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A NEW PROOF OF THE EXISTENCE OF EQUILIBRIUM IN INCOMPLETE MARKETS ECONOMIES

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I. Introduction

The theory of incomplete markets economies has obvious importance. It is simply more general than the fully equipped Arrow-Debreu general equilibrium model. However it lacks a well developed equilibrium existence theory. The first important work was made by Radner in 1972 for the finite dimensional case. He proved the existence of equilibrium in incomplete markets economies with a finite number of assets and a lower bound on uncovered sales. This unpleasant hypothesis is needed as Hart showed in 1975 by means of a counter-example. The realm of validity of Hart's example is satisfactorily determined. In Duffie and Shafer (1985) it was shown that the set of initial endowments and asset returns for which, without a lower bound on short sales equilibrium does not exist in a null set. Also Werner (1985) and Geanakoplos-Polemarchakis (1986) proved that, respectively, for the nominal returns and for the "gold" returns case equilibrium exists in general without a lower bound on asset trades. Thus in conclusion the finite dimensional theory is well understood.

The case with a continuum of states was studied by Mas-Colell and Monteiro (1990) and then by Hellwig (1991) and Mas-Colell and Zame (1991). To fix the ideas let me suppose the consumption set to be the positive octant of the commodity space (the last two works include this case and the first can be modified to cover it and then the comment that follows is true). Hence in this situation, in the three works, the assumption of non-negativity of ex-post endowments is needed. This assumption imply a limit on short sales and is much stronger than this. However its necessity was settled by a counter-example in Mas-Colell and Zame (1991). This counter-example shows that a lower bound on short sales is not a hypothesis strong enough to prove incomplete markets equilibrium existence even in sequence spaces. The countable number of states case was studied by Green and Spear (1987) and by Zame (1988).

In this paper I prove an incomplete markets equilibrium existence theorem with hypotheses similar to Hellwig (1991) and Mas-Colell and Zame (1991) papers. The main novelty of my work is its proof. I use finite-dimensional truncations of utilities, endowments and asset returns. This finite-dimensional economy has an equilibrium and each consumer maximization problem satisfy Slater's condition. Therefore, to each

budget constraint there is a Lagrange multiplier. The consumer's set of Lagrangean multipliers can be thought as truncations of infinite dimensional multipliers of the limit economy (i.e. the original economy).

In the second section I present the model and enunciate the main theorem. In the third section I present some preliminary mathematical work. The fourth section contains some lemmas and the main theorem's proof. In the fifth section I discuss the works of Hellwig. Mas-Collel and Zame.

II. The model and the main theorem

The economy has I agents and two periods. The state space is a complete atomless probability space $(T, \mathcal{A}, P)$.

There are G commodities and J assets. In the first period each agent choose a consumption vector $x \in \mathbf{R}^G_+$ and a portfolio of assets $\theta \in \mathbf{R}^J$ and second period consumption $y: T \to \mathbf{R}^G_+$. There are no initial endowments of securities. The assets returns are given by a measurable function $A: T \to (\mathbf{R}^G)^J$. There is a lower bound on short sales v_i, $v_i << 0$.

For a fixed p, $1 \leq p \leq \infty$ the consumption set of consumer i, $1 \leq i \leq I$ is $X^i = \mathbf{R}^G_+ \times L^p_+(T; \mathbf{R}^G)$ and his endowment is $(w^i, W^i) \in X$. The utility function $U^i: X^i \to \mathbf{R}$ is separable:

$$U^i(x, y) = \int_T u_{it}(x, y_t) dP(t) \qquad (x, y) \in X^i$$

where the function $u_{it}: \mathbf{R}^G_+ \times \mathbf{R}^G_+ \to \mathbf{R}$ is continuous, concave and increasing.

It may be useful to explicity conditions that ensure U^i to be well defined and finite. This will be done in the next section.

If $B \subset \mathbf{R}^\ell$ I define $S(B)$ as the set $\left\{x \in B; \sum_{t=1}^{\ell} |x_t| = 1\right\}$.

An equilibrium for the economy $\mathcal{E} = (X^i, U^i, w^i, W^i, A)_{i \leq I}$ is a vector $(\bar{p}, \bar{\pi}, \bar{q}, \bar{\theta}, \bar{x}, \bar{y})$ such that

a) $(\bar{p}, \bar{\pi}) \in S(\mathbf{R}^G_+ \times \mathbf{R}^J)$

b) $\bar{q}: T \to S(\mathbf{R}^G_+)$ is measurable

c) $\bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_I) \in (\mathbf{R}^J)^I, \quad \sum_{i \leq I} \bar{\theta}_i = 0$

d) $\bar{x} = (\bar{x}^1, \dots, \bar{x}^I),\ \bar{y} = (\bar{y}^1, \dots, \bar{y}^I),\ (\bar{x}^i, \bar{y}^i) \in X^i$ for every i

e) $\sum_{i \leq I} (\bar{x}^i, \bar{y}^i) = \sum_{i \leq I} (w^i, W^i)$

f) $(\bar{x}^i, \bar{y}^i, \bar{\theta}_i)$ solves

$$\max U^i(x, y) \qquad (x, y) \in X^i, \quad \theta \in \mathbf{R}^J \quad \text{subject to}$$

$$\theta \geq v_i$$

$$\bar{p}x + \bar{\pi}\theta \leq \bar{p} \cdot w^i$$

$$\bar{q}_t \cdot y_t \leq \bar{q}_t(W^i_t + \theta A_t) \quad \text{for almost every } t \text{ in } T.$$

Theorem 1. Suppose that $(T, \mathcal{A}, P)$ is a complete atomless probability space. Suppose also that

i) for every $\theta \geq v_i$

$W^i_t + \theta A_t \geq 0$ for almost every t in T

ii) there is a $c > 0$ such that

$W^i_t \geq c(1, \dots, 1)$ for almost every t in T

iii) w^i is a strictly positive vector

Then for any p, $1 \leq p \leq \infty$, there is an incomplete markets equilibrium for $\mathcal{E}$.

Remarks

1. Note that the measure space does not need to be separable

2. To add, in theorem 1, an atomic part in P is possible. The main steps to do that are the same as in Hellwig (1991) and Mas-Colell and Monteiro (1990). I refer the interested reader to these papers.

