develop an heuristic argument that suggests that the value function satisfy equation 20 below. The argument is heuristic because it assumes a priori that $V(x, y, k) \in C^2$. It turns out that the value function will satisfy 20 only in a weak (viscosity) sense.

Consider the following strategy: Sell ϵ worth of stock instantly and continue optimally thereafter. This implies that we jump from position (x, y, k) to $(x + \epsilon(1 + k)^{-1}, y - \epsilon, k)$. Hence

$$V(x, y, k) \geq V(x + \epsilon(1 + k)^{-1}, y - \epsilon, k)$$

Dividing by ϵ and letting $\epsilon \downarrow 0$, I get

$$\frac{\partial V}{\partial y} - (1 + k)^{-1} \frac{\partial V}{\partial x} \geq 0 \tag{17}$$

Analogously, buying ϵ worth of stock instantly and proceeding optimally thereafter yields

$$V(x, y, k) \geq V(x - \epsilon(1+k), y + \epsilon, k) \text{ which implies,}$$

$$(1+k)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} \geq 0 \tag{18}$$

Before I derive the last equation, let me introduce some notation. Let

$$g(x, y, k) = \begin{bmatrix} rx \\ \mu y \\ \beta(\alpha - k) \end{bmatrix},$$

$$dV = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial k} \end{bmatrix},$$

$$\Sigma(y, k) = \begin{bmatrix} \sigma^2 y^2 & \lambda\rho\sigma\sqrt{k}y \\ \lambda\rho\sigma\sqrt{k}y & \lambda^2 k \end{bmatrix} \quad \text{and}$$

$$D^2 V = \begin{bmatrix} \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial k} \\ \frac{\partial^2 V}{\partial y \partial k} & \frac{\partial^2 V}{\partial k^2} \end{bmatrix}.$$

Notice that $D^2 V$ is not the Hessian matrix of V, since it does not include partials with respect to x.

Last, consider the strategy S: In $[0, t)$ do not transact and consume at a constant rate c; thereafter proceed optimally. This yields

$$V(x, y, k) \geq \int_0^t e^{-\delta s}\, U(c)\, ds + \mathrm{E}^S\, [e^{-\delta t}\, V^S(X(t), Y(t), K(t))]$$

where $\mathrm{E}^S[e^{-\rho t}\, V(X(t), Y(t), K(t))]$ means that we let the system 13–14 evolve freely, just draining the money market account at rate c. By Ito's lemma

$$\begin{aligned} &\mathrm{E}^S[e^{-\delta t}\, V^S(X(t), Y(t), K(t))] - V(x, y, k) \\ &= \textstyle\int_0^t e^{-\delta s}(-\delta\, V + g^\top dV + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2 V - c\frac{\partial V}{\partial x})ds \end{aligned}$$

Hence

$$\int_0^t e^{-\delta s}(-\delta\, V + g^\top dV + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2 V + U(c) - c\frac{\partial V}{\partial x})ds \leq 0$$

dividing by t and letting $t \downarrow 0$, I get $\forall c \geq 0$,

$$-\delta\, V + g^\top dV + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2 V + U(c) - c\frac{\partial V}{\partial x} \leq 0$$

which implies

$$-\delta\, V + g^\top dV + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2 V + \max_{c \geq 0}\{U(c) - c\frac{\partial V}{\partial x}\} \leq 0 \tag{19}$$

Finally we define the operator $\mathcal{L}$ by

$$\mathcal{L}\,V(x,y,k) \equiv -\delta\,V + g^{\top}dV + \frac{1}{2}\mathrm{Tr}\,\Sigma.D^2V + \max_{c\geq 0}\{U(c) - c\frac{\partial V}{\partial x}\},$$

combine equations 17–19 above to get

$$\min\{-\mathcal{L}\,V\,,\,(1+k)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\,,\,\frac{\partial V}{\partial y} - (1+k)^{-1}\frac{\partial V}{\partial x}\} = 0 \tag{20}$$

The form of the equation 20 suggests that the (x,y,k) space will split in three regions.

(NT) Where $L\,V = 0$. Here it is optimal not to transact.

(BS) Where $(1+k)V_x - V_y = 0$. Here it is optimal to buy stock immediately, forcing (x,y,k) to return to the boundary of the (NT) region.

(SS) Where $V_y - (1+k)^{-1}V_x = 0$. Here it is optimal to sell stock immediately, forcing (x,y,k) to return to the boundary of the (NT) region.

The boundaries between the regions are not known a priori and have to be determined together with $V(x,y,k)$.

It is natural to conjecture that the optimal strategy involves singular controls. At time 0, if the controller finds himself either at the (BS) or the (SS), he jumps immediately to the boundary of the (NT) region along the lines $x(1+k) = y$ and $x(1+k)^{-1} = y$, respectively. Afterwards he exercises the controls $L(t), M(t)$ just enough to keep (x,y,k) inside the (NT) region. The controls therefore behave like the local time of the Brownian motions $(W_{1,t}, W_{2,t})$. I will show below that the optimal controls never jumps, except, perhaps at $t = 0$.

To fully characterize the optimal controls, one would have to actually solve the variational inequality 20 above. This task is considered near to impossible with today's knowledge. It is feasible to prove numerically that the space (x,y,k) splits in the three regions $L\,V = 0$, $(1+k)V_x - V_y = 0$ and $V_y - (1+k)^{-1}V_x = 0$. However, the behaviour of the optimal control described above will remain a conjecture.

4 The Viscosity Property

In this section I introduce the notion of viscosity solutions. I prove that the value function defined by 16 is the unique viscosity solution of the variational inequality 20. For a first class survey in the theory, the reader is referred to the "User's guide" by Crandall, Ishii and Lions [6]. Another useful source of results is Ishii and Lions [20]. The standard reference concerning control problems is the book by Fleming and Soner [12].

The theory applies to PDE's of the form $F(x,u,Du,D^2u) = 0$ where $F : R^N \times R \times R^N \times R^{N\times N} \to R$, where $R^{N\times N}$ denotes the set of symmetric $N \times N$ matrices and u is a real valued function defined in a open subset $\Omega \subseteq R^N$. The theory allows F to be fully non-linear and the solution u to be merely continuous. However F is required to satisfy the monotonicity condition

$$F(x,r,p,X) \leq F(x,s,p,Y) \quad \text{whenever } r \leq s \text{ and } Y \leq X$$

where $r, s \in R$; $x, p \in R^N$; $X, Y \in R^{N\times N}$ and $R^{N\times N}$ is equipped if its usual order. The condition on r and s is named properness, while the condition on X and Y is referred as degenerate elliptic.

In control problems where state constraints are present the relevant notion is that of constrained viscosity solution. It was introduced by Capuzzo-Dolcetta and Lions [4]. See also Katsoulakis [21].

Definition 1 *A continuous function $u : \Omega \to R$ is a constrained viscosity solution of*

$$F(x,u(x),Du(x),D^2u(x)) = 0$$

if

1. *u is a viscosity subsolution of $F = 0$ on $\overline{\Omega}$; that is, if for all $\psi \in C^2(\Omega)$ and all points x_0 such that $u(x_0) - \psi(x_0)$ attains a local maximum we have*

$$F(x_0,u(x_0),D\psi(x_0),D^2\psi(x_0)) \leq 0$$

2. *u is a viscosity supersolution of $F = 0$ on Ω; that is, if for all $\psi \in C^2(\Omega)$ and all points x_0 such that $u(x_0) - \psi(x_0)$ attains a local minimum we have*

$$F(x_0,u(x_0),D\psi(x_0),D^2\psi(x_0)) \geq 0$$

The reader should notice the judicious use of Ω and $\overline{\Omega}$ in the definition.

With this definition in hands I can now establish the following proposition, the proof of which follow the lines of [8, 31, 32]. The argument is nowadays standard.

Proposition 3 *The value function $V(x,y,k)$ defined by equation 16 is a constrained viscosity solution of*

$$\min\{-\mathcal{L}V\ ,\ (1+k)\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\ ,\ \frac{\partial V}{\partial y} - (1+k)^{-1}\frac{\partial V}{\partial x}\} = 0$$

Proof. By definition we have to establish the sub and supersolution properties. In my case $\Omega = R^3_{++} = \{x,y,k > 0\}$. I also denote $R^3_+ = \overline{R}^3_{++}$.

subsolution property Let $\psi \in C^2(R^3_+)$ and $z_0 = (x_0, y_0, k_0)$ be a local maximum of $V - \psi$. Assume, without loss of generality that

$$V(z_0) = \psi(z_0) \text{ and } V \leq \psi \text{ } on \text{ } R^3_+ \tag{21}$$

I need to show that

$$\min\{-L\psi(z_0)\,,\,(1+k_0)\frac{\partial\psi(z_0)}{\partial x} - \frac{\partial\psi(z_0)}{\partial y}\,,\,\frac{\partial\psi(z_0)}{\partial y} - (1+k_0)^{-1}\frac{\partial\psi(z_0)}{\partial x}\} \leq 0 \tag{22}$$

Suppose to the contrary that

$$(1+k_0)\frac{\partial\psi(z_0)}{\partial x} - \frac{\partial\psi(z_0)}{\partial y} > 0\,, \tag{23}$$

$$\frac{\partial\psi(z_0)}{\partial y} - (1+k_0)^{-1}\frac{\partial\psi(z_0)}{\partial x} > 0\,, \tag{24}$$

and

$$\begin{aligned} \delta\,\psi(z_0) - g^\top(z_0)d\psi(z_0) \quad &- \quad \frac{1}{2}\mathrm{Tr}\,\Sigma(y_0, k_0).D^2\psi(z_0) \\ &- \quad \max_{c\geq 0}\{U(c) - c\frac{\partial\psi(z_0)}{\partial x}\} > \theta \end{aligned} \tag{25}$$

for some $\theta > 0$.

Since ψ and $U(c)$ are smooth these inequalities also hold in a neighborhood of z_0. That is, there exists $\theta > 0$ and a radius $r > 0$ such that for all $z = (x, y, k)$ with $z \in B(z_0, r)$

$$(1+k)\frac{\partial\psi(z)}{\partial x} - \frac{\partial\psi(z)}{\partial y} > 0\,, \tag{26}$$

$$\frac{\partial\psi(z)}{\partial y} - (1+k)^{-1}\frac{\partial\psi(z)}{\partial x} > 0 \tag{27}$$

and

$$\begin{aligned} \delta\,\psi(z) - g^\top(z)d\psi(z) \quad &- \quad \frac{1}{2}\mathrm{Tr}\,\Sigma(y, k).D^2\psi(z) \\ &- \quad \max_{c\geq 0}\{U(c) - c\frac{\partial\psi(z)}{\partial x}\} > \theta \end{aligned} \tag{28}$$

Next I need the following auxiliary result, which says that the optimal trajectory has no jumps at $t > 0^+$. Let the optimal trajectory be $Z_0^\star(t) = (X_0^\star(t), Y_0^\star(t), K_0(t))$; where optimal controls $(C^\star(t), L^\star(t), M^\star(t))$ are being used. For issues of existence of optimal control , the reader is referred to Zhu [33] and Hausmann and Suo [17]. Tradition demands it to be called lemma 1. (see [8, 29, 31, 32])

Lemma 1 *Assume inequality 23 holds (resp. 24). Let $A(\omega)$ be the event that the optimal trajectory has a jump of size ϵ along the direction $(1+k_0, 1, 0)$ (resp. direction $((1+k_0)^{-1}, -1, 0)$).*

Assume that after the jump the state is $(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0)$. Then,

$$((1+k_0)\frac{\partial\psi(z_0)}{\partial x} - \frac{\partial\psi(z_0)}{\partial y})P(A) \leq 0.$$

which implies $P(A)=0$.

Proof. By the dynamic programming principle

$$\begin{aligned} V(x_0,y_0,k_0) &= \mathrm{E}V(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0) \\ &= \int_A V(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0)\,dP \\ &+ \int_{\Omega-A} V(x_0,y_0,k_0)\,dP \end{aligned}$$

so,

$$\begin{aligned} 0 &= \int_A V(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0) - V(x_0,y_0,k_0)\,dP \\ &\leq \int_A \psi(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0) - \psi(x_0,y_0,k_0)\,dP \end{aligned}$$

hence,

$$0 \leq \limsup_{\epsilon\to 0} \int_A \frac{\psi(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0) - \psi(x_0,y_0,k_0)}{\epsilon}\,dP$$

By Fatou's lemma

$$\begin{aligned} 0 &\leq \int_A \limsup_{\epsilon\to 0} \frac{\psi(x_0-(1+k_0)\epsilon, y_0+\epsilon, k_0) - \psi(x_0,y_0,k_0)}{\epsilon}\,dP \\ &= \int_A -(1+k_0)\frac{\partial\psi(z_0)}{\partial x} + \frac{\partial\psi(z_0)}{\partial y}\,dP \\ &= (-(1+k_0)\frac{\partial\psi(z_0)}{\partial x} + \frac{\partial\psi(z_0)}{\partial y})\,P(A) \end{aligned}$$

The same argument holds for the other inequality.

