

"MORAL HAZARD AND NONLINEAR PRICING IN A GENERAL EQUILIBRIUM MODEL"

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Moral Hazard and Nonlinear Pricing in a General Equilibrium Model*

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Abstract

The paper analyzes a two period general equilibrium model with individual risk and moral hazard. Each household faces two individual states of nature in the second period. These states solely differ in the household's vector of initial endowments, which is strictly larger in the first state (*good state*) than in the second state (*bad state*). In the first period households choose a non-observable *action*. Higher levels of *action* give higher probability of the good state of nature to occur, but lower levels of utility. Households have access to an insurance market that allows transfer of income across states of nature. I consider two models of financial markets, the *price-taking behavior model* and the *nonlinear pricing model*.

In the price-taking behavior model suppliers of insurance have a belief about each household's *action* and take asset prices as given. A variation of standard arguments shows the existence of a rational expectations equilibrium. For a generic set of economies every equilibrium is *constrained sub-optimal*: there are commodity prices and a reallocation of financial assets satisfying the first period budget constraint such that, at each household's optimal choice given those prices and asset reallocation, markets clear and every household's welfare improves.

In the nonlinear pricing model suppliers of insurance behave strategically offering nonlinear pricing contracts to the households. I provide sufficient conditions for the existence of equilibrium and investigate the optimality properties of the model. If there is a single commodity then every equilibrium is constrained optimal. If there is more than one commodity, then for a generic set of economies every equilibrium is constrained sub-optimal.

*Preliminary. Do not quote.

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1. Introduction

The standard Walrasian general equilibrium model has remarkable optimality properties summarized in the fundamental theorems of welfare. While requiring surprisingly few assumptions, the Pareto optimality of a competitive equilibrium rely strongly on the existence of complete financial markets. One could conjecture, however, that once markets are incomplete a weaker version of this theorem should remain true; a version that would take into account precisely the available financial assets.

An optimality definition that takes into account the available financial assets was proposed by Diamond (1967). An equilibrium is *constrained optimal* if there are no commodity prices and reallocation of financial assets satisfying households' first period budget constraint such that, at every household's optimal choice given those prices and assets reallocation, markets clear and every household gets strictly better off. An equilibrium is constrained sub-optimal if it is not constrained optimal. Stiglitz (1981) and Greenwald and Stiglitz (1986) argue that in economies with more than one commodity and incomplete financial markets, every equilibrium must be constrained sub-optimal. This claim is formally proved by Geanakoplos and Polemarkachis (1986). They show that in the numéraire asset model with incomplete markets generically every equilibrium is constrained sub-optimal, provided the existence of more than one commodity and a upper bound on the number of households.^{1,2}

The incompleteness of financial markets is usually attributed in the general equilibrium literature to market imperfections such as moral hazard.³ In a typical example, which will be exploited in this paper, there are two periods and a household face two states of nature in the second period. The household's vector of endowments in the first state is strictly larger than in the second state. In the first period the household chooses an *action* level. Higher levels of action imply a higher probability of the first state, but lower utility for the household.⁴ For an extreme parameterization of this example, financial markets must be incomplete in equilibrium. Let the probability of the first state be zero when the household chooses the lowest level of *action*. Suppose

¹Generically means for an open and dense set of economies, where the set of economies is parameterized by utility functions and the allocation of initial endowments.

²The inefficiency of the competitive equilibrium when markets are incomplete was already noticed by Hart (1975). He constructed an example showing that the introduction of a new financial asset may result in an equilibrium that reduces every household's welfare. Cass and Citanna (1994) and Elul (1995) show that Hart's result hold for a generic set of economies.

³For a survey on the literature of general equilibrium models with incomplete financial markets, see Geanakoplos (1990) and Magill and Shaffer (1991).

⁴This example corresponds to a variation of the moral hazard model, that has been developed during the last 20 years. See Tirole (1988, chap.1) for a presentation of the model and further references.

there are complete financial markets in equilibrium. Then, the household can get arbitrarily large amounts of Arrow securities with payoff contingent on the second state, which will happen for sure, and finance this position by selling Arrow securities with payoff contingent on the first state, a zero probability event. But this corresponds to an arbitrage opportunity, and thus one cannot have equilibrium with complete financial markets.