3. Hypothesis (i) is strong. It implies $A \geq 0$ and $A \leq nW^i$ for some n. I commented in the introduction about its necessity.

4. Despite the appearance of generality, the set defined by (ii) is a first category subset of $L^p_+(T;\mathbf{R}^G)$. I refer the reader to Araujo-Monteiro (1991) for more information on non-genericity in equilibrium models.

III. Mathematical preliminaries

Before going into the preparatory work for the main theorem's proof I will explain its idea. First I truncate the endowments, asset returns and utilities with respect to the net Λ'_n (defined in lemma 2 below). The pointwise convergence of the truncations is proved in lemma 2. With these truncations in hand a finite dimensional economy is naturally defined. This economy has by Radner's (1972) an equilibrium. The existence of Lagrange multipliers, corresponding to each consumer's budget constraints, is easily verified. Next considering the multipliers so obtained as functions on the whole probability space I prove that they converge to Lagrangean multipliers of the original economy. To prove the multipliers convergence and the existence of a limit the multi-dimensional Fatou's lemma is used.

I begin the preparatoy work by giving necessary and sufficient conditions for U^i to be well defined and finite. The continuity of each u_{it} is supposed.

Proposition 1. U^i is well defined if and only if for every $(x,y) \in \mathbf{R}^G_+ \times \mathbf{R}^G_+$ the function $t \mapsto u_{it}(x,y)$ is measurable.

Proof: The proof is easy and will not be done.

Proposition 2. Suppose P atomless and $1 \leq p < \infty$. Suppose also that U^i is finite everywhere. Then for every $x \in \mathbf{R}^G_+$ there are an $\alpha \in L^1(P)$ and a $c > 0$ such that

$$u_{it}(x,y) \leq \alpha(t) + c|y|^p_S \quad \text{for every } y \in \mathbf{R}^G_+ \text{ and almost every } t.$$

The numbers $|y|_S$ is the sum norm of y.

Proof: This proposition is an easy consequence of theorem 2.3 in Krasnoselskii (1964) pp. 27-28,32.

Q.E.D.

I need some results on differentiation in abstract spaces. This topic was introduced by R. de Possel in 1936. In the following I use Zaanen (1967).

Definition 1:

i) A net $\mathcal{N}$ is a countable and measurable partition of T

ii) A sequence of nets $(\mathcal{N}_n)_n$ is monotone if each element of $\mathcal{N}_{n+1}$ is contained in some element of $\mathcal{N}_n$.

Lemma 1. Suppose $(\mathcal{N}_n)_n$ is a monotone sequence of nets. Then for any $\mathcal{M} \subset \mathcal{N}_o$, $\mathcal{N}_o = \bigcup\limits_n \mathcal{N}_n$, there is a sequence $\{I_n; n \in L\}$, $L \subset \mathbf{N}$, of pairwise disjoint sets such that $\bigcup\limits_{n \in L} I_n = \cup\{M; M \in \mathcal{M}\}$.

Proof: This is lemma 1 in page 268 in Zaanen (1967).

Definition 2. Suppose $(\mathcal{N}_n)_n$ is a sequence of nets in T and $\mathcal{N}_o = \bigcup\limits_n \mathcal{N}_n$. This sequence is regular with respect to the measure P if whenever is given an $\varepsilon > 0$ and a measurable set E, there exists a sequence $(I_n)_n$ of sets from $\mathcal{N}_o$ such that $P(E - \bigcup\limits_n I_n) = 0$ and $P(\bigcup\limits_n I_n - E) < \varepsilon$.

Proposition 3 (Differentiation of an indefinite integral).

Suppose $(\mathcal{N}_n)_n$ is a monotone sequence of nets in T, regular with respect to P and $f \in L^1(P)$. Then for almost every t in T

$$f(t) = \lim_{n \to \infty} \frac{1}{P(I_{n'})} \int_{I_{n'}} f \, dP$$

where $n' = n'(t, n)$ is (uniquely) defined by $t \in I_{n'}$, $I_{n'} \in \mathcal{N}_n$.

Proof: This is theorem 3, page 272 in Zaanen (1967).

Definition 3. A function $f = (f_1, \dots, f_\ell): T \to \mathbf{R}^\ell$ is integrable if each $f_k: T \to \mathbf{R}$ is integrable, $1 \le k \le \ell$. The integral of f in this case is the vector $\left(\int_T f_1\, dP, \dots, \int_T f_\ell\, dP\right)$ and is denoted by $\int_T f\, dP$.

Corollary 1. For every integrable $f: T \to \mathbf{R}^\ell$ define $E(f|\mathcal{N}_n): T \to \mathbf{R}^\ell$ by $E(f|\mathcal{N}_n)(t) = \frac{1}{P(I)} \int_I f\, dP$ if $t \in I$, $I \in \mathcal{N}_n$.

Then:

a) $\lim_{n \to \infty} E(f|\mathcal{N}_n)(t) = f(t)$ for almost every t in T

b) $(E(f|\mathcal{N}_n))_n$ is uniformly integrable.

Proof: Define $f_n = E(f|\mathcal{N}_n)$.

a) This is just a rephrasing of proposition 3 for vector valued functions.

b) Suppose $r > 0$

$$\int_{|f_n|_S > r} |f_n|_S\, dP = \sum_{I \in \mathcal{N}_n} \frac{P([|f_n|_S > r] \cap I)}{P(I)} \left| \int_I f\, dP \right|_S .$$

Define $\mathcal{N}' = \{I \in \mathcal{N}_n : \left|\int_I f\, dP\right|_S > r \cdot P(I)\}$. So we have that

$$\int_{|f_n|_S > r} |f_n|_S\, dP = \sum_{I \in \mathcal{N}'} \left| \int_I f\, dP \right|_S \le \sum_{I \in \mathcal{N}'} \int_I |f|_S\, dP = \int_{U\mathcal{N}'} |f|_S\, dP.$$

But $\int_I |f|_S\, dP \ge \left|\int_I f\, dP\right|_S > rP(I)$ for every $I \in \mathcal{N}'$, so $\int_{U\mathcal{N}'} |f|_S\, dP > rP(U\mathcal{N}')$. Therefore $P(U\mathcal{N}') < \int_T |f|_S\, dP/r$ and this ends the proof.

