

“Adverse Selection Without the Single Crossing Condition”

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Adverse Selection Problems without the Single Crossing Property

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We relax the single crossing property (SCP) for the adverse selection problem and new conditions for the incentive compatibility are derived: the marginal rate of substitution identities. The set of implementable and the optimal contracts are characterized. The SCP is no longer valid and the characterization of the solution depends on this new conditions. A new type of equilibrium appears: the discrete pooling equilibrium.

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1. INTRODUCTION

In the one-dimensional parameter case, the SCP is by definition the monotonicity of the marginal rate of substitution between the decision taken by the agent and the money transfer given by the principal with respect to the parameter (the asymmetric information).

The SCP permits the second order approach for the problem. In this case, the problem is a concave one: in the presence of positive (respectively negative) SCP, a decision path is implementable if and only if it is non-decreasing (respectively non-increasing) in the parameter.

The SCP enables also a full characterization of the optimal solution: in the presence of positive (respectively negative) SCP, if the optimal decision solution is increasing (respectively decreasing) in the parameter, then it should be equal to the relaxed solution.¹ Moreover, a maximal interval where it is constant is such that the marginal welfare of the principal should be zero. These properties are sufficient to provide an algorithm allowing the computation of optimal solutions (see Guesnerie and Laffont (1984)).

However, the lack of SCP allows us to find examples where it is not satisfied. Therefore, it is natural to study a more general characterization of the adverse selection problem. The question is: what does it happen when the SCP is not valid anymore? In this case, there are at least two regions in the plane of the parameter versus the decision variable: the positive and the negative single crossing regions. An implementable decision path should preserve the monotonicity property in each region and it can cross or not the curve that separates the two regions (the frontier). If the decision path crosses the frontier, what is the condition for feasibility? First,

¹ This solution is obtained by imposing the first order condition of incentive compatibility constraint to reduce the problem only to the decision variable, i.e., we eliminate the money transfer variable.

it is necessary that the decision path crosses the frontier in a U -shaped form (or an inverted U -shaped form) because of the monotonicity condition. Besides, a necessary condition is: if two types have the same decision, then their marginal rate of substitution should be the same. In economic terms, if two types are pooling in a given contract, then the principal guarantees truth telling only if the marginal rate of substitution of the two types is the same. We will call this condition the *marginal rate of substitution identity*. Moreover, there exist analogous marginal condition with respect to the type. We can interpret this as a duality between the decision and the type.

We use the second order condition of the incentive compatibility (IC) constraint and the marginal rate of substitution identities as the constraints of the adverse selection problem and derive the first order conditions for the optimal contract. The constraints will be equality and inequality type ones and our problem is not concave anymore. However, we can compute the optimal contract.

Chassagnon and Chiappori (1995) study the insurance market competitive equilibrium with adverse selection and moral hazard such that the SCP is not valid. However, they study the two type case.

The following example shows the importance of the marginal rate of substitution identity:

Example. (Labor Market)

Suppose that an employer (the principal) has to hire a worker (the agent). However, the worker is more or less productive depending on his type (unknown for the employer) and the employer has to design the salary schedule.

Let x be the units of output, θ be the worker's productivity, y be the worker's effort and t be the reward (salary). We assume that $x = \theta y$. The employer is risk neutral with utility

function given by:

$$U(x, t, \theta) = x - t.$$

The set of types is $\Theta = [\frac{3}{2}, 3]$ and the distribution of the worker's types is represented by a distribution whose density function is $p(\theta) = 3\theta^{-2}$ on Θ . The worker knows his type but the employer just knows the distribution of the types. The employer only observes the output (the worker's productivity and effort are unknown to him).

The worker's utility function is:

$$V(y, t, \theta) = t - (\theta - 1)y^2.$$

The type θ worker has the utility of the reward minus the disutility of the effort. The main property of this utility function is that the marginal disutility of effort is an increasing function of the type. This will produce a trade off between productivity and disutility of effort which implies no SCP. In the literature, this marginal disutility is assumed to be a constant function of the type.

For simplicity, we will do the following change of parameter: $\eta = \frac{1}{\theta}$. An easy computation gives the new distribution in terms of η : it is uniform on $[\frac{1}{3}, \frac{2}{3}]$. Therefore, the worker's utility function in (x, t, η) is:

$$V(x, t, \eta) = (\eta^2 - \eta)x^2 + t.$$

It is not difficult to show that x is an implementable decision if and only if x is U -shaped symmetric with respect to $\frac{1}{2}$ (see section 3 for the details). Therefore, the optimal output and reward are going to be U -shaped with respect to the parameter. The intuition is that two workers that choose the same contract should have the same marginal rate of substitution between output and reward. While the marginal rate of substitution is increasing (respectively decreasing) with respect to the productivity, the optimal contract is decreasing

(respectively increasing) in the productivity. The optimal solution is given by

$$x^*(\eta) = \frac{1}{-6\eta^2 + 6\eta - 1}.$$

The figure 1 shows the optimal output in terms of the parameter θ .

(Fig. 1)

The optimal salary as a function of x is:

$$t^*(x) = \frac{1}{6}x^2 + \frac{1}{3}x - \frac{5}{6}$$

The first-best problem (the problem with symmetric information) has the following solution:

$$x_{FB}(\eta) = \frac{1}{2\eta(1-\eta)}$$

and

$$t_{FB}(x) = \frac{1}{2}x.$$

Observe that the first best output is implementable but not by t_{FB} . We can easily check that the second best output is less than the first best one, except at $\frac{1}{2}$ where they have the same value.

The distortions with respect to the first best solution are the following: the second best contract is less than the first best and they coincide only at the intermediate type (no distortion for this type); the rent of the intermediate type is equal to zero and every other type has non zero rent. The equilibrium can be thought as a discrete pooling equilibrium, i.e., given that the worker chooses a certain level of output and reward, the

employer knows that the worker might be one of the two types: a low or a high type, but the employer does not know if he is contracting the more or the less productive type.

The existence of an optimal contract is provided. Page (1991) has an existence result for the case where the contracts are lotteries. However, we show the existence of an optimal deterministic contract. Also, Athey (1997) shows the existence of a pure strategy in games with incomplete information under some generalized single crossing condition. The strategies can be monotone or have “limited complexity” form (i.e., they have a finite number of peaks). These properties have a straightforward relationship with our case.

The paper is organized as follows. In section 2 we present the adverse selection model. In section 3 we relax the SCP. Section 4 presents an example. Section 5 gives the extensions and the final conclusions.

2. THE ADVERSE SELECTION MODEL

The relationship between the principal and the agent(s) involves only two types of variables: The first type is associated with a decision (or action) variable, denoted by x which is observable. The variable of the second type, denoted by t has generally the meaning of money transfer from the principal to the agent.

The principal and the agent interact through these two variables and the asymmetry of information can be described as follows: there is an one-dimensional parameter θ which is known to the agent but unobservable to the principal. This parameter belongs to some compact interval $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$. The principal has some *a priori* probability distribution on Θ which is associated to a continuous density $p: \Theta \rightarrow \mathbb{R}^{++}$. We can interpret this function as the principal’s subjective assessment of the probability of θ when there is only one agent or the objective distribution of their types when there are many agents.