Despite the existence of motivating examples for endogenously incomplete financial markets, market imperfections are usually not explicitly incorporated into the standard incomplete markets model.⁵ The relevance of the sub-optimality property in the incomplete markets model has been frequently questioned precisely on the grounds that in a general equilibrium model the existent financial assets should be an endogenous variable, determined in equilibrium, and not a primitive of the model.

One may argue that the standard general equilibrium models with incomplete markets are simply a reduced form of a more general model in which the markets imperfection are actually taken into account and the financial assets are endogenously determined. This argument, however, does not lead to a generalization of the constrained suboptimality result for a model with endogenous financial markets. The difficulty lies in the functional dependence between asset payoffs and the remaining equilibrium variables that this more general framework may imply. It is possible that once financial markets are endogenized, a competitive equilibrium may be constrained, or even Pareto, optimal. The investigation of this question requires a model that endogenizes the financial structure or at least a model that explicitly takes into account the market imperfections that lead to the existence of incomplete financial markets.

This paper presents a general equilibrium model with moral hazard and strategic behavior in financial markets. The available financial assets are determined endogenously in equilibrium. Contrarily to the example presented above, in equilibrium households have access to as many assets as states of nature in the second period. They face, however, endogenous constraints on the amount of insurance they can buy. The paper investigates the optimality properties of an equilibrium, and in particular it shows that the constrained suboptimality properties of economies with incomplete financial markets generalizes to this model, provided there is more than one commodity.

I consider an extension of Malinvaud's model of individual risk.⁶ I analyze a general equilibrium model with two periods, C commodities in the second period, finitely many

⁵There are few exceptions. Allen and Gale (1988, 1991) proposed a model with endogenous financial assets. Their model, however, assumes a single commodity, which precludes a discussion about the constrained suboptimality property of equilibrium in the incomplete markets model. Bisin (1994) also endogenizes the financial assets in order to investigate indeterminacy of equilibria in models with nominal assets.

⁶See Malinvaud (1972, 1973). An extension of Malinvaud's model with emphasis on financial markets is provided by Cass, Chichilnisky and Wu (1995).

types of households and a large number of identical households of each type. Each household faces an individual risk in the second period. With probability π_{th} in the second period household h of type t faces an individual state s_{th}^1 (*good state*) and with probability $1 - \pi_{th}$ the household will face state s_{th}^2 (*bad state*). The only difference between both states is the amount of endowments received by the household. In state s_{th}^1 the household receives e_t^1 , while in state s_{th}^2 the household receives e_t^2 , $e_t^1 \gg e_t^2$. Since preferences are assumed to be monotonic, in the absence of insurance the household strictly prefers the good state to the bad state.

Contrary to Malinvaud's model, it is assumed that the probability of the good state depends on an *action* chosen by the household in the first period, $a_{th} \in [0, 1)$. This *action* is non-observable by any of the remaining agents. As in the moral hazard literature, higher levels of *action* imply a higher probability of the good state to occur, but lower utility for the household. Under standard assumptions of risk aversion, households would rather smooth consumption across states of nature, provided that the price of transferring income across states is not much higher than the fair price. If *actions* were observable, in any equilibrium households would get full insurance and the price of transferring income would be precisely the fair price.

If households' *actions* are not observable what should the asset prices be? As an example, suppose there are countably many identical households, and the individual risk is independent across households. Therefore, there is no aggregate risk. A natural agreement between households would be to find the optimal level of consumption and *action* that maximizes the representative household welfare. It is simple to verify that at such an agreement households get full insurance in the second period. However, since the *action* levels are not observable, this agreement is not enforceable, except if the optimal level of *action* correspond to $a_t = 0$. In fact, any agreement that requires $a_t > 0$ with full insurance in the second period is not enforceable: choosing $a_t = 0$ increases the household utility in the first period with no individual cost in the second period. Therefore, any enforceable agreement that involves *action* levels $a_t > 0$ must be associated with partial insurance. But this immediately implies that any enforceable agreement cannot be Pareto optimal. One may refer to an enforceable agreement that maximizes households welfare as *constrained sub-optimal*, *second best* or *optimal given the informational constraints*.