Continuing the proof of the proposition, define the stopping time

$$\tau = \inf\{t \geq 0 : Z_0^\star(t) \notin B(z_0, r)\}$$

the lemma implies that $\tau > 0$ a.s. Now combine inequalities 26 , 27 and

28 to get

$$
\begin{aligned}
\mathrm{E}\int_0^\tau \theta\, e^{-\delta t} dt \quad < \quad \mathrm{E} \quad & \int_0^\tau e^{-\delta t}\, [\delta\, \psi(Z_0^\star(t)) - g^\top(Z_0^\star(t)) d\psi(Z_0^\star(t)) \\
& -\tfrac{1}{2}\mathrm{Tr}\ \Sigma(Y_0^\star(t), K_0(t)).D^2\psi(Z_0^\star(t)) \\
& -U(C^\star(t)) + C^\star(t)\tfrac{\partial\psi(Z_0^\star(t))}{\partial x}]\, dt \\
+ \quad \mathrm{E} \quad & \int_0^\tau e^{-\delta t}[(1+K_0(t))\tfrac{\partial\psi(Z_0^\star(t))}{\partial x} - \tfrac{\partial\psi(Z_0^\star(t))}{\partial y}]\, dL^\star(t) \\
+ \quad \mathrm{E} \quad & \int_0^\tau e^{-\delta t}[\tfrac{\partial\psi(Z_0^\star(t))}{\partial y} - (1+K_0(t))^{-1}\tfrac{\partial\psi(Z_0^\star(t))}{\partial x}]\, dM^\star(t)
\end{aligned}
\tag{29}
$$

Denote the terms on the left hand side by

$$
\mathrm{E}\, I_1(\tau) - \mathrm{E}\int_0^\tau U(C^\star(t))\, dt + \mathrm{E}\, I_2(\tau) + \mathrm{E}\, I_3(\tau) \tag{30}
$$

Applying Ito's formula for semi-martingales yields

$$
\mathrm{E}\{e^{-\delta\tau}\psi(Z_0^\star(\tau))\} = \psi(z_0) - \mathrm{E}\, I_1(\tau) - \mathrm{E}I_2\,(\tau) - \mathrm{E}I_3\,(\tau) \tag{31}
$$

To get the desired result, just combine 29 , 31 and $\psi(z_0) = V(z_0)$

$$
\begin{aligned}
\theta\frac{1-\mathrm{E}e^{-\delta\tau}}{\delta} \quad &< \quad \psi(z_0) - \mathrm{E}\,\{e^{-\delta\tau}\psi(Z_0^\star(\tau))\} - \mathrm{E}\int_0^\tau U(C^\star(\tau))\, dt \\
&= \quad V(z_0) - \mathrm{E}\,\{e^{-\delta\tau}V(Z_0^\star(\tau))\} - \mathrm{E}\int_0^\tau U(C^\star(\tau))\, dt \\
&= \quad 0
\end{aligned}
$$

which is a contradiction with $\theta > 0$ and $\tau > 0$ a.s. The equalities follow from the optimality of $Z^\star(t)$ and the dynamic programming principle. So, at least one of the arguments inside the min operator must be non positive, what establishes the subsolution property.

supersolution property Let $\psi \in C^2(R^3_{++})$ and $z_0 = (x_0, y_0, k_0)$ be a local minimum of $V - \psi$. Assume, without loss of generality that

$$
V(z_0) = \psi(z_0) \ \text{ and } \ V \geq \psi \ \ on \ \ R^3_{++} \tag{32}
$$

I need to show that

$$
\min\{-L\psi(z_0)\,,\,(1+k_0)\frac{\partial\psi(z_0)}{\partial x} - \frac{\partial\psi(z_0)}{\partial y}\,,\,\frac{\partial\psi(z_0)}{\partial y} - (1+k_0)^{-1}\frac{\partial\psi(z_0)}{\partial x}\} \geq 0 \tag{33}
$$

I will prove that each term inside the min operator is nonnegative.

Consider the following strategy: Starting at z_0 , sell ϵ worth of stock instantly and continue optimally thereafter. This implies that we jump

from position (x_0, y_0, k_0) to $(x_0 + \epsilon(1+k_0)^{-1}, y_0 - \epsilon, k)$. Hence

$$\begin{aligned}\psi(x_0, y_0, k_0) = V(x_0, y_0, k_0) &\geq V(x_0 + \epsilon(1+k_0)^{-1}, y_0 - \epsilon, k_0) \\ &\geq \psi(x_0 + \epsilon(1+k_0)^{-1}, y_0 - \epsilon, k_0)\end{aligned}$$

dividing by ϵ and letting $\epsilon \downarrow 0$, I get

$$\frac{\partial \psi(z_0)}{\partial y} - (1+k_0)^{-1} \frac{\partial \psi(z_0)}{\partial x} \geq 0$$

Analogously, starting at z_0 , buying ϵ worth of stock instantly and proceeding optimally thereafter yields

$$\begin{aligned}\psi(x_0, y_0, k_0) = V(x_0, y_0, k_0) &\geq V(x_0 - \epsilon(1+k_0), y_0 + \epsilon, k_0) \\ &\geq \psi(x_0 - \epsilon(1+k_0), y_0 + \epsilon, k_0)\end{aligned}$$

which implies

$$(1+k_0)\frac{\partial \psi(x_0)}{\partial x} - \frac{\partial \psi(z_0)}{\partial y} \geq 0$$

Last, consider the strategy S: In $[0, t)$ do not transact and consume at a constant rate c; thereafter proceed optimally. This yields

$$\begin{aligned}\psi(z_0) = V(z_0) &\geq \int_0^t e^{-\delta s}\, U(c)\, ds + \mathrm{E}^S\left[e^{-\delta t}\, V^S(X(t), Y(t), K(t))\right] \\ &\geq \int_0^t e^{-\delta s}\, U(c)\, ds + \mathrm{E}^S\left[e^{-\delta t}\, \psi^S(X(t), Y(t), K(t))\right]\end{aligned}$$

where $E^S[e^{-\rho t}\, V^S(X(t), Y(t), K(t))]$ means that we let the system 13–14 evolve freely, just draining the money market account at rate c. By Ito's lemma

$$\begin{aligned}&\mathrm{E}[e^{-\delta t}\, \psi^S(X(t), Y(t), K(t))] - \psi(x_0, y_0, k_0) \\ &= \textstyle\int_0^t e^{-\delta s}(-\delta\, \psi + g^\top d\psi + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2\psi - c\frac{\partial \psi}{\partial x})ds\end{aligned}$$

hence

$$\int_0^t e^{-\delta s}(-\rho\, \psi + g^\top d\psi + \frac{1}{2}\mathrm{Tr}\, \Sigma.D^2\psi(z_0) + U(c) - c\frac{\partial \psi}{\partial x})ds \leq 0$$

dividing by t and letting $t \downarrow 0$, I get $\forall c \geq 0$,

$$-\delta\, \psi(z_0) + g^\top(z_0)d\psi + \frac{1}{2}\mathrm{Tr}\, \Sigma(z_0).D^2\psi + U(c) - c\frac{\partial \psi(z_0)}{\partial x} \leq 0$$

which implies

$$-\delta\, \psi(z_0) + g^\top(z_0)d\psi + \frac{1}{2}\mathrm{Tr}\, \Sigma(z_0).D^2\psi + \max_{c\geq 0}\{U(c) - c\frac{\partial \psi(z_0)}{\partial x}\} \leq 0$$

or,

$$-L\psi(z_0) \geq 0.$$

The last result in this section proves that the value function as defined by equation 16 is the unique constrained viscosity solution of the variational inequality 20. As is common in this theory, the result is stated in the form of a comparison result.

Proposition 4 *Let u be a continuous viscosity subsolution of 20 on R^3_+ and v be a bounded below, uniformly continuous viscosity supersolution of 20 on R^3_{++}. Then $u \leq v$ in R^3_+.*

First I construct a strict supersolution of 20 in R^3_{++}.

Let $w_M(x,y)$ be the solution of the Merton problem, with utility function $U_1(c)$. Assume $U_1(c) \geq U(c)$ with strict inequality for $c > 0$. Then, $w_M(x,y) = v(z)$ where $z = x + y$ and for $z > 0$, v satisfies

$$\begin{cases} \delta v(z) = -\frac{(\mu-r)^2}{2\sigma^2}\frac{v'^2(z)}{v''(z)} + rzv'(z) + \max_{c\geq 0}\{U_1(c) - cv'(z)\} \\ v(z) > 0\ ,\ v'(z) > 0\ ,\ v''(z) < 0 \end{cases} \tag{34}$$

now, define

$$W_M(x,y,k) = w_M(x+ay) \qquad \text{for } (1+k)^{-1} < a < (1+k)$$

then,

$$\begin{aligned} (1+k)\frac{\partial W_M}{\partial x}(x,y,k) - \frac{\partial W_M}{\partial y}(x,y,k) &= (1+k-a)v'(x+ay) \\ &= f(x,y,k) > 0 \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\partial W_M}{\partial y}(x,y,k) - (1+k)^{-1}\frac{\partial W_M}{\partial x}(x,y,k) &= (a-(1+k)^{-1})v'(x+ay) \\ &= g(x,y,k) > 0 \end{aligned} \tag{36}$$

also , since $W_M(x,y,k)$ does not depend on k,

$$\begin{aligned} & \delta\, W_M - g^{\top} dW_M - \frac{1}{2}\mathrm{Tr}\,\Sigma.D^2W_M - \max_{c\geq 0}\{U(c) - c\frac{\partial W_M}{\partial x}\} \\ = {} & \delta v(z) - (rx+\mu ay)v'(z) - \frac{1}{2}\sigma^2(ay)^2v''(z) + \max_{c\geq 0}\{U(c) - cv'(z)\} \\ = {} & \left[-\frac{(\mu-r)^2}{2\sigma^2}\frac{v'^2(z)}{v''(z)} - (\mu-r)ayv'(z) - \frac{1}{2}\sigma^2(ay)^2v''(z)\right] \\ + {} & \left[\max_{c\geq 0}\{U_1(c) - cv'(z)\} - \max_{c\geq 0}\{U(c) - cv'(z)\}\right] \\ = {} & h_1(x,y,k) - h_2(x,y,k) \end{aligned}$$

The first term in brackets is quadratic in (ay). Inspection reveals that it is always nonnegative. The second term is strictly positive given the choice of $U_1(c)$. Therefore

$$h_1(x,y,k) + h_2(x,y,k) > 0 \tag{37}$$

Now, combining 35, 36 and 37

$$\min\{-L\, W_M\,,\, (1+k)\frac{\partial W_M}{\partial x} - \frac{\partial W_M}{\partial y}\,,\, \frac{\partial W_M}{\partial y} - (1+k)^{-1}\frac{\partial W_M}{\partial x}\} \;\geq$$

$$\min\{f(x,y,k), g(x,y,k), h_1(x,y,k) + h_2(x,y,k)\} = H(x,y,k) \;>\; 0$$

To conclude the theorem I will need the following lemma which is stated without proof. The reader is referred to Theorem VI.5 in Ishii and Lions [20] or Theorem 5.1 in the User's Guide [6].

Lemma 2 *Let u be continuous with sublinear growth viscosity subsolution of 20 on R^3_+. Let $\tilde{v}$ be a bounded below uniformly continuous viscosity supersolution of $F(z,\tilde{v},d\tilde{v},D^2\tilde{v}) - H(z) = 0$, where $H(z) > 0$ on R^3_{++}. Then $u \leq \tilde{v}$ on R^3_+.*

To conclude the proposition, define the function $\tilde{v}^\theta = \theta v + (1-\theta)W_M$.

It is assumed that v is a supersolution of $F(z,v,dv,D^2v) = 0$, and that W_M is a supersolution of $F - H = 0$. F given by 20. I claim that $\tilde{v}^\theta$ is a supersolution of $F - (1-\theta)H = 0$.

In fact, let $\psi \in C^2(R^3_{++})$ and assume that $\tilde{v}^\theta - \psi$ has a minimum at z_0. Then, $v - \phi$ also has a minimum at z_0, where

$$\phi = \frac{\psi - (1-\theta)W_M}{\theta}$$

hence,

$$F(z_0,\psi(z_0),d\psi(z_0),D^2\psi(z_0)) \;\geq$$

$$\theta F(z_0,\phi(z_0),d\phi(z_0),D^2\phi(z_0)) \;+$$

$$(1-\theta)F(z_0,W_M(z_0),dW_M(z_0),D^2W_M(z_0)) \;\geq\; (1-\theta)H(z_0)$$

what establishes the supersolution property. The first inequality follows from the fact that $F(z,u,du,D^2u)$ is jointly concave in (u,du,D^2u).

Now apply the lemma to conclude $u \leq \tilde{v}^\theta$. Finally finish the proof by sending $\theta \to 1$.

Now that the reader has an idea of the difficulties involved in the problem, I will comment on the complications that the alternative setup 7–8 introduces.

Concerning the elementary properties in proposition 2, the non linearity of the term $\sigma/\alpha Y_t K_t dW_{1,t}$ will complicate the proofs considerably. The straightforward arguments used to prove concavity in (x,y) and monotonicity in k no

longer work. I am not interested in these properties per se, but they come in handy to prove the uniform continuity.

The viscosity property naturally still holds with the obvious modifications of the H-J-B equation. The comparison result also would have to be modified, probably in a non trivial way.

Finally, the next section will show that in the simple case with HARA utility function, it is possible to reduce one variable in the value function. This is not possible in the stochastic volatility case.

5 The HARA Case

In this section I deal with the case where the utility function is HARA. This will allow a reduction in the number of variables of the value function and a further characterization of the boundaries of the three regions.

To simplify the notation, I assume the value function to be smooth. The differentials of $V(x, y, k)$ should be understood in the viscosity sense. For $0 < \gamma < 1$, let

$$U(c) = c^\gamma / \gamma$$

The linearity of the system 13–14 with respect to $(C(t), L(t), M(t))$ implies that if $\theta > 0$ and $(C, L, M) \in A_{(x,y,k)}$ then $(\theta C, \theta L, \theta M) \in A_{(\theta x, \theta y, k)}$

$$\begin{aligned} V(\theta x, \theta y, k) &= \sup_{A_{(\theta x, \theta y, k)}} \mathrm{E}^{(\theta x, \theta y, k)} \left[\int_0^\infty e^{-\delta t} \, c_t^\gamma / \gamma \, dt \right] \\ &= \sup_{A_{(x,y,k)}} \mathrm{E}^{(x,y,k)} \left[\int_0^\infty e^{-\delta t} \, (\theta c_t)^\gamma / \gamma \, dt \right] \\ &= \theta^\gamma v(x, y, k). \end{aligned}$$

Hence, $V(x, y, k)$ inherits the homothetic property of $U(c)$. The homothetic property implies that

$$V_x(\theta x, \theta y, k) = \theta^{\gamma - 1} V_x(x, y, k)$$

and

$$V_y(\theta x, \theta y, k) = \theta^{\gamma - 1} V_y(x, y, k)$$

thus, if

$$(1 + k) \frac{\partial V}{\partial x}(x, y, k) - \frac{\partial V}{\partial y}(x, y, k) = 0$$

or

$$\frac{\partial V}{\partial y}(x, y, k) - (1 + k)^{-1} \frac{\partial V}{\partial x}(x, y, k) = 0$$

the same holds for all points $(\theta x, \theta y, k)$. This strongly suggests that for fixed k, the boundaries between the no transaction and the transaction regions are straight lines through the point $(0, 0, k)$.

To get the reduction of variables define $W(x, k) = V(x, 1, k)$. The homothetic property implies that $V(x, y, k) = y^\gamma W(x/y, k)$. Furthermore, if my conjecture is correct, for fixed k , there will be points $0 < x_S(k) \leq x_B(k) \leq \infty$ such that: For $x < x_S$ it is optimal to sell stock immediately and for $x > x_B$ it is optimal to buy.