The principal's utility function is $U: I \times \mathbb{R} \times \Theta \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, $U(x, t, \theta) = u(x, \theta) - ct$ (quasi-linearity) where $c = 1$ or $c = 0$ (a benevolent principal that does not care about the money transfer) and u is C^3 . The agent's utility function is $V: I \times \mathbb{R} \times \Theta \rightarrow \mathbb{R}$ such that $V(x, t, \theta) = v(x, \theta) + t$ and v is C^3 .

A mechanism (contract or allocation) is a pair of functions $(x, t): \Theta \rightarrow \mathbb{R}^2$. A mechanism can be viewed as a procedure giving the decision to the principal who commits himself to a decision rule relating the choice of x and t to messages sent by the agent. By the revelation principle², any mechanism can be mimicked by a direct truthful one in the sense that there is no loss of welfare to the principal.

A decision function $x: \Theta \rightarrow \mathbb{R}$ is implementable if there exists a money transfer function $t: \Theta \rightarrow \mathbb{R}$ such that the allocation $x(\theta) \in I$, for all $\theta \in \Theta$ and satisfies the incentive compatibility constraint

$$V(x(\theta), t(\theta), \theta) \geq V(x(\hat{\theta}), t(\hat{\theta}), \theta), \quad \text{for all } (\theta, \hat{\theta}) \in \Theta^2. \quad (\text{IC})$$

We will say that the allocation (x, t) is implementable or that x implements t . In other words, given an implementable allocation (x, t) , the announcement of the truth is an optimal strategy for the agent whatever the truth.³

We say that an allocation (x, t) satisfies the individual-rationality constraint if

$$V(x(\theta), t(\theta), \theta) \geq 0, \quad \text{for all } \theta \in \Theta. \quad (\text{IR})$$

An implementable allocation that satisfies the IR constraint is called feasible. We assume that the agent's reser-

² See Fudenberg and Tirole (1991).

³ (x, t) is sometimes called the truth telling mechanism.

vation utility is independent of his type⁴ and, without loss of generality, we normalize it as zero.

The principal's (or the adverse selection) problem is to choose a feasible allocation with the highest expected payoff, i.e., the principal maximizes his expected utility subject to the agent's IR and IC constraints:

$$\max_{x,t} \int_{\underline{\theta}}^{\bar{\theta}} U(x(\theta), t(\theta), \theta) p(\theta) d\theta$$

s.t.

$$V(x(\theta), t(\theta), \theta) \geq V(x(\hat{\theta}), t(\hat{\theta}), \theta), \quad \forall (\theta, \hat{\theta}) \in \Theta^2 \quad (\text{IC})$$

$$V(x(\theta), t(\theta), \theta) \geq 0, \quad \forall \theta \in \Theta \quad (\text{IR})$$

$$x(\theta) \in I, \quad \forall \theta \in \Theta.$$

Consider the space \mathcal{C} of all càlàg contracts, i.e., the space of all $x: \Theta \rightarrow \mathbb{R}$ right continuous and such that $\lim_{\tilde{\theta} \rightarrow \theta, \tilde{\theta} < \theta} x(\tilde{\theta})$ exists for each $\theta \in \Theta$ (and, in this case, it will be denoted by $x_-(\theta)$), with the topology of pointwise limit at every continuous parameter of the limit decision function.

(i) Let x be a bounded decision such that the set of its discontinuity points has zero Lebesgue measure. If t implements x , then the agent's value (or rent) function is given by

$$\mathcal{V}(\theta) = v(x(\theta), \theta) + t(\theta) = \mathcal{V}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_{\theta}(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}, \quad \forall \theta \in \Theta.$$

(ii) If x is a bounded càdlàg implementable decision, then x is non-decreasing on the region where $v_{x\theta} > 0$ (respectively non-increasing on the region where $v_{x\theta} < 0$).

⁴ We do this for simplicity. However, we can consider the case where the agent's reservation utility depends on the type. See Maggi and Rodriguez-Clare (1995), for instance.

Proof. See the appendix.

Lemma 2.1 shows that for each implementable càdlàg x , there exist a unique càdlàg money transfer that implements x defined by⁵

$$t(\theta) = \mathcal{V}(\theta) - v(x(\theta), \theta), \quad \forall \theta \in \Theta.$$

Then, we define

$$\begin{aligned} \Phi_x(\theta, \hat{\theta}) &= V(x(\theta), t(\theta), \theta) - V(x(\hat{\theta}), t(\hat{\theta}), \theta) \\ &= \int_{\theta}^{\hat{\theta}} \left[\int_{x(\bar{\theta})}^{x(\hat{\theta})} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} \end{aligned}$$

and, after an integration by parts,

$$\begin{aligned} f(x(\theta), \theta) &= u(x(\theta), \theta) - ct(\theta) \\ &= \left(u(x(\theta), \theta) + cv(x(\theta), \theta) \right. \\ &\quad \left. - c \frac{(1 - P(\theta))}{p(\theta)} v_{\theta}(x(\theta), \theta) \right) p(\theta) \end{aligned}$$

where $P(\theta) = \int_{\underline{\theta}}^{\theta} p(\tilde{\theta}) d\tilde{\theta}$ is the cumulative distribution.

The principal's optimization program becomes

$$\begin{aligned} \max_{x \in \mathcal{C}} \int_{\underline{\theta}}^{\bar{\theta}} f(x(\theta), \theta) d\theta \\ \text{s.t. } \Phi_x(\theta, \hat{\theta}) \geq 0, \quad \forall \theta, \hat{\theta} \in \Theta \end{aligned} \tag{P}$$

⁵ $t(\underline{\theta})$ is chosen such that the IR constraint is satisfied and it is binding at least for one type.

If we ignore the IC constraint, then the problem is called the relaxed problem and also its solution (first order approach). The first order condition of the relaxed problem is given by⁶

$$f_x(x(\theta), \theta) = 0, \quad \text{for all } \theta \in \Theta$$

when $x(\theta)$ is in the interior of I .

It is well known in the literature of adverse selection problems that sufficient conditions for implementation is the constant sign of the partial derivative of the marginal rate of substitution with respect to the parameter:

$$\partial_\theta \left(\frac{V_x}{V_t} \right) = v_{x\theta} > 0 \quad \text{on } I \times \Theta, \quad (\text{CS}_+)$$

or

$$\partial_\theta \left(\frac{V_x}{V_t} \right) = v_{x\theta} < 0 \quad \text{on } I \times \Theta. \quad (\text{CS}_-)$$

This is known as the single crossing property (SCP) or sorting condition.⁷

In the presence of CS_+ (respectively CS_-), it is easy to show (see the proof of lemma 2.1) that if a càdlàg decision is non-decreasing (respectively non-increasing), then it is implementable. Therefore, the adverse selection problem is equivalent to

$$\max_{x \in \mathcal{C}} \int_{\underline{\theta}}^{\bar{\theta}} f(x(\theta), \theta) d\theta$$

s.t. x is non-decreasing (respectively non-increasing)

⁶ The sub index in the function represents the partial derivative of the function with respect to that sub index. Also, the superior order derivative will be represented in a multi-index notation.

