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"GENERAL EQUILIBRIUM MODEL WITH RESTRICTED PARTICIPATION IN FINANCIAL MARKETS"

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A General Equilibrium Model with Restricted Participation in Financial Markets*

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Abstract

The paper analyses a general equilibrium model with financial markets in which households may face restrictions in trading financial assets such as borrowing constraints and collateral (restricted participation model). However, markets are *not* assumed to be incomplete. We consider a standard general equilibrium model with $H > 1$ households, 2 periods and S states of nature in the second period. We show that generically the set of equilibrium allocations is indeterminate, provided the existence of at least one nominal asset and one household for who some restriction is binding. Suppose there are $C > 1$ commodities in each state of nature and assets pays in units of some commodity. In this case for each household with binding restrictions it is possible to reduce the set of feasible assets trading and obtain a new equilibrium that *utility improve* all those households. There is however an upper bound on the number of households to be improved related to the number of states of nature and the number of commodities. In particular, if the number of households is smaller than the number of states of nature *it is possible to Pareto improve any equilibrium by reducing the feasible choice set for each household.*

*Preliminary. Comments welcome.

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1. Introduction

The general equilibrium model with incomplete markets (GEI) has been developed in the last twenty years following the contributions of Arrow (1964), Radner (1972) and Hart (1975). The equilibrium set in the (smooth) GEI model presents several differences with respect to the equilibrium set in the standard (smooth) Walrasian model. First, the real asset GEI model may not have an equilibrium under standard convexity and continuity assumptions. The remarkable example is due to Hart (1975). Duffie and Shaffer (1985), however, show that equilibrium does exist for a generic choice of endowments and asset payoffs.

Second, the equilibrium set in a GEI model with nominal assets generically contains at least a continuum of distinct equilibrium allocations.[1] The original example was provided by Cass (1984), and it was generalized by Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989).

Third, Geanakoplos and Polemarchakis (1986) show that the equilibrium allocation is constraint sub-optimal in the GEI model with numéraire assets for a weakly generic set of economies.[2] They consider an exchange model with two periods, asset markets in the first period and consumption only in the second period. In this set up, constraint sub-optimality means that the following property holds. Fix an equilibrium associated with an economy in the weakly generic set. Consider a reallocation of financial assets holding that satisfy each household first period budget constraint. Given each consumer new assets holdings, consider the vector of commodity spot prices that clear the second period markets. We refer to this reallocation of financial holdings and market clearing prices as a *pseudo-equilibrium.* An equilibrium is constraint sub-optimal if it is Pareto dominated by a pseudo-equilibrium. Geanakoplos and Polemarchakis (1986) result shows that in equilibrium the existent asset structure is not efficiently used in a GEI model.

An even stronger sub-optimality property of equilibrium was proved by Cass and Citanna (1994) and Elhu (1994). As in Geanakoplos and Polemarchakis (1986), the property holds for a weakly generic set of economies. For every equilibrium there is a financial innovation whose introduction makes every household strictly worse off. In particular, increasing the set of financial assets available for every household may result in an equilibrium that Pareto impairs the economy.

[1] Generically means for an open and full measure set of endowments.

[2] We say that a property holds for a weakly generic set of economies if it holds for an open and dense of endowments and utility functions.

The relevance of those results is not widely accepted. The inexistence of complete financial markets in GEI models is frequently attributed to imperfections such as moral hazard.[3] However, such imperfections are usually not embedded in the model specification. The financial structure and, in particular, the fact that markets are incomplete, is taken as a primitive in GEI models. It is not clear that the equilibrium properties (indeterminacy and sub-optimality) will continue to be true in a GEI model with endogenous asset structure. Moreover, it is possible that in a model with endogenous financial assets, markets will be complete.

Given the lack of theoretical justification for the assumption of incomplete markets, one could try to justify this assumption based on empirical grounds. However, it is not empirically obvious that financial markets are incomplete (or for what matters, dynamically incomplete). The fact that financial assets contingent upon certain states of nature are not observed in reality may not mean that markets are incomplete. It may simple mean that even though markets are complete, at equilibrium there is no trade in some financial assets.[4] Therefore, one can object to the assumption that in reality households are constraint to transact assets that do not provide complete insurance against uncertainty.

The lack of empirical evidence in favor of the incomplete markets assumption does not generalize to other types of restrictions one may face in financial markets. Casual empiricism shows that households have restrictions on the amount of available credit (borrowing constraints), and that in some cases those credits require a collateral. These are the most common example of household's restrictions in trading financial assets, but certainly not the unique.

The existence of restrictions in financial markets can be incorporated in a general equilibrium model as a natural generalization of the GEI model. Cass, Siconolfi and Villanacci (1992) referred to this generalization as the *restricted participation model.* In this model financial markets may be complete, but each household is only allowed to buy or sell financial assets that belong to a certain convex set, which may be household specific. In the particular case the convex set is large enough, the household faces no constraints and the model reduces to the standard Walrasian model. The interesting problem, however, seems to be to analyze the equilibrium properties of the model when some household constraint is biding.

The paper basic motivation is to investigate the following question. Do the equilibrium properties of the GEI model generalize to the restricted participation model if instead of missing markets we assume that some household constraint

[3]For a survey, see Geanakoplos (1990).

[4]More intuitively, the price that induces someone to sell some insurance may be too high compare to the price the remaining households are willing to pay.

is biding? We show that under certain conditions the equilibrium set is indeterminate, provided there is at least one nominal asset. The notion of constraint sub-optimality has a natural generalization to the restricted participation model. In this case, we say that an equilibrium is constraint sub-optimal if it is possible to reduce the set of feasible financial trading for each household and Pareto improve the economy. We show that for a weakly generic set of economies every equilibrium is constraint sub-optimal. More precisely, the following results are obtained. Consider a GEI model with $H > 1$ households, 2 periods, I financial assets and $S \geq I$ states of nature in the second period.

a) Suppose assets pay in units of accounts (nominal asset model). Then:

ai) for a generic set of endowments, the set of equilibrium allocations contains a $(S - I + k)$ –dimensional manifold, where k is the maximum number of binding restrictions faced by a household;

aii) for a weakly generic set of economies, either no household is restricted or the set of equilibrium allocations contains a S–dimensional manifold.[5]

b) Let $C > 1$ be the number of commodities in every state of nature. Suppose assets pay in units of a single commodity (numéraire asset model). Then:

bi) for a weakly generic set of economies, it is possible to *reduce* the set of feasible financial transactions and utility improve every restricted household up to $S + 1$ households. Symmetrically it is possible to enlarge the set of feasible financial transactions and utility impairs every restricted household up to $S + 1$ households.

bii) The upper bound in the last result can be weakened in the following way: it is possible to reduce the set of feasible financial transactions and utility improve every restricted household up to $H^* \leq (C-1)(S+1)$ households if the total number of households is $H > H^*$. Symmetrically it is possible to enlarge the set of feasible financial transactions and utility impairs every restricted household up to $H^* \leq (C-1)(S+1)$ households, provided that $H > H^*$.

The restricted participation model has been developed in the last years as an extension of the GEI model. The financial structure, and in particular the restrictions faced by the households, is taken as a primitive and not as an endogenous variable. Siconolfi (1989) proves existence for a large class of restrictions. In the particular case households only face linear restrictions, the restricted participation model is equivalent to a GEI model. Balasko, Cass and Siconolfi (1990) exploits this equivalence and show that in the case several households face similar linear restrictions the indeterminacy result of Balasko and Cass (1989) generalizes

[5] An economy is parameterized by each household's utility function, initial endowments and restrictions on financial markets.

to this model. Finally, Cass, Siconolfi and Villanacci (1992) show how the standard smooth analysis can be extended to the restricted participation model. The major difficulty in using smooth analysis in this model relies on the possibility of having corner solutions at equilibrium. If at equilibrium some household constraint is binding and the corresponding Kuhn Tucker multiplier is equal to zero then the solution to the consumer problem is not differentiable. Cass, Siconolfi and Villanacci (1992) show that for a generic set of economies at equilibrium this problem does not appear and the consumer demand is differentiable.

The model presented in this paper slightly differs from the one used by Siconolfi (1989) and Cass, Siconolfi and Villanacci (1992). In order to make the paper as self contained as possible, we have included an alternative proof of Siconolfi's existence theorem, whose proof relies on a fairly sophisticated application of the standard fixed point argument. Our proof, simpler than Siconolfi's, uses a degree argument for continuous functions, which generalizes the proof of existence of equilibrium in the Walrasian model provided by Balasko (1975) and Smale (1974). We also included in the appendix a proof of the Cass, Siconolfi and Villanacci (1992).

2. The Basic Model and The Space of Economies

In this section we present the basic model and assumptions that will be used throughout this paper. In the next section we give a proof of existence of equilibrium in the restricted participation model that easily generalizes to utility functions and restricted functions satisfying weaker assumptions.[6] In fact, in order to prove indeterminacy and sub-optimality of equilibrium we use differential techniques that require differential assumptions on the primitives of the model. As it is well known, such assumptions are not necessary in proving existence of equilibrium in most variations of the standard Walrasian model.

Consider a general equilibrium model with 2 periods, S states of nature in the second period and C commodities in the first period and in each state of nature in the second period. As standard, we index the first period as $s = 0$, and each state of nature in the second period as $s = 1, \ldots, S$. There are $H > 1$ households and I financial assets in this economy. Each household is completely characterized by a pair (e_h, u_h), where $e_h \in \Re^C_{++}$ and $u_h : \Re^C_{++} \rightarrow \Re \in C^2$ satisfies the usual smoothness assumptions:

$(H1)\ \forall\, y \in \Re^C_{++} : i)\ Du_h(y) \gg 0;\ ii)\ d^T Du_h(y) = 0,\ d \neq 0 \ \Rightarrow\ d^T D^2 u_h(y) d < 0;$

[6] The generalization of the existence result is provided in the appendix.

$$iii)\ cl\left\{x \in \Re^C_{++} /\ u_h(x) \geq u_h(y)\right\} \subset \Re^C_{++}$$

The asset payoffs are described by the matrix

$$R = \begin{bmatrix} -q \\ Y \end{bmatrix} \in \Re^{(S+1)\times I}, \quad Y = [y_{s,i}]^{i=1,\ldots,I}_{s=1,\ldots,S}$$

where q represents the vector of asset prices, and $y_{s,i}$ describes asset i's payoff in state s. We assume all assets pay in units of the first commodity in each state of nature (numéraire asset model).

$$(H2)\ Y\ has\ full\ column\ rank$$

Notice that in particular we allow $S = I$. Contrary to the standard GEI model, the existence of redundant assets may alters the consumption set of some household, given the restrictions in trading financial assets to be described below. Therefore, assumption $(H2)$ is *not* without loss of generality. Each household faces $J_h \leq I$ additional restrictions when demanding financial assets. These restrictions are described as follows. For each h, there is $a_h : \Re^I \longrightarrow \Re^{J_h} \in C^2$. Each consumer assets' demand is required to satisfy $a_h(b_h) \geq 0$, where

$$(H3)\ a_h(0) \geq 0\ \ and\ a^j_h\ is\ quasi\text{-}concave\ for\ all\ h\ and\ j.$$

The most natural example of restriction in trading financial assets is the existence borrowing constraints

$$b_h \geq -k, \quad where\ k \geq 0$$

In this simple case

$$a_h\left(b_h\right) = b_h + k$$

which clearly satisfies $(H3)$. It is simple to provide several alternative examples.