This coupled with the homothetic property of V implies that

$$\begin{cases} W(x,k) = W(x_S(k), k)(\frac{x+(1+k)^{-1}}{x_S+(1+k)^{-1}})^\gamma & \text{for} \quad x < x_S \\ W(x,k) = W(x_B(k), k)(\frac{x+(1+k)}{x_B+(1+k)})^\gamma & \text{for} \quad x > x_B \end{cases} \tag{38}$$

To complete the characterization of the variational inequality in the HARA case, notice that $V(x, y, k) = y^\gamma W(x/y, k)$ implies that

$$\begin{aligned} V_y(x,1,k) &= \gamma W(x,k) - xW_x(x,k) \\ V_{yk}(x,1,k) &= \gamma W_k(x,k) - xW_{xk}(x,k) \\ V_{yy}(x,1,k) &= x^2 W_{xx}(x,k) - (2\gamma+2)xW_x(x,k) - \gamma(1-\gamma)W(x,k) \end{aligned}$$

and the partials of $V(x, 1, k)$ involving only x and k equals those of $W(x, k)$. Finally, with the HARA utility function

$$\max_{c\geq 0}\{c^\gamma/\gamma - cW_x\} = (\frac{1-\gamma}{\gamma})W_x^{-\frac{\gamma}{1-\gamma}}.$$

Everything considered, the term $L\,V = 0$ in 20 becomes in the HARA case

$$aW + f^\top dW + \frac{1}{2}\Lambda.D^2W + (\frac{1-\gamma}{\gamma})W_x^{-\frac{\gamma}{1-\gamma}} = 0 \tag{39}$$

where

$$a = \mu\gamma - \delta - \frac{1}{2}\sigma^2\gamma(1-\gamma) \ ,$$

$$f(x,k) = \begin{bmatrix} (r - \mu - \frac{1}{2}\sigma^2(2\gamma+2))x \\ \beta(\alpha - k) + \gamma\rho\lambda\sigma\sqrt{k} \end{bmatrix}$$

and

$$\Lambda(x,k) = \begin{bmatrix} \sigma^2 x^2 & -\lambda\rho\sigma\sqrt{k}x \\ -\lambda\rho\sigma\sqrt{k}x & \lambda^2 k \end{bmatrix}$$

In the present case, the value function is expected to be a constrained viscosity solution of 39 in $x_S(k) \leq x \leq x_B(k)$ and satisfy 38 in $x \leq x_S$ and $x \geq x_B$. Of course, the boundaries still have to be determined jointly with $W(x, k)$.

6 Numerical Procedure

This section describes a methodology to discretize the control problem. The essence is to replace the continuous time process (X_t, Y_t, K_t) by a sequence of Markov chains. The original continuous domain is to be replaced by a sequence of discrete and bounded grids. The spacing of the grids has to converge to zero and the bound has to converge to infinity. In this sense the grid will approximate the original domain. The most important feature of the approximating scheme is what is called local consistency. One has to find transition probabilities for the chain such that, as the step size converges to zero, the drift and the volatility of the Markov chain converge to those of the original process, for all control policies.

The general methodology for the discretization scheme can be found in Kushner and Dupuis [25]. For a problem similar to ours in mathematical character the reader can consult Hindy, Huang and Zhu [18]. For the convergence of the methodology there are too branches in the literature. Barles and Souganidis [1] use a viscosity formulation whereas Kushner and Dupuis follow a more probabilistic formulation. We use the Markov chain structure of the latter and the viscosity ideas of the former. The proof of convergence of the procedure is deferred to the next subsection. The reader is warned that the Barles and Souganidis ideas require a comparison theorem that we do not have. They require that comparison holds for upper and lower semicontinuous functions, while we have proved it only for uniformly continuous functions.

6.1 The Markov Chain

It turns out that the covariance structure of the original process lacks a key property required by the numerical approximation scheme. The property is called diagonal dominance. Basically it is required that the principal components of the matrix is stable. In a 2×2 matrix, the property required is that the diagonal terms of the matrix are bigger in absolute value than the off diagonal terms, for all values of y and k. That is $\forall y, k > 0$

$$\begin{aligned} \sigma^2 y^2 &> |\rho| \lambda \sigma y \sqrt{k} \\ \lambda^2 k &> |\rho| \lambda \sigma y \sqrt{k} \end{aligned}$$

Even if we try rotating the coordinate system, so as the matrix becomes closer to a diagonal matrix, this properties still refuses to hold. There are no linear transformation of the variables y and k that satisfy the property in the whole domain. We are thus forced to rewrite the control problem.

First we introduce a change of the variables X_t ,Y_t and K_t. Next we obtain the Bellman equation of the new control problem. The discrete version of this

new equation will then be solved by a iterative procedure.

We start by making the transformations:

$$\begin{aligned} x_t &= \log X_t \\ y_t &= \log Y_t \\ k_t &= a\sqrt{K_t} \end{aligned}$$

where a is a positive constant to be chosen later. The original system 13 – 15 has the following form, after the transformation

$$\begin{aligned} dx_t &= (r - c_t)dt - (1 + \frac{k_t^2}{a^2})e^{-x_t}dL(t) + (1 + \frac{k_t^2}{a^2})^{-1}e^{-x_t}dM(t) \\ dy_t &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_{1,t} + e^{-y_t}dL(t) - e^{-y_t}dM(t) \\ dk_t &= \{\frac{a^2}{2k_t}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k_t}{2}\}dt + \frac{a\lambda}{2}\, dW_{2,t} \end{aligned}$$

A check on the boundary classification for the process k_t show that zero is an entrance boundary as long as $\lambda^2 < 2\beta\alpha$. This is exactly the restriction imposed on the parameters of the square root process so as to make it stationary. Notice that the transaction processes $dM(t)$ and $dL(t)$ are in dollar units. The exponentials e^{-y_t} and e^{-x_t} make the transformation for percentage units. Finally, for the same reason, a choice of c_t at a given moment, will give rise to a utility of $U(e^{x_t}c_t)$.

For the numerical implementation, next we need to replace the unbounded processes $(x_t, y_t, k_t) \in [-\infty, \infty]\times[-\infty, \infty]\times[0, \infty]$ by a reflected version $(\tilde{x}_t, \tilde{y}_t, \tilde{k}_t) \in [\underline{M}^x, \overline{M}^x] \times [\underline{M}^y, \overline{M}^y] \times [0, \overline{M}^k]$. For that purpose we introduce reflection processes $\underline{Z}^x(t)$, $\overline{Z}^x(t)$, $\underline{Z}^y(t)$, $\overline{Z}^y(t)$ and $\overline{Z}^k(t)$ active at the boundaries of the region. Dropping the tilde, the process suitable for numerical implementation becomes :

$$\begin{aligned} dx_t &= (r - c_t)dt - (1 + \frac{k_t^2}{a^2})e^{-x_t}dL(t) + (1 + \frac{k_t^2}{a^2})^{-1}e^{-x_t}dM(t) + d\underline{Z}^x(t) - d\overline{Z}^x(t) \\ dy_t &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_{1,t} + e^{-y_t}dL(t) - e^{-y_t}dM(t) + d\underline{Z}^y(t) - d\overline{Z}^y(t) \\ dk_t &= \{\frac{a^2}{2k_t}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k_t}{2}\}dt + \frac{a\lambda}{2}\, dW_{2,t} - d\overline{Z}_t^k \end{aligned}$$

Again, the processes $\underline{Z}^x(t)$, $\overline{Z}^x(t)$, $\underline{Z}^y(t)$, $\overline{Z}^y(t)$ and $\overline{Z}^k(t)$ are nondecreasing and increase only when the state variables hit the boundaries of the region $\{[\underline{M}^x, \overline{M}^x] \times [\underline{M}^x, \overline{M}^x] \times [0, \overline{M}^k]\}$.

In this new setting, the control problem is restated as

$$V(x, y, k) = \sup_{A_{x,y,k}} \mathrm{E}^{x,y,k}\,[\int_0^\infty e^{-\delta\, t}\, U(x_t, c_t)\, dt\,] \tag{40}$$

given the dynamics and the set of admissible controls. Next let

$$g(x,y,k;c) = \begin{bmatrix} r-c \\ \mu - \frac{\sigma^2}{2} \\ \frac{a^2}{2k}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k}{2} \end{bmatrix}$$

,

$$\Sigma(y,k) = \begin{bmatrix} \sigma^2 & \frac{a}{2}\lambda\rho\sigma \\ \frac{a}{2}\lambda\rho\sigma & \frac{\lambda^2 a^2}{4} \end{bmatrix}$$

and

$$\mathcal{L}_c\, V(x,y,k) \equiv g^{\top}(c)dV + \frac{1}{2}\mathrm{Tr}\,\Sigma.D^2V,$$

It is also useful to define the operators

$$\begin{aligned} \mathcal{B}V(x,y,k) &\equiv (1+\frac{k^2}{a^2})^{-1}e^{-y}\frac{\partial V}{\partial y} - e^{-x}\frac{\partial V}{\partial x} \\ \mathcal{S}V(x,y,k) &\equiv (1+\frac{k^2}{a^2})^{-1}e^{-x}\frac{\partial V}{\partial x} - e^{-y}\frac{\partial V}{\partial y} \end{aligned}$$

then the H-J-B equation becomes:

$$\max\{-\delta V + \max_{c\geq 0}\{\mathcal{L}_c\, V + U(x,c)\}\,,\, \mathcal{B}V\,,\, \mathcal{S}V\} = 0 \tag{41}$$

Before we proceed, let us comment on the viscosity property of the new H-J-B equation. By inspecting the proofs of the comparison result, the reader will see that the argument goes on unchanged for the new processes. This is because the value functions of the problems have the relation $V(x,y,k) = W(e^x, e^y, \frac{k^2}{a^2})$. To be more precise, a similar relation will hold for the solution of the Merton problems with the two classes of variables. We could proceed with the arguments in the Viscosity session, using the solution of the corresponding Merton problem as the source for the comparison result.

Also, due to the continuity of the consumption process, we can always add a nonbinding upper bound of the form $0 \leq c \leq \bar{c}$ on the H-J-B equation above on any given compact domain. Finally let me add that signs have been reversed between the formulation 20 and 41, so now subsolutions satisfy $\max\{\cdots\} \geq 0$ and supersolutions satisfy $\max\{\cdots\} \leq 0$.

I will work with sequence of grids indexed by the step size h. Abusing the notation on the k variable slightly:

$$\begin{aligned} G^h \;=\; & \{(i,j,k) : x = i\times h\,,\, y = j\times h\,,\, k = k\times h \\ & ; i = -\underline{N}^x, \ldots, -1, 0, 1, \ldots, \overline{N}^x \\ & ; j = -\underline{N}^y, \ldots, -1, 0, 1, \ldots, \overline{N}^y \\ & ; k = 0, 1, \ldots, \overline{N}^k\} \end{aligned}$$

where $\underline{N}^x = \underline{M}^x/h$, $\overline{N}^x = \overline{M}^x/h$ $\underline{N}^y = \underline{M}^y/h$, $\overline{N}^y = \overline{M}^y/h$, $\overline{N}^k = \overline{M}^k/h$. Each grid point (i, j, k) clearly correspond to a state (x, y, k) where $x = i \times h$,$y = j \times h$, $k = k \times h$.

The set of allowed strategies:

$$A^h = \{(L, M, c); \Delta L = 0 \,\text{or}\, h\,,\ \Delta M = 0 \,\text{or}\, h\,,\ 0 \leq c \leq \bar{c}\}$$

or

$$A^h = \{(L, M, C); \Delta L = 0 \,\text{or}\, h\,,\ \Delta M = 0 \,\text{or}\, h\,,\ C = l \times h \,\text{for}\, l = 0, \ldots, N(l)\}$$

The choice will be dictated by the utility function. If one can solve the maximization problem in closed form, the first choice is to be preferred. The second form, simplifies the numerical procedure, looking for the solution of the maximization problem in a grid. In both cases, the upper bound in consumption should be chosen so as it is never binding. Again, such a $\bar{c}$ always exist for a given compact domain.

The continuous time process (x_t, y_t, k_t) is to be approximated by Markov chains $\{(x_n, y_n, k_n); n = 1, 2, \ldots\}$ where the index n denotes time. Next I describe the set of allowed transitions and their probabilities. These will differ depending on the transaction regions. I start with the case where it is optimal not to transact.

(NT) No Transaction Region ($\Delta L = \Delta M = 0$)

In this case, transitions will be allowed for the 6 closest neighbors, two diagonal states depending on the sign of the correlation of the Brownian motions and also a transition to the current state. The directions of transitions are represented by 9 vectors in R^3. In the case $\rho < 0$ we define.

$$\begin{array}{lcccl}
v_0 = (0,0,0) & : & (i,j,k) & \mapsto & (i,j,k) \\
v_1 = (1,0,0) & : & (i,j,k) & \mapsto & (i+1,j,k) \\
v_2 = (-1,0,0) & : & (i,j,k) & \mapsto & (i-1,j,k) \\
v_3 = (0,1,0) & : & (i,j,k) & \mapsto & (i,j+1,k) \\
v_4 = (0,-1,0) & : & (i,j,k) & \mapsto & (i,j-1,k) \\
v_5 = (0,0,1) & : & (i,j,k) & \mapsto & (i,j,k+1) \\
v_6 = (0,0,-1) & : & (i,j,k) & \mapsto & (i,j,k-1) \\
v_7 = (0,1,-1) & : & (i,j,k) & \mapsto & (i,j+1,k-1) \\
v_8 = (0,-1,1) & : & (i,j,k) & \mapsto & (i,j-1,k+1)
\end{array}$$

If one is interested in the case $\rho \geq 0$, the only modification required in all that follows is to substitute the vectors v_7 and v_8 by

$$\begin{array}{lcccl}
\tilde{v}_7 = (0,1,1) & : & (i,j,k) & \mapsto & (i,j+1,k+1) \\
\tilde{v}_8 = (0,-1,-1) & : & (i,j,k) & \mapsto & (i,j-1,k-1)
\end{array}$$

Next we define quantities $q_i^{0,h}, q_i^{1,h}$ for $i = 1, \ldots, 8$. These are the building blocks of the transition probabilities:

$$\begin{aligned}
q_1^{0,h}((i,j,k);c) &= (r-c)^+ \\
q_2^{0,h}((i,j,k);c) &= (r-c)^- \\
q_3^{0,h}(i,j,k) &= (\mu - \frac{\sigma^2}{2})^+ \\
q_4^{0,h}(i,j,k) &= (\mu - \frac{\sigma^2}{2})^- \\
q_5^{0,h}(i,j,k) &= \{\frac{a^2}{2kh}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta kh}{2}\}^+ \\
q_6^{0,h}(i,j,k) &= \{\frac{a^2}{2kh}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta kh}{2}\}^- \\
q_7^{0,h}(i,j,k) &= 0 \\
q_8^{0,h}(i,j,k) &= 0
\end{aligned}$$

with the usual notation $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.

$$\begin{aligned}
q_1^{1,h}(i,j,k) &= 0 \\
q_2^{1,h}(i,j,k) &= 0 \\
q_3^{1,h}(i,j,k) &= \frac{\sigma^2}{2} - \frac{a}{4}|\rho|\sigma\lambda \\
q_4^{1,h}(i,j,k) &= \frac{\sigma^2}{2} - \frac{a}{4}|\rho|\sigma\lambda \\
q_5^{1,h}(i,j,k) &= \frac{\lambda^2 a^2}{8} - \frac{a}{4}|\rho|\sigma\lambda \\
q_6^{1,h}(i,j,k) &= \frac{\lambda^2 a^2}{8} - \frac{a}{4}|\rho|\sigma\lambda \\
q_7^{1,h}(i,j,k) &= \frac{a}{4}|\rho|\sigma\lambda \\
q_8^{1,h}(i,j,k) &= \frac{a}{4}|\rho|\sigma\lambda
\end{aligned}$$

These quantities are non negative for a suitable choice of the parameter a. For instance $a = \frac{2\sigma}{\lambda}$ [4]. The last piece needed to construct the transition probabilities is the normalization factor:

$$\begin{aligned}
Q^h(i,j,k) &= \max_{0 \leq c \leq \bar{c}} \sum_{m=1}^{8} [q_m^{1,h} + h q_m^{0,h}] \\
&= h|r - \bar{c}| + h|\mu - \frac{\sigma^2}{2}|
\end{aligned}$$

[4]This is the exact place where the procedure for the original process would break down. It is impossible to guarantee that the corresponding quantities are positive in all the relevant region.

$$+h\left|\frac{a^2}{2kh}(\beta\alpha-\frac{\lambda^2}{4})-\frac{\beta kh}{2}\right|$$
$$+\sigma^2+\frac{a^2\lambda^2}{4}-\frac{a}{2}|\rho|\sigma\lambda$$

Notice that with this definition, the quantity $Q^h(i,j,k)$ does not depend on the control police. The transition probabilities are:

$$P_m^h(i,j,k;c)=\frac{q_m^{1,h}+hq_m^{0,h}}{Q^h(i,j,k)}$$

for $m=1,\ldots,8$, and

$$P_0^h(i,j,k;c)=1-\sum_{m=1}^{8}P_m^h(i,j,k)$$

The interpolation time of the chain is defined as:

$$\Delta t^h(i,j,k)=\frac{h^2}{Q^h(i,j,k)}$$

which also does not depend on the controls.