Q.E.D.

Example: Suppose $\mathcal{F} = \{f_j; j \le J\}$ is a family of functions $f_j \in L^p(T; \mathbf{R}^L)$. And define $\mathcal{B}$ to be the smallest σ-algebra such that each f_j is measurable. Then using that f_j is the pointwise limit of simple functions it is easy to see that there is a countable family $\{A_n; n \ge 1\} = \mathcal{A}'$ such that $\mathcal{B} = \sigma(\mathcal{A}')$. Without loss of generality we can suppose $\mathcal{A}'$ to be a ring of sets [since the ring generated by a countable set is countable: Halmos (1950) p. 23].

Now define the nets $\mathcal{N}_1 = \{A_1, A_1^c\}$, $\mathcal{N}_2 = \{A_1A_2, A_1A_2^c, A_1^cA_2, A_1^cA_2^c\}$ and so on. Then since $\sigma(\bigcup_n \mathcal{N}_n) = \mathcal{B}$ the monotone sequence of nets $(\mathcal{N}_n)$ is regular.

Now I apply the above example to the initial endowments and the assets returns of the economy.

Lemma 2. There is a monotone sequence of nets $(\mathcal{N}_n)$ in $(T, \mathcal{A}, P)$ such that $W_{in} = E(W^i|\mathcal{N}_n)$ and $A_n = E(A|\mathcal{N}_n)$ satisfy:

i) $\lim_{n\to\infty} W_{in}(t) = W^i(t)$ for almost every t in T

ii) $\lim_{n\to\infty} A_n(t) = A(t)$ for almost every t in T.

Proof: Choose the nets $(\mathcal{N}_n)$ as in the above example for $\mathcal{F} = \{A, W^1, \ldots, W^I\}$. Then the sequence of nets $(\mathcal{N}_n)$ is regular for $\sigma(\mathcal{F})$. Now (i) and (ii) follows directly from Corollary 1 (a).

IV. The main theorem's proof

From now on a sequence of nets as in lemma 2 is fixed. For each N define $M = M(N)$ as the number of elements of the net $\mathcal{N}_N = \{I_1, \ldots, I_{MN}\}$. Define $W_{imN} = \frac{1}{P(I_{mN})}\int_{I_{mN}} W^i\, dP$, $A_{mN} = \frac{1}{P(I_{mN})}\int_{I_{mN}} A\, dP$ and define $\mathcal{X}_{mN}$ as the characteristic function of I_{mN}, $1 \le m \le M$. Define the utility function

$$U^{iN}(x, y_1, \ldots, y_M) = U^i\left(x, \sum_{m\le M} y_m \mathcal{X}_{mN}\right).$$

Thus a finite dimensional economy $\mathcal{E}^N = (U^{iN}, W_{imN}, A_{mN})_{\substack{i\le I\\ m\le M}}$ is defined.

Theorem 2. For each N the economy $\mathcal{E}^N$ has an equilibrium $(p^N, \pi^N, x_{iN}, y_{imN}, q_{mN}, \theta_{iN})$ and there are Euler-Lagrange multipliers $(\mu_{iN}, \lambda_{imN})$ such that for every $x \ge 0$, $y_m \ge 0$, $\theta \ge v_i$, $1 \le m \le M$, $1 \le i \le I$

$$U^{iN}(x, y_1, \ldots, y_M) - \mu_{iN}(p^N(x - w^i) + \pi^N\theta) -$$
$$- \sum_{m\le M} \lambda_{imN} q_{mN}(y_m - W_{imN} - \theta A_{mN}) \le U^{iN}(x_{iN}, y_{iN})$$

where $y_{iN} = \sum_{m \leq M} y_{imN} \mathcal{X}_{mN}$.

Moreover the inequalities below are true:

$$p^N x_{iN} + \pi^N \theta_{iN} \leq p^N w^i \qquad \mu_{iN}(p^N x_{iN} + \pi^N \theta_{iN}) = \mu_{iN} p^N w^i$$

$$\theta_{iN} \geq v_i, \qquad \sum_i \theta_{iN} = 0$$

$$x_{iN} \geq 0, \quad \sum_i x_{iN} = \sum_i w^i, \quad y_{imN} \geq 0, \quad \sum_i y_{imN} = \sum_i W_{imN}$$

$$q_{mN} \cdot y_{imN} \leq q_{mN}(W_{imN} + \theta_{iN} A_{mN}) \quad \lambda_{imN} q_{mN}(y_{imN} - W_{imN} - \theta_{iN} A_{mN}) = 0$$

$$(p^N, \pi^N) \geq 0, \quad |(p^N, \pi^N)|_S = 1$$

Proof: The finite dimensional economy $\mathcal{E}^N$ has by Radner (1972) an incomplete markets equilibrium $(p^N, \pi^N, q_{mN}, \theta_{iN}, x_{iN}, y_{imN})$. The Slater's condition for the consumer's maximization problem of this economy is satisfied. Therefore there are multipliers as above.

Q.E.D.

Lemma 3. The sequence $h_N = U^i\left(\sum_j w^j, \sum_k W_{kN}\right)$ is bounded.

Proof: First I consider the case $p = \infty$.