The assumption that the restriction function is quasi-concave implies that the household feasible set of asset portfolios is convex. This assumption, together with $(H1)$ and $(H4)$, permits to characterize the solution of the consumer problem using the Kuhn-Tucker conditions. However, a household may face in reality non-convex restrictions. An alternative approach, used by Smale (1974), is to drop the convexity requirement and still use the Kuhn-Tucker conditions, which is this case are necessary, but not sufficient, to characterize the solution of the consumer problem. One does not know if the solution to the problem exists, but if it does then it must satisfy those conditions, and hence the equilibrium properties implied by then.

Let be given $p^s \in \Re^C_{++}$ for each s and $q \in \Re^I$.[7] Each household solves the problem

$$\max u_h(x_h) \ s.t.$$

$$\begin{aligned} p^0\left(x_h^0 - e_h^0\right) &= -qb_h \\ p^s\left(x_h^s - e_h^s\right) &= p^{s1}y^s b_h \quad s > 0 \\ b_h \in B\left(a_h\right) := \Big\{b \in \Re^I / \ a_h\left(b_h\right) &\geq 0\Big\} \end{aligned}$$

Dividing each budget constraint in the second period by p^{s1} if necessary, we can restrict the analysis to commodity prices $p^s \in \Re^C_{++}$ such that $p^{s1} = 1$ for every s. Let

$$\Psi = \begin{bmatrix} p^0 & & 0 \\ & \ddots & \\ 0 & & p^S \end{bmatrix} \in \Re^{(S+1)\times(S+1)C}_{++}$$

We can then write the household h as follows

$$\max u_h(x_h) \ s.t.$$

$$\Psi\left(x_h - e_h\right) = Rb_h$$

$$b_h \in B\left(a_h\right) := \Big\{b \in \Re^I / \ a_h\left(b_h\right) \geq 0\Big\}$$

For every h and b_h we define the set

$$J'_h := \Big\{j = 1, \ldots, J_h / \ a^j_h(b_h) = 0\Big\}$$

We make the following final assumption:

$$(H4) \ If \ \sum_h b_h = 0 \ then \ \left[Da^j_h(b_h)\right]^{j\in J'_h} \ has \ full \ rank \ for \ every \ h$$

Let $J' := \{(j, h) / \ j \in J'_h.\}$

Notice that under $(H1) - (H4)$ an equilibrium for each household in the restricted participation model is completely characterized by the Kuhn-Tucker conditions

$$\begin{aligned} Du_h\left(x_h\right) - \lambda_h\Psi &= 0 \\ \lambda_h R + \mu_h Da_h\left(b_h\right) &= 0 \\ -\Psi\left(x_h - e_h\right) + Rb_h &= 0 \\ \min\Big\{a^j_h\left(b_h\right), \mu^j_h\Big\} &= 0 \ \ for \ all \ j \end{aligned}$$

[7]For purposes of calculations, we treat prices and Lagrange multipliers as row vectors.

where λ_h^i denotes household h Lagrangian associated with b_h^i, and μ_h^j denotes household h Kuhn-Tucker multiplier associated with constraint $a_h^j(b_h) \geq 0$.

Fix commodity prices p and asset prices q. Let each household primitives be given by (e_h, u_h, a_h) and let the payoff matrix be given by Y. Suppose $(x_h, b_h, \lambda_h, \mu_h)$ satisfies the Kuhn-Tucker conditions for each household h, and both commodity and asset market clear

$$\sum_h (x_h - e_h) = 0 \textit{ and } \sum b_h = 0$$

Then we say $((x_h, b_h, \lambda_h, \mu_h)_h, p, q)$ is a *competitive equilibrium* for the economy $((e_h, u_h, a_h)_h, Y)$.

In the next sections we shall consider perturbations of the restriction functions as well as perturbations of utility functions. The set of restriction functions is given as follows

$$\mathcal{A}_h := \left\{a_h : \Re^I \to \Re^{J_h} \in C^2 / \ a_h \ satisfies \ \ (H3)\right\}$$

The set of utility functions is given as follows

$$\mathcal{U}_h := \left\{u_h : \Re^G_{++} \to \Re \in C^2 / \ u_h \ satisfies \ (H1)\right\}$$

We endowed the set $\mathcal{A}_h$ with the C^2 topology of convergence in compact sets, which is a metrizable topology. See appendix for further details on this topology. Let

$$\mathcal{A} := \left\{(a_h) \in \times_{h=1}^H \mathcal{A}_h / \ (H4) \ is \ satisfied\right\}$$

be endowed with the product topology. For each h, the set $\mathcal{U}_h$ also be endowed with the C^2 topology of convergence in compact sets and

$$\mathcal{U} = \times_{h=1}^H \mathcal{U}_h$$

be endowed with the product topology as well.

In the following sections we consider alternative spaces of economies. In section 4 we fix a profile of utility functions $(u_h') \in \mathcal{U}$, a profile of restriction functions $(a_h') \in \mathcal{A}$ and a payoff matrix, Y. An economy is completely characterized by the specification of a distribution of initial endowments and the space of economies is identified with E, the space of conceivable initial endowments.

Let

$$\mathcal{Y} = \left\{Y \in \Re^{(S+1)\times I} / \ Y \ has \ full \ column \ rank\right\}$$

In the second result we present in section 4, we fix a profile of utility and restrictions functions and consider perturbations of the initial endowments and the payoff matrix. In this case the space of economies is identified with $E \times \mathcal{Y}$.

In the end of section 4 and in section 5 we consider perturbations of the restricted functions and eventually perturbations of utility functions as well. In such cases we do not perturb the payoff matrices and the space of economies is identified with $E \times \mathcal{A} \times \mathcal{U}$, endowed with the product topology.

Let $G = (S+1)C$, $J = \sum_h J_h$ and

$$\Xi = \left(\Re^G_{++} \times \Re^{S+1}_{++} \times \Re^J \times \Re^J\right)^H \times \Re^G_{++} \times \Re^I$$

be the set of endogenous variables. Let $n = \dim \Xi$. An element $\xi \in \Xi$ is also written as

$$\xi = \left((x_h, \lambda_h, b_h, \mu_h)_{h=1}^H, p, q\right)$$

If $z \in \Re^G$, then $z^{\backslash} \in \Re^{G-(S+1)}$ refers to the vector of all components of z but the first commodity in each state of nature

$$z^{\backslash} = \left(z^{0\backslash}, z^{1\backslash}, \ldots, z^{S\backslash}\right) \ \ where \ \ z^{s\backslash} = \left(z^{s2}, \ldots, z^{sC}\right) \ \ for\ all\ s$$

Consider the function $F : \Xi \times \mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y} \to \Re^n$ given by:

$$F(\xi, u, a, e, Y) = \begin{bmatrix} \vdots \\ Du_h(x_h) - \lambda_h \Psi \\ \lambda_h R + \mu_h Da_h(b_h) \\ -\Psi z_h + Rb_h \\ \vdots \\ \min\left\{\mu_h^j, a_h^j(b_h)\right\} \\ \vdots \\ \sum_h z_h^{\backslash} \\ \sum b_h \\ p^{11} - 1 \\ \vdots \\ p^{S1} - 1 \\ \| p^0, q \| - 1 \end{bmatrix}$$

where $\| \cdot \|$ is the Euclidean norm. We refer to the function $F(\cdot)$ as the *extended system of equations.* Notice that under conditions $(H1) - (H4)$ a competitive

equilibrium is completely characterized by the solutions to the system of equations $F(\xi, u, a, e, Y) = 0$, where the price of the first commodity in each state of nature in the second period has been normalized. The *equilibrium set* is then given by

$$M = \{(\xi, u, a, e, Y) \in \Xi \times \mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y} / \ F(\xi, u, a, e, Y) = 0\}$$

3. Existence and Regularity of Equilibrium

In this section we prove the existence of equilibrium for every economy satisfying the assumptions $(H1)-(H4)$. In the appendix we show how this proof generalizes to economies satisfying weaker assumptions. We also present a result due to Cass, Siconolfi and Villanacci (1992) that establish generic regularity of equilibrium in the restricted participation model.

The following lemma establish properness of the projection mapping

$$\pi : M \to \mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y}$$

$$\pi(\xi, u, a, e, Y) = (u, a, e, Y)$$

This lemma is essential in proving both existence and regularity of equilibrium.

Properness Lemma: *The projection π is a proper mapping: if $K \subset (\mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y})$ is compact then $\pi^{-1}(K)$ is compact as well.*

Proof: Recall that in metric spaces a set is compact if and only if every sequence has a convergent subsequence. Let $\{\xi(n), u(n), a(n), e(n), Y(n)\} \subset M$ be an arbitrary sequence such that

$$(u(n), a(n), e(n), Y(n)) \to (u, a, e, Y) \in \mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y}$$

We have to show, taking a subsequence if necessary, that there is $\xi \in \Xi$ such that

$$\xi(n) \to \xi$$

Since $F : \Xi \times \mathcal{U} \times \mathcal{A} \times E \times \mathcal{Y} \to \Re^{n+J} \in C^1$, we then have $F(\xi, u, a, e, Y) = 0$, which completes the proof.

From the market clearing equations and the fact that $e(n) \to e$ the sequence $\{x_h(n)\}$ is bounded and hence it has a convergent subsequence for each h. Boundary behavior of demand $(H1(iii))$ and $u(n) \to u$ guarantees that, taking a subsequence if necessary, $\lim x_h(n) \gg 0$ for every h. The first order conditions for each consumer and the last equations then gives that $p(n)$, $q(n)$ and $\lambda_h(n)$ for every

h also have a convergent subsequence. Since $Y(n)$ and Y have full column rank and the budget constraint holds, $b_h(n)$ converges for each h. Finally, assumption $(H4)$ guarantees the convergence of $\mu_h(n)$. The Lemma is then proved. □

In order to prove the existence of equilibrium we shall use the following mathematical fact from degree theory:[8]

Let $B \subset \Re^n$ be a bounded and open set, and let $H : clB \times [0,1] \longrightarrow \Re^n \in C^0$ satisfy:
i) $H^{-1}(0) \subset B \times [0,1]$;
ii) the restriction $H(\cdot,1) : B \longrightarrow \Re^n$ is continuously differentiable;
iii) 0 is a regular value of $H(\cdot,1)$;
iv) the equation $H(\cdot,1) = 0$ has an odd number of solutions.
Then, the system of equations $H(x,0) = 0$ also has a solution.