This generate a chain that is locally consistent with the reflected process. To show this, define $\Delta x_n^h \equiv x_{n+1}^h - x_n^h$, $\Delta y_n^h \equiv y_{n+1}^h - y_n^h$, $\Delta k_n^h \equiv k_{n+1}^h - k_n^h$. Let E_n^h denote the expectation conditional on the nth-time and state (x_n^h, y_n^h, k_n^h). It is then easy to check that:

$$\begin{aligned}
E_n^h[\Delta x_n^h] &= (r-c)\Delta t^h(i,j,k)\\
E_n^h[\Delta y_n^h] &= (\mu-\frac{\sigma^2}{2})\Delta t^h(i,j,k)\\
E_n^h[\Delta k_n^h] &= \{\frac{a^2}{2k_n^h}(\beta\alpha-\frac{\lambda^2}{4})-\frac{\beta k_n^h}{2}\}\Delta t^h(i,j,k)\\
E_n^h[\Delta x_n^h]^2 &= o(\Delta t^h(i,j,k))\\
E_n^h[\Delta y_n^h]^2 &= \sigma^2\Delta t^h(i,j,k)+(\mu-\frac{\sigma^2}{2})h\Delta t^h(i,j,k)\\
&= \sigma^2\Delta t^h(i,j,k)+o(\Delta t^h(i,j,k))\\
E_n^h[\Delta k_n^h]^2 &= \frac{\lambda^2a^2}{4}\Delta t^h(i,j,k)+\{\frac{a^2}{2k_n^h}(\beta\alpha-\frac{\lambda^2}{4})-\frac{\beta k_n^h}{2}\}h\Delta t^h(i,j,k)\\
&= \frac{\lambda^2a^2}{4}\Delta t^h(i,j,k)+o(\Delta t^h(i,j,k))\\
E_n^h[\Delta k_n^h\Delta y_n^h] &= -\frac{a}{2}|\rho|\sigma\lambda\Delta t^h(i,j,k)
\end{aligned}$$

This estimation, combined with the fact that

$$\begin{aligned}
(E_n^h[\Delta X_n^h])^2 &= o(\Delta t^h(i,j,k))\\
(E_n^h[\Delta Y_n^h])^2 &= o(\Delta t^h(i,j,k))\\
(E_n^h[\Delta K_n^h])^2 &= o(\Delta t^h(i,j,k))
\end{aligned}$$

make the first and second moments of the chain close to those of the continuous process (x_t, y_t, k_t). This property is called local consistency. Next the transaction cases.

(SS) Sell Stock Region ($\Delta L = 0$, $\Delta M = h$) In this case, the chain should jump along the direction $((1+\frac{(kh)^2}{a^2})^{-1}e^{-x}, -e^{-y}, 0)$. However, the new state will not belong to the grid. To overcome this difficulty, we introduce a randomization scheme. The allowed transition directions are:

$$\begin{array}{lcccc} u_1 = (1,0,0) & : & (i,j,k) & \mapsto & (i+1,j,k) \\ u_2 = (0,-1,0) & : & (i,j,k) & \mapsto & (i,j-1,k) \end{array}$$

associated to this directions we have the following probabilities:

$$\begin{aligned} Ps_1^h(i,j,k) &= \frac{(1+\frac{(kh)^2}{a^2})^{-1}e^{-ih}}{e^{-ih}(1+\frac{(kh)^2}{a^2})^{-1}+e^{-jh}} \\ Ps_2^h(i,j,k) &= \frac{e^{-jh}}{e^{-ih}(1+\frac{(kh)^2}{a^2})^{-1}+e^{-jh}} \end{aligned}$$

(BS) Buy Stock Region ($\Delta L = h$, $\Delta M = 0$) In this case, the chain should jump along the direction $(-e^{-x}, (1+\frac{(kh)^2}{a^2})^{-1}e^{-y}, 0)$. Again, we need a randomization scheme. The allowed transition directions are:

$$\begin{array}{lcccc} w_1 = (-1,0,0) & : & (i,j,k) & \mapsto & (i-1,j,k) \\ w_2 = (0,+1,0) & : & (i,j,k) & \mapsto & (i,j+1,k) \end{array}$$

associated to this directions we have the following probabilities:

$$\begin{aligned} Pb_1^h(i,j,k) &= \frac{e^{-ih}}{e^{-ih}+(1+\frac{(kh)^2}{a^2})^{-1}e^{-jh}} \\ Pb_2^h(i,j,k) &= \frac{(1+\frac{(kh)^2}{a^2})^{-1}e^{-jh}}{e^{-ih}+(1+\frac{(kh)^2}{a^2})^{-1}e^{-jh}} \end{aligned}$$

The next step is to discretize the consumers maximization problem:

$$V^h(i,j,k) \equiv \max_{A^h} E^h_{i,j,k} \sum_{n=0}^{\infty} e^{-\delta t_n^h} U(e^{x_n} c_n^h) \Delta t_n^h$$

where $t_n^h = \sum_{0 \leq m \leq n} \Delta t_m^h$. Notice that this is the discrete time analogous of (??).

We are now able to describe the discrete time dynamic programming equation. Given a current state $s = (i,j,k)$ it has the following iterative form:

$$\begin{aligned} V^h(s) = \; & \max\{Pb_1^h(s)V(s+w_1) + Pb_2^h(s)V(s+w_2), \\ & Ps_1^h(s)V(s+u_1) + Ps_2^h(s)V(s+u_2), \\ & \max_{0 \leq c \leq \bar{c}} \{e^{-\delta \Delta t^h} \sum_{m=0}^{8} P_m^h V(s+v_m) + U(e^{hi}c)\Delta t^h\}\} \end{aligned} \quad (42)$$

The behavior of the chain in the boundaries will be described shortly.
Let us denote:

$$
\begin{aligned}
D_i^- V(i,j,k) &= \frac{V(i,j,k) - V(i-1,j,k)}{h} \\
D_i^+ V(i,j,k) &= \frac{V(i+1,j,k) - V(i,j,k)}{h} \\
D_j^- V(i,j,k) &= \frac{V(i,j,k) - V(i,j-1,k)}{h} \\
D_j^+ V(i,j,k) &= \frac{V(i,j+1,k) - V(i,j,k)}{h} \\
D_k^- V(i,j,k) &= \frac{V(i,j,k) - V(i,j,k-1)}{h} \\
D_k^+ V(i,j,k) &= \frac{V(i,j,k+1) - V(i,j,k)}{h} \\
d_{jj}^2 V(i,j,k) &= \frac{V(i,j+1,k) - 2V(i,j,k) + V(i,j-1,k)}{h^2} \\
d_{kk}^2 V(i,j,k) &= \frac{V(i,j,k+1) - 2V(i,j,k) + V(i,j,k-1)}{h^2} \\
d_{jk}^2 V(i,j,k) &= -\frac{2V(i,j,k) + V(i,j+1,k-1) + V(i,j-1,k+1)}{2h^2} \\
&\quad + \frac{V(i,j+1,k) + V(i,j-1,k) + V(i,j,k+1) + V(i,j,k-1)}{2h^2} \\
D_{jk}^2 V(i,j,k) &= \begin{bmatrix} d_{jj}^2 V(i,j,k) & d_{jk}^2 V(i,j,k) \\ d_{jk}^2 V(i,j,k) & d_{kk}^2 V(i,j,k) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_c^h V(i,j,k) \equiv\ & (r-c)^+ D_i^+ V(i,j,k) + (r-c)^- D_i^- V(i,j,k) \\
& + (\mu - \frac{\sigma^2}{2})^+ D_j^+ V(i,j,k) + (\mu - \frac{\sigma^2}{2})^- D_j^- V(i,j,k) \\
& + \{\frac{a^2}{2kh}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k h}{2}\}^+ D_k^+ V(i,j,k) \\
& + \{\frac{a^2}{2kh}(\beta\alpha - \frac{\lambda^2}{4}) - \frac{\beta k h}{2}\}^- D_k^- V(i,j,k) \\
& + \frac{1}{2}\mathrm{Tr} D_{jk}^2 V(i,j,k)\Sigma
\end{aligned}
$$

Using this notation we can expressed the Bellman equation as:

$$
\begin{aligned}
0 &\geq -\frac{1 - e^{-\delta \Delta t^h}}{\Delta t^h} V^h(s) + \max_{0 \leq c \leq \bar{c}} \{\mathcal{L}_c^h V^h(s) + U(s,c)\} \\
0 &\geq P s_1^h(s) D_i^+ V(s) - P s_2^h(s) D_j^- V(s) \\
0 &\geq P b_2^h(s) D_i^+ V(s) - P b_1^h(s) D_j^- V(s)
\end{aligned}
$$

where one of this inequalities hold as an equality for each state s. Clearly the quantity $\frac{1-e^{-\delta \Delta t^h}}{\Delta t^h}$ approximates the discount factor δ as h approaches zero. This

is the discrete time analog of the original variational inequality. The numerical scheme is complete with the specifications of the boundary conditions.

Given the reflected processes

$$(x_t, y_t, k_t) \in [\underline{M}^x, \overline{M}^x] \times [\underline{M}^y, \overline{M}^y] \times [0, \overline{M}^k]$$

we divide the specifications in two classes. The first one deals with the natural boundaries of the original domain, that is the region where one of the original processes $X(t), Y(y), K(t)$ reaches zero. The behavior of the processes (x_t, y_t, k_t) at the lower boundaries $\underline{M}^x$, $\underline{M}^y$ and 0, will mimic the behavior of the original processes at zero.

The second one deals with the upper boundaries $\overline{M}^x, \overline{M}^y, \overline{M}^k$. Again the behavior of the processes (x_t, y_t, k_t) at this points will mimic the behavior of $X(t), Y(y), K(t)$ at (∞, ∞, ∞).

Before we proceed, it is worth commenting that the convergence result to be presented holds for any specification of the boundary behavior. While true in theory, one should look for a non distorting boundary specification, for it will affect the actual numerical implementation. In the actual implementation, one start with an arbitrary value function. Then the computation follows the scheme in 42 for all points internal to the grid. Next the boundary values of the new value function is computed as described below.

- Natural Boundaries:

    1. $(i, j, k) = (-\underline{N}^x, -\underline{N}^y, 0)$. We set

    $$V^h(-\underline{N}^x, -\underline{N}^y, 0) = \frac{U(0)}{\delta}$$

    2. $(i, j, k) = (\cdot, \cdot, 0)$. The fact that zero is an entrance boundary for the k_t process dictates

    $$V^h(\cdot, \cdot, 0) = V^h(\cdot, \cdot, 1)$$

    3. $(i, j, k) = (-\underline{N}^x, \cdot, \cdot)$. It is assumed that it is optimal to sell stock. Hence, for $j > -\underline{N}^y$.

    $$\begin{aligned} V^h(-\underline{N}^x, j, k) &= V^h(-\underline{N}^x + 1, j, k) Ps_1^h(-\underline{N}^x + 1, j, k) \\ &\quad + V^h(-\underline{N}^x, j-1, k) Ps_2^h(-\underline{N}^x, j-1, k) \end{aligned}$$

    4. $(i, j, k) = (\cdot, -\underline{N}^y, \cdot)$. It is assumed that it is optimal to buy stock. Hence, for $i > -\underline{N}^x$.

    $$\begin{aligned} V^h(i, -\underline{N}^y, k) &= V^h(i-1, -\underline{N}^y, k) Ps_1^h(i-1, -\underline{N}^y, k) \\ &\quad + V^h(i, -\underline{N}^y + 1, k) Ps_2^h(i, -\underline{N}^y + 1, k) \end{aligned}$$

- Upper Boundaries:

  1. $(i,j,k) = (\cdot,\cdot,\overline{N}^k)$. We impose

  $$V(\cdot,\cdot,\overline{N}^k) = V(\cdot,\cdot,\overline{N}^k - 1).$$

  For the other two upper boundaries, first notice that if we write $W(X,Y,K)$ for the value function of the original processes, then it should be the case that at the infinity

  $$\frac{\partial W}{\partial X} = \frac{\partial W}{\partial Y} = 0$$

  because the marginal utility of wealth should decrease to zero. Using the change of variables relating V and W, one concludes that $V(\log X, \log Y, k) = W(X,Y,K)$.