In this case $W_{kN}(t) \leq |W^k|_\infty(1, \ldots, 1)$ for almost every t. So it is true for every N that

$$h_N \leq U^i\left(\sum_j w^j, \sum_{k \leq I} |W^k|_\infty(1, \ldots, 1)\right) < \infty.$$

Now I consider the case $p < \infty$. By proposition 2 for $x = \sum_j w^j$ there is an $\alpha \in L^1(p)$ and a $c > 0$ such that

$$u_{it}\left(\sum_j w^j, z\right) \leq \alpha(t) + c|z|_S^p \quad \text{for every } z \in \mathbf{R}_+^G \text{ for a.e. } t \text{ in } T.$$

Therefore

$$h_N - \int_T \alpha(k)dP(t) \le c \cdot \int_T \left| \sum_k W_{kN} \right|_S^p dP = c \sum_{m \le M} P(I_{mN})^{1-p} \left| \int_{I_{mN}} \sum_k W_t^k \, dP(t) \right|_S^p =$$

$$\le c \cdot \sum_{m \le M} P(I_{mN})^{1-p} \left(\int_{I_{mN}} \left| \sum_k W_t^k \right|_S dP(t) \right)^p \le$$

$$\le c \sum_{m \le M} P(I_{mN})^{1-p} \left[\left(P(I_{mN})^{\frac{p-1}{p}} \left(\int_{I_{mN}} \left| \sum_k W_t^k \right|_S^p dP \right)^{1/p} \right) \right]^p =$$

$$= c \int_T \left| \sum_k W_t^k \right|_S^p dP < \infty.$$

Lemma 4. $(\mu_{iN})_N$ is bounded.

Proof: Since $v_i << 0$ there is a number $a < 0$ such that $v_i \le a(1, \ldots, 1)$. Define $\theta_i = a(1, \ldots, 1)$. Now make $x = 0$ and choose $\bar{y}_m \ge 0$ such that $q_{mN}\bar{y}_m = q_{mN}(W_{imN} + \theta_i A_{mN})$. Substituting those values in theorem 2: $\mu_{iN}(p^N w^i + a|\pi^N|_S) \le U^{iN}(x_{iN}, \bar{y}_1, \ldots, \bar{y}_M) \le U^i \left(\sum_j w^j, \sum_k W_{kN} \right) = h_N$. From lemma 3 (h_N) is bounded.

Since $\inf_N (p^N w^i + A|\pi^N|_S) > 0$ the proof is done.

Q.E.D.

The next propostion is an easy consequence of theorem 2.

Proposition 4. The inequalities below are true for $x \ge 0$, $y \ge 0$, $\theta \ge v_i$.

i) $\sum_{m \le M} \int_{I_{mN}} [u_{it}(x, y_m) - u_{it}(x_{iN}, y_{imN})] dP \le \mu_{iN} p^N (x - x_{iN})$

$+ \sum_{m \le M} \lambda_{imN} q_{mN} (y_m - y_{imN})$.

ii) $\left(\sum_{m \le N} \lambda_{imN} q_{mN} A_{mN} - \mu_{iN} \pi^N \right) (\theta - \theta_{iN}) \le 0$.

Proof: (i) make $\theta = \theta_{iN}$ in theorem 2.

(ii) make $x = x_{iN}$ and $y_n = y_{imN}$, $1 \le n \le M$ in theorem 2.

Q.E.D.

In order to obtain Euler-Lagrange multipliers for the limit economy it will first be necessary to obtain subgradients of the instantaneous utility u_{it} at the "equilibrium". The next lemma is an important step in this direction.

Lemma 5. There is a sequence $(\mu_m^{iN})_{m\le M}$ in $\mathbf{R}_+^G$ such that

i) $\sum_{m\le M} \mu_m^{iN} = \mu_{iN} p^N$

ii) for every $(x, y) \ge 0$, $1 \le m \le M$

$$\int_{I_{mN}} u_{it}(x,y)dP(t) - \int_{I_{mN}} u_{it}(x_{iN}, y_{imN})dP(t) \le \mu_m^{iN}(x - x_{iN}) + \\ + \lambda_{imN} q_{mN}(y - y_{imN}).$$

Proof: Define for each $1 \le m \le M$

$$f_m(x, y_1, \ldots, y_M) = \int_{I_{mN}} u_{it}(x, y_m)dP(t)$$

and $f = f_1 + \cdots + f_M$.

So it is true by proposition 4(i) that the vector $(\mu_{iN} \cdot p^N, (\lambda_{imN} q_{mN})_{m\le M})$ is a subgradient of f at $(x_{iN}, (y_{imN})_{m\le M}) = z_{iN}$.

Hence from the continuity of f_i, $i \le M$ we conclude by a result of Moreau (1964) that there are, for each m, a vector $\left(\mu_m^{iN}, (\lambda_{im}^n)_{n\le M}\right)$ such that

a) $f_m(x, y_1, \ldots, y_M) - f_m(z_{iN}) \le \mu_m^{iN}(x - x_{iN}) + \sum_{n\le M} \lambda_{im}^n (y_n - y_{imN})$

b) $\sum_{m\le M} \mu_m^{iN} = \mu_{iN} p^N, \quad \sum_{n\le M} \lambda_{in}^m = \lambda_{imN} q_{mN}.$

The first part of (b) proves (i). Now fix $m' \le M$ and define $y_n = y_{inN}$ if $n \ne m'$, $y_{m'} = y \ge 0$. So from (a) it follows that

$$f_{m'}(x, y_1, \ldots, y_M) - f_{m'}(z_{iN}) \le \mu_m^{iN}(x - x_{iN}) + \lambda_{im'}^{m'}(y - y_{imN}).$$

Also (a) implies for $m \neq m'$ that $0 \leq \lambda_{im}^{m'}(y - y_{im'N})$. Therefore

$$\lambda_{im'}^{m'}(y - y_{im'N}) \leq \sum_{m \leq M} \lambda_{imN}^{m'}(y - y_{im'N}) = \lambda_{im'N} q_{mN}(y - y_{im'N})$$

and this proves (ii).

Q.E.D.

In the following I use Hellwig's adaptation of Hildenbrand-Mertens (1971) proof of the multi-dimensional Fatou's lemma. The next lemma is the fundamental building block of this adaptation.

Lemma 6 (Hildenbrand and Mertens (1971))

Suppose $(T, \mathcal{A}, P)$ is a complete atomless probability space and $(f_n)_n$ is a sequence of integrable functions $T \to \mathbf{R}_+^\ell$.