A variation of the next proposition was first proved by Siconolfi (1989) using a fixed point argument. In the appendix we generalize the result to standard continuity assumptions.

Theorem (Siconolfi) : *There is an equilibrium for every economy satisfying (H1)-(H4).*

Proof:[9] The proof consists essentially in verifying the conditions of the mathematical fact for a suitable constructed function.

Suppose initially $a_h(0) \gg 0$. Let e^* be a Pareto optimal allocation and fix any initial distribution of endowments e. Let $H : \Xi \times [0,1] \longrightarrow \Re^{n+J}$ be given by

$$H(\xi,t) = F(\xi,u,a,te^* + (1-t)e,Y)$$

We have to show that there is $\xi' \in \Xi$ such that $H(\xi',0) = 0$. Clearly, $H \in C^0$ and by the properness lemma there exists a compact set K such that if (ξ,t) solves $H(\xi,t) = 0$ then $(\xi,t) \in K \times [0,1]$. Take an open set $B \subset \Xi$ such that $K \subset B$. Consider the restriction $H : B \times [0,1] \longrightarrow \Re^n$. Then, condition (i) of the mathematical fact is satisfied by construction.

[8] Nagumo (1951) presents a proof of this fact using basic real analysis. Heinz (1959) provides an alternative proof also using real analysis. His argument is simpler then Nagumo's, but less intuitive. A general presentation of degree theory can be found either in Lloyd (1978) or in Schwartz (1969).

[9] Degree theory was introduced in economic theory to prove existence of equilibrium by Balasko (1975) and Smale (1974).

Following the same reasoning used in the Walrasian model, it is simple to verify that there is a unique equilibrium, ξ^*, associated with the economy e^*. Moreover, at this equilibrium $x_h = e_h^*$ for every h and hence $b_h = 0$ for every h. Then, at this equilibrium ξ^* we necessarily must have $\mu = 0$. By continuity, $a_h(b_h) \gg 0$ for every asset's demand in a neighborhood of 0, which implies that locally

$$\min\left\{\mu_h^j, a_h^j(b_h)\right\} = \mu_h^j$$

Then, $H(\cdot, 1) \in C^1$ in a neighborhood of ξ^*, the unique solution of the system $H(\xi, 1) = 0$. Moreover, it is simple to verify that 0 is a regular value. Therefore, all conditions of the mathematical fact are satisfied, which implies the existence of $\xi' \in \Xi$ such that $H(\xi', 0) = 0$.

Now suppose $a_h(0) \geq 0$. For each j, let $a_{h,k}^j(b_h) = a_h^j(b_h) + 1/k$, and let $H_k(\cdot)$ be the constructed function associated with the restriction $a_{h,k}(\cdot)$ for each h. For k large enough, $(\xi, e) \in B$ and $H_k(\xi, e) = 0$ imply

$$\det\left[Da_{h,k}^j(b_h)\right]^{j \in J_h'} \neq 0$$

for every h. Moreover, the set $H_k^{-1}(0)$ is compact. By the previous existence argument, for each k large enough there is (ξ_k) such that $H_k(\xi_k, e) = 0$. But then, taking a subsequence if necessary, there is $\xi \in \Xi$ such that $\xi_k \to \xi$ and $F(\xi, u, a, e, Y) = 0$. The argument is then complete. □

Let $B \subset \Re^n$ be an open set. A subset $A \subset B$ is said to be **generic** if A is open and $B \backslash A$ has Lebesgue measure zero. If B is a topological space then $A \subset B$ is said to be **weakly generic** if A is and open and dense. In order to prove generic regularity of equilibrium, an additional assumption is required:

$$(H5)\ \textit{If}\ \sum_h b_h = 0\ \textit{then}\ \forall\, i\ \exists\, h\ \textit{such that}\ D_i a_h^j(b_h) = 0\ \ \forall j \in J_h'$$

One can verify that under $(H1) - (H5)$, the properness lemma holds even if we use the normalization $p^{01} = 1$. In the remaining of this paper we shall use this normalization. Moreover, in order to simplify the arguments we take the space of prices to be given by $P = \left\{p \in \Re_{++}^G / \ p^{s1} = 1 \ for\ every\ s\right\}$.

Theorem (Cass-Siconolfi-Villanacci): *Fix* $((u_h, a_h)_h, Y)$ *satisfying (H1)-(H5). There is a generic set* $E^* \subset E$ *such that* $e \in E^*$ *and* $F(\xi, \mu, e, a, Y) = 0$ *imply*

$$i)\ \max\left\{a_h^j(b_h), \mu_h^j\right\} > 0\ \ for\ every\ pair\ (h, j),\ \ and$$

ii) $D_{\xi,\mu}F$ *is non-singular.*

This proposition is proved in Cass, Siconolfi and Villanacci (1992). For completeness, we present a variation of their proof in the appendix using our notation. Notice that, as in the proof of the properness lemma, assumption $(H5)$ is not necessary to establish that the desired set E^* is open, but only to guarantee that a certain matrix has full row rank. An investigation of the proposition's proof shows, in particular, that $(H5)$ can be substituted as follows. Suppose every household's restriction on financial markets depends on some parameter α_h. Let $O^h \subset \Re^{J_h}$ be an open set and let $a_h : \Re^I \times O^h \to \Re^{J_h} \in C^2$.

$$(H5')\ \textit{If}\ \sum_h b_h = 0\ \textit{and}\ J_h' \neq \emptyset\ \textit{for all}\ h$$

$$\textit{then}\ \exists\ h'\ \textit{s.t.}\ \left[D_{\alpha_{h'}} a_{h'}^j(b_{h'}, \alpha_{h'})\right]^{j \in J_{h'}}\ \textit{has full row rank.}$$

Let $O = \times_{h=1}^{H} O_h$.

Corollary: *There is a generic set* $(E \times O)^*$ *such that* $(e, \alpha) \in (E \times O)^*$ *and*

$$F(\xi, u, \alpha, e, Y) = 0$$

imply

$$i)\ \max\left\{a_h^j(b_h, \alpha_h), \mu_h^j\right\} > 0\ \textit{for every pair}\ (h, j),\ \textit{and}$$

$$ii)\ D_\xi F\ \textit{is non-singular.}$$

4. Indeterminacy of Equilibria

In this section we consider a variation of the model presented so far. Suppose each asset pays in units of accounts instead of units of the first commodity in every state of nature. Let $u = (u_1, \ldots, u_H)$ be a given profile of utility functions satisfying $(H1)$, and (a_h) be a given profile of restriction functions satisfying $(H3) - (H4)$. Fix $p_n \in \Re^G_{++}$ and $q \in \Re^I$. The household's problem is given by

$$\max u_h(x)\ s.t.$$

$$\begin{aligned} p_n^0 \left(x_h^0 - e_h^0\right) &= -q b_h \\ p_n^s \left(x_h^s - e_h^s\right) &= y^s b_h \quad s > 0 \end{aligned}$$

$$b_h \in B(a_h)$$

where y^s is the vector of assets payoffs in units of accounts in state s. Suppose the payoff matrix

$$Y = \begin{bmatrix} y^1 \\ \vdots \\ y^S \end{bmatrix}$$

has full column rank. Dividing the budget constraint in each state in the second period by p_n^{s1} we get

$$\frac{p_n^s}{p_n^{s1}}\left(x_h^s - e_h^s\right) = \frac{1}{p_n^{s1}} y^s b_h \quad s > 0$$

Let

$$p := \left(p_n^0, \ldots, \frac{p_n^s}{p_n^{s1}}, \ldots\right), \; \alpha^s := \frac{1}{p_n^{s1}} \; \textit{for each } s$$

$$\hat{\alpha}^1 = \begin{bmatrix} 1 & & \\ & \alpha^1 & & 0 \\ & & \ddots & \\ & 0 & & \alpha^S \end{bmatrix} \; \textit{and } \hat{\alpha} = \begin{bmatrix} \alpha^1 & & 0 \\ & \ddots & \\ 0 & & \alpha^S \end{bmatrix}$$

Therefore, for a given vector of relative prices, households solve the problem

$$\max u_h(x) \;\; s.t.$$

$$\Psi\left(x_h - e_h\right) = \hat{\alpha} R b_h$$

$$b_h \in B(a_h)$$

where the payoff matrix $\hat{\alpha}Y$ pays in units of the numéraire commodity. In the nominal asset model the equilibrium set cannot be determined independently of each sate of nature normalization of relative prices. In fact, for a given p and q, changes in α implies changes in the consumer demand for both commodities and assets. Let

$$M(\alpha) = \{(\xi, u, a, e, Y, \alpha) \,/\, F(\xi, u, a, e, Y, \alpha) = 0\}$$

denote the set of equilibrium for the numéraire asset model associated with payoff matrix $\hat{\alpha}Y$ and

$$M = \{(\xi, u, a, e, Y) \,/\, F(\xi, u, a, e, Y) = 0\}$$

denote the set of equilibrium for the nominal asset model associated with payoff matrix Y. Then

$$M = \bigcup_{\alpha} M(\alpha)$$

By the previous section we known that for a generic set of initial endowments, each α is associated with finitely many equilibrium allocations. However, since α belongs to an open set of dimension S, the set of equilibrium allocations in the nominal asset model may have dimension S as well. In this case, we say that the equilibrium set is *indeterminate.*

Let ξ^* be an equilibrium associated with a regular economy $(u^*,a^*,e^*,Y^*,\alpha^*)$:

$$F(\xi^*,u^*,a^*,e^*,Y^*,\alpha^*) = 0$$

Then $F \in C^1$ in a neighborhood of this equilibrium and

$$D_\xi F(\xi^*,u^*,a^*,e^*,Y^*,\alpha^*)$$

is non-singular. In particular, by the Implicit Function Theorem, there is a neighborhood of α^*, V_α, a neighborhood of ξ^*, V_ξ, and a function $\Phi : V_\alpha \rightarrow V_\xi \in C^1$ such that

$$F_{|V_\xi \times V_\alpha}(\xi,u^*,e^*,a^*,Y^*,\alpha) = 0 \ \textit{if and only if} \ \xi = \Phi(\alpha)$$

Let

$$\pi_x : M \rightarrow \Re^{HG}_{++}$$

be the projection,

$$\pi_x(\xi,u^*,e^*,a^*,Y^*,\alpha) = (x_1,\ldots,x_H)$$

We shall refer to the composite function $\pi_x \circ \Phi(\alpha)$ as $x(\alpha)$. Suppose we show that the mapping $D(x(\alpha))$ is injective. Then, taking a smaller neighborhood if necessary, the function $x(\alpha)$ is also injective and thus if $\alpha' \neq \alpha$ implies $x(\alpha') \neq x(\alpha)$. This implies that changes in α imply changes in the equilibrium allocations. *This result is formally equivalent to show that the set of equilibrium allocations contains a manifold of dimension S when assets pays in units of accounts instead of units of the numéraire commodity.*

It might be the case, however, that the function α is only injective when restricted to a sub-manifold $V \subset \Re^S_{++}$, $\dim V = v$. In this case the set of equilibrium allocations in the nominal asset model contains a manifold of dimension v. We refer the reader to Lisboa (1995) for further details. The next assumptions give alternative conditions to establish indeterminacy of equilibrium. Notice that a particular case of $(H6)$ occurs when the first asset is inside money, $y^{1s} = 1$ for every s, and if bounds markets clear then at least two households can locally trade inside money without changing the value of the (if any) binding constraints.