  2. $(i,j,k) = (\overline{N}^x,\cdot,\cdot)$. We set

  $$V(\overline{N}^x,\cdot,k) = V(\overline{N}^x - 1,\cdot,k)e^h$$

  3. $(i,j,k) = (\cdot,\overline{N}^y,\cdot)$. We set

  $$V(\cdot,\overline{N}^y,k) = V(\cdot,\overline{N}^y - 1,k)e^h$$

  4. $(i,j,k) = (\overline{N}^x,\overline{N}^y,\cdot)$. We set

  $$V(\overline{N}^x,\overline{N}^y,k) = V(\overline{N}^x - 1,\overline{N}^y - 1,k)e^{2h}$$

6.2 Convergence of the Numerical Procedure

Let $B(G^h)$ denote the space of real valued functions on the lattice G^h. For what follows we will introduce the mappings

$$\begin{aligned}
\mathcal{N}_c^h &: B(G^h) \mapsto B(G^h) \quad \text{for } 0 \le c \le \bar{c} \\
\mathcal{N}^h &: B(G^h) \mapsto B(G^h) \\
\mathcal{B}^h &: B(G^h) \mapsto B(G^h) \\
\mathcal{S}^h &: B(G^h) \mapsto B(G^h)
\end{aligned}$$

defined as

$$\begin{aligned}
\mathcal{N}_c^h V(s) &\equiv e^{-\delta \Delta t^h(s)} \sum_{m=0}^{8} P_m^h(s,c) V(s+v_m) + U(s,c)\Delta t^h(s) \\
\mathcal{N}^h V(s) &\equiv \max_{0 \le c \le \bar{c}} \mathcal{N}_c^h V(s) \\
\mathcal{B}^h V(s) &\equiv Pb_1^h(s)V(s+w_1) + Pb_2^h(s)V(s+w_2) \\
\mathcal{S}^h V(s) &\equiv Ps_1^h(s)V(s+u_1) + Ps_2^h(s)V(s+u_2)
\end{aligned}$$

define also $\Delta^h = \min_{s\in G^h} \Delta t^h(s)$. Clearly $\Delta^h > 0$. Define also the usual operator norm, using the L^∞ norm on the space of bounded functions $B(G^h)$.

$$||\mathcal{T}|| = \sup_{V\in B} \frac{||\mathcal{T}V||_\infty}{||V||_\infty}$$

In terms of this operator the discrete Bellman equation can be written as:

$$V^h(s) = \max\{\mathcal{N}^h V^h(s)\,,\, \mathcal{B}^h V^h(s)\,,\, \mathcal{S}^h V^h(s)\}$$

Naturally, a fixed point of this operator equation will be the solution of the Bellman equation. In order to prove the convergence of the numerical scheme we need to establish the existence of a fixed point for this equation and to establish some properties of the operators $\mathcal{N}_c^h$, $\mathcal{N}^h$, $\mathcal{B}^h$ and $\mathcal{S}^h$.

We say that $V \leq V'$ if $V(x,y,k) \leq V'(x,y,k)$ for all $(x,y,k) \in G^h$. An operator $\mathcal{T}$ is called monotone if $V \leq V'$ implies $\mathcal{T}V \leq \mathcal{T}V'$.

Proposition 5 *$\mathcal{N}_c^h$, $\mathcal{N}^h$, $\mathcal{B}^h$ and $\mathcal{S}^h$ are monotone operators.*

Proof: The proposition follows immediately from the nonnegativity of $P_m^h(s,c)$, Pb_m^h and Ps_m^h

Proposition 6 *$\mathcal{B}^h$ and $\mathcal{S}^h$ are contractions with norms $||\mathcal{B}^h|| \leq 1$ and $||\mathcal{S}^h|| \leq 1$.*

Proof: A direct estimation yields the result.

$$\begin{aligned}
|\mathcal{B}^h V(s) - \mathcal{B}^h V'(s)| &\leq \sum_{m=1}^{2} Pb_m^h(s)|V(s+u_m) - V'(s+u_m)| \\
&\leq \max_{s\in G^h} |V(s) - V'(s)|
\end{aligned}$$

$$\begin{aligned}
|\mathcal{S}^h V(s) - \mathcal{S}^h V'(s)| &\leq \sum_{m=1}^{2} Ps_m^h(s)|V(s+u_m) - V'(s+u_m)| \\
&\leq \max_{s\in G^h} |V(s) - V'(s)|
\end{aligned}$$

Proposition 7 *$\mathcal{N}^h$ is a strict contraction.*

Proof :Let a be a constant function on G^h. Then, since probabilities add up to one for all $0 \leq c \leq \bar{c}$

$$\begin{aligned}
\mathcal{N}^h[V(s)+a] &= \max_{0\leq c\leq \bar{c}} \{e^{-\delta\Delta t^h(s)} \sum_{m=0}^{8} P_m^h(s,c)[V(s+v_m)+a] + U(s,c)\Delta t^h(s)\} \\
&\leq \mathcal{N}^h V(s) + e^{-\Delta^h} a.
\end{aligned}$$

This discounting property together with the monotonicity of $\mathcal{N}^h$ form the Blackwell sufficient condition for a strict contraction.

The strict contraction property of $\mathcal{N}^h$, the contraction property of B^h and S^h, combined with the fact that there are states where it is optimal not to transact, imply that there is a unique fixed point in the operator equation.

The next property we are concerned is the stability of the scheme. To establish this property we have to show that given a compact domain and a sequence of grids G^h in this domain, there exists a $h_o > 0$, such that for all $0 < h < h_0$ the solution of the operator equation has a bound independent of h. For that to hold it suffices that the operators $\mathcal{N}^h$, $\mathcal{B}^h$ and $\mathcal{S}^h$ have a bound independent of h. It remains to show the property for the operator $\mathcal{N}^h$.

Proposition 8 *For all $0 < h < h_0$, the operator $\mathcal{N}^h$ has a bound independent of h.*

Proof: We estimate for $||V|| = 1$,

$$\begin{aligned}
|e^{-\delta\Delta t^h(s)}\mathcal{L}_c^h V(s) + U(s,c)\Delta t^h(s)| &= |e^{-\delta\Delta t^h(s)}\sum_{m=0}^{8} P_m^h(s,c)V(s+v_m) \\
&+ U(s,c)\Delta t^h(s)| \\
&\leq e^{-\delta\Delta^h}\max_{s\in G^h}|V(s)| + U(s,c)o(h^2) \\
&\leq e^{-\delta\Delta^h} + U(e^{\overline{M}^x},\bar{c})o(h_0^2).
\end{aligned}$$

The last property of the scheme we have to deal with is consistency. Usually, a scheme $T^hV = f$ is said to be consistent with a partial differential equation $T^hV = f$ if for any smooth function ϕ we have $\lim_{h\to 0} T^h\phi - T\phi = 0$ pointwise for each grid point. In the presence of gradient constraints, the property needed for the convergence result is a slightly modified version of the usual property.

Proposition 9 *The numerical scheme is consistent in the following sense: Given a smooth function $\phi^h(i,j,k) \equiv \phi(hi,hj,hk)$ in a grid point,*

$$\begin{aligned}
\mathcal{N}^h\phi^h(s) - \phi^h(s) &= \Delta t^h(s)\{-\delta\phi^h(s) + \max_{0\leq c\leq\bar{c}}\{\mathcal{L}_c\,\phi^h(s) + U(x,c)\} + O(h)\phi^h(s)\} \\
\mathcal{B}^h\phi^h(s) - \phi^h(s) &= [\mathcal{B}\phi^h(s) + o(h)]ha(s) \\
\mathcal{S}^h\phi^h(s) - \phi^h(s) &= [\mathcal{S}\phi^h(s) + o(h)]hb(s)
\end{aligned}$$

for positive constants $a(s)$ and $b(s)$.

Proof: First estimate

$$\begin{aligned}
D_i^{\pm}\phi &= \phi_x + o(h) \\
D_j^{\pm}\phi &= \phi_y + o(h)
\end{aligned}$$

$$
\begin{aligned}
D_k^{\pm}\phi &= \phi_k + o(h) \\
D_{jk}^2\phi &= D^2\phi + o(h)
\end{aligned}
$$

Next by the definition of the probabilities $P_m^h(s,c)$, Pb_m^h and Ps_m^h compute,

$$
\begin{aligned}
\mathcal{B}^h\phi^h(s) - \phi^h(s) &= [Pb_2^h(s)D_i^+\phi^h(s) - Pb_1^h(s)D_j^-\phi^h(s)]h \\
&= [\mathcal{B}\phi^h(s) + o(h)]\frac{h}{e^{-ih} + (1+\frac{(kh)^2}{a^2})^{-1}e^{-jh}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^h\phi^h(s) - \phi^h(s) &= [Ps_1^h(s)D_i^+\phi^h(s) - Ps_2^h(s)D_j^-\phi^h(s)]h \\
&= [\mathcal{S}\phi^h(s) + o(h)]\frac{h}{e^{-ih}(1+\frac{(kh)^2}{a^2})^{-1} + e^{-jh}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}^h\phi^h(s) - \phi^h(s) &= \max_{0\le c\le \bar{c}}\{-[1-e^{-\delta\Delta t^h}]\phi^h(s) + \mathcal{L}_c^h\phi^h(s)\Delta t^h + U(s,c)\Delta t^h\} \\
&= \Delta t^h \max_{0\le c\le \bar{c}}\{-\frac{1-e^{-\delta\Delta t^h}}{\Delta t^h}\phi^h(s) + [\mathcal{L}_c + O(h)]\phi^h(s) + U(s,c)\} \\
&= \Delta t^h\{-\delta\phi^h(s) + \max_{0\le c\le \bar{c}}\{\mathcal{L}_c\phi^h(s) + U(s,c)\} + O(h)\phi^h(s)\}
\end{aligned}
$$

The last equality following from the Theorem of the Maximum.

We are now in a position to state the main result of this section, which says that the solution of the discrete problem converge to the solution of the continuous problem.

Theorem 1 *The sequence $V^h(i,j,k)$ of solutions of the operator equation converge to the value function $V(x,y,k)$ of the H-J-B equation uniformly in any compact subset of $Q = (-\infty,\infty)\times(-\infty,\infty)\times(0,\infty)$ as $h\downarrow 0$, $\underline{M}^x \downarrow -\infty$, $\underline{M}^y \downarrow -\infty$, $\overline{M}^x \uparrow \infty$ and $\overline{M}^y \uparrow \infty$ such as $hi \to x$, $hj \to y$ and $hk \to k$* [5].

Proof: Given a point $s = (x,y,k) \in Q$ define

$$
V^*(s) \equiv \limsup_{\substack{s' \to s \\ s' \in G^h, h\downarrow 0}} V^h(s') \tag{43}
$$

$$
V_*(s) \equiv \liminf_{\substack{s' \to s \\ s' \in G^h, h\downarrow 0}} V^h(s') \tag{44}
$$

Since the scheme is stable, V^* is finite in all the domain Q. It is clear that V^* is upper semicontinuous, V_* is lower semicontinuous and $V_* \le V^*$ on Q. We

[5] with a little abuse of notation on the k variable.

will show that V^* is a viscosity subsolution and V_* is a viscosity supersolution of the H-J-B equation. The comparison result asserts that $V_* \geq V^*$ on Q. Hence, by uniqueness both are equal to V [6].

Take a point $s_0 = (x_0, y_0, k_0)$ on Q and let it be a local maximum of $V^* - \phi$, for a smooth function ϕ. Without loss assume that $V^* - \phi = 0$ at s_0. Since we are taking the limit $h \downarrow 0$, $(\underline{N}^x, \underline{N}^y) \downarrow -\infty$ and $(\overline{N}^x, \overline{N}^y, \overline{N}^k) \uparrow \infty$ in such a way as $h \times (\underline{N}^x, \underline{N}^y) \downarrow -\infty$ and $h \times (\overline{N}^x, \overline{N}^y, \overline{N}^k) \uparrow \infty$, there exists some $h_0 > 0$ such that for $h < h_0$, s_0 is an interior point of the grid G^h. By the definition of lim sup, there exists a sequence indexed by h such that , as $h \downarrow 0$

$$s_h \to s_0 \text{ and } V^h(s_h) \to V^*(s_0)$$

For $h < h_0$, $V^h - \phi$ has a local maximum at s_h. Hence, for some neighborhood $\mathcal{N}_0$ of s_0,

$$V^h(s_h) - \phi(s_h) \geq V^h(s) - \phi(s)$$

$\forall (s) \in \mathcal{N}_0$ Modifying ϕ outside $\mathcal{N}_0$ if necessary , we can assume that the inequality holds globally. Let

$$\xi_h = V^h(s_h) - \phi(s_h) \tag{45}$$

so that $\xi_h \to 0$ as $h \downarrow 0$. 6.2 and 45 imply

$$V^h(s) \leq \phi(s) + \xi_h \tag{46}$$

But V^h is the solution of the operator equation

$$V^h(s_h) = \max\{\mathcal{N}^h V^h(s_h) , \mathcal{B}^h V^h(s_h) , \mathcal{S}^h V^h(s_h)\}$$

Using the monotonicity of the operators $\mathcal{N}^h$, $\mathcal{B}^h$ and $\mathcal{S}^h$ together with 46 we conclude that at (s_h)

$$0 \leq \max\{\mathcal{N}^h[\phi + \xi_h] - [\phi + \xi_h] , \mathcal{B}^h[\phi + \xi_h] - [\phi + \xi_h] , \mathcal{S}^h[\phi + \xi_h] - [\phi + \xi_h]\}$$

But recall that:

$$\begin{aligned}
\mathcal{N}^h[\phi + \xi_h] - [\phi + \xi_h] &= \Delta t^h(s)\{-\delta[\phi + \xi_h] \\
&+ \max_{0 \leq c \leq \bar{c}}\{\mathcal{L}_c [\phi + \xi_h] + U(x, c)\} + O(h)[\phi + \xi_h]\} \\
\mathcal{B}^h[\phi + \xi_h] - [\phi + \xi_h] &= [\mathcal{B}[\phi + \xi_h] + o(h)]ha(s) \\
\mathcal{S}^h[\phi + \xi_h] - [\phi + \xi_h] &= [\mathcal{S}[\phi + \xi_h] + o(h)]hb(s)
\end{aligned}$$

[6] For the result to be tight, one would need a comparison theorem for upper and lower semicontinuous functions or to prove that V_* and V^* are uniformly continuous with sublinear growth.

Just divide the first equation by $\Delta t^h(s)$, the second by $ha(s)$ and the third by $hb(s)$ and send $h \downarrow 0$ and conclude that:

$$0 \leq \max\{-\delta\phi(s_0) + \max_{0 \leq c \leq \bar{c}}\{\mathcal{L}_c\, \phi(s_0) + U(x_0, c)\}\, ,\, \mathcal{B}\phi(s_0)\, ,\, \mathcal{S}\phi(s_0)\}$$

So V^* is a viscosity subsolution. The same argument can be applied to show that V_* is a viscosity supersolution.