In a model with moral hazard there is an obvious motivation for introducing strategic behavior in the trade of insurance: a seller of insurance can always use the amount of insurance bought by a household as a signal of the *action* chosen by the household. In this case, it seems natural to allow sellers of insurance to set either asset prices or borrowing constraints as a way of providing incentives to the household to take higher levels of *action* when there is a net gain from it. While I allow for strategic behavior in the financial markets, I keep the standard price-taking behavior assumption in the commodity markets. However, if there are finitely many households, equilibrium prices

depend on the aggregate state of nature, which in turn depends on the *action* chosen by each household. Therefore, price-taking behavior assumption on commodity markets implies bounded rationality for the households. In this case one does not know if the equilibrium properties obtained depend on the bounded rationality assumption for some, but not all, agents, or if it results uniquely from the moral hazard assumption.

In order to differentiate between the consequences of introducing moral hazard and of imposing bounded rationality, I assume the existence of countably many identical households of each type, and finitely many types.⁷ This modeling has the advantage of preventing the well known problem of using the law of large number with a continuum of random variables and, at the same time, it makes precise the argument that the aggregate values do not depend on an individual household behavior.⁸

The introduction of financial assets with payoffs contingent on the individual state of nature presents some problems in this framework. Since the number of households is infinite, and there are two individual states per household, if financial markets are complete then the number of financial assets is also infinite. This creates several additional technical problems in analyzing the household's problem. Each household now can potentially trade infinitely many assets. However, precisely because there are countably many households of each type, each one facing an individual risk, there is no aggregate uncertainty. Each household, then, really only cares about obtaining insurance against her individual risk. Nevertheless, in order to buy such an insurance, the household must buy assets from infinitely many distinct households of the same type.

In order to avoid these difficulties, I introduce a pooling risk technology. I assume the existence of finitely many firms (*insurance companies*) that offer the following contract to each household of each type t : if the good state s_{th}^1 happens then the household pays bq units of numéraire to the firm, if the bad state s_{th}^2 happens then the firm pays b units of numéraire to the household. This contract is equivalent to the household trading Arrow securities contingent on the individual risk, the relative price between those securities being q . I assume that firms maximize expected profits. If each firm sells the same amount of contract to countably many households of each type then the firm faces no uncertainty in the second period. Firms are equally owned by the households. Since, as I show below, in equilibrium firms always make zero profits, this assumption is without loss of generality. I consider two alternative models for the insurance market, the *price-taking behavior model* and the *nonlinear pricing model*.

In the *price-taking behavior model* firms behave competitively, taking asset prices

⁷In this paper, a set is said to have countably many elements if there is a bijection between this set and the set of natural numbers. Therefore, a countable set is, necessarily, an infinite set.

⁸Judd (1985) and Feldman and Giles (1985) present the basic problems of using the law of large numbers in models with a continuum of households. The alternative of using countably many households was proposed in the literature of random matching. Gilboa and Matsui (1992) present a model in this spirit and also provide further references.

as given. Each firm also has an expectation about the *action* level chosen by each type of household. Expectations are required to be fulfilled in equilibrium. A variation of standard arguments show existence of equilibrium. At equilibrium households get full insurance and choose the minimum level of action, $a_t = 0$. For a generic set of economies, parameterized by initial endowments, every equilibrium is constrained sub-optimal: there is a reallocation of assets' demand and relative prices such that commodities markets clear and every household gets strictly better off.⁹

The result changes, however, once one allows strategic behavior in the supply of insurance. In the *nonlinear pricing model* for the financial market firms behave strategically, choosing a contract to offer to a household of each type. This contract specifies the household's transfer to the firm in case the good state happens to occur, and the firm's transfer to the household otherwise. Each firm has a belief about the other firms' offers and the households' behaviors, and chooses its own offer in order to maximize expected profits. A household of type t then chooses the contract that maximizes utility. An equilibrium requires every belief to be fulfilled and markets to clear. Under suitable assumptions, there is a nonlinear pricing equilibrium for every economy. Moreover, if there is a unique commodity in the second period, every equilibrium is constrained optimal. If there is more than one commodity, however, for a generic set of economies every equilibrium is constrained sub-optimal.