(H6) For every s we have $y^{s,1} > 0$. Moreover, if

$$\sum_h b_h = 0$$

there are two distinct households h' and h'' such that

$$D_1 a_h^j (b_h) = 0 \;\; for\ all\ j \in J'_h,\ h = h', h''$$

(H7) Let $a_h(0) \neq 0$ for every h. Suppose also that of the alternatives hold:
i) if $Y \in \mathcal{Y}$ then Y is in general position;
ii) if $\sum_h b_h = 0$ then there is a unconstrained household;
iii) if $\sum_h b_h = 0$ then either no household is constrained or there is a constrained household h' such that $\forall\, i\ \exists\ h(i) \neq h'$ such that $D_i a_h (b_h) = 0$.

Proposition 4.1: *i) Let $(H1) - (H5)$ and $(H6)$ hold. There is a generic set $E^* \subset E$ such that the set of equilibrium allocation contains a manifold of dimension $(S - I + k)$ for every economy in E^*, where*

$$k = \max_h \dim J'_h$$

ii) Let $(H1) - (H5)$ and $(H7)$ hold. There is a generic set $(E \times \mathcal{Y})^* \subset E \times \mathcal{Y}$ such that the set of equilibrium allocation contains a manifold of dimension $(S - I + k)$ for every economy in this set.

iii) Let $(H1) - (H5)$ and $(H7)$ hold. There is a weakly generic set $(E \times \mathcal{A})^* \subset (E \times \mathcal{A})$ such that for any economy in this set either no household is restricted at any equilibrium or the set of equilibrium allocation contains a $S-$dimensional manifold.

Proof: *"Openness"* That the desired set is open follows from the continuity of $F(\cdot)$ and the properness lemma.

"Denseness" Notice that the mapping $Dx(\alpha)$ is given by

$$Dx = \pi_x \left(- [D_\xi F]^{-1} D_\alpha F \right)$$

The mapping $Dx(\alpha)$ is injective if $Dx(\alpha)\Delta = 0$ implies $\Delta = 0$, where. Since π_x is the projection, $Dx\Delta = 0$ implies $\Delta = 0$ if and only if

$$D_\xi F d_\xi + D_\alpha F d_\alpha = 0 \;\; and \;\; D_x F d_\xi = 0$$

implies $(d_\xi, d_\alpha) = 0$.[10] Let

$$\xi^{\setminus} := ((b_h, \lambda_h)_h, p, q)$$

stand for the vector of endogenous variables but the consumption allocations. Let
The derivative $D_{\xi^{\setminus},\upsilon,\alpha}F$ is given below.

$$\begin{bmatrix} \cdots & (b_h & \lambda_h & \mu'_h) \cdots & p^{\setminus} & q & \alpha \end{bmatrix}$$

$$\begin{bmatrix}
 & \vdots & & \vdots & \vdots & \vdots \\
0 & -\Psi^T & 0 & \Lambda_h & 0 & \\
\mu_h D^2 a_h & (\hat{\alpha} R)^T & 0 & 0 & -\lambda_h^0 I & 0 \\
 & & & & & \lambda_h^1 y^1, \ldots, \lambda_h^S y^S \\
\hat{\alpha} R & 0 & 0 & \Pi_h & \Upsilon_h & \Gamma_h \\
 & & & & & 0 \\
Da'_h & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & \\
 & \vdots & & \vdots & \vdots & 0 \\
I & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

where

$$\Gamma_h = \begin{bmatrix} y^1 b_h & & 0 \\ & \ddots & \\ 0 & & y^S b_h \end{bmatrix} \textit{ and } \Lambda_h = \begin{bmatrix} -\lambda_h^0 \begin{bmatrix} 0 \\ I \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & -\lambda_h^S \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix}$$

Let $d = (db_1, d\lambda_1, \ldots, db_h, d\lambda_h, d\mu'_h, \ldots, dp, dq, d\alpha)$ be such that

$$D_{\xi^{\setminus},\upsilon,\alpha} F d = 0$$

The system $D_{\xi^{\setminus},\upsilon,\alpha} F d = 0$ can be written as

$$\begin{aligned}
\lambda_h^s \begin{bmatrix} 0 \\ dp^{s\setminus} \end{bmatrix} + d\lambda_h^s p^s &= 0 \;\forall\, h, s \Rightarrow d\lambda_h = 0,\ dp^{\setminus} = 0 \\
\mu'_h D^2 a'_h db_h - \lambda_h^0 dq + \left(\lambda_h^1 y^1, \ldots, \lambda_h^S y^S\right) d\alpha &= 0 \quad (h.1)
\end{aligned}$$

[10] See Lisboa (1994) for further details.

$$\hat{\alpha} Y db_h + \begin{bmatrix} y^1 b_h d\alpha^1 \\ \vdots \\ y^S b_h d\alpha^S \end{bmatrix} = 0 \quad (h.2)$$

$$Da'_h db_h = 0 \quad (h.3)$$

If $k = I$ we then immediately have $db_h = 0$ by $(h.3)$. But by $(h.1) - (h.2)$ this implies $(d\alpha, dq) = 0$, which gives $d = 0$. So, suppose $k < I$.

Proof of cases (i) **and** (ii).

In order to prove case (ii) we need the following fact:

Let $(H1) - (H4)$ *hold. There is a generic set* $(E \times \mathcal{Y})^* \subset (E \times \mathcal{Y})$ *such that for every equilibrium and for every constraint household* h *we have* $y^s b_h \neq 0$ *for every* s. *If in addition (H7) is satisfied then for every* $Y \in \mathcal{Y}$ *there is a generic set* $E^* \subset E$ *such that the desired property holds.*

The proof consists in showing that the system

$$F(\xi, u, a, e, Y) = 0$$
$$y^s b_h = 0$$

has no solution for a generic set of economies. Openness is standard. Suppose we show that

$$\begin{bmatrix} D_{\xi,e,Y} F \\ D_{\xi,e,Y} (y^s b_h) \end{bmatrix}$$

is surjective at any solution of the system. Then, by the Transversality theorem, for a dense set of parameters at any solution of the system the derivative

$$\begin{bmatrix} D_\xi F \\ D_\xi (y^s b_h) \end{bmatrix}$$

is also surjective. But this is impossible since the above matrix has more rows than columns. Therefore, the system does not have a solution in a dense set of parameters, which is the desired result.

Fix a regular economy e and a corresponding equilibrium ξ such that

$$a_h^j (b_h) = 0$$

for some (h, j). By regularity of equilibrium, there is a neighborhood of e, V_e, and a neighborhood of ξ, V_ξ, such that $e \in V_e$, $\xi \in V_\xi$, and

$$F(\xi, u, a, e, Y) = 0$$

imply $a_h^j(b_h) = 0$. By the transversality theorem it is sufficient to show that

$$D\left[\begin{array}{c} F(\xi, u, a, e, Y) \\ y^s b_h \end{array}\right]$$

is surjective in this neighborhood. In this case the desired result holds for every $e \in V_e \backslash V^n$, where V^n is a measure zero set. Since this is true for every regular economy, every regular economy has finitely many equilibria, and both the number of households and restrictions is finite, we get the desired result. Following essentially the same computations as in the proof of Cass-Siconolfi-Villanacci proposition, proves the claim.

Let h be the household with most restrictions bindings. Since $\hat{\alpha} \gg 0$ the matrix $\hat{\alpha}Y$ has full column. Without loss of generality suppose the last $k = J_h'$ columns of Da_h', denoted by Da_h^2, are linearly independent and let $db_h = (db_h^1, db_h^2)$, where $db_h^2 \in \Re^k$.

$$Da_h' db_h = 0 \Rightarrow db_h^2 = -\left[Da_h^2\right]^{-1} Da_h^1 db_h^1$$

The system

$$\hat{\alpha}Y db_h + \left[\begin{array}{c} y^1 b_h d\alpha^1 \\ \vdots \\ y^S b_h d\alpha^S \end{array}\right] = 0$$

then implies

$$A db_h^1 = d\alpha \quad where\ A = \left[\begin{array}{ccc} \ddots & & 0 \\ & -\frac{\alpha^s}{y^s b_h} & \\ 0 & & \ddots \end{array}\right] Y \left[Da_h^1\right]^{-1} Da_h^2$$

where A is a $S \times (I-k)$ matrix. Therefore, in order the system $D_{\xi\backslash,v,\alpha}F(d, d\alpha) = 0$ to have a solution different from zero we must have $d\alpha \in spanA$, which has at most dimension $I - k$. The proof of cases (i) and (ii) is then complete.

Proof of case (iii).

From $(h.1) - (h.2)$ if there is h such that $db_h = 0$ then $d\alpha = 0$ and $dq = 0$ which readily implies $d = 0$. Therefore, it is sufficient to show that the system

$$\sum_s \left(\frac{\lambda_h^s}{\lambda_h^0} - \frac{\lambda_1^s}{\lambda_1^0}\right) d\alpha^s y^s + \frac{1}{\lambda_h^0}\mu_h' D^2 a_h' db_h - \frac{1}{\lambda_1^0}\mu_1' D^2 a_1' db_1 \quad = \quad 0$$

$$\begin{pmatrix} \vdots \\ \sum_i y_i^s db_h^i \alpha^s \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ \sum_i y_i^s b_h^i d\alpha^s \\ \vdots \end{pmatrix} = 0$$
$$\begin{aligned} Da_h' db_1 &= 0 \\ Da_h' db_h &= 0 \\ db_1 &\neq 0 \\ db_h &\neq 0 \end{aligned}$$

has no solution for a weakly generic choice of endowments and restriction functions. For easy of notation, we denote the above system as

$$B(\xi, a) \begin{pmatrix} d\alpha \\ db_h \\ db_1 \end{pmatrix} = 0, \textit{ where}$$

$$B(\xi, a) = \begin{bmatrix} \left(\ldots, \left(\lambda_1^{s\backslash} - \lambda_h^{s\backslash}\right) y^s, \ldots\right) & \frac{1}{\lambda_h^0} \mu_h' D^2 a_h' & -\frac{1}{\lambda_1^0} \mu_1' D^2 a_1' \\ \begin{pmatrix} \ddots & & 0 \\ & \sum_i y_i^s b_h^i & \\ 0\cdot & & \ddots \end{pmatrix} & \begin{pmatrix} \vdots \\ \alpha^s y^s \\ \vdots \end{pmatrix} & 0 \\ 0 & Da_h' & 0 \\ 0 & 0 & Da_1' \end{bmatrix}$$

We need the following claim.