⁷ This property implies that the indifference curves of two different types cross only one time.

This problem is known as the second order approach because, under the SCP, the monotonicity of the decision is equivalent to the local second order condition of the IC constraint. Using the Hamiltonian approach, as in Guesnerie and Laffont (1984), one can obtain a full characterization of the solution.

3. RELAXING THE SINGLE CROSSING ASSUMPTION

We make the following assumption:

- A1.** $v_{x\theta}(x, \theta) = 0$ defines x as a function of θ on Θ named x_0 ; $v_{x^2\theta} < 0$ and $v_{x\theta^2} > 0$ on $I \times \Theta$.

By the Implicit Function Theorem, x_0 is C^1 and increasing.⁸ Moreover, if $x < x_0(\theta)$, $v_{x\theta}(x, \theta) > 0$ and if $x > x_0(\theta)$, $v_{x\theta}(x, \theta) < 0$, for all $\theta \in \Theta$. Therefore, the assumption A1 generalizes the SCP, because $\Theta \times I$ is separated into two parts: above (respectively below) x_0 , $v_{x\theta} > (<) 0$ on $I \times \Theta$.

There are three other possibilities: $v_{x^2\theta} > 0$ and $v_{x\theta^2} < 0$, with x_0 increasing, but reverting the regions where $v_{x\theta} > 0$ and $v_{x\theta} < 0$; $v_{x^2\theta} < 0$ and $v_{x\theta^2} < 0$; $v_{x^2\theta} > 0$ and $v_{x\theta^2} > 0$, for the respective cases where x_0 is decreasing.

(Fig. 2)

We can relax the second part of A1: instead of assuming that x_0 is increasing, we could say that x_0 has a finite number of peaks. However, the analysis would be more difficult without any substantial gain in the results.

The next lemma shows when we can extend the definition of an implementable càdlàg to the associated correspondence. In what follows, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

⁸ $\dot{x}_0(\theta) = -\frac{v_{x\theta^2}(x_0(\theta), \theta)}{v_{x^2\theta}(x_0(\theta), \theta)}$.

Let x be an implementable càdlàg decision and define $X(\theta) = [x_-(\theta) \wedge x(\theta), x_-(\theta) \vee x(\theta)]$, for all $\theta \in \Theta$, the induced correspondence.

(i) If $\theta, \hat{\theta} \in \Theta$, $\theta \leq \hat{\theta}$, then $\Phi_X(\theta, \hat{\theta}) \geq 0$, i.e.,

$$\int_{\theta}^{\hat{\theta}} \left[\int_{x(\tilde{\theta})}^y v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} \geq 0, \quad \forall y \in X(\hat{\theta}).$$

(ii) x is in the closure of the set of all continuous implementable decision if and only if X is implementable.

Proof: See the appendix.

Observe that x is discontinuous where its inverse x^{-1} is constant and vice-versa. This represents a kind of duality between the variables x and θ .

If X is implementable, then x crosses x_0 in continuous way one time at most. In this case, x should be non-increasing or non-decreasing or U -shaped. From now on, we will consider only the decision $x \in \mathcal{C}$ such that the associated X is implementable. We do this for the following reasons:

(i) economic meaning: if x is discontinuous at $\theta \in \Theta$, then it might not matter how x is defined in θ between $x_-(\theta)$ and $x(\theta)$.

(ii) tractability: if we weak this assumption, there are implementable decisions that cross x_0 many times.

(iii) when the SCP is valid, this space is the same of the implementable decisions.

The next theorems will give the necessary and sufficient conditions for the feasibility. First, we say that x is right increasing at $\theta \in \Theta$ if $x(\theta) < x(\theta + \epsilon)$, for every sufficiently small $\epsilon > 0$. Analogously, we define left increasing and right and left decreasing.

Assume A1. If x is an implementable càdlàg decision, then

(a) If x is right (left) increasing at $\hat{\theta}$ and $\Phi_x(\theta, \hat{\theta}) = 0$, then

$$v_x(x(\hat{\theta}), \hat{\theta}) \underset{(\leq)}{\geq} v_x(x(\hat{\theta}), \theta).$$

and reverting the inequalities when x is right (left) decreasing.

(b) If $\Phi_x(\theta, \hat{\theta}) = 0$, then

$$v_\theta(x(\hat{\theta}), \theta) \leq v_\theta(x(\theta), \theta) \text{ and } v_\theta(x(\hat{\theta}), \theta) \geq v_\theta(x_-(\theta), \theta)$$

and with equality when x is continuous at θ and $\theta > \underline{\theta}$.

(c) If x is right or left increasing at $\hat{\theta}$ and $y \in X(\theta) \cap X(\hat{\theta})$, then

$$v_x(y, \hat{\theta}) = v_x(y, \theta).$$

Moreover, x is continuous at $\hat{\theta}$ (i.e., $y = x(\hat{\theta}) = X(\hat{\theta})$).

Proof: See the appendix.

Remark 1. The item (b) can be interpreted as a dual condition when we interchange θ by x , i.e., instead of looking the direct decision (x as a function of θ), we look the inverse function (θ as a function of x). Also, we can use the duality between x and θ and $v_{x\theta^2} > 0$ to show that $\Phi_x(\cdot, \hat{\theta}) \geq 0$ need to be checked only in extremities of an interval where x is constant.

Item (c) implies that if $v_{x\theta}(x(\hat{\theta}), \hat{\theta}) > 0$, then x is continuous at $\hat{\theta}$. Otherwise, $y(\theta)$ defined implicitly by the equality in (c) would be increasing in CS_- for a fixed $\hat{\theta} \in \Theta$ where x is discontinuous (see also the proof of theorem 3.3).

Remark 2. We have the following economic interpretation for lemma 2.1 (ii) and theorem 3.2 (c): In order to provide truth telling, the principal should offer a contract that

(1) is non-decreasing (respectively non-increasing) in θ if the marginal rate of substitution is decreasing (respectively increasing) in θ .

(2) if two agents (θ and $\hat{\theta}$) choose the same contract and the agents cannot locally misrepresent their types, then the principal should equalize the marginal rate of substitution of the two agents ($MRS^\theta = MRS^{\hat{\theta}}$).

(Fig. 3)

Remark 3. Every implementable decision x should not decrease (respectively increase) on CS_+ (respectively on CS_-). This property is a direct consequence of the local second order condition of the IC constraint.