Suppose also that $\lim_{n \to \infty} \int f_n \, dP$ exists. Then:

i) there is a vector valued measure $\mu : \mathcal{A} \to \mathbf{R}^\ell$ such that μ is a limit (in the product topology $\sigma^\ell(ba, L^\infty)$) of $(P_n)_n$, P_n is defined by:

$$P_n(E) = \int_E f_n \, dP \qquad E \in \mathcal{A}.$$

ii) $\mu = g\, dP + \mu^p$, where $g : T \to \mathbf{R}_+^\ell$ is integrable and μ^p is purely finitely additive. Moreover

$$\lim_{m \to \infty} \int_T f_n \, dP - \int_T g \, dP = \mu^p(T)$$

and there is a sequence of $E_n \supseteq E_{n+1}$ decreasing to the empty set such that $\mu^p(T) = \mu^p(E_n)$ for every n

iii) for almost every t in T there are sequences $(\delta^{in})_n$ and $(z^{in})_n$ $0 \leq i \leq \ell$ and numbers δ^i such that

$$\delta^{in} \geq 0, \ \sum_{i=0}^{\ell} \delta^{in} = 1, \ \delta^i = \lim_{n \to \infty} \delta^{in}, \ z^{in} \in \{f^{n+m}(t), m \geq 0\} \text{ for every } n$$

and $(\delta^{in} z^{in})_n$ is convergent (z^{in} may be not).

iv) $g(t) = \sum_{i=0}^{\ell} \lim_n (\delta^{in} z^{in})$.

Proof: For a proof I refer the reader to pages 151-153 of Hildenbrand and Mertens (1971).

In the next lemma the integrability conditions needed to apply Fatou's lemma are proved.

Lemma 7. The following sequences of functions are integrable and the sequence of integrals is bounded.

i) $q_N : T \to \mathbf{R}^G$,

$q_N(t) = q_{mN}$ if $t \in I_{mN}$ $\quad 1 \le m \le M$

ii) $y_{iN} : T \to \mathbf{R}^G$

$y_{iN}(t) = y_{imN}$ if $t \in I_{mN}$ $\quad 1 \le m \le M$

iii) $\Lambda_{iN} q_N A_N$

where $\Lambda_{iN}(t) = \frac{1}{P(I_{mN})} \lambda_{imN}$ if $t \in I_{mN}$. $\quad 1 \le m \le M$

iv) $\Lambda_{iN} q_N$ and Λ_{iN}

v) $O^{iN} : T \to \mathbf{R}$

$O^{iN}(t) = \frac{1}{P(I_{mN})} \int_{I_{mN}} u_{it}(x_{iN}, y_{imN}) dP(t) \quad t \in I_{mN}, 1 \le m \le M$

vi) $\tilde{\mu}^{iN} : T \to \mathbf{R}$

$\tilde{\mu}^{iN}(t) = \frac{1}{P(I_{mN})} \mu_m^{iN} \quad t \in I_{mN}, 1 \le m \le M$

vii) the sequence in (v) is pointwise bound d and uniformly integrable.

Proof: (i) is obvious

(ii) From the feasability of (y_{iN}) we have $0 \le y_{iN} \le \sum_j W_{jN}$. Now $\int_T \sum_j W_{jN} = \int_T \sum_j W^j \, dP$ proves (ii).

To prove (iii) use that $(\mu_{iN})_N$ is bounded (lemma 4) in the inequality:

$$\int_T \Lambda_{iN} q_N A_N \, dP = \sum_{m \le M} \lambda_{imN} q_{mN} A_{mN} \le \mu_{iN} \pi^N.$$

The inequality above is a consequence of proposition 4(ii). To prove (iv) I will first prove that $\sum_{m \leq M} \lambda_{imN} q_{mN}$ is bounded. From hypothesis (iii) in theorem 1 it is true that

$$
\begin{aligned}
c\left|\sum_{m} \lambda_{imN} q_{mN}\right|_S \leq \sum_{m} \lambda_{imN} q_{mN} W_{imN} = \sum_{m \leq M} \lambda_{imN} q_{mN} y_{imN} - \\
- \sum_{m \leq M} \lambda_{imN} q_{mN} A_{mN} \theta_{iN} \leq \int_T u_{it}(x_{iN}, y_{iN}) dP + \\
+ \sum_{m \leq M} \lambda_{imN} q_{mN} A_{mN} |v_i|.
\end{aligned}
$$

The last term of the last inequality is by (iii) above, bounded. The integral is bounded by lemma 3. This proves the first part of (iv). The equalities that follows proves the second part:

$$
\int_T \Lambda_{iN}\, dP = \int_T \Lambda_{iN} q_N (1, \ldots, 1) dP = \left(\int_T \Lambda_{iN} q_N \right) (1, \ldots, 1).
$$

Now (v) is a consequence of lemma 3

$$
\int_T O^{iN}\, dP = \int_T u_{ik}(x_{iN}, y_{iN}) dP \leq h_N.
$$

Item (vi) follows from lemma 5(i)

$$
\int_T \tilde{\mu}^{iN}\, dP = \sum_{m \leq M} \mu_m^{iN} = \mu_{iN} p^N.
$$

Finally I will prove that (O^{iN}) is pointwise bounded and uniformly integrable. The case $p = \infty$ is easy and will not be done. Suppose now $p < \infty$. From proposition 2 for $x = \sum_i w^i$ it's true that

$$
u_{it}\left(\sum_i w^i, z \right) \leq \alpha(t) + c|z|_S^P \quad \text{for every } z \in \mathbf{R}_+^G \text{ and a.e. } t \in T.
$$

Hence for $t \in I_{mN} \quad 1 \leq m \leq M$

$$O^{iN}(t) = \frac{1}{P(I_{mN})} \int_{I_{mN}} u_{it}(x_{iN}, y_{imN}) dP \leq$$

$$\leq \frac{1}{P(I_{mN})} \int_{I_{mN}} \alpha \, dP + c \frac{1}{P(I_{mN})} \int_{I_{mN}} \left| \sum_j W_{jmN} \right|_S^p dP =$$

$$= \frac{1}{P(I_{mN})} \left[\int_{I_{mN}} \alpha \, dP + c \, P(I_{nM})^{1-p} \left| \int_{I_{mN}} \sum_j W^j \, dP \right|_S^p \right] \leq$$

$$\leq \frac{1}{P(I_{mN})} \left[\int_{I_{mN}} \alpha \, dP + c \, P(I_{mN})^{1-p} \left(\int_{I_{mN}} \left| \sum_j W^j \right|_S dP \right)^p \right] \leq$$

$$\leq \frac{1}{P(I_{mN})} \left[\int_{I_{mN}} \alpha \, dP + c \, P(I_{mN})^{1-p} \left(P(I_{mN})^{\frac{p-1}{p}} \left(\int_{I_{nM}} \left| \sum_j W_j \right|_S^p dP \right)^{1/p} \right)^p \right] =$$

$$= \frac{1}{P(I_{mN})} \int_{I_{mN}} \left(\alpha + c \left| \sum_j W_j \right|_S^p \right) dP.$$

Now using corollary 1 we finish the proof.