Let $(H1) - (H5)$. For a weakly generic set $(E \times \mathcal{A})^* \subset (E \times \mathcal{A})$ then either no household is constraint at any equilibrium associated with an economy in this set, or there is h' such that $y^s b_h \neq 0$ for every s.

Once more, openness is immediate. As is the previous proof, we have to show that

$$\begin{bmatrix} D_{\xi,\mu,a,e} F \\ D_{b_h}\left(y^s b_h\right) \end{bmatrix}$$

is surjective at any solution of the system.

Following the discussion in the appendix, we can restrict our analysis to restriction functions that are differentiably strict quasi-concave and that satisfy $a_h(0) \gg 0$ for all h. Fix a regular equilibrium and assume that some household, without loss of generality $h = 1$, is restricted at this equilbrium. By Cass-Siconolfi-Villanacci theorem, household 1 is restricted in neighborhood of this

equilibrium as well. Consider a local perturbation in each restriction function in a neighborhood of a regular equilibrium given by:

$$a_1^j(b_1, h^j) = a_1^j(b_1) + \Phi_1(b_1)\left(b_1^T h^j\right)$$

where Φ_h is a bump function. In the appendix we provide further details on this perturbation and we also show that $a_1^j(\cdot, h^j) \in \mathcal{A}_1$ for h^j small enough.

Following the same procedure used in the proof of Cass-Siconolfi-Villanacci theorem, it is simple to verify that this derivative

$$\begin{bmatrix} D_{\xi,h}F \\ D_{b_1}y^s b_1 \end{bmatrix}$$

is surjective. The proof of this claim is complete.

The proof will be complete once we establish the following fact.

Let $(H1)-(H5)$ and $(H7)$ hold. There is an open and dense set $(E \times \mathcal{A})^* \subset (E \times \mathcal{A})$ such that the system

$$\begin{aligned} F(\xi, u, a, e) &= 0 \\ B(\xi, a)\begin{pmatrix} d\alpha \\ db_h \\ db_1 \end{pmatrix} &= 0 \\ db_1^T db_1 - 1 &= 0 \\ db_h^T db_h - 1 &= 0 \end{aligned}$$

has no solution for every $(e, a) \in (E \times \mathcal{A})^*$.

That the desired set is open follows once more from the continuity of the functions defining the above system and the properness lemma.

The denseness argument follows the same procedure used in the proof of the previous claim. Fix a regular equilibrium and consider a perturbation of household h restricted function given by

$$a_h^j\left(b_h, h^j, H^j\right) = a_h^j(b_h) + \Phi_h(b_h)\left[(b_h^* - b_h)^T h^j + \frac{1}{2}(b_h^* - b_h)^T H^j (b_h^* - b_h)\right]$$

where without loss of generality a_h is assumed differentiably strict quasi-concave and $a_h(0) \gg 0$.[11] In particular, it is sufficient to show that the following derivative is surjective at any solution of the system.

[11] See appendix for further details.

$$
\begin{array}{c}
\left[\cdots \left(x_h \quad b_h \quad \lambda_h \quad \mu_h' \quad \mu_h'' \quad e_h^1 \quad e_h^{\backslash}\right) \quad p \quad d\alpha \quad db_h \quad dH_i \quad dh_h^i \quad db_1 \quad dh\right. \\
\begin{bmatrix}
 & & \vdots & & & & & \vdots & & & & \vdots & & \\
D^2u_h & 0 & -\Psi^T & 0 & 0 & 0 & 0 & \Lambda_h & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_h' D^2 a_h' & (\hat{\alpha}R)^T & Da_h' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
-\Psi & \hat{\alpha}R & 0 & 0 & 0 & I & \Psi^{\backslash} & \Pi_h & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Da_h' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & \vdots & & & & \vdots & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I^{\backslash} & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & * & 0 & \mu_j db_h^i I & 0 & 0 & * \\
0 & d\hat{\alpha}Y & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_h & * & 0 & 0 & 0 & 0 \\
0 & D^2 a_h' 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Da_h' & 0 & [db_h^i I] & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & [db_1^i I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2db_h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2db_1 & 0
\end{bmatrix}
\end{array}
$$

But for the case in which $(H1)-(H4)$ and $(H6)$ hold this is easily verified, given the previous lemma. For the case in which $(H1)-(H5)$ hold and Y is in general position the computation is a bit more subtle. Pre-multiply the matrix by the vector

$$\left(\ldots(\Delta x_h, \Delta b_h, \Delta\lambda_h), \ldots, \Delta p^{\backslash}, \Delta q, \Delta\chi, \Delta\omega, \Delta\epsilon, \Delta\rho, \Delta\tau, \Delta\theta\right)$$

where the dimension of each of the first components of this vector should be obvious, $\Delta\chi \in \Re^S$, $\Delta\omega \in \Re^S$, $\Delta\epsilon \in \Re^{J_h'}$, $\Delta\rho \in \Re^{J_1'}$, $\Delta\tau \in \Re$, $\Delta\theta \in \Re$.

Since both db_h and db_1 are different from 0, there is i and i' such that $db_1^{i'}$ and db_h^i are both different from 0. Then, from the columns corresponding to $dh_1^{i'}$, dH^i, dh_h^i we immediately get $\Delta\chi$, $\Delta\epsilon$ and $\Delta\rho$ are equal to 0. From the columns corresponding to $dh_1^{i'}$, $d\alpha$ and db_h, we then get $\Delta\theta = 0$, $\Delta\omega^s y^s b_1 = 0$ for every s and $\Delta\omega\hat{\alpha}Y + \Delta\tau 2\,(db_1)^T = 0$. Then

$$\Delta\omega\hat{\alpha}Y db_1 + \Delta\tau 2\,(db_1)^T db_1 = 0$$

Suppose $\Delta\tau \neq 0$. Then, $\Delta\tau 2\,(db_1)^T db_1 \neq 0$ which implies $\Delta\omega\hat{\alpha}Y db_1 \neq 0$. But from

$$B\begin{pmatrix} d\alpha \\ db_h \\ db_1 \end{pmatrix} = 0 \Rightarrow y^s db_1 \alpha^s + y^s b_1 d\alpha^s = 0, \ \textit{where}\ \alpha^s > 0\ \forall s$$

we get $y^s b_1 = 0$ implies $y^s db_1 = 0$ for all s. But then, $\Delta\omega\alpha^s y^s db_1 \neq 0$ implies $y^s db_1 \neq 0$, which implies $y^s b_1 \neq 0$ and $\Delta\omega^s \neq 0$ for some s, contradicting $\Delta\omega^s y^s b_1 = 0$ for every s. Therefore, $\Delta\tau = 0$. It remains to show that $\Delta\omega = 0$. Since Y is in general position and $b_1 \neq 0$ then Yb_h has at most $I-1$ components equal to 0. Hence, $\Delta\omega^s\alpha^s y^s b_1 = 0$, $\alpha^s > 0$ for every s, implies that $\Delta\omega$ has at most $I-1$ components different from 0, and hence at least $S-I$ components equal to 0. Let $(\Delta\omega)^{S-I}$ denote $S-I$ components of $\Delta\omega$ that are equal to 0. But then $\Delta\omega\hat{\alpha}Y = 0$ implies

$$(\Delta\omega)^I = (\Delta\omega)^{S-I}(\hat{\alpha}Y)^{S-I}\left[(\hat{\alpha}Y)^I\right]^{-1}$$

and hence $\Delta\omega = 0$.

This concludes both the claim and the proposition. □

5. Constraint Optimality

In this section we investigate the optimality properties of the restricted participation model. We show that for a generic set of parameters the following results are true: one can relax the restrictions and utility impairs all restricted households. Symmetrically, one can utility improve every restricted household by making the restrictions stronger. Recall that we defined the set of feasible asset trading, given restrictions a_h as follows

$$B(a_h) = \{b_h / \ a_h(b_h) \geq 0\}$$

Proposition 5.1: *Let $(H1)-(H5)$ hold and $C > 1$. Then there is an open and dense set $(E \times \mathcal{U} \times \mathcal{A})^* \subset (E \times \mathcal{U} \times \mathcal{A})$ such that at any equilibrium for an economy in this set the following properties hold.*

a) Fix any collection of restricted households $\mathcal{H}$, $dim\mathcal{H} \leq S+1-I$. Then for every $h \in \mathcal{H}$

ai) there is $a'_h \in \mathcal{A}_h$ and a new equilibrium, $F(\xi', u, a', e) = 0$, such that

$$B(a'_h) \subset B(a_h) \ \ and \ \ u_h(x'_h) > u_h(x_h)$$

aii) there is $a'_h \in \mathcal{A}_h$ and a new equilibrium, $F(\xi', u, a', e) = 0$, such that

$$B(a'_h) \supset B(a_h) \ \ and \ \ u_h(x'_h) < u_h(x_h)$$

b) Fix any collection of restricted households $\mathcal{H}$, such that

$$\dim \mathcal{H} \leq \min \{(C-1)(S+1), H-1\}$$

Conclusions (ai) and (aii) hold for every $h \in \mathcal{H}$.