If x hits the curve x_0 , then it should cross x_0 in a constant way or preserving the marginal utility for the types that choose the same level of x .⁹ This last condition is new and, when the SCP is not valid, it can play an important role in order to characterize the optimal solution of the adverse selection problem as the example of section 4 will show. We will call this condition **the marginal rate of substitution identity**.¹⁰

Observe that, if (x, t) is feasible and $x(\theta) = x(\hat{\theta})$, then $t(\theta) = t(\hat{\theta})$. Thus, if two types are pooling in a feasible contract, then they should have the same marginal rate of substitution or a continuum of types between them should also pool.

⁹ If x was identical to x_0 in an interval, then the IC constraint would not hold locally.

¹⁰ Observe that the marginal rate of substitution identity is equivalent to v_x be constant on every level set of a feasible decision x .

Remark 4. Theorem 3.2 is also valid for a more general agent's utility functions. This is why we are calling our condition the marginal rate of substitution identity and not marginal utility identity. See section 5 for more details.

It is important to note that we are dealing with a non-concave problem because the set of feasible decision for the agent is not a convex set when the agent's utility function does not satisfy the SCP. (The SCP guarantees concavity: the IC constraint is substituted by its second order condition.)

A natural question is: Are the conditions above sufficient for the characterization of an implementable decision? Theorem 3.3 gives a partial answer of this question.

Assume A1. Let x be a bounded càdlàg decision that satisfies the necessary condition of lemma 2.1 (ii) and theorem 3.2 (c). If $x(\underline{\theta}) \geq x(\bar{\theta})$, then x is implementable.

Proof: See the appendix.

The Existence and The Optimality Conditions

We will investigate the necessary conditions for optimality. First, we will characterize the relaxed solution. Let x_1 be the relaxed solution for (P). By the Maximum Theorem, x_1 is a continuous function of θ . Let us assume that:

A2. x_1 has a finite number of peaks on Θ ; if $x < x_1(\theta)$, $f_x(x, \theta) > 0$ and if $x > x_1(\theta)$, $f_x(x, \theta) < 0$ for all $\theta \in \Theta$.

The assumption A2 is a standard one and a sufficient condition for the first part of A2 (besides the concavity of u and v) is $v_{x^2\theta} > 0$ or $c = 0$. In this case, if $x_1(\theta)$ belongs to the interior of I , then $f_x(x_1(\theta), \theta) = 0$.

Under A2, the principle of optimality for the adverse selection problem is to find an implementable decision as close as possible to x_1 . The finite number of peaks of x_1 is to provide

an analytical treatment of the problem and it is well known in the literature (see Guesnerie and Laffont (1984)).

A natural question is the existence of an optimal contract. Page (1991) provides a general result for the existence of an optimal contract in the case where the contracts are lotteries. In the case of deterministic contracts, Athey (1997) gives the existence of a pure strategy equilibrium for games with incomplete information under a generalized single crossing property or a limited complexity condition (i.e., the strategies have a finite number of peaks as a function of the parameter). In our case, we have the following:

Assume that A1 and A2 hold. Then, there exist a solution of (P) in the set of all decision $x \in \mathcal{C}$ such that the associated correspondence X is implementable.

Proof: See the appendix.

The next theorem gives the characterization of the optimal decision in a special case. Let $x^* \in \mathcal{C}$ be an optimal decision for (P).

Assume that A1 and A2 hold. If $x^(\underline{\theta}) \geq x^*(\bar{\theta})$, let θ_0 be the minimum parameter for x^* and $\theta_1 \leq \theta_0$ such that $x^*(\bar{\theta}) \in X^*(\theta_1)$. Then*

(a) *If x^* is right and left decreasing at $\hat{\theta}$, then*

$$f_x(x, \hat{\theta}) + \delta \left(f_x(x, \theta) + \frac{f_\theta(x, \theta)}{v_{x\theta}(x, \theta)} (v_{xx}(x, \hat{\theta}) - v_{xx}(x, \theta)) \right) = 0$$

where $x = x^(\hat{\theta})$ and if $\hat{\theta} < \theta_1$, then $\delta = 0$ and if $\theta_1 \leq \hat{\theta} \leq \theta_0$, then $\delta = 1$, $v_x(x, \hat{\theta}) = v_x(x, \theta)$ and $x = x^*(\theta)$.*

(b) *If $[a, b] \subset [\underline{\theta}, \theta_0]$ is a maximal interval where x^* is constant, then*

$$\int_a^b \left[f_x(x, \hat{\theta}) + \delta \left(f_x(x, \theta) + \frac{f_\theta(x, \theta)}{v_{x\theta}(x, \theta)} (v_{xx}(x, \hat{\theta}) - v_{xx}(x, \theta)) \right) \right] d\hat{\theta} = 0$$

where $x = x^*(\hat{\theta})$ and if $b < \theta_1$, then $\delta = 0$ and if $\theta_1 \leq a \leq \theta_0$, then $\delta = 1$ and, in the integral, x and θ are functions of $\hat{\theta}$ implicitly defined by $v_x(x, \hat{\theta}) = v_x(x, \theta)$ and $x = x^*(\theta)$.

Proof: See the appendix.

Remark 1. In theorem 3.5 (a), if we interchange θ for $\hat{\theta}$ when $\hat{\theta} \in [\theta_1, \theta_0]$, then, subtracting the two equations, we have that

$$\frac{f_{\theta}(x, \hat{\theta})}{v_{x\theta}(x, \hat{\theta})} + \frac{f_{\theta}(x, \theta)}{v_{x\theta}(x, \theta)} = 0.$$

Remark 2. The part (b) is analogous to the principle of optimality when a continuous pooling occurs like in Guesnerie and Laffont (1984). However, in our case this set may be disconnected.

Under the same assumptions of theorem 3.5, if x^* crosses x_0 at θ_0 , then $f_x(x^*(\theta_0), \theta_0) = \frac{v_{x2\theta}(x^*(\theta_0), \theta_0)}{v_{x\theta^2}(x^*(\theta_0), \theta_0)}$.

Let us consider a particular case of assumption A2.

If the relaxed solution x_1 is non-increasing, then the optimal solution x^* for (P) satisfies $x^*(\underline{\theta}) \geq x^*(\bar{\theta})$.

Proof: See the appendix.

The new feature of the solution that appears in theorem 3.5 is the possibility of discrete pooling, i.e., in the optimal contract some isolated types can choose the same level of the contract. In the literature there exist just two types of equilibria: separating and continuous pooling equilibrium. In the first the agent's type is known *ex-post* by the principal and in the second the principal knows a range of types where the agent is. When the SCP is not valid we can have discrete pooling

equilibria besides separating and continuous pooling. In this case the principal does not know the true type between two types or between two ranges of types. Therefore, the optimal solution can have these three characteristics: separating and continuous or discrete pooling.

Under SCP, the pooling interval of the optimal contract is characterized by the marginal welfare of the principal to be zero along this interval. Theorem 3.5 shows that this property is no longer valid when there exists discrete pooling.