7 Numerical Results

This section describes the results of the algorithm presented in the last section. We are going to emphasize the distinction between the optimal policies in the Random Transaction Costs case (RTC for short) and those obtained in the Fixed Transaction Cost case (FTC). Notice that the FTC case or the Constantinides [5] case can be viewed as the system 13- 15 with $\lambda = 0$ and $k_0 = \alpha$. Thus the same algorithm apply to both cases. This is important for the exercise we present, since any adjustments needed in the algorithm can be applied to both cases. For instance, we mentioned in the last section that the convergence of the algorithm takes place irrespective of the outward boundary condition. However the choice there will affect the actual result. So by imposing the same condition for both cases we hope that both results are affected in a similar way.

We implemented the algorithm in Fortran 90 in a 300 MHz Pentium II workstation. It takes about 24 hours or 6000 steps for the algorithm to converge. We proceeded as follows:

1. Set the initial guess. We set always $V_{old} = V_{Merton}$, the solution to the Merton problem.

2. Using the algorithm and V_{old} produce V_{new}.

3. If $\max\{\frac{V_{new} - V_{old}}{V_{old}}\} < 10^{-6}$, stop. Else, set $V_{old} = V_{new}$ and go to 2.

We used the utility function $U(c) = \frac{c^\gamma}{\gamma}$. This choice reduces the computational time in about 10 times as compared to a general utility function. The reason is that the maximization problem $\max_{c \geq 0}\{U(c) - cV_x\}$ has a closed form expression in the HARA case. For the main exercise of the thesis we set the time unit to be 1 year and used the following parameters:

γ	0.4	r	0.03	μ	0.1
σ	0.4	α	0.03	β	40.0
λ	0.5	ρ	-0.8	δ	0.075

Table 1: Parameters

We used a step of size $h = 0.025$. The grid spanned a stock and bond holdings from US\$ 0.17 to US\$90.02 in 251 steps. The transaction costs span was from 0.03% to 18% in 28 steps. The nonlinear transformation we used in the previous section implies that the grid points are not evenly spaced in those regions. Please look at figures 3–2.

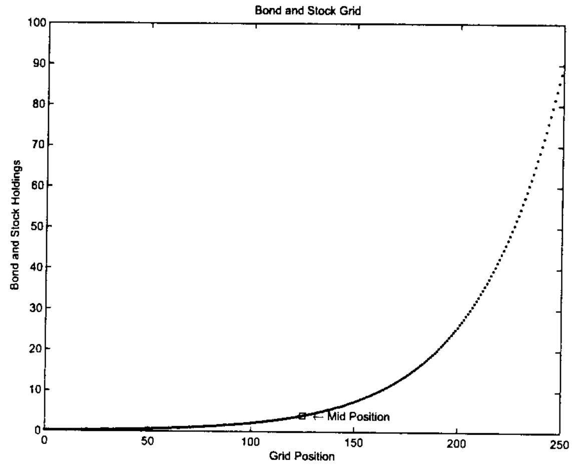

Figure 2: Stock and Bond grid

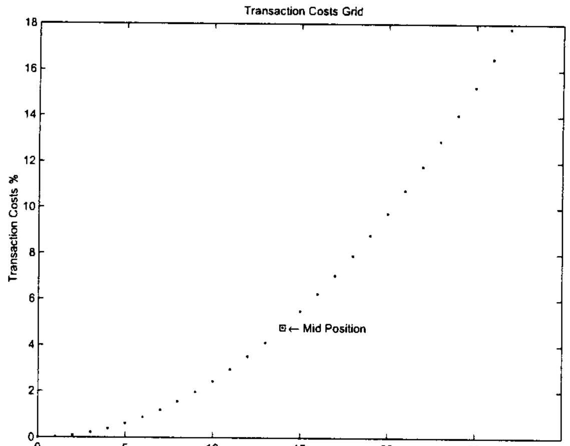

Figure 3: Transaction Costs grid

The transaction costs grid includes the value of 2.95% in the position 12. This is roughly the mean reverting level and is close to the middle of the grid. The parameters we have chosen for the transaction costs process implies a reasonable stationary distribution. Please refer to figure 4.

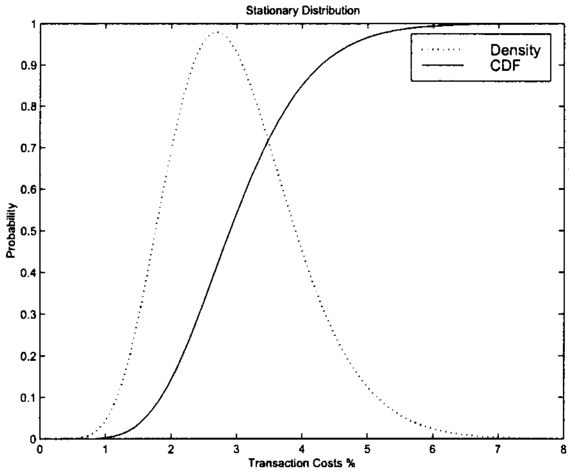

Figure 4: Stationary Distribution

7.1 Main Results

Next we present the main findings of the paper. That is, the best way to cope with random transaction costs is to look for bargains. Pictures 5– 9 clearly demonstrate the situation. Figure 5 is just a reminder of the general setup. It depicts a contour plot of the value function and the 3 distinct transaction regions for a given level of transaction cost.

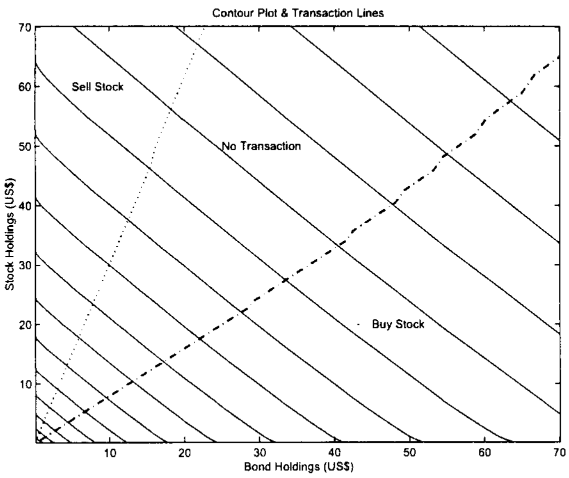

Figure 5: Transaction Regions

Figure 6 shows the various transaction lines for the Constantinides case. Notice that even for a very high transaction cost (18%) transactions do take

place.

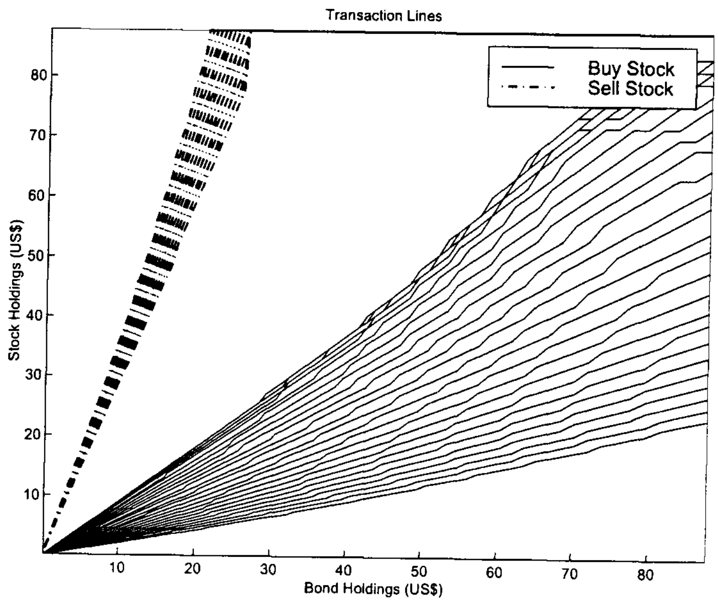

Figure 6: FTC case. k ranges from 0.03% to 18%

Contrast this with the RTC situation shown in picture 7 below. The values of transaction costs depicted are $k1 = 0.03\%$, $k2 = 1.56\%$ and $k3 = 1.98\%$, all below the mean reverting value of 3%. For the last value, no stock is ever purchased, the Buy Stock line degenerates to the x axis. Stocks are sold in this case only under extreme unbalanced positions, roughly with the Stock holdings over 10 times the bond holdings.

The whole picture is shown in the figures 8– 9. These are plots of the angles of the lines obtained using least squares. They clearly show the optimal way to deal with Random Transaction Costs. Do not engage in transactions even if the costs are fair. It is best to wait for bargains.

Let me also add that the waiting period to come back to the market is non trivial. Figure 8 shows that nearly all transactions take place when the transaction cost is below a value somewhere between $k2 = 1.56\%$ and $k3 = 1.98\%$. Figure 4 implies that this happens only about 10% of the time. Furthermore, figure 10 shows that the waiting period until transaction costs fall back to these levels are significant. Facing a 5% transaction cost, the consumer should wait on average, 3 to 4.5 weeks for it to go to the range 1.56% – 1.98%.

Before we proceed with more quantitative results, we want to present a better description of the transaction lines produced by the algorithm. Remember that we proved that in the HARA case, the transaction regions are separated by

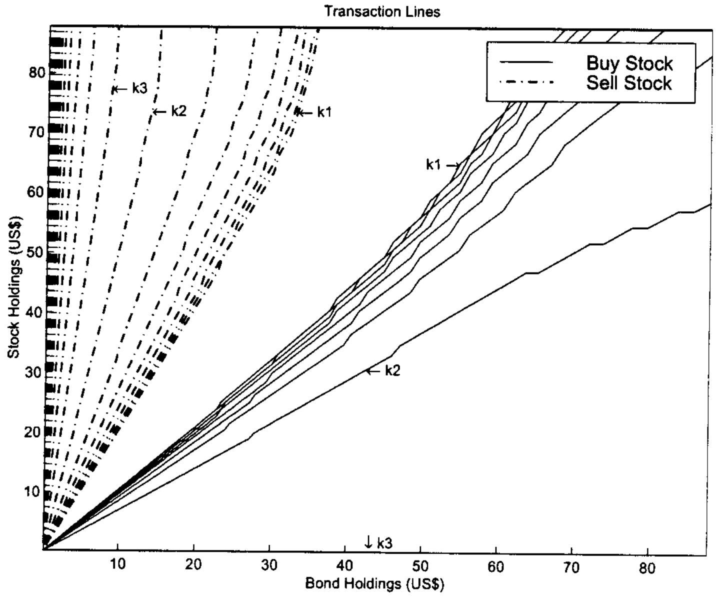

Figure 7: RTC case. k ranges from 0.03% to 18%

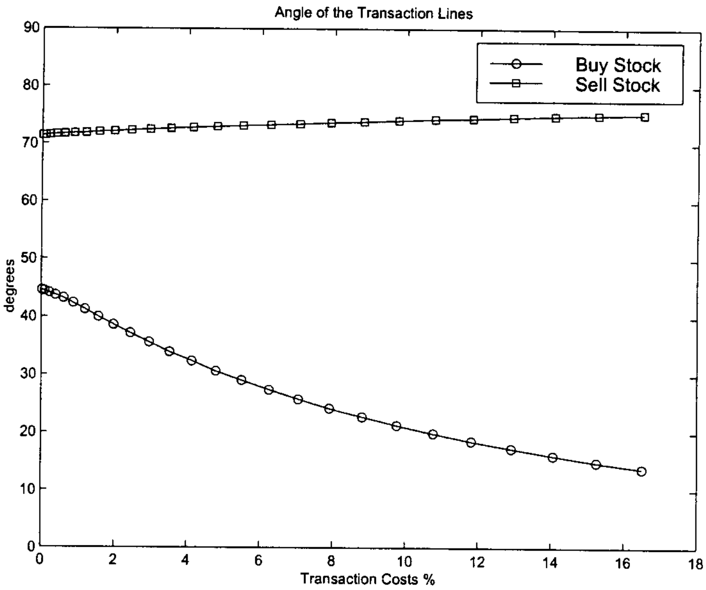

Figure 8: FTC case.

straight lines through the origin. First notice that figure 7 show a slightly strange behaviour near the outward boundaries. This is probably caused by boundary condition imposed, together with the relative coarseness of the grid

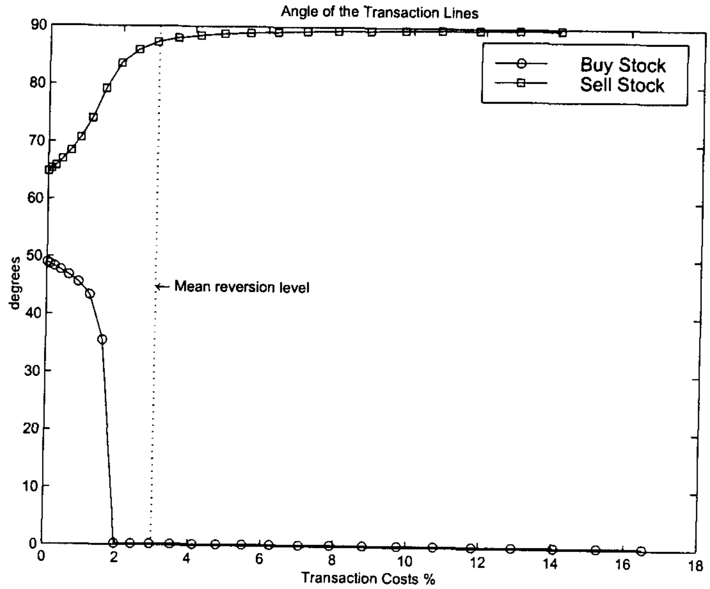

Figure 9: FTC case.

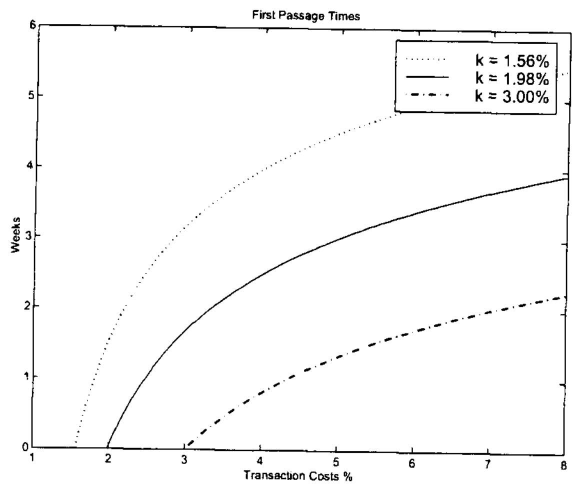

Figure 10: Expected First Passage Times

there. Figure 11 and 12 show the lines closer and closer to the origin. Figure 11 depicts nearly straight lines, avoiding the irregularities of figure 7. Figure 12 show that the behaviour is still very reasonable near the origin. The transaction regimes are still clearly separated.