Q.E.D.

From now on I suppose, without loss of generality, that $(x_{iN}, p^N, \pi^N, \theta_{iN}, \mu_{iN}) \to (\bar{x}_i, \bar{p}, \bar{\pi}, \overline{\theta}_i, \bar{\mu}_i)$. It is clear that $\sum_i (\bar{x}_i, \overline{\theta}_i) = \sum_i (w^i, 0)$ and $|(\bar{p}, \bar{\pi})|_S = 1$. The items (3) and (4) below are based on Heilwig's adaptation of Hildenbrand and Mertens (1971) proof of Fatou's lemma.

Theorem 3. Define the sequence $f^N = (f^{oN}, f^{ijN})_{\substack{j \leq 5 \\ i \leq I}}$ by the equality

$$(f^{oN}, f^{ijN})_{\substack{j \leq 5 \\ i \leq I}} = (q_N, \Lambda_{iN}, \Lambda_{iN} q_N A_N, \tilde{\mu}^{iN}, y_{iN}, O^{iN})_{i \leq I}.$$

The lemma 7 permits to suppose, without loss of generality, that $\lim \int_T f_n \, dP$ exists. Then there is a function $\bar{f}: T \to \mathbf{R}^\ell$ (it is not necessary to specify ℓ) such that.

1) $\bar{f}(t) \in \limsup f^N(t)$ for almost every t in T

$$\int_T \bar{f}\, dP \leq \lim \int_T f^N\, dP$$

2) Defining $\overline{\Lambda}_i = \bar{f}^{i1}$, $\tilde{\mu}^i = \bar{f}^{i3}$, $\bar{y}_i = \bar{f}^{i4}$, $\overline{O}^i = \bar{f}^{i5}$, $\bar{q} = \bar{f}^o$ $1 \leq i \leq T$ it is true that

$$\bar{f} = (\bar{q}, \overline{\Lambda}_i, \overline{\Lambda}_i \bar{q} A, \tilde{\mu}^i, \bar{y}_i, \overline{O}^i)_{i \leq I} \quad \text{for almost every } t \text{ in } T$$

3) for every $\theta \geq v_i$

$$\left(\lim \int_T \Lambda_{iN} q_N A_N\, dP - \int_T \overline{\Lambda}_i \bar{q} A\, dP\right)(\theta - \bar{\theta}_i) \geq 0$$

4) for every $x \geq 0$

$$\left(\lim \int_T \tilde{\mu}^{iN}\, dP - \int_T \tilde{\mu}^i\, dP\right)(x - \bar{x}_i) \geq 0$$

Proof: We can apply lemma 6 since its hypotheses are true by lemma 7. We have that:

i) there are a purely finitely additive measure $\mu^p = (\mu^{op}, \mu^{kip})_{\substack{k \leq 5 \\ i \leq I}}$ such that $g\, dP + \mu^p$ is a weak limit of $(P^N)_N$, where $P^N(E) = \int_E f^N\, dP$

ii) for almost every t in T there are sequences (δ^{jn}) and (z^{jn})

$$z^{jN} = \left(z^{jn}, z_{ki}^{jn}\right)_{\substack{k \leq 5 \\ i \leq I}} \qquad 0 \leq j \leq \ell$$

such that

$$\delta^{in} \geq 0,\ \sum_j \delta^{jn} = 1,\ \delta^i = \lim_n \delta^{jn},\ z^{jn} \in \{f^{n+m}(t); n \geq 0\} \text{ for every } n$$

iii) $g(t) = \sum_{j \leq \ell} \lim_n (\delta^{jn} . z^{jn})$.

Define the sets

$$H_1 = \{(t,(x_1,x_{ki})) \leq (t,g(t)); (g_{3i}(t) - x_{3i})(\theta - \overline{\theta}_i) \geq 0$$
$$\text{and} \quad (g_{4i}(t) - x_{4i})(x - \bar{x}_i) \geq 0 \text{ for every } (x,\theta) \geq (0, v_i)\}$$

$$H_2 = \{(t,(x_1,x_{ki}) \in T \times R^2; (x_1,x_{ki}) \in \text{con} \limsup f^N(t)\}$$

and $C = H_1 \cap H_2$.

Both sets above belongs to $\mathcal{A} \times \mathcal{B}(R^1)$ (for this see Hildenbrand and Mertens (1971) p. 153).

Claim: $P(\text{proj}_T C) = 1$.

Proof of the claim: Fix t in T and $(x',\theta) \geq (0,v_i)$. For each N and j there is an $N' = m(N,j)$ such that $z^{jN} = f^{N'}(t)$. Substituting in lemma 4(iv) and choosing y_{mN} so that $q_{mN}y_{mN} = q_{mN}(W_{imN} + \theta A_{mN})$ and dividing by $P(I_{mN})$

$$-O^{iN'}(t) \leq \tilde{\mu}^{iN'}(t)(x' - x_{iN'}) + (\Lambda_{iN}.q_{N'}.A_{N'}(t)(\theta - \theta_{iN'})$$

or

$$-O^{iN'}(t) \leq z_{4i}^{jN'}(x' - x_{iN'}) + z_{3i}^{jN'}(\theta - \theta_{iN'}) \qquad 0 \leq j \leq \ell, \quad 1 \leq i \leq I$$

Therefore multiplying by $\delta^{jN'}$ and summing for those j, $\delta^j = 0$

$$-\sum_{\delta^j=0} \delta^{jN'} O^{iN'}(t) \leq \left(\sum_{\delta^j=0} \delta^{jN'} z_{4i}^{jN'}\right)(x' - x_{iN'}) + \left(\sum_{\delta^j=0} \delta^{jN'} z_{3i}^{jN'}\right)(\theta - \theta_{iN'})$$

In the limit with $N \to \infty$ we have by lemma 7

$$0 \leq \sum_{\delta^j=0} \lim(\delta^{jN} z_{4i}^{jN})(x' - \bar{x}_i) + \left(\sum_{\delta^j=0} \lim(\delta^{jN} z_{3i}^{jN})\right)(\theta - \overline{\theta}_i)$$

Now since $\sum_{\delta^j>0} \lim(z^{jN}\delta^{jN}) = \sum_{\delta^j>0} \delta^j \lim z^{jM} \in \text{con} \limsup f^N(t)$.