2. The Basic Model

Consider an exchange economy with 2 periods and C commodities in the second period. There are finitely many types of households indexed by $t \in \{1, \dots, T\}$, and countably many identical households of each type.^{10,11} In the first period each household h of type t chooses an *action* $a_{th} \in [0, 1]$. I assume that this *action* is not observable by any other agent in the economy. In the second period each household of type t faces two individual states of nature, and in each state she receives a bundle of endowments $e_t^s \in \mathbb{R}_{++}^C$. The probability associated with these states of nature depends on the *action* chosen by the

⁹The equilibrium properties of the price-taking behavior model is well known in the partial equilibrium literature. See, for example, Stiglitz (1993), and the references given there. I am not aware of a formulation of this model in the general equilibrium framework. In any case, the only role of this model in the paper is to serve as a benchmark to analyze the nonlinear pricing model.

¹⁰Recall that in this paper a set has countably many elements if there is a bijection between this set and the set of natural numbers.

¹¹This construction implies a uniform distribution of households across types. One can easily relax this assumption by introducing an additive (but not σ -additive) measure as in Gilboa and Matsui (1992). The relative size of each type is then taken as a primitive of the model. This additive measure has the advantage of attributing measure zero for each individual household and being defined for every subset of households. All results remain unchanged if this measure is introduced and types of different sizes are allowed.

household in the first period and it is independent across households of the same type.¹² Each household is characterized by a utility function $U_t : \mathbb{R}_{++}^{2C} \times [0, 1] \rightarrow \mathbb{R} \in C^\infty$, an initial distribution of endowments $e_t \in \mathbb{R}_{++}^C \times \mathbb{R}_{++}^C$ and a function that for each *action* taken in the first period gives the probability of state 1 happening in the second period $\pi_t : [0, 1] \rightarrow (0, 1) \in C^\infty$. It is assumed that for every type t the following holds:

- (H1) $U_t(x, a) = \pi_t(a)u_t(x^1) + (1 - \pi_t(a))u_t(x^2) - v_t(a)$. For all $(x, a) \in \mathbb{R}_{++}^C \times [0, 1]$:
- i) $D_x u_t(x) \gg 0$;
 - ii) $D_a v_t(0) = 0$, $D_a v_t(a) > 0$ for $a > 0$, $\lim_{a \rightarrow 1} D_a v_t(a) = \infty$;
 - iii) $D^2 u_t$ is negative definite and $D^2 v_t$ is strictly positive;
 - iv) $cl \left\{ y \in \mathbb{R}_{++}^C / u_t(y) \geq u_t(x) \right\} \subset \mathbb{R}_{++}^C$;
 - v) $D\pi_t(a) > 0$, $D^2\pi_t(a) < 0$;
 - vi) $e_h^2 \ll e_h^1$.

Assumptions (i), (iii)–(v) are standard. The first part of assumption (ii) guarantees that higher levels of *action* reduces the household utility. Assumption $D_a v_t(0) = 0$ prevents discontinuities on the household optimal choice. The second part of assumption (ii) can be relaxed if instead one assume $a_t \in [0, 1]$. The separability of utility function can be relaxed as well. The only cost is a significant increase in the complexity of the computations.

There are 2 financial assets available in this economy for each household. These asset's payoffs are contingent on the realization of the household individual state of nature in the second period. To simplify the analysis, I assume that all assets pay in units of the first commodity. Let y_t^s be the asset that pays one unit of commodity 1 if state s happens for a household of type t and 0 otherwise. Let q_t^s be the first period asset price, and b_t^s be the household demand for the asset. Suppose there is no aggregate uncertainty, and household expect second period commodity prices p .¹³ Let p^c denote the price of the c -th commodity. In this case, the budget constraint of a household of type t can be written as follows

$$\begin{aligned} q_t^1 b_t^1 + q_t^2 b_t^2 &= 0 \\ p(x_t^1 - e_t^1) - p^1 b_t^1 &= 0 \\ p(x_t^2 - e_t^2) - p^1 b_t^2 &= 0 \end{aligned}$$

Therefore, without any loss of generality, we can restrict the analysis to commodity

¹²The independence assumption is strong and can be significantly relaxed. The arguments presented in the paper only require the individual risk to satisfy De Finetti's notion of exchangeability. See Kingman (1978).