Since the rent of a type θ is

$$\mathcal{V}(\theta) = \mathcal{V}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v_{\theta}(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta},$$

then θ has zero rent if and only if it is a minimum of \mathcal{V} . In this case, $v_{\theta}(x(\theta), \theta) \geq 0$ and $v_{\theta}(x_{-}(\theta), \theta) \leq 0$. If v_{θ} is non-negative everywhere, it is possible that the low type has a positive rent. In the case of example of the introduction, only the intermediate type has zero rent.

4. EXAMPLES

Example 1. (Comparative Statics)¹¹

$$\Theta = [0, 1]; \quad p \equiv 1 \text{ (uniform distribution); } c = 0$$

$$u(x, \theta) = (1 - \theta)x - \frac{x^2}{2}; \quad v(x, \theta) = (k\theta^2 + 1)x - \theta \frac{x^2}{2},$$

$$x, \theta \in [0, 1]; \quad k > 0.$$

The curve x_0 is $x_0(\theta) = 2k\theta$. The relaxed solution x_1 is given by $x_1(\theta) = 1 - \theta$.

¹¹ We are assuming that the types are observable but not verifiable.

Suppose that the optimal solution x^* crosses the curve x_0 on a maximal interval $[\theta_1, \theta_2]$ and $\theta_0 \in [\theta_1, \theta_2]$ the minimal point of x .

Applying theorem 3.5 and 3.6, we have

$$\begin{cases} x^*(\theta) = 1 - \theta - \mu(\theta)\theta \\ \varphi(\theta) = \frac{x(\theta)}{k} - \theta. \end{cases} \quad (S)$$

where $\mu(\theta) = \frac{1 - \sqrt{1 - 4\theta(1 - \theta - 2k\theta)}}{2\theta}$ and $x^*(\theta) = x^*(\varphi(\theta))$ for all $\theta \in [\theta_1, \theta_2]$.

We have the following possibilities:

- (1) $k > 1$: The solution is constant equals to $\frac{1}{2}$.
- (2) $0 < k \leq 1$: The solution satisfies the system (S) on an interval and it is identical to the relaxed solution on the rest of Θ (see theorem 3.5).

Observe that if $k \rightarrow 0$, $x^* \rightarrow x_1$ and if $k \rightarrow \infty$, $x^* \rightarrow \frac{1}{2}$. The limit situations correspond to CS_+ and CS_- , respectively, and the limit solutions correspond to the expected solutions in each case.

Example 2. (Labor Market - Continuation)

The example in the introduction could be solved using the single crossing approach: just doing another change of parameter identifying η and $1 - \eta$. Thus, that example is trivial, although it illustrates our method. In order to transform the problem in a non-trivial one, we are going to make an perturbation in the preferences of the workers.

Let us assume the same framework of the example in the introduction, except that the worker's utility function now is:

$$V(y, t, \theta) = t - (\theta - 1)y^2 - \frac{h}{3}\theta^2 y^3.$$

where $h > 0$ is a disturbance parameter.

Doing the change of parameter $\eta = \frac{1}{\theta}$, the worker's utility function in (x, t, η) is:

$$V(x, t, \eta) = (\eta^2 - \eta)x^2 - \frac{h}{3}\eta x^3 + t.$$

The curve x_0 is given by

$$x_0(\eta) = \frac{4}{3h}(\eta - \frac{1}{2})$$

The relaxed solution x_1 is the implicit solution of

$$x_1(\eta) = \frac{9\eta^2 - 10\eta + 2 + ((9\eta^2 - 10\eta + 2)^2 + 18h(3\eta - 1))^{1/2}}{6h(3\eta - 1)}$$

and it is not a feasible output contract. It is easy to show that if $h > 0$ is sufficiently small, $x_1(\frac{1}{3}) \geq x_1(\frac{2}{3})$.

Using theorem 3.5 and 3.6,¹² it is not difficult to obtain the second best output: for each η , $x(\eta)$ is the implicit solution of

$$1 + (6\eta^2 - 6\eta + 1)x + (-\frac{9}{2}\eta + \frac{11}{2})hx^2 - \frac{3}{2}h^2x^3 = 0$$

for $\eta \in [\eta_1, \frac{2}{3}]$, where $x(\eta_1) = x(\frac{2}{3})$ and x is identical to x_1 on $[\frac{1}{3}, \eta_1]$.

5. CONCLUSIONS AND EXTENSIONS

In this paper we studied a general characterization of optimal solutions for the adverse selection problems when the set of parameter is an interval of \mathbb{R} . We gave a generalization of the SCP. The characterization of the IC constraint depends

¹² Observe that the relaxed solution is not decreasing. However, we can show that the same kind of conclusion is valid in this case.

on a new condition: the marginal rate of substitution identity. When the SCP does not hold a new type of equilibrium appears: the discrete pooling equilibrium. Two examples illustrated the discrete pooling equilibrium.

Some extensions about the cases that were not covered follow:

(1) More general agent utility function: Assume that the agent's utility function can be any C^2 function with the same assumptions of section 2 except the quasi-linearity one. Given a feasible contract (x, t) continuously differentiable piecewise denoted $V(x(\hat{\theta}), t(\hat{\theta}), \theta)$ by $V(\hat{\theta}, \theta)$, for simplicity. Then

$$V(\theta, \theta) = \int_{\underline{\theta}}^{\theta} V_{\theta}(\tilde{\theta}, \tilde{\theta}) d\tilde{\theta} + V(\underline{\theta}, \underline{\theta}) \text{ and}$$

$$V(\hat{\theta}, \theta) = \int_{\theta}^{\hat{\theta}} [V_x(\tilde{\theta}, \theta) \dot{x}(\tilde{\theta}) + V_t(\tilde{\theta}, \theta) \dot{t}(\tilde{\theta})] d\tilde{\theta} + V(\theta, \theta).$$

By the first order condition of (IC),

$$\dot{t}(\theta) = -\frac{V_x(\theta, \theta)}{V_t(\theta, \theta)} \dot{x}(\theta).$$

Then

$$V(\theta, \theta) - V(\hat{\theta}, \theta) = \int_{\hat{\theta}}^{\theta} \left[V_x(\tilde{\theta}, \theta) - V_t(\tilde{\theta}, \theta) \frac{V_x(\tilde{\theta}, \tilde{\theta})}{V_t(\tilde{\theta}, \tilde{\theta})} \right] \dot{x}(\tilde{\theta}) d\tilde{\theta}.$$

If $\theta, \hat{\theta} \in \Theta$ is such that $V(\theta, \theta) = V(\hat{\theta}, \theta)$, then the derivative of the above expression with respect to $\hat{\theta}$ should be zero, i.e.,

$$\dot{x}(\hat{\theta}) \left(V_x(\hat{\theta}, \theta) - V_t(\hat{\theta}, \theta) \frac{V_x(\hat{\theta}, \hat{\theta})}{V_t(\hat{\theta}, \hat{\theta})} \right) = 0.$$

If $\dot{x}(\hat{\theta}) \neq 0$, then

$$-\frac{V_x(\hat{\theta}, \theta)}{V_t(\hat{\theta}, \theta)} = -\frac{V_x(\hat{\theta}, \hat{\theta})}{V_t(\hat{\theta}, \hat{\theta})}.$$

This condition means that if the agent θ is indifferent between his bundle $(x(\theta), t(\theta))$ and the agent $\hat{\theta}$ bundle $(x(\hat{\theta}), t(\hat{\theta}))$ and the agent $\hat{\theta}$ can not locally misrepresent his type ($\dot{x}(\hat{\theta}) \neq 0$), then agents θ and $\hat{\theta}$ should have the same marginal rate of substitution at the bundle $(x(\hat{\theta}), t(\hat{\theta}))$.