Proof: We provide a proof of both (ai) and (bi). The other cases are symmetric. Fix a regular economy (e^*, a^*) and an equilibrium associated with this economy ξ^*. Consider a local perturbation of the restricted functions in a neighborhood small enough of the equilibrium given by

$$a_h^j(b_h, \beta_h^j) = a_h^j(b_h) + \Phi_h(b_h)\left(b_h^1 - \beta_h^j b_h^{*1}\right)^2$$

where $\beta_h^{j,1} = 1$ and j is chosen among the binding restrictions (see appendix for further details on this perturbation). Notice that

$$b_h \in V_{b_h^*} \backslash \{b_h^*\} \ \ \textit{implies} \ \ a_h^j(b_h, \beta_h^j) = a_h^j(b_h) + \left(b_h^1 - \beta_h^j b_h^{*1}\right)^2 > a_h^j(b_h)$$

$$D_{b_h} a_h^j(b_h, \beta_h^j) = D_{b_h} a_h^j(b_h) + 2\left(b_h^1 - \beta_h^j b_h^{*1}, 0, \ldots, 0\right)$$

$$D^2_{b_h, \beta_h} a_h^j(b_h, \beta_h^j) = -2 \begin{bmatrix} b_h^{*1} & 0 \\ 0 & 0 \end{bmatrix} \equiv \hat{b}_h$$

We claim that the system

$$\begin{bmatrix} F(\xi, \mu, a, u) = 0 \\ D_{\xi,\beta} F^T d_F + D_x G^T d_G = 0 \\ d^T d - 1 = 0 \end{bmatrix} \quad (*)$$

has no solution for a weakly generic set. Suppose the claim is true. Then the matrix

$$\left[D_{\xi,\beta} F^T, D_x G^T\right]$$

is surjective for every equilibrium associated with an economy in the weakly generic set. This means that the function

$$G(x)_{|F(\cdot)=0}$$

is locally surjective since $D_{\xi,\mu} F$ is surjective. Thus, we can locally perturb the level of utility function for every perturbed household in any desired direction by

perturbing (β_h). In particular, we can make every perturbed household worse off, which is the desired result.[12]

"Openness" The proof of this step is standard.

"Denseness" We shall need the following fact, whose proof is an immediate variation of the last claim proved in the previous proposition.

Let $(H1)-(H5)$ hold. There is an open and dense set $(E\times\mathcal{A})^*\subset(E\times\mathcal{A})$ such that the system

$$\begin{aligned} F(\xi,\mu,a,e) &= 0 \\ b_h^1 &= 0 \end{aligned}$$

has no solution for every $(e,a)\in(E\times\mathcal{A})^*$.

The system $D_{\xi,\mu,\beta}F^T d_F + DGd_G$ can be written as follows

$$\begin{aligned} D^2u_h dx_h - \Psi^T d\lambda_h + dp^\backslash + Du_h d\alpha_h &= 0 \\ \mu_h^j b_h^1 db_h^1 &= 0 \Rightarrow db_h^1 = 0 \\ \left(D^2a_h'\right)^T \mu_h' db_h^\backslash + R^T d\lambda_h + \left(Da_h'\right)^T d\mu_h' + dq &= 0 \\ \Psi^T dx_h + Rdb_h^\backslash &= 0 \\ Da_h' db_h^\backslash &= 0 \\ \sum_h \left(\lambda_h^s dx_h^{s\backslash} + z_h^{s\backslash} d\lambda_h^s\right) &= 0 \\ \sum_h \left(d\lambda_h^0 b_h + db_h^\backslash \lambda_h^0\right) &= 0 \end{aligned}$$

Computing derivatives of the system $(*)$ we get

$$\begin{bmatrix} \xi & dx_h & db_h^\backslash & d\lambda_h & d\alpha_h & d\mu_h' & dp & dq & H_h^i \end{bmatrix}$$

[12] This approach to study optimality problems was proposed by Samle (1974) and generalized to GEI models by Geanakoplos and Polemarchakis (1986) and Cass and Citana (1994).

$$
\left[\begin{array}{ccc}
D_\xi F & 0 & 0 \\
* & \left[\begin{array}{ccccccc}
D^2u_h & 0 & -\Psi^T & Du_h & 0 & I^\backslash & 0 \\
0 & \mu'_h\left(D^2a_h\right)^T & R^T & 0 & \left(Da'_h\right)^T & 0 & I \\
\Psi & R & 0 & 0 & 0 & 0 & 0 \\
0 & Da'_h & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & \\
\Lambda_h^T & 0 & \hat{z}_h^\backslash & 0 & 0 & 0 & 0 \\
0 & \lambda_h^0 I & [b_h \quad 0] & 0 & 0 & 0 & 0
\end{array}\right] & \begin{array}{c} dx_h^i I \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{array} \\
0 & \left(\begin{array}{ccccccc} dx_h & db_h^\backslash & d\lambda_h & d\alpha_h & d\mu'_h & dp & dq \end{array}\right) & 0
\end{array}\right] \qquad (**)
$$

Let $(\Delta x_h, \Delta b_h, \Delta\lambda_h, \Delta l, \Delta) \in \Re^{H(G+I+S+1)+(C-1)(S+1)+1}$ and consider the system of equations:

$$
\begin{aligned}
D^2u_h\Delta x_h - \Psi^T\Delta\lambda_h + \left[\begin{array}{c} \vdots \\ \left(\begin{array}{c} 0 \\ \lambda_h^s \Delta l^s \end{array}\right) \\ \vdots \end{array}\right] + \Delta dx_h &= 0 \quad (h.1) \\
\mu'_h D^2a_h\Delta b_h + R^T\Delta\lambda_h + \left(Da'_h\right)^T\Delta\mu'_h + \lambda_h^0\Delta q^\backslash + \Delta db_h^\backslash &= 0 \quad (h.2) \\
-\Psi\Delta x_h + R\Delta b_h + \left[\begin{array}{c} \vdots \\ \left(\Delta l^s\right)^T z_h^s \\ \vdots \end{array}\right] + \left[\begin{array}{c} b_h^{\backslash T}\Delta q^\backslash \\ 0 \end{array}\right] + \Delta d\lambda_h^0 &= 0 \quad (h.3) \\
Du_h^T\Delta x_h + \Delta d\alpha_h &= 0 \quad (h.4) \\
Da'_h\Delta b_h &= 0 \quad (h.5) \\
dx_h^i\Delta x_h &= 0 \quad (h.6.i) \\
\sum_h \Delta x_h^\backslash + \Delta dp^\backslash &= 0 \quad (m.1) \\
\sum_h \Delta b_h + \Delta dq^\backslash &= 0 \quad (m.2)
\end{aligned}
$$

Suppose $dx_h \neq 0$ for every h. Then, for every h there is i such that $(h.6.i)$ implies $\Delta x_h = 0$ and hence by $(h.4)$, $\Delta = 0$. $(h.1)$ then gives $\Delta\lambda_h = 0$ and hence $\Delta l = 0$. Since Y has full column rank, $(h.3)$ gives $\Delta b_h = 0$ and $(h.2)$ and the assumption that for every i there is h such that $D_i a'_h = 0$ then gives $dq^\backslash = 0$. Since Da'_h has full row rank, $\Delta\mu'_h = 0$. In this case it follows immediately from regularity of equilibrium that the derivative $(**)$ is surjective and the proof is complete.

Suppose for some h', $dx_{h'} = 0$. The original system of equation then gives

$$\begin{aligned}
D^2 u_h dx_h - \Psi^T d\lambda_h + Du_h d\alpha_h + dp^{\backslash} &= 0 \quad (h.1)\\
\mu_h' D^2 a_h db_h^{\backslash} + R^T d\lambda_h + \left(Da_h'\right)^T d\mu_h' + dq^{\backslash} &= 0 \quad (h.2)\\
\Psi dx_h + R db_h &= 0 \quad (h.3)\\
Da_h' db_h &= 0 \quad (h.4)\\
&\vdots\\
-\Psi^T d\lambda_{h'} + Du_{h'} d\alpha_{h'} + dp^{\backslash} &= 0 \quad (h'.1)\\
\mu_{h'}' D^2 a_{h'} db_{h'}^{\backslash} + R^T d\lambda_{h'} + \left(Da_{h'}'\right)^T d\mu_{h'}' + dq^{\backslash} &= 0 \quad (h'.2)\\
R db_{h'} &= 0 \quad (h'.3) \Rightarrow db_{h'} = 0\\
&\vdots\\
\sum_h \lambda_h^s dx_h^{s\backslash} + z_h^{s\backslash} d\lambda_h^s &= 0 \quad (m.1)
\end{aligned}$$

From $(h'.1)$ and using the first order conditions of household h' we get

$$\Psi^T \left(\lambda_{h'} d\alpha_{h'} - d\lambda_{h'}\right) + \begin{bmatrix} \vdots \\ \begin{pmatrix} 0 \\ dp^s \end{pmatrix} \\ \vdots \end{bmatrix} = 0$$

But then $\lambda_{h'} d\alpha_{h'} = d\lambda_{h'}$ which implies $dp^{\backslash} = 0$. Pre-multiplying $(h.1)$ by dx_h^T,we get

$$dx_h^T D^2 u_h dx_h + Du_h dx_h = 0$$

Since by $(h.3)$ and the fact that we take derivatives at an equilibrium, and in particular

$$\lambda_h R = -\mu_h' Da_h$$

we have, using $(h.3)$,

$$Du_h dx_h = \lambda_h \Psi dx_h = -\lambda_h R db_h = \mu_h' Da_h' db_h = 0$$

which gives $dx_h^T = 0$. Therefore, for every h we have $dx_h = 0$ and $\lambda_h d\alpha_h = d\lambda_h$. From $(m.1)$ we then obtain

$$\sum_h z_h^{s\backslash} \lambda_h^s d\alpha_h = 0$$

That for a weakly generic set of economies this equation implies $d\alpha_h = 0$ for every h follows from the next lemma. Then, regularity of equilibrium implies $d = 0$, which contradicts $d^T d - 1 = 0$. Therefore, for a weakly generic set of economies the last equation is possible only if $dx_h = 0$ for every h. But in such a case we already know by case 1 that, taking a weakly generic subset if necessary, that the desired system of equations does not have a solution. The proof is then complete once we prove the next result.