It is true that $g(t) = \sum_{\delta^j>0} \lim(\delta^{jN} z^{jN}) + \sum_{\delta^j>0} \lim(\delta^{jN} z^{jN})$ is such that $b_t = (t, \sum_{\delta^j>0} \lim(\delta^{jN} z^{jN})) \in H_1$. Now it is clear that the first term above belongs to con limsup $f^N(t)$. Hence $b_t \in C$. This proves the claim.

By the measurable choice theorem (see Hildenbrand p. 54) there is a $\bar{g}: T \to R^\ell$ such that $\bar{g}(t) \in \text{com}\limsup f^N(t)$. Now from graf $\bar{g} \subset H_1$ it follows that

$$0 \leq \left(\int_T g_{3i} dP - \int_T \bar{g}_{3i} dP\right)(\theta - \overline{\theta}_i) \quad \text{for every } \theta \geq v_i \text{ and}$$

$$\left(\int_T g_{4i} dP - \int_T \bar{g}_{4i} dP\right)(x' - \bar{x}_i) \geq 0 \quad \text{for every} \quad x' \geq 0.$$

Also it is clearly true that $\int_T \bar{g}\, dP \leq \int_T g\, dP$. Finally to end the proof of (1) using theorem 4 p. 64 of Hildenbrand (1971) there is a $\bar{\bar{g}}: T \to R^\ell$ such that $\bar{\bar{g}}(t) \in \limsup f^N(t)$ for almost every t in T and $\int_T \bar{\bar{g}}\, dP = \int_T \bar{g}\, dP$. Define $\bar{f} = \bar{\bar{g}}$. Obviously $\bar{f}$ satisfy (1) since $\int_T g\, dP + \mu^p(T) = \lim \int_T f_n dP$. Now (2) is a simple consequence of (1) for the third coordinate and for the others it is by definition. To verify (3) and (4), first note that since graf $\bar{g} \subset H_1$ it suffices to prove that for any $(x', \theta) \geq (0, v_i)$

$$\lim \left(\int_T \Lambda_{iN} q_N A_N dP - \int_T g_{3i}(t) dP\right)(\theta - \overline{\theta}_i) \geq 0 \qquad \text{and}$$

$$\left(\lim \int_T \tilde{\mu}^{iN}(t) dP - \int_T g_{4i}(t) dP\right)(x' - \bar{x}_i) \geq 0.$$

To prove this first note that by lemma 5(i) there is a sequence $E_n \supset E_{n+1}$ decreasing to the empty set such that $\mu^p(T) = \mu^p(E_n)$ for every n. From lemma 4(iv) dividing by $P(I_{mN})$ where $t \in I_{mN}$

$$-O^{iN}(t) \leq \tilde{\mu}^{iN}(t)(x' - x_{iN}) + \Lambda_{iN} q_N (y_m - y_{imN}).$$

Now choose y_m such that $q_{mN} y_m = q_{mN}(W_{imN} + \theta A_{mN})$. Therefore

$$-O^{iN}(t) \leq \tilde{\mu}^{iN}(t)(x' - x_{iN}) + (\Lambda_{iN} q_N A_N)(t)(\theta - \theta_{iN}).$$

Integrating in E_n

$$-\int_{E_n} O^{iN}(t) dP \leq \int_{E_n} \tilde{\mu}^{iN}(t) dP(x' - x_{iN}) + \int_{E_n} \Lambda_{iN} q_N A_N(t) dP(\theta - \theta_{iN})$$

or

$$-\int_{E_n} O^{iN}(t)dP \leq P_{4i}^{N}(E_n)(x' - x_{iN}) + P_{3i}^{N}(E_n)(\theta - \theta_{iN}) \quad \text{for every } N$$

Now make $N \to \infty$.

$$-\sup_{N'} \int_{E_n} O^{iN'}(t)dP \leq \int_{E_n} g_{4i}(t)dP(x' - \bar{x}_i) + \int_{E_n} g_{3i}(t)dP(\theta - \overline{\theta}_i) +$$
$$+ \mu_{3i}^{p}(T)(\theta - \overline{\theta}_i) + \mu_{4i}^{p}(T)(x' - \bar{x}_i).$$

Now making n go to infinity ends the proof of (3) and (4).

Only one more lemma is needed to prove theorem 1.

Lemma 8. For almost every t in T for every $(y, x) \geq (0, 0)$

$$u_{it}(x, y) \leq u_{it}(\bar{x}_i, \bar{y}_i(t)) + \tilde{\mu}^i(t)(x - \bar{x}_i) + \overline{\Lambda}_{it}\bar{q}_t(y - \bar{y}_i(t)).$$

Proof: I first prove that $\overline{O}^i(t) \leq u_{it}(\bar{x}_i, \bar{y}_i(t))$ for almost every t. Note that from propositon 3, $\mathbf{1} = (1, \ldots, 1)$

$$\lim_{t \in I_{mN}} \frac{1}{P(I_{mN})} \int_{I_{mN}} u_{it}[\bar{x}_i + \left(\frac{1}{s}\right)\mathbf{1}, \bar{y}_i(t) + \left(\frac{1}{s}\right)\mathbf{1}]dP = u_{it}(\bar{x}_i + \left(\frac{1}{s}\right)\mathbf{1}, \bar{y}_i(t) + \left(\frac{1}{s}\right)\mathbf{1})$$

for almost every t. Now for almost every t,

$$(\tilde{\mu}^i, \overline{O}^i, \bar{y}_i)(t) \in \limsup(\tilde{\mu}^{iN}, O^{iN}, y_{iN})(t).$$