(2) Multidimensional decision variable case: Assume that x is n -dimensional vector and θ is one-dimensional. A similar argument as above shows that

$$\dot{x}(\hat{\theta}) \cdot \left(-\frac{V_x(\hat{\theta}, \theta)}{V_t(\hat{\theta}, \theta)} \right) = \dot{x}(\hat{\theta}) \cdot \left(-\frac{V_x(\hat{\theta}, \hat{\theta})}{V_t(\hat{\theta}, \hat{\theta})} \right)$$

for all $\theta, \hat{\theta} \in \Theta$ such that $V(\theta, \theta) = V(\hat{\theta}, \theta)$. However, observe that the dot in the above equation is the inner product of the respective vectors. The interpretation is that if the type θ is indifferent between his bundle and the type $\hat{\theta}$ bundle, then the covariation of the marginal increasing of type $\hat{\theta}$ in x with his marginal rate of substitution at the bundle $(x(\hat{\theta}), t(\hat{\theta}))$ should be equal to the covariation of the same marginal increasing with the marginal rate of substitution of type θ at the same bundle.

Observe also that the second order condition of the IC constraint is

$$\frac{d}{d\theta} \left(\frac{V_x(\theta, \theta)}{V_t(\theta, \theta)} \right) \cdot \dot{x}(\theta) \geq 0, \forall \theta \in \Theta.$$

This two conditions characterize the (IC) constraint.

(3) Discontinuous crossing when x_0 is constant: If we permit discontinuous crossing, the solution can be improved by a

discontinuous contract. The example is the following¹³

$$\Theta = [0, 1], \quad p \equiv 1, \quad c = 0$$

$$u(x, \theta) = (1 - \theta)x - \frac{x^2}{2}$$

$$v(x, \theta) = \frac{\theta}{2}(x - x^2)$$

where we are considering quasi-linear utility functions for the principal and for the agent.

The relaxed solution is $x_1(\theta) = 1 - \theta$ and $x_0(\theta) = 1/2$, for all $\theta \in \Theta$. If we only admit continuous crossing, the optimal solution will be $x^*(\theta) = 1/2$. However, if we allows us to discontinuous crossing, then the optimal solution will be

$$x^*(\theta) = \begin{cases} \frac{3}{4} & \text{if } 0 \leq \theta < \frac{1}{2} \\ \frac{1}{4} & \text{if } \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

APPENDIX

Proof of Lemma 2.1: The IC constraint implies that for $\theta > \hat{\theta}$

$$\frac{v(x(\theta), \theta) - v(x(\theta), \hat{\theta})}{\theta - \hat{\theta}} \geq \frac{\mathcal{V}(\theta) - \mathcal{V}(\hat{\theta})}{\theta - \hat{\theta}} \geq \frac{v(x(\hat{\theta}), \theta) - v(x(\hat{\theta}), \hat{\theta})}{\theta - \hat{\theta}}$$

Since v is C^3 and x is bounded, the inequality above shows that \mathcal{V} is a Lipschitz function. Moreover, if x is continuous at θ , then

$$\frac{d}{d\theta} \mathcal{V}(\theta) = v_\theta(x(\theta), \theta).$$

By the Fundamental Theorem of Calculus, we get (i). \square

¹³ The parameter is observable but not verifiable.

(ii) From (i), $t(\theta) = \mathcal{V}(\theta) - v(x(\theta), \theta)$, for all $\theta, \hat{\theta} \in \Theta$. Thus, it is easy to see that

$$V(x(\theta), t(\theta), \theta) - V(x(\hat{\theta}), t(\hat{\theta}), \theta) = \int_{\hat{\theta}}^{\theta} \left[\int_{x(\hat{\theta})}^{x(\tilde{\theta})} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta}$$

for all $\theta, \hat{\theta} \in \Theta$.

Let $\hat{\theta}_0 \in [\underline{\theta}, \bar{\theta})$ such that $v_{x\theta}(x(\hat{\theta}_0), \hat{\theta}_0) > 0$. By the right continuity, x restrict to a small interval $I = [\hat{\theta}_0, \theta_0)$ has its graphic on CS_+ . Let $A = (a, b)$ be a maximal interval in I such that $x(\hat{\theta}_0) > x(\theta)$, for all $\theta \in A$. If $a = \hat{\theta}_0$, then the double integral above will be negative when $\hat{\theta} = a$ and $\theta = b$. If $a > \hat{\theta}_0$, then the left limit of the double integral will be also negative when $\hat{\theta}_0 = a$ and $\theta = b$ (since $x_-(a) \geq x(\hat{\theta}_0)$). In both cases we have a contradiction with the (IC) constraint. Therefore, $x(\hat{\theta}) \leq x(\theta)$, for all $\theta \in I$. \square

Proof of Lemma 3.1:

(i) Define the function

$$\varphi(y) = \int_{\theta}^{\hat{\theta}} \left[\int_{x(\tilde{\theta})}^y v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta}$$

for $y \in X(\hat{\theta})$. Taking the second derivative, we have

$$\varphi''(y) = v_{xx}(y, \hat{\theta}) - v_{xx}(y, \theta).$$

Thus, $\hat{\theta} \geq \theta$ and $v_{x^2\theta} < 0$ implies that $\varphi''(y) \leq 0$, for all $y \in X(\hat{\theta})$. Since $\varphi(x_-(\hat{\theta})) \geq 0$ and $\varphi(x(\hat{\theta})) \geq 0$, we have that $\varphi(y) \geq 0$, for all $y \in I_{\hat{\theta}}$.

(ii) In this case, x can cross from CS_- to CS_+ just one time at most. Thus, lemma 2.1 (ii) implies that x is non-increasing or non-decreasing or U -shaped. Observe that if $y \in$

$X(\theta) \cap X(\hat{\theta})$, with $\theta, \hat{\theta} \in \Theta$, then

$$\begin{aligned}\Phi_X(\hat{\theta}, \theta) &= \int_{\hat{\theta}}^{\theta} \left[\int_{x(\tilde{\theta})}^y v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} \\ &= - \int_{\theta}^{\hat{\theta}} \left[\int_{x(\tilde{\theta})}^y v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} = -\Phi_X(\theta, \hat{\theta}).\end{aligned}$$

From (i), this means that both integrals above are zero, i.e., $\Phi_X(\theta, \hat{\theta}) = 0$.