The next study shows the overall loss incurred by the consumer in different settings. The studies consider an initial holding of US$4.48 in the bond account

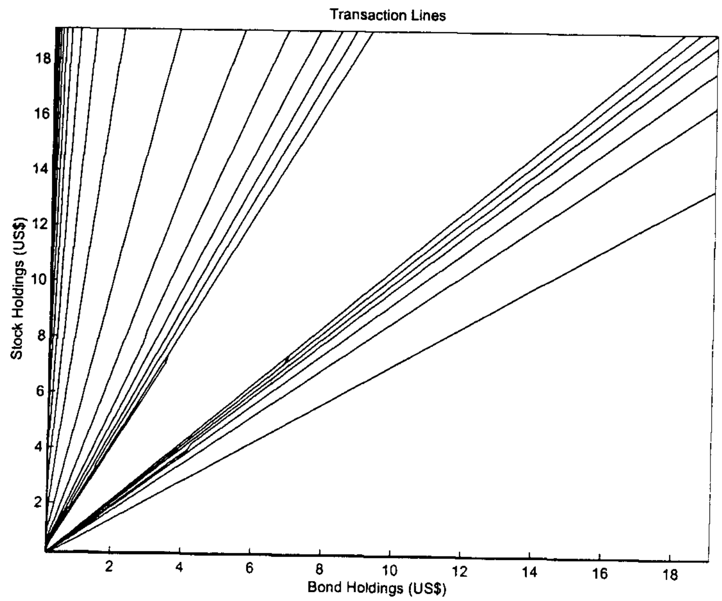

Figure 11: RTC case. Zoom 1

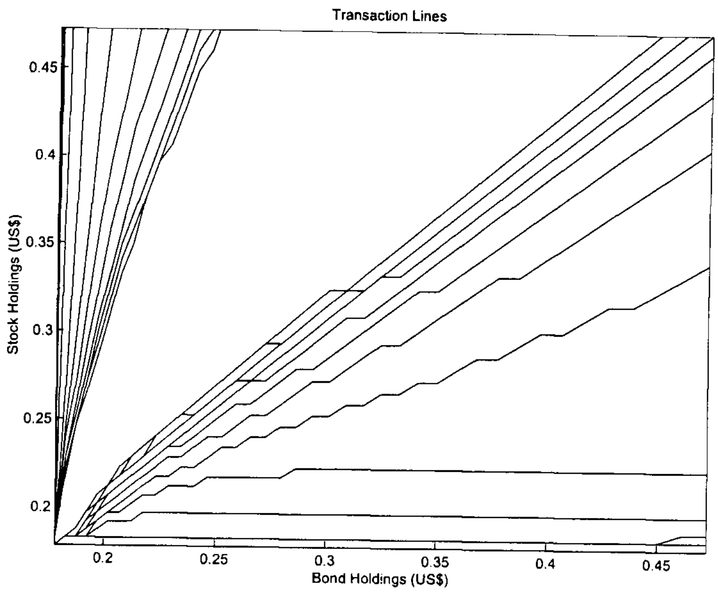

Figure 12: RTC case. Zoom 2

and US$5.75 in the stock account. Where applicable RTC start at 2.95%. We show the value function and the consumption loss in a certainty equivalent

formulation. By that we mean that if we consider the problem

$$V(x_0) = \max_{c \geq 0} \int_0^\infty e^{-\delta t} U(c_t) dt$$

subject to

$$dx_t = (r - c_t) x_t dt \quad x_0 \text{ given}$$

then optimal consumption is a constant and since $V(x_0)$ is increasing, it can be written as

$$c^* = c(x_0^{-1}(V))$$

The consumption loss is computed using the value functions obtained in the algorithm, benchmarked against

$$c^0 = c(x_0^{-1}(V_{Merton}))$$

Figures 13– 14 depicts the situation for two cases of RTC. Namely, the case of negative correlation ($\rho = -0.8$) and the zero correlation case. The difference agrees with the intuition that consumers are worse off in the negative correlation case. However, negative correlation appears to be a minor distraction.

The unavoidable conclusion is that the consumer is much better off in the RTC case. Notice the nearly zero slope of the RTC case[7] . Consumers do not seem to be bothered by a temporarily high transaction cost for they know to wait.

In these pictures we also show an example of naive behaviour. This is a Monte Carlo study we describe next.

In the processing of the algorithm, we obtain the optimal consumption police and the transaction lines. The system then becomes suitable to simulations. Notice that the algorithm ties a given consumption process in the grid to transition probabilities of the Markov chain. Evolving the chain, starting in the no transaction region, we compute the utility achieved in each step and also check if one of the transaction lines have been crossed. If this is the case, the algorithm also implies a randomization scheme to bring the process back to the no transaction region. Meanwhile, we record that as an instance of a transaction. We did 1 million simulations, stopping the system at a time t such that:

$$\int_0^t e^{-\delta s} ds = \frac{0.999}{\delta}$$

What we call naive behaviour is to ignore that transaction costs are random. We solved the FTC case for the various transaction costs in the grid shown in figure 3. Thus, obtaining the consumption process and transaction lines for several Constantinides problems. Next we introduced these into the dynamics

[7]They are equal to zero within the precision set for the algorithm

of RTC. That is, the naive consumer checks every time his/her holdings and the current transaction cost. Believing the cost will remain fixed at the current level forever, he/she acts optimally. However, transaction costs are random. The table 2 below show the results. The Monte Carlo started with US$4.48 in the bond account, US$5.75 in the stock account and RTC at 2.95%. The $\pm$ intervals have a coverage of 99%.

	Value	Consumption Loss
Merton	27.2489	0 %
Naive	26.7381 ± 0.0179	$4.62 \pm 0.16\%$
Optimal (MC)	27.2135 ± 0.0195	$0.32 \pm 0.18\%$
Optimal (Value)	27.1252	1.13 %

Table 2:

We see that the naive behaviour can hurt the consumer rather seriously. The two values for the optimal police corresponds to the value found in the Monte Carlo study and the value function computed with the algorithm. We see a slight upward bias in the Monte Carlo setting. This implies that if one uses these values as a control variate for the naive police, the consumption loss will be even stronger. Figures 13– 14 show that a consumer facing a RTC averaging 3%, but despising the RTC effect, ends up being as well off a consumer facing a FTC of around 7%.

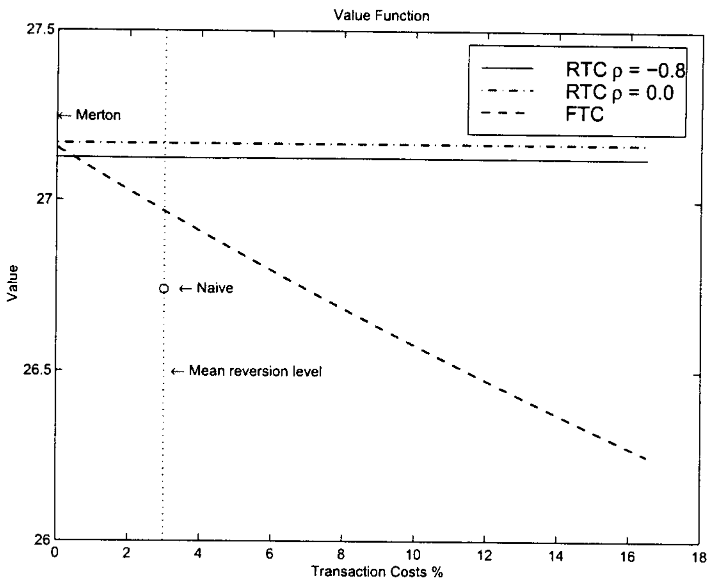

Figure 13: Value Function Loss

The next exhibit is the table below. It sheds light on the source of the

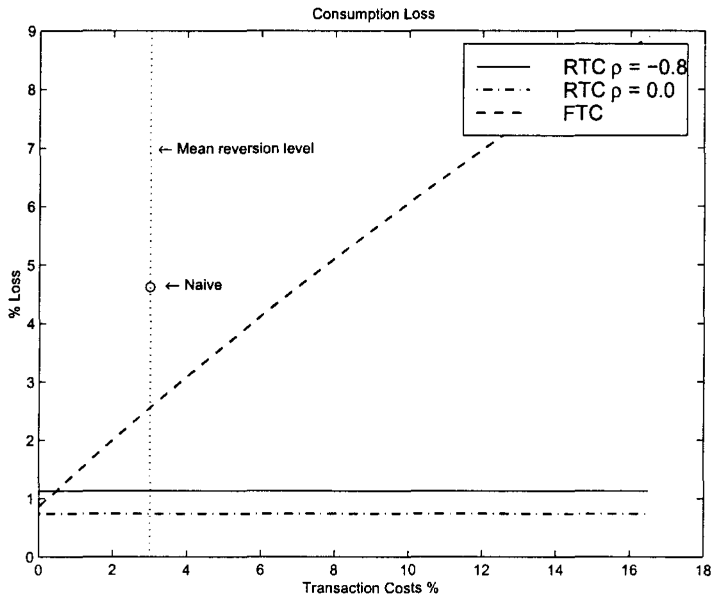

Figure 14: Consumption Loss

well being of the consumer in the RTC case. Glancing at figures 6- 7 one could imagine that the consumer transacts rather less in the RTC case. This conclusion is not warranted. The Monte Carlo study shows that the optimal police for the RTC case transacts more often than the naive police. The source of well being comes from placing the buy/sell orders at bargain spreads. The table below show the yearly average of both strategies.

	Optimal Police	Naive Police
Stock Revenue (US$)	0.632 ± 0.002	0.599 ± 0.002
Stock Expenditures (US$)	0.01823 ± 0.00007	0.01215 ± 0.00006
Number of Sells	10.035 ± 0.007	9.823 ± 0.007
Number of Buys	0.551 ± 0.002	0.409 ± 0.002

Table 3: MC Yearly averages

We finish this section by presenting some comparative statics. We showed that he ability to choose the best moment to transact is very valuable to the consumer. However,the consumer may have to keep an unbalanced position while the RTC is temporarily high. The bigger the volatility of the RTC process the fastest the consumer will see bargain spreads again. Figure 15–16 confirms this intuition. The consumer is better off with high volatility in the spreads, and again, the higher the volatility, the wider is the no transaction region, since the incentive to wait for bargains is larger.

The last comparative statics study deals with correlation. We found that the transaction lines vary slightly with correlation in a reasonable way. Please look at figure 17. With negative correlation, the Sell Stock line is a little above than in the zero correlation case. Close to the line, the consumer reasons: should stock prices go up I will have to sell. But if ρ is negative, RTC will be smaller than in the $\rho = 0$ case, so I can wait a little longer. The reverse reasoning applies in the Buy Stock line, making the consumer less willing to wait.

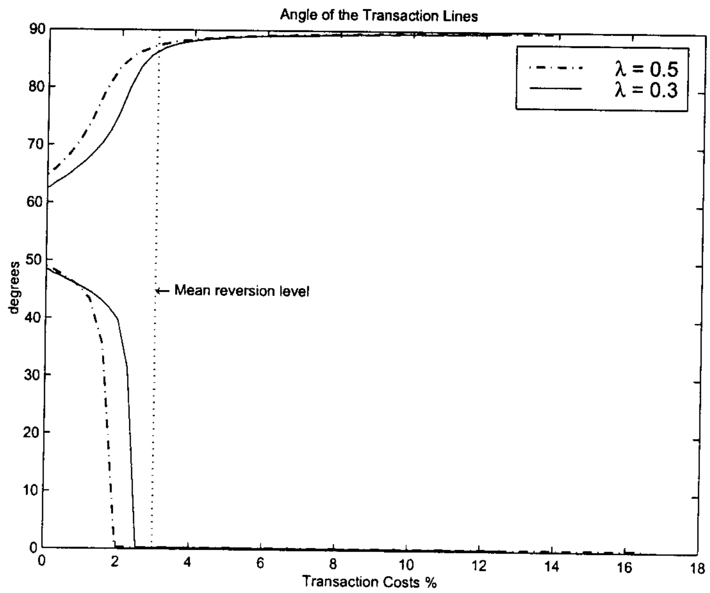

Figure 15: Volatility Comparison

7.2 Conclusion and Extensions

We have shown that the best way to cope with bad terms of trade is to wait. This findings have a theoretical interest on its own, but also raises questions in two other areas. Namely, option pricing and market microstructure. We saw that in the balance between keeping a desirable portfolio and waiting for better terms of trade, the consumer prefers to wait. However, in a option hedging situation like the one described in Davis, Panas and Zariphopoulou [8], the urge to rebalance the portfolio is probably much stronger, so one cannot tell beforehand which effect will dominate.

Concerning the information based branch of market microstructure literature, we usually see models where the noise traders are much less responsive to bid-ask spreads fluctuations than what is predicted in this paper. Increasing the elasticity of the response will impair the ability of the informed traders to

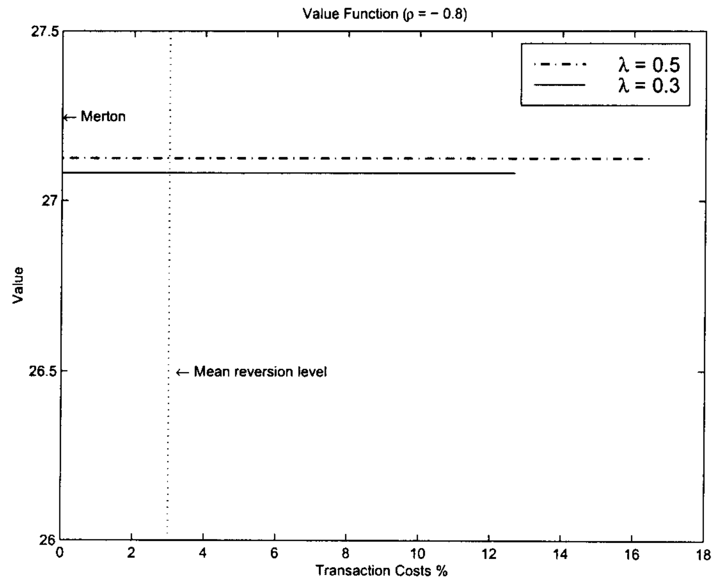

Figure 16: Volatility Comparison

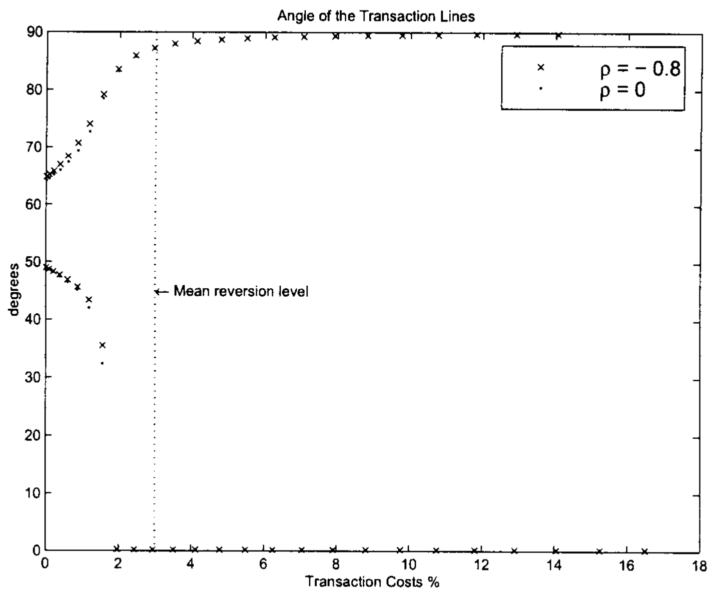

Figure 17: Correlation Comparison

disguise themselves into noise traders. The result is likely to be an improvement in the information flow to the markets.