There is a generic set $(E \times \mathcal{U} \times \mathcal{A})^* \subset (E \times \mathcal{U} \times \mathcal{A})$ such that:

i) for every equilibrium and for any collection of locally restricted households, $H^* \subset \{1, \ldots, H\}$, $\dim H^* \leq S + 1$, *we have*

$$rank \left[\lambda_h^s z_h^{1,s}\right]_{h \in H^*}^{s=0,\ldots,S} = \dim H^*$$

ii) for every equilibrium and for any collection of locally restricted households $H^* \subset \{1, \ldots, H\}$, $\dim H^* \leq (C-1)(S+1)$, *we have*

$$rank \left[\lambda_h^s z_h^{\setminus s}\right]_{h \in H^*}^{s=0,\ldots,S} = \dim H^*$$

Once more openness is immediate. The denseness result uses the same perturbation of endowments and restricted functions used in lemmas 1 and 2 used in the of the previous proposition. Given those perturbations, one simple add to the system

$$F(\xi, u, a, e) = 0$$

the equations for case (i)

$$\begin{aligned} \left[\lambda_h^s z_h^{1,s}\right] a &= 0 \\ a^T a - 1 &= 0 \end{aligned}$$

and for case (ii)

$$\begin{aligned} \left[\lambda_h^s z_h^{\setminus s}\right] a &= 0 \\ a^T a - 1 &= 0 \end{aligned}$$

We have to show that the derivative of the equilibrium and additional equations is surjective. We show the desired result for case (i). Case (ii) is easier. Let $h = 1 \notin H^*$.Computing derivatives we obtain

$$
\begin{bmatrix} x_1 & b_1 & \lambda_1 & \mu_1' & \mu_1'' & e_1^1 & e_1^\backslash & p^\backslash & q \end{bmatrix}
$$

$$
\begin{bmatrix}
D^2u_1 & 0 & -\Psi^T & 0 & 0 & 0 & 0 & \Lambda_1 & 0 \\
0 & \mu_1' D^2 a_1' & R^T & Da_1' & 0 & 0 & 0 & 0 & -\lambda_1^0 I \\
-\Psi & R & 0 & 0 & 0 & I & \Psi^\backslash & \Pi_1 & \Upsilon_1 \\
0 & Da_1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & \vdots & \vdots \\
 & & 0 & & & & & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I^\backslash & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & & & & & 0 & 0 \\
 & & 0 & & & & & 0 & 0
\end{bmatrix} (\ldots)
$$

$$
\begin{bmatrix} \Big(x_h & b_h & \lambda_h & \mu_h' & \mu_h'' & e_h^1 & e_h^\backslash \Big) \ldots & p & q & \beta_h^j & a \end{bmatrix}
$$

$$
(\ldots) \begin{bmatrix}
D^2u_h & 0 & -\Psi^T & 0 & 0 & 0 & 0 & \Lambda_h & 0 & 0 & 0 \\
0 & \mu_h' D^2 a_h' & R^T & Da_h' & 0 & 0 & 0 & 0 & -\lambda_h^0 I & \mu_h^j \hat{b}_h & 0 \\
-\Psi & R & 0 & 0 & 0 & I & \Psi^\backslash & \Pi_h & \Upsilon_h & 0 & 0 \\
0 & Da_h' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \vdots & & & & & \vdots & \vdots & \vdots & 0 \\
0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
I^\backslash & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_h \Lambda_h^T & 0 & a_h Z_h & 0 & 0 & 0 & a_h \Lambda_h^T & 0 & 0 & 0 & * \\
 & & 0 & & & & & 0 & 0 & 0 & 2a
\end{bmatrix}
$$

where

$$
Z_h = \begin{bmatrix} \left(z_h^0\right)^T \\ \vdots \\ \left(z_h^S\right)^T \end{bmatrix}
$$

Pre-multiplying the matrix by

$$
\Big((dx_1, db_1, d\lambda_1), \ldots, (dx_h, db_h, d\lambda_h, d\mu_h), \ldots, dp^\backslash, dq, dv, da \Big)
$$

we immediately obtain $\left(d\lambda, db, d\mu, dp^\backslash\right) = 0$. Since for every i there is $h(i)$ such that $D_i a_h = 0$ then the column corresponding to $b_{h(i)}$ gives $dq^i = 0$ for every i.

If $a_h = 0$ then $dx_h = 0$. If $a_h \neq 0$, which is true for at least one component since $a \neq 0$, then by the column corresponding to $e_h^\backslash$ we have $dv = 0$ and hence by the column corresponding to x_h, $dx_h = 0$. Then, $da = 0$ and the proof is complete. □

6. Appendix

6.1. Generalization of the Existence Result

In order to prove existence of equilibrium assumptions $(H1)$ and $(H3)$ can be relaxed as follows.

$(H1')$ $u_h : \Re^G_{++} \to \Re \in C^0$ *is strictly increasing, quasi-concave and satisfies assumption* $(H1)(iii)$.

$(H3')$ $a^j_h : \Re^I \to \Re \in C^1$ *is quasi-concave,* $a^j_h(b_h) \geq 0$ *and satisfies* $(H4)$.

One can also generalizes the existence result to utility functions defined and continuous on the positive orthand and that do not satisfy $(H1)(iii)$. In this case one simple add the traditional non-negative constraints and Kuhn-Tucker multipliers to the consumer problem. The space of utility functions is taken to be the set of functions satisfying $(H1')$ endowed with the C^0 uniform topology on compact sets. The space of restricted functions is taken to be the set of functions satisfying $(H2')$ endowed with the C^1 uniform topology on compact sets. The generalization of the properness lemma for this case is a variation of the standard argument, and known in the literature as *upper-heme-continuity of the equilibrium correspondence.*[13] The existence result is then shown in this case by exploiting this lemma.

Let (u_h, a_h) satisfies $(H1')-(H2')$ for every h. Choose a compact set $K \subset \Re^G_{++}$ large enough so that every feasible equilibrium bundle

$$\left\{ x \in \Re^G_{++} / \ u_h(x) \geq u_h(e_h) \ \ for \ all \ h, \ x \leq \sum_h e_h \right\}$$

lies in the interior of K. That such a compact set exists follows from $(H1(iii))$. Let $K_b \subset \Re^I$ be a compact set large enough so that the set

$$\left\{ b \in \Re^I / \ \Psi(x - e_h) = Rb \ \ for \ some \ h \ and \ some \ x \in K \right\}$$

[13]See, for example, Hildenbrand and Mertens (1972).

is contained in the interior of K_b. The existence of K_b follows from $(H2)$. By Mas-Colell (1985, p.90), for every h there is a sequence $u_h(n)_{|K} \to u_{h|K}$ in the C^0 uniform topology that satisfies $(H1)$ and a sequence $a_h(n)_{|K_b} \to a_{h|K_b}$ in the C^1 uniform topology that satisfies $(H3)$ and hence, for n large enough, $a_h(n)_{|K_b}$ also satisfies $(H4)$. By the previous argument, for each n the economy $((u_h(n), a_h(n), e_h), Y)$ has an equilibrium, $\xi(n)$. By the properness lemma, taking a convergent sub-sequence if necessary, $\xi(n) \to \xi$. By continuity, ξ is an equilibrium of the limit economy, which concludes the proof.

6.2. Proof of Cass-Siconolfi-Villanacci Theorem

"Openness"

Let e be an economy satisfying (i) and (ii). Suppose for every neighborhood of e there is an economy e' that violates one of these restrictions. Since equilibrium exists for every economy, this implies the existence of a sequence $\{(\xi(n), e(n))\}$, $e(n) \to e$ such that either (i) or (ii) are violated for every n. By the properness lemma, taking a convergent subsequence if necessary, there is (ξ, μ) such that $\xi(n) \to \xi$ and $F(\xi, e) = 0$. By continuity of both the maximum and the determinant function, either (i) or (ii) are also violated for this limiting equilibrium. But this implies that the economy e does not satisfies either (i) or (ii), which is the desired contradiction.

"Full measure"

$$\text{Let } \mathcal{C} = \left\{ (J'_h, J''_h)_{h=1}^{H} \ s.t.\ J''_h \subset J'_h \subset \{1, \ldots, J_h\} \ \ for\ all\ h \right\}$$

Fix $C \in \mathcal{C}$ and let $J' = \sum_h \#J'_h$, $J'' = \sum_h \#J''_h$,

$$a'_h(b_h) = \left(a^j_h(b_h)\right)_{j \in J'_h}, \quad a'''_h(b_h) = \left(a^j_h(b_h)\right)_{j \in J'_h \setminus J''_h}, \quad \mu''_h = \left(\mu^j_h\right)_{j \in J''_h}, \quad \mu'''_h = \left(\mu^j_h\right)_{j \in J'_h \setminus J''_h}$$

Consider the function $G^C : \Xi \times \Re^J \to \Re^{n-J+J'+J''} \in C^1$ given by

$$G^C(\xi, e) = \begin{bmatrix} \vdots \\ Du_h(x_h) - \lambda_h \Psi \\ \lambda_h R + \mu_h''' Da_h'''(b_h) \\ -\Psi z_h + R b_h \\ a_h'(b_h) \\ \mu_h'' \\ \vdots \\ \sum_h z_h^{\backslash} \\ \sum b_h \\ p^{11} - 1 \\ \vdots \\ p^{S1} - 1 \\ \sum_c \left(p^{0c}\right)^2 + \sum_i \left(q^i\right)^2 - 1 \end{bmatrix}$$

The derivative $D_{\xi,e}G^C$ contains the following matrix

$$
\begin{array}{c}
\left[\cdots\left[\begin{array}{cccccc} x_h & b_h & \lambda_h & \mu_h'' & e_h^1 & e_h^\backslash \end{array}\right] \quad \cdots \quad p \right] \\
\left[\begin{array}{cc}
 & \vdots \\
\begin{bmatrix} D^2u_h & 0 & -\Psi^T & 0 & 0 & 0 \\ 0 & \mu_h''' D^2 a_h''' & R^T & 0 & 0 & 0 \\ -\Psi & R & 0 & 0 & I & \Psi^\backslash \\ 0 & Da_h' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} & \begin{matrix} -\Lambda_h \\ 0 \\ -Z_h^T \\ 0 \\ I \end{matrix} \\
\vdots & \vdots \\
\begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ I^\backslash & 0 & 0 & 0 & 0 & I^\backslash \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\
0 & \begin{bmatrix} 0 & & 0 \\ & [1\ 0] & \\ 0 & & [1\ 0] \end{bmatrix} \\
0 & \begin{bmatrix} 2p^{0T} & 0 & 0 \end{bmatrix}
\end{array}\right]
\end{array}
$$

where

$$
\Lambda_h = \begin{bmatrix} \lambda_h^0 I & & 0 \\ & \lambda_h^s I & \\ 0 & & \lambda_h^S I \end{bmatrix} \quad and \quad Z_h = \begin{bmatrix} z_h^0 & & 0 \\ & z_h^s & \\ 0 & & z_h^S \end{bmatrix}
$$

We claim this matrix has full row rank. Let

$$
d = \left(\ldots, dx_h, db_h, d\lambda_h, d\mu_h', d\mu_h'', \ldots, dq, dp^\backslash, \ldots, d^s, \ldots, d^0\right) \in \Re^{n+J'+J''}
$$

and assume $dDG^C = 0$. We have to show that this implies $d = 0$. The system of equations is given as follows

$$
\begin{aligned}
dx_h D^2 u_h - d\lambda_h \Psi + dp^\backslash &= 0 \quad (1_h) \\
db_h \mu_h''' D^2 a_h''' + d\lambda_h R + d\mu_h' Da_h' + dq &= 0 \quad (2_h) \\
-dx_h \Psi^T + db_h R^T &= 0 \quad (3_h) \\
db_h Da_h' &= 0 \quad (4_h) \\
d\mu_h'' &= 0 \quad (5_h) \\
d\lambda_h &= 0 \quad (6_h) \\
d\lambda_h \Psi^\backslash - dp^\backslash &= 0 \quad (7_h)
\end{aligned}
$$

$$\sum_h (-dx_h\Lambda_h - d\lambda_h\Pi_h) + \begin{pmatrix} d^0 2p^{0T} \\ \vdots \\ d^s \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vdots \end{pmatrix} = 0 \quad (p)$$

$$\sum_h \left(-\lambda_h^0 db_h - d\lambda_h \Upsilon_h\right) + d^0 2q^T = 0 \quad (q)$$

From $(5_h) - (7_h)$ we get $d\mu_h'' = 0$, $d\lambda_h = 0$ for every h and $dp^\backslash = 0$. Therefore, from (1_h) we have $dx_h = 0$, and using (3_h) and assumption $(H2)$, $db_h = 0$ for every h. By assumption, for every i there is $h(i)$ such that $D_i a_h^j = 0$ for every $j \in J_h'$ and hence by equation $(2_{h(i)})$ we have $dq^i = 0$. Therefore, we get $dq = 0$. Condition $(H4)$ and equation $(2_{h'})$ implies $d\mu_h' = 0$ for every h. Finally, (p) implies $d^0 = d^s = 0$ for every s, which completes the proof.