Assume that $\hat{\theta} < \theta$. From what we have just seen above, the case $x_-(\hat{\theta}) \leq x(\theta)$ follows from monotonicity. Therefore, we only have to check the case $x_-(\hat{\theta}) \geq x(\theta)$. In this case, using Fubini's Theorem

$$\varphi(y) = \int_y^{x_-(\hat{\theta})} \left[\int_{\hat{\theta}}^{\theta} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{\theta} \right] d\tilde{x}.$$

Therefore, it is easy to see that the signal of this double integral is preserved if we considered a sequence of decisions converging to x at every continuous parameter of x . \square

Proof of Theorem 3.2:

(a) Define a local inverse for x at $\hat{\theta}$. Applying Fubini's Theorem (as we did in lemma 3.1) and taking right (left) derivative at $x(\hat{\theta})$ and observing that $x(\hat{\theta})$ is a minimum point for the function defined, we get our result.

(b) Observe that if we fix $\hat{\theta}$, θ is a minimum point of $\Phi_x(\hat{\theta}, \cdot)$. Then, the result is a direct consequence of the first order conditions.

(c) From the proof of lemma 3.1 (ii),

$$\int_{\theta}^{\hat{\theta}} \left[\int_{x(\tilde{\theta})}^y v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} = 0, \quad \forall y \in X(\theta) \cap X(\hat{\theta}).$$

If x is right and left increasing at $\hat{\theta}$, applying (a), we get the result. If x is just right (or left) increasing at $\hat{\theta}$, use the continuity of v_x and the previous case to conclude the proof. \square

Proof of Theorem 3.3:

Let $\theta_0 \in \Theta$ be the minimum point of x and $\theta_1 \in \Theta$ such that $x(\bar{\theta}) \in X(\theta_1)$. Consider the following cases for $\hat{\theta}, \theta \in \Theta$:

(1) $\theta_1 \leq \theta \leq \theta_0 \leq \hat{\theta} \leq \bar{\theta}$ such that $x(\hat{\theta}) \in X(\theta)$.

Using Fubini's Theorem,

$$\Phi_x(\theta, \hat{\theta}) = \int_{x_0}^{x(\hat{\theta})} \left[\int_{\varphi_1(\tilde{x})}^{\varphi_2(\tilde{x})} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{\theta} \right] d\tilde{x}$$

where $x_0 = x(\theta_0)$ and φ_1, φ_2 are the two inverses of x on $[x_0, x(\underline{\theta})]$, where $\varphi_2(x) = \bar{\theta}$, for all $x \in [x(\bar{\theta}), x(\underline{\theta})]$.

From theorem 3.2 (c), $v_x(\tilde{x}, \varphi_1(\tilde{x})) = v_x(\tilde{x}, \varphi_2(\tilde{x}))$, for all $\tilde{x} \in [x_0, x(\hat{\theta})]$. Thus, $\Phi_x(\theta, \hat{\theta}) = 0$.

(2) $\theta \leq \hat{\theta}$

If $x(\theta) < x(\hat{\theta})$, then $\hat{\theta} \leq \theta_0$ or $\varphi_1(\hat{\theta}) \leq \theta \leq \theta_0 \leq \hat{\theta}$. In the first case,

$$\Phi_x(\theta, \hat{\theta}) = - \int_{\theta}^{\hat{\theta}} \left[\int_{x(\hat{\theta})}^{x(\bar{\theta})} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta},$$

and since the region delimited in the integral is negative, we have that $\Phi_x(\theta, \hat{\theta}) \geq 0$.

In the second case, using Fubini's Theorem again and (1) above

$$\begin{aligned} \Phi_x(\theta, \hat{\theta}) &= \int_{x(\theta)}^{x(\hat{\theta})} \left[\int_{\theta}^{\varphi_2(\tilde{x})} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{\theta} \right] d\tilde{x} \\ &= \int_{x(\theta)}^{x(\hat{\theta})} (v_x(\tilde{x}, \varphi_2(\tilde{x})) - v_x(\tilde{x}, \theta)) d\tilde{x}. \end{aligned}$$

since the function $v_x(\tilde{x}, \cdot)$ is U -shaped and $\varphi_1(\tilde{x}) \leq \theta$, $v_x(\tilde{x}, \varphi_2(\tilde{x})) \geq v_x(\tilde{x}, \theta)$, for all $\tilde{x} \in [x(\theta), x(\hat{\theta})]$. Thus, $\Phi_x(\theta, \hat{\theta}) \geq 0$.

If $x(\theta) \geq x(\hat{\theta})$, then $\hat{\theta} \leq \theta_0$ or $\theta \leq \varphi_1(\hat{\theta}) \leq \theta_0 \leq \hat{\theta}$. With analogous proof above, $\Phi_x(\theta, \hat{\theta}) \geq 0$.

(3) $\theta > \hat{\theta}$

If $x(\theta) > x(\hat{\theta})$, then $\theta_0 \leq \hat{\theta} < \theta$ or $\theta_1 \leq \hat{\theta} \leq \theta_0 < \theta$. The proof is analogous to the case (2). If $x(\theta) < x(\hat{\theta})$, then, using Fubini's Theorem and (1)

$$\begin{aligned}\Phi_x(\theta, \hat{\theta}) &= - \int_{x(\theta)}^{x(\hat{\theta})} \left[\int_{\varphi_1(\tilde{x})}^{\theta} v_{x\theta}(\tilde{x}, \tilde{\theta}) d\tilde{\theta} \right] d\tilde{x} \\ &= - \int_{x(\theta)}^{x(\hat{\theta})} (v_x(\tilde{x}, \theta)) - v_x(\tilde{x}, \varphi_1(\tilde{x})) d\tilde{x}.\end{aligned}$$

If $\tilde{\varphi}_1$ is identical to φ_1 on $[x_0, x(\bar{\theta})]$ and $v_x(\tilde{x}, \varphi_1(\tilde{x})) = v_x(\tilde{x}, \varphi_2(\tilde{x}))$, for all $\tilde{x} \in [x(\bar{\theta}), x(\underline{\theta})]$, then $\varphi_1(\tilde{x}) \leq \tilde{\varphi}_1(\tilde{x})$, for all $\tilde{x} \in [x_0, x(\underline{\theta})]$. Since $v_x(\tilde{x}, \cdot)$ is U -shaped, $v_x(\tilde{x}, \varphi_1(\tilde{x})) \geq v_x(\tilde{x}, \tilde{\varphi}_1(\tilde{x})) \geq v_x(\tilde{x}, \theta)$. Thus, $\Phi_x(\theta, \hat{\theta}) \geq 0$. \square

Proof of Theorem 3.4: Consider the topology of the pointwise convergence at the continuous points of the limit, i.e., x_n converges to x if and only if $x_n(\theta) \rightarrow x(\theta)$ for every $\theta \in \Theta$ where x is continuous. It is well known that every bounded and closed set in \mathcal{C} is compact with respect to this topology.