8 Appendix

The appendix gives the proof of propositions 1 and 2. Proposition 1 is a straightforward application of theorem 3.1 in Nelson [27]. The proof of proposition 2 is not deep, but differ significantly from usual arguments. What makes it more demanding then usual is the lack of joint concavity in the value function. I chose to give a direct proof, using as much as possible the economics of the problem. For more general results, which are not fully applicable here, the reader is referred to Hausmann and Suo [17].

Proposition 1 *$(S_t^h, K_t^h) \Longrightarrow (S_t, K_t)$ (weakly) as $h \downarrow 0$. Where (S_t, K_t) satisfy:*

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_{1,t} \tag{47}$$

$$dK_t = \beta(\alpha - K_t) \, dt + \sqrt{K_t} \, \lambda \, dW_{2,t} \tag{48}$$

$$S_0, K_0 \ \text{ given}$$

and $[W_{1,t} \, W_{2,t}]$ is a 2-dimensional Brownian Motion satisfying

$$\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \begin{bmatrix} dW_{1,t} & dW_{2,t} \end{bmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt$$

Proof: Since Z_{mh}^h is i.i.d. $N(0, h)$ we have $\mathrm{E}|Z_{mh}^h| = \sqrt{2h/\pi}$ and $\mathrm{Var}|Z_{mh}^h| = (1 - 2/\pi)h$. It is easy to compute that

$$\begin{aligned} &\mathrm{E}\begin{bmatrix} Z_{mh}^h \\ \rho \, Z_{mh}^h + \gamma \, (\, |Z_{mh}^h| - (2\,h/\pi)^{1/2}) \end{bmatrix} \\ &\times \begin{bmatrix} Z_{mh}^h & \rho \, Z_{mh}^h + \gamma \, (\, |Z_{mh}^h| - (2\,h/\pi)^{1/2}) \end{bmatrix} \\ &= h \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \end{aligned}$$

then, by theorem 3.1 of Nelson it suffices to verify that the drift

$$b(s,k) = \begin{bmatrix} \mu s \\ \beta(\alpha - k) \end{bmatrix}$$

and covariance

$$a(s,k) = \begin{bmatrix} \sigma^2 s^2 & \lambda \rho \sigma s \sqrt{k} \\ \lambda \rho \sigma s \sqrt{k} & \lambda^2 k \end{bmatrix}$$

uniquely determine the distribution of the process (S_t, K_t) with initial conditions (S_0, K_0). By theorem A-1 of that paper, it is enough to verify the following conditions:

Condition C. $a(s,k)$ and $b(s,k)$ are locally bounded and measurable. Let $\sigma(s,k)$ satisfying $a(s,k) = \sigma(s,k)'\sigma(s,k)$ be also locally bounded and measurable. Then, for every $R > 0$ there exists $A_R > 0$ such that

$$\begin{aligned}\sup_{\|(s,k)\|<R, \|(s',k')\|<R} \|\sigma(s,k) - \sigma(s',k')\| &+ \|b(s,k) - b(s',k')\| \\ &- A_R\|(s,k) - (s',k')\| \leq 0\end{aligned}$$

Non explosion Condition. There exists a nonnegative function $\phi(s,k) \in C^2$ and a nonnegative constant $A > 0$ such that

$$\lim_{\|(s,k)\|\to\infty} \phi(s,k) = \infty$$

$$b(s,k).D\phi(s,k) + \frac{1}{2}\mathrm{Tr}\, a(s,k).D^2\phi(s,k) \leq A\phi(s,k)$$

both conditions are easily verified.

Proposition 2 **(a)** *$V(x,y,k)$ is concave in (x,y).*

(b) *$V(x,y,k)$ is strictly increasing in (x,y), non-increasing in k.*

(c) *$V(x,y,k)$ is uniformly continuous.*

proof:

(a) By the linearity of the system 13–15 with respect to $(C(t), L(t), M(t))$, it follows that if

$$(C^1, L^1, M^1) \in A_{(x_1,y_1,k)}$$

and

$$(C^2, L^2, M^2) \in A_{(x_2,y_2,k)}$$

then

$$\lambda(C^1, L^1, M^1) + (1-\lambda)(C^2, L^2, M^2) \in A_{(\lambda(x_1,y_1,k)+(1-\lambda)(x_2,y_2,k))}.$$

This, together with the concavity of $U(c)$ yields the desired result.

(b) If $x > x'$ and $(C(t), L(t), M(t)) \in A_{(x',y,k)}$ then,

$$(C(t) + \frac{\delta}{\delta - r}(x - x'), L(t), M(t)) \in A_{(x,y,k)}.$$

this is because it is feasible to behave as if you were poorer. The extra cash is then consumed at a constant rate. $\frac{\delta}{\delta-r}x$ is the optimal consumption for the trivial problem where the only asset is a money market account, whose value starts at x. Hence $V(x,y,k) > V(x',y,k)$.

If $y > y'$ it is feasible to sell the extra stock, what strictly increases your money market holdings and apply the argument above.

For the monotonicity in k, suppose $k' > k$. A standard comparison principle for SDE's yields that if $K(t)$ is a solution of 15 with $K(0) = k$ and $K'(t)$ is a solution with $K'(0) = k'$ then $K'(t) \geq K(t)$ almost surely. Hence, $k' > k$ implies $A_{(x,y,k')} \subseteq A_{(x,y,k)}$. Now, let $J(x, y, k; k')$ be the value of acting as if transaction costs started at k' while in fact they started at k. It follows that

$$V(x, y, k) \geq J(x, y, k; k') \geq V(x, y, k')$$

Notice that it is feasible never to engage in any transactions and just consume optimally from the bond. Thus

$$\frac{1}{\delta}U(\frac{\delta}{\delta - r}x) \leq V(x, y, k) \leq W_M(x, y)$$

where $W_M(x, y)$ is the solution to the Merton Problem. This shows that $V(x, y, k)$ cannot be concave in k.

(c) I first establish continuity in (x, y). Consider sequences $(x_n, y_n) \to (x, y)$.

If $x, y > 0$, $V(x_n, y_n, k) \to V(x, y, k)$ by concavity.

If $x = 0$, $y > 0$ then $V(0, y_n, k) \to V(0, y, k)$ also by concavity. Hence, if $x_n \downarrow 0$ then $V(0, y + \frac{x_n}{1+k}, k) \downarrow V(0, y, k)$. But due to the possibility of lump transactions

$$V(0, y + \frac{x_n}{1 + k}, k) \geq V(x_n, y, k) \geq V(0, y, k)$$

The case $y = 0, x > 0$ is handled similarly.

If $x = y = 0$, I appeal to the solution to the Merton problem to get

$$0 \leq V(0, 0, k) \leq W_M(x, y) \to 0 \text{ as } (x, y) \to (0, 0).$$

For the continuity of $V(x, y, k)$ in k , it is convenient to treat first the case k large. Let $Z(t) = \frac{2}{\lambda}\sqrt{K(t)}$. By Ito's lemma

$$dZ(t) = (\frac{\beta}{\lambda}(\alpha - K(t)) - \frac{\lambda}{2})K^{-1/2}(t)\, dt + dW_{2,t}$$

now find $k^\star$ such that the drift is smaller than -1 for $k \geq k^\star$. Notice that the drift is decreasing is k. Let $d\tilde{Z}(t) = -dt + dW_{2,t}$, fix $k \geq k^\star$ and for $0 < a_n \downarrow 0$, define the stopping times

$$\tau_n(k) = \inf\{t \geq 0 \,;\, Z(t) = \frac{2\sqrt{k}}{\lambda} \,,\, Z(0) = \frac{2\sqrt{k + a_n}}{\lambda} \,\}$$

$$\tilde{\tau}_n = \inf\{t \geq 0 \; ; \; \tilde{Z}(t) = \frac{2\sqrt{k}}{\lambda} \; , \; \tilde{Z}(0) = \frac{2\sqrt{k+a_n}}{\lambda} \; \}$$

A standard comparison result for SDE's yields that $\tau_n(k) \leq \tilde{\tau}_n$ a.s. and that $\tilde{\tau}_n$ is independent of k. Now notice that the monotonicity of $V(x,y,k)$ in k implies that if $V(x,y,k+a_n) \uparrow V(x,y,k)$ for some sequence $0 < a_n \downarrow 0$, then V is continuous in k.

Now consider the following strategies for the initial points $(x, y, k + a_n)$. Between $[0, \tau_n)$ do nothing, consume zero; proceed optimally thereafter. Let the value of such strategy be $J(x, y, k + a_n)$. Then

$$V(x,y,k+a_n) \geq J(x,y,k+a_n) = \mathrm{E}\left\{e^{-\delta\tau_n} \, V(X(\tau_n), Y(\tau_n), k)\right\}$$

$X(t), Y(t)$ is the solution of equations 13–14 with $C = L = M = 0$. Now, since $\tilde{\tau}_n \xrightarrow{P} 0$ as $a_n \downarrow 0$ it follows that $\tau_n \xrightarrow{P} 0$ and consequently

$$e^{-\delta\tau_n} \, V(X(\tau_n), Y(\tau_n), k) \xrightarrow{P} V(x,y,k)$$

thanks to the continuity of V in (x, y).

To finish this part, just claim the dominated convergence theorem, noting that $\tilde{\tau}_n$ is almost surely bounded along some subsequence $\tilde{\tau}_{n_j}$.

The proof implies that the convergence is uniform for $k > k^\star$. Thanks to the concavity of $V(\cdot, \cdot, k)$ the convergence is also uniform in (x, y).

The same proof applies to $k > 0$ arbitrary (with the loss of uniformity). The reason is that, thanks to the stationarity of $K(t)$, it is still true that $\tau_n(k) \xrightarrow{P} 0$ (e.g. Karlin and Taylor [23]). The case $k = 0$, however, cannot be handled similarly, because $\mathrm{E}\,\tau_n(0) = \infty$ (0 is an entrance boundary).

For continuity in $k = 0$, let $(C(t), L(t), M(t))$ be a optimal strategy for the initial condition $(x, y, 0)$. Lemma 1 shows that $L(t), M(t)$ are continuous except perhaps at $t = 0$. Since at $t = 0$ transactions are costless, all initial jumps with $|L(0^+) - M(0^+)| = |\tilde{L}(0^+) - \tilde{M}(0^+)|$ are equivalent. I normalize things setting either $L(0^+) = 0$ or $M(0^+) = 0$. Consider the case $L(0^+) = \alpha \geq 0$ (so $M(0^+) = 0$).

$$V(x,y,0) = EV(x-\alpha, y+\alpha, 0) = V(x-\alpha, y+\alpha, 0)$$

For $\epsilon > 0$, define the stopping time

$$\tau_\epsilon = \inf\{t \geq 0 \; ; \; K(t) = \epsilon \; , \; K(0) = 0 \; \}$$

then,

$$\begin{aligned} V(x-\alpha, y+\alpha, 0) &= \int_0^{\tau_\epsilon} e^{-\delta t} U(C(t))dt \quad + \mathrm{E}\left\{e^{-\delta\tau_\epsilon} V(X(\tau_\epsilon), Y(\tau_\epsilon), \epsilon)\right\} \\ &= \int_0^{\tau_\epsilon} e^{-\delta t} U(C(t))dt \quad + \mathrm{E}\left\{e^{-\delta\tau_\epsilon} (V(X(\tau_\epsilon), Y(\tau_\epsilon), \epsilon)\right. \\ &\qquad \left. - V(x-\alpha, y+\alpha, \epsilon))\right\} \\ &\qquad + V(x-\alpha, y+\alpha, \epsilon)\, \mathrm{E}\, e^{-\delta\tau_\epsilon} \end{aligned}$$

$X(t), Y(t)$ is the solution of equations 13–14 with initial conditions $x-\alpha, y+\alpha$. Arguing as above I get

$$V(x-\alpha, y+\alpha, 0) = \lim_{\epsilon\downarrow 0} V(x-\alpha, y+\alpha, \epsilon)$$

Now, it suffices to show that

$$\lim_{\epsilon\downarrow 0} V(x-\alpha, y+\alpha, \epsilon) = \lim_{\epsilon\downarrow 0} V(x, y, \epsilon)$$

by contradiction, suppose not. Suppose that for ϵ sufficient small

$$\begin{aligned} V(x-\alpha, y+\alpha, \epsilon) &= V(x, y, \epsilon) + a_\epsilon \qquad , a_\epsilon > a > 0 \\ &\geq V(x-\alpha(1+\epsilon), y+\alpha, \epsilon) + a_\epsilon \end{aligned}$$

or

$$\begin{aligned} V(x, y, \epsilon) &= V(x-\alpha, y+\alpha, \epsilon) + b_\epsilon \qquad , b_\epsilon > a > 0 \\ &\geq V(x, y+\alpha\epsilon, \epsilon) \end{aligned}$$

again, the inequalities are due to the feasibility of lump transactions. Both statements contradict the continuity of $V(\cdot, \cdot, \epsilon)$ for positive ϵ. The argument with $M(0^+) > 0$ is entirely analogous.

Most of the work has been done. For the uniform continuity of $V(x, y, k)$, just evaluate

$$|V(x, y, k) - V(\tilde{x}, \tilde{y}, \tilde{k})| \leq |V(x, y, k) - V(x, y, \tilde{k})| + |V(x, y, \tilde{k}) - V(\tilde{x}, \tilde{y}, \tilde{k})|$$

I proved above that the first term is uniform continuous. For the second term just write

$$\lim_{k\uparrow\infty} V(x, y, k) = V(x, y, \infty) \geq 0$$

then $V(x, y, \infty)$ is concave and for k sufficiently large

$$|V(x, y, \tilde{k}) - V(\tilde{x}, \tilde{y}, \tilde{k})| \leq |V(x, y, \infty) - V(\tilde{x}, \tilde{y}, \infty)| + 2\epsilon$$

Uniform continuity follows from the concavity of $V(x, y, \infty)$

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