By the Transversality Theorem, there is a full measure set $E^C \subset E$ such that if $e \in E^C$ and $G^C(\xi, e) = 0$ then $D_{\xi,\mu}G^C$ has full rank. For $C \in \mathcal{C}$ such that $J_h = \emptyset$ for every h this implies that any solution (ξ, e) is locally unique. The Lemma then implies that for each $e \in E^C$ there at most finitely many ξ such that $G^C(\xi, e) = 0$. For the remaining choice of C, the matrix $D_{\xi,\mu}G^C$ has more rows than columns. Therefore, for any such C the system $G^C(\xi, e) = 0$ has no solution. Let $M = \Xi \times E$. Notice that

$$\{(\xi, e) \in M / \ F(\xi, e) = 0\} \subset \bigcup_{C \in \mathcal{C}} \left\{(\xi, e) \in M / \ G^C(\xi, e) = 0\right\}$$

Let $E^{**} = \cap_{c \in \mathcal{C}} E^C$. Since $\mathcal{C}$ is finite, and each $E^C \subset E$ is a full measure set, $E^{**} \subset E$ also has full measure.

Fix $e \in E^{**}$ and consider an equilibrium associated with this economy ξ. By the previous argument for each pair (h, j) we must have

$$\max\left\{a_h^i(b_h), \mu_h^i\right\} > 0$$

Since the number of equilibria is finite, by continuity there is a neighborhood $V_e \subset \Xi \times E$ such that

$$\max\left\{a_h^i(b_h), \mu_h^i\right\} > 0$$

for every $(\xi, e) \in V_e$. Define the open set

$$D = \cup_{e \in E^*} V_e$$

and consider the extended system of equations restricted to the set D, $F : D \rightarrow \Re^{n+J}$. Then, $F \in C^1$ by construction. Following essentially the same argument used to show that DG^C has full row rank, one can show that DF also has full row rank. Then, by the Transversality theorem, $D_{\xi,\mu}F$ has full rank for a full measure set $E^* \subset E$, which completes the proof. □

6.3. Perturbation of Restrict and Utility Functions

We endow the set $\mathcal{A}_h$ with the topology of uniform convergence on compact sets: let $a_h \in \mathcal{A}_h$ and $K \subset \Re^I$ be a compact set for every h. Define the set $V(a_h, K, \varepsilon)$ to be

$$\left\{ a_h' \in \mathcal{A}_h / \sup_{b_h \in K} \sum_{s=0}^{2} \|D^s a_h(b_h) - D^s a_h'(b_h)\| < \varepsilon \right\}$$

The C^2 topology of convergence in compact sets is the topology generated by the collection of all possible sets $V(a_h, K, \varepsilon)$. This topology is compatible with a metric. Fix the sequence of compact sets $\{B_n\}$

$$B_n = \left\{ b \in \Re^I / \|b\| < n \right\}$$

Let $d : \mathcal{A}_h \times \mathcal{A}_h \rightarrow \Re_+$

$$d(a_h, a_h') = \sum_n \frac{1}{2^n} \min \left\{ \sup_{b_h \in B_n} \sum_{s=0}^{2} \|D^s a_h(b_h) - D^s a_h'(b_h)\|, 1 \right\}$$

It is simple to verify that d is a metric. Moreover, let $K \subset \Re^I$ be a compact, non-empty, set, and

$$\mathcal{A}_h(K) = \left\{ a_h : K \rightarrow \Re^{J_h} \in C^2 / \ a_h \in \mathcal{A}_h \right\}$$

Then, the C^2 topology of convergence in compact sets restricted to the set $\mathcal{A}_h(K)$ is compatible with the norm:

$$\|a_h\|_2 = \sup_{b_h \in K} \sum_{s=0}^{2} \|D^s a_h(b_h)\|$$

We also endow the set $\mathcal{U}_h$ with the same topology.

Let (ξ^*, e^*) be a regular equilibrium associated with a given restriction for each household $a_h(\cdot)$. Let $b^* = (b_1^*, \ldots, b_H^*) \in \Re^{HI}$. Fix an open ball, $B(1)$ centered at b^* and for a given b let $J_h'(b_h) \subset \{1, \ldots, J_h\}$ denote the set of indexes

j such that $a_h^j(b_h) = 0$. By taking this neighborhood small enough, we can assure that for every h and b in this neighborhood the following hold

i) the set $J_h'(b_h)$ *does not depend on* b_h^j;

ii) $\left[Da_h^j(b_h)\right]^{j \in J_h'}$ *is non-singular;*

Let $B(2)$ be an open ball also centered at b^*, $clB(2) \subset B(1)$ and let $\pi_h : B(1) \to \Re^I$ be the projection, $\pi_h(b) = b_h$.

Consider the set

$$\Theta = \left\{a : clB(1) \to \Re^{J_h} \in C^\infty / \ a \text{ is differentiably strict quasi-concave}\right\}$$

By Mas-Colell (1985, p.90) there is a sequence $\{a_h(n)\} \subset \Theta$, $a_h(n) \to a_h$ in the C^1 uniform topology. For n large enough $a_h(n)_{|clB(1)}$ satisfies $(H4)$. Moreover, by taking

$$a_h(n,k) = a_h(n) + \frac{1}{k}$$

if necessary, we can assume $a_h(n)$ satisfies $(H3)$. Since we only use perturbations of restriction functions to get denseness, we can assume without loss of generality that $a_h' \in C^\infty$ is differentiable strictly quasi-concave. By the same argument, we can assume $u_h \in C^\infty$.In what follows we shall refer to a_h' as a_h.

Consider the following perturbation of a_h:

$$a_h^j(b_h, h_h, H_h) = a_h^j(b_h) + \Phi_h(b_h)\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right]$$

where $\Phi_h : \Re^I \to [0,1] \in C^\infty$ satisfies

$$\Phi_h(b_h) = \begin{cases} 1 \ \ if \ \ b_h \in \pi_h(B(2)) \\ 0 \ \ if \ \ b_h \notin \pi_h(B(1)) \end{cases}$$

$h_h^j \in \Re^I$, $H_h^j \in \Re^{I \times I}$ for every h and j.

Lemma: *There is* $\delta > 0$ *such that* $\|h_h^j\|, \|H_h^j\| < \delta$ *for every* j *implies*

$$a_h^j(b_h, h_h, H_h) \in \mathcal{A}_h$$

Proof: Let

$$m = \min_{h, b_h \in clB(1)} \| \det \left[Da_h^j\right]^{j \in J_h'} \|$$

By construction of the neighborhood $B(1)$, $m > 0$. By taking δ small enough, condition $(H4)$ is trivially satisfied. Let S denote the unit sphere. We claim that there is $\eta > 0$ such that $\|d^T Da_h^j\| < \eta$ implies $d^T \left(D^2 a_h^j\right) d < -\eta$. In fact, suppose that this is not true. Then, for every $n > 0$ there is $d(n) \in S$ and $b_h(n) \in B(1)$ such that $\|d^T(n) Da_h^j(b_h(n))\| < 1/n$ and $d^T(n)\left[D^2 a_h^j(b_h(n))\right] d(n) \geq -1/n$. Since S and $clB(1)$ are compact, taking a subsequence if necessary, there is $d \in S$ and $b_h \in clB(1)$ such that $d(n) \to d \neq 0$, $b_h(n) \to b_h$, $d^T Da_h^j(b_h) = 0$ and $d^T\left(D^2 a_h^j(b_h)\right) d = 0$, which contradicts the differentiably strict quasi-concavity of $a_h^j(\cdot)$.

By definition

$$\begin{aligned} Da_h^j(b_h, h_h, H_h) &= Da_h^j(b_h) + D\Phi_h(b_h)\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right] \\ &\quad -\Phi_h(b_h)\left[h_h^j + H_h^j(b_h^* - b_h)\right] \end{aligned}$$

$$\begin{aligned} D^2 a_h^j(b_h, h_h, H_h) &= D^2 a_h^j(b_h) + D^2\Phi_h(b_h)\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right] \\ &\quad -2D\Phi_h(b_h)\left[h_h^j + H_h^j(b_h^* - b_h)\right] + \Phi_h(b_h)\left[H_h^j\right] \end{aligned}$$

Choose $\delta_h^j > 0$ small enough such that $\|h_h^j\|, \|H_h^j\| < \delta_h^j$ implies

$$\sup_{b_h \in clB(1), d \in S} \|d^T D\Phi_h(b_h)\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right]$$
$$-d^T \Phi_h(b_h)\left[h_h^j + H_h^j(b_h^* - b_h)\right]\| < \eta$$
$$\sup_{b_h \in clB(1)} \det\left(D^2\Phi_h\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right]\right.$$
$$\left.-2D\Phi_h\left[h_h^j + H_h^j(b_h^* - b_h)\right] + \Phi_h(b_h)\left[H_h^j\right]\right) < \eta/2$$

Let $d \neq 0$ such that $d^T Da_h^j(b_h, h_h, H_h) = 0$. This implies

$$\|d^T Da_h^j(b_h)\| = \| - d^T D\Phi_h(b_h)\left[(b_h^* - b_h)^T h_h^j + \frac{1}{2}(b_h^* - b_h)^T H_h^j (b_h^* - b_h)\right]$$
$$-d^T \Phi_h(b_h)\left[-b_h^T h_h^j + H_h^j(b_h^* - b_h)\right]\| < \eta$$

and thus $d^T \left(D^2 a_h^j(b_h)\right) d < -\eta$, which gives the desired result. □

By a similar argument, we can use the same perturbations in the space of utility functions.

7. References

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