Hence there is a subsequence (one for each t) $k_n \to \infty$ such that

$$\lim(\tilde{\mu}^{iN}, O^{iN}, y_{iN})(t) = (\tilde{\mu}^i, \overline{O}^i, \bar{y}_i)(t) \text{ and } t \in I_{k_n n} \text{ for every } n.$$

Hence eventually

$$O^{iN}(t) = \frac{1}{P(I_{k_n n})} \int_{I_{k_n n}} u_{it}(x_{in}, y_{ik_n n})dP \leq$$
$$\leq \frac{1}{P(I_{k_n n})} \int_{I_{k_n n}} u_{it}(\bar{x}_i + (\frac{1}{s})\mathbf{1}, \bar{y}_{it} + (\frac{1}{s})\mathbf{1})dP$$

and therefore $\overline{O}^i(t) \leq u_{it}(\bar{x}_i + (\frac{1}{s})\mathbf{1}, \bar{y}_{it} + (\frac{1}{s})\mathbf{1})$.

Making s go to infinity $\overline{O}^i(t) \leq u_{it}(\bar{x}_i, \bar{y}_{it})$ almost every t in T. Now using for each (x, y) the proposition 3 implies it is true for almost every t that

$$\frac{1}{P(I_{k_n n})} \int_{I_{k_n n}} u_{it}(x, y) dP \to u_{it}(x, y).$$

Therefore this is true for a countable dense set D. For each $(x, y) \in D$

$$u_{it}(x, y) \leq u_{it}(\bar{x}_i, \bar{y}_i(t)) + \tilde{\mu}^i(t)(x - \bar{x}_i) + \overline{\Lambda}_{it}\bar{q}(y - \bar{y}_i(t)).$$

Now the continuity of u_{it} and the density of D proves the result.

Q.E.D.

Finally I can prove theorem 1.

Proof of theorem 1: I will prove that $(\bar{x}_i, \overline{\theta}_i, \bar{q}, \bar{p}, \bar{\pi}, \bar{y}_i)$ is an equilibrium. The only point that is not obvious is that $(\bar{x}_i, \overline{\theta}_i, \bar{y}_i)$ maximize utility. Therefore suppose $\bar{p}x + \bar{\pi}\theta \leq \bar{p}w^i$, $\theta \geq v_i$ $\bar{q}_t y_t \leq \bar{q}_t(W_t^i + \theta A_t)$

$$\int_T u_{it}(x, y_t) dP \leq \int_T u_{it}(\bar{x}_i, \bar{y}_{it}) dP + \int_T \tilde{\mu}^i(t) dP (x - \bar{x}_i) + \int_T \overline{\Lambda}_{it}\bar{q}_t A_t dP (\theta - \overline{\theta}_i).$$

Hence from theorem 3 (3) and (4)

$$U^i(x, y) \leq U^i(\bar{x}_i, \bar{y}_t) + (x - \bar{x}_i) \lim \int_T \tilde{\mu}^{iN}(t) dP + (\theta - \overline{\theta}_i) \lim \int_T \Lambda_{iN} q_N A_n dP.$$

Using proposition 3(ii) and lemma 4(iv) we have

$$U^i(x, y) - U^i(\bar{x}_i, \bar{y}_i) \leq \bar{\mu}_i \bar{p}(x - \bar{x}_i) + \left(\lim \int_T \Lambda_{iN} q_N A_N dP - \bar{\mu}_i \bar{\pi}\right)(\theta - \overline{\theta}_i) +$$

$$+ \bar{\mu}_i \bar{\pi}(\theta - \overline{\theta}_i) \leq \bar{\mu}_1(\bar{p}(x - \bar{x}_i) + \bar{\pi}(\theta - \overline{\theta}_i)) \leq 0$$

and this ends the proof.

Q.E.D.

V. Related works

In this section I discuss the relationship between this paper and Mas-Colell and Monteiro (1990), Hellwig (1991) and Mas-Colell and Zame (1991).

In my joint work with Mas-Colell (1990) we prove the incomplete markets existence theorem for a general state space. We suppose the utility function to be separable. This is basic to our proof (and the other's as well). The special tractability of separable utilities for equilibrium existence was explored by Araujo and Monteiro (1989) where a complete markets equilibrium existence theorem is proved with assumption of properness at only one point. Besides separability, it is supposed that for any asset trade, almost every non-atomic state has a regular spot equilibrium. This condition allows we to construct self-fulfilled conditional on any asset trades price expectations with a strong continuity property. After this the argument runs along usual routes.

The next work I discuss is Mas-Colell and Zame (1991). They don't need the above regularity hypothesis, however they need strong restrictions on the utilities which precludes for example Inada's type of conditions. Their methodology is based on recursivity: they solve the equilibrium at each state at the terminal date as a function of earlier choices. This enables them to construct a finite-dimensional range correspondence which is convex by Lyapunov's theorem. Then an induction process allows them to prove the theorem for the multiple period case.

The last work I discuss is Hellwig's. In his work he does not make any special assumptions on the utilities or endowments (besides separability or strictly positivity respectively). The initial steps are as in Mas-Colell and Monteiro (1990). The maximum instantaneous utility is parametrized (for fixed prices) by the asset trades. Since he doesn't make the regularity assumption as in Mas-Colell and Monteiro he cannot parametrize the prices in the same fashion. Here Fatou's lemma is essential to guarantee some convexity. However several problems appear and in their solution the need for a modification of Hildenbrand-Mertens Fatou's lemma proof appears. In the final steps he approximates the economy by means of atomic measures, then a key closedness property ensures the limit to be an equilibrium. The only discomfortable hypothesis Hellwig does is the continuity in the euclidean topology of the asset returns.

In this work in terms of hypotheses I, essentially, eliminate the above continuity assumption. Also the approximation that I use is in terms of partitions. I also need to combine Hellwig's adaptation of Hildenbrand and Mertens Fatou's lemma proof and their proof since the sequence I have is a combination of both.

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