Let \underline{x} and \bar{x} be an inferior and superior bound for x_1 . It is easy to see that if $x \in \mathcal{C}$ is implementable, then y defined by $y(\theta) = \underline{x} \vee x(\theta) \wedge \bar{x}$, for all $\theta \in \Theta$ is implementable and the principal weakly prefers y than x .

It is easy to see that if x_n is a sequence of implementable decisions converging to x such that the associated sequence X_n is also implementable, then the associated X is implementable. In particular, if x crosses x_0 , it should cross in a continuous

way. Therefore, the space of implementable decision such that the associated correspondence is also implementable is closed.

Finally, the objective function of (P) is continuous with respect to the considered topology. Then, by Weierstrass Theorem, there exist an optimal contract for (P) as described in the theorem. \square

Proof of Theorem 3.5: From theorem 3.2 and 3.3, we can rewrite (P) as

$$\max_x \int_{\underline{\theta}}^{\theta_1} f(x(\bar{\theta}), \bar{\theta}) d\bar{\theta} + \int_{\theta_1}^{\theta_0} \left[f(x(\bar{\theta}), \bar{\theta}) + f(x(\bar{\theta}), \varphi(x(\bar{\theta}), \bar{\theta})) \right] d\bar{\theta}$$

where $x: [\underline{\theta}, \theta_0] \rightarrow \mathbb{R}$ is non-increasing and φ is implicitly defined by $v_x(x, \hat{\theta}) = v_x(x, \varphi)$ as a function of x and $\hat{\theta}$, for $\hat{\theta} \in [\theta_1, \theta_0]$. From the Implicit Function Theorem

$$\frac{\partial \varphi}{\partial x}(x, \hat{\theta}) = \frac{v_{xx}(x, \hat{\theta}) - v_{xx}(\theta, \varphi(x, \hat{\theta}))}{v_{x\theta}(x, \varphi(x, \hat{\theta}))}$$

The non-decreasing condition can be identified as $dx \leq 0$ (in a distributional sense). It is easy to see that the interiority condition for the existence of a Lagrange multiplier is satisfied. Then using the same approach of Guesnerie and Laffont (1984), we have our result. \square

Proof of Theorem 3.6: From theorems 3.3 - 3.5, we only have to show that for every implementable càdlàg x , there exist an implementable càdlàg y that the principal prefers and $y(\underline{\theta}) \geq y(\bar{\theta})$. We have the following cases for x :

- (i) if $x(\underline{\theta}) \geq x(\bar{\theta})$, then define $y = x$.
- (ii) if $x(\underline{\theta}) < x(\bar{\theta})$, then x is non-decreasing or x is U -shaped. If x is non-decreasing, then define y as the constant equals to the value of the intersection between x and x_1 or the minimal or maximal value of x_1 depending on whether x is

below or above x_1 respectively. If x is U -shaped and x crosses x_1 , define $y(\theta) = x(\theta) \wedge x(\underline{\theta})$ when the value of the intersection is less than $x(\underline{\theta})$ or define y constant equal to the value of the intersection in other case. Otherwise, define y constant equal to the minimum value of x_1 . \square

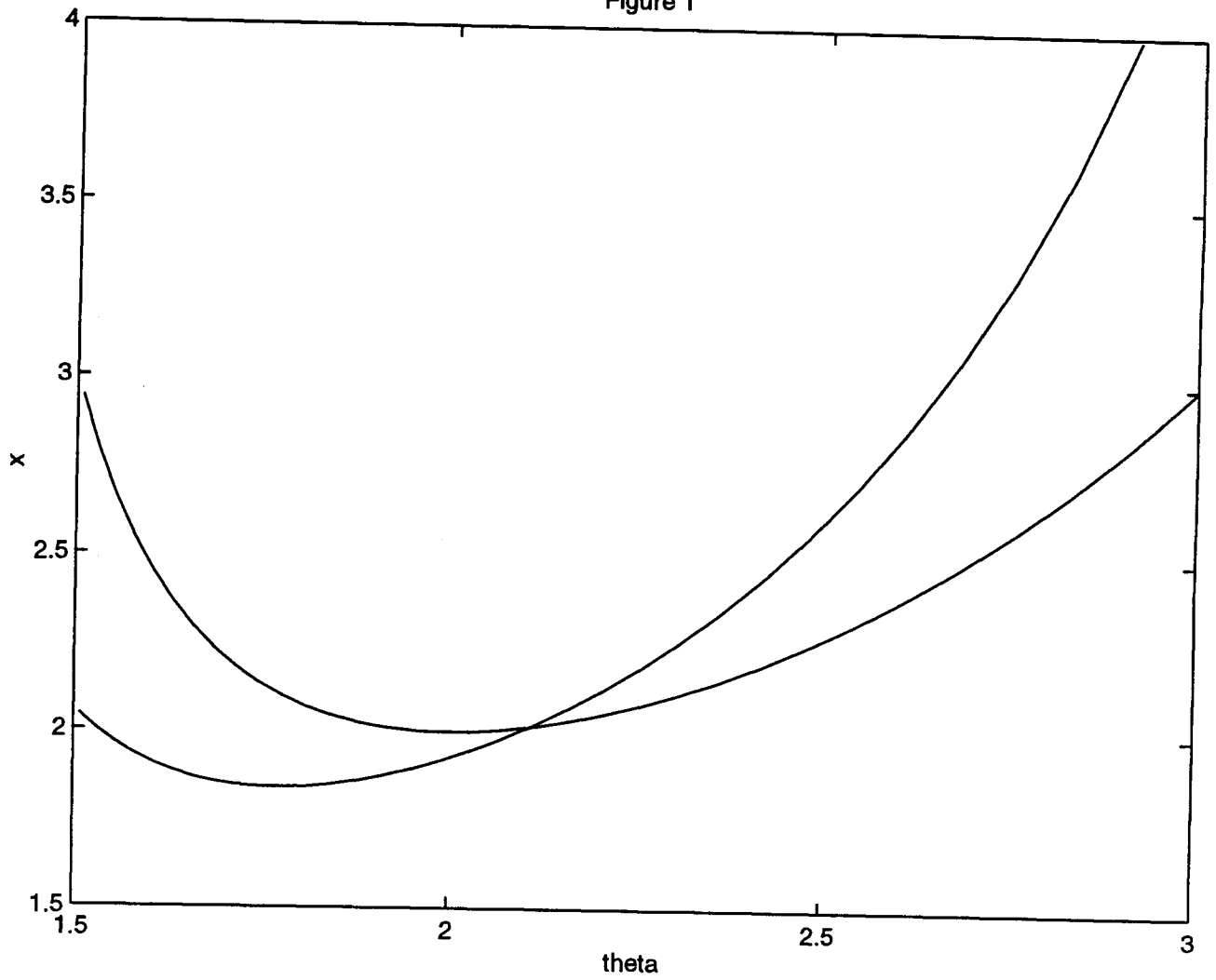
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Figure 1



BIBLIOTECA

ESTE VOLUME DEVE SER DEVOLVIDO À BIBLIOTECA
NA ÚLTIMA DATA MARCADA

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Título: Adverse selection without the single crossing



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