¹³For purposes of computations, I treat prices and Lagrangian multipliers as row vectors.

prices in the set

$$\begin{aligned} P & : = \{p \in \mathfrak{R}_{++}^C / p^1 = 1\} \\ Q & : = \{q_t \in \mathfrak{R}_{++}^2 / q_t^1 = 1\} \end{aligned}$$

With some abuse of notation, I will use the notation $q_t \in \mathfrak{R}_{++}$ for the price of the second asset, b_t for b_t^2 and write the budget constraint in following, equivalent way

$$\begin{aligned} p(x_t^1 - e_t^1) + q_t b_t &= 0 \\ p(x_t^2 - e_t^2) - b_t &= 0 \end{aligned}$$

Notice that by strict concavity of the utility function, all households of the same type have the same decision rules. Let

$$\{x_t(p, q_t), a_t(p, q_t), b_t(p, q_t)\}$$

denote a household of type t 's optimal decision when facing commodity prices p and asset prices q_t . In this case, commodity market clearing is given by

$$\sum_t \left[\pi_t(a_t(p, q_t)) (x_t^1(p, q_t) - e_t^1) + (1 - \pi_t(a_t(p, q_t))) (x_t^2(p, q_t) - e_t^2) \right] = 0$$

By Walras' law, one can drop the market clearing equation for the first commodity. If $z \in \mathfrak{R}^C$ we write $z^\backslash \in \mathfrak{R}^{C-1}$ for vector with the same components as z but the first one. Therefore, the market clearing equation can be written as

$$\sum_t \left[\pi_t(a_t(p, q_t)) (x_t^\backslash(p, q_t) - e_t^\backslash) + (1 - \pi_t(a_t(p, q_t))) (x_t^{2\backslash}(p, q_t) - e_t^{2\backslash}) \right] = 0$$

What is the equivalent of market clearing equation for financial assets in this model? If in the second period individual risks are perfectly shared across households of the same type then we must have in the second period

$$\pi_t(a_t(p, q_t)) q_t b_t(p, q_t) = (1 - \pi_t(a_t(p, q_t))) b_t(p, q_t)$$

which gives, provided $b_t(p, q_t) > 0$,

$$q_t = \frac{(1 - \pi_t(a_t(p, q_t)))}{\pi_t(a_t(p, q_t))}$$

I refer to this asset price as *fair price*. Notice that in the nonlinear pricing model where suppliers of insurance behave strategically I cannot restrict the analysis to asset prices

satisfying the above conditions. Agents in this model cares about their profit function and *not* about aggregate feasibility. I will show, however, that even in this model competition will lead sellers of insurance to set asset prices equal to the fair prices. This result will be equivalent to a non-profit condition for a supplier of insurance that trades with countably many households of the same type.

Finally, I need an extra assumption relating household primitives and feasible allocations. This assumption is necessary in order to characterize the solutions of the household's problem by the Kuhn-Tucker equations and it guarantees the concavity of households' utility function on the feasible set. Consider the compact set

$$\Delta : = \left\{ \begin{array}{l} x \in \mathbb{R}_{++}^{2CT} / u_t(x^1) \geq u_t(x^2), U_t(x_t, a_t) \geq U_t(e_t, 0) \text{ and} \\ \sum_t \{ \pi_t(a_t) (x_t^1 - e_t^1) + (1 - \pi_t(a_t)) (x_t^2 - e_t^2) \} = 0 \text{ for some } a_t \text{ for all } t \end{array} \right\}$$

$$\Delta_t : = \text{Pr}_t \Delta, \text{ where } \text{Pr}_t : \Delta \rightarrow \Delta_t \text{ is given by } \text{Pr}_t(x) = x_t$$

(H2) $U_t : \Delta_t \times [0, 1) \rightarrow \mathbb{R}$ has negative definite Hessian.

Simple computations give the following proposition, which guarantees the existence of utility functions satisfying (H1) – (H2):

Let (u_t, v_t, e_t) satisfy (i) – (iv) and (vi) in assumption (H1). There is $\varepsilon > 0$ such that if a probability function π_t satisfies (v) in (H1) and $D_a \pi_t(a) \leq \varepsilon$ for every $a \in (0, 1)$ then U_t satisfies (H2).

3. Price-Taking Behavior Model

In the price-taking behavior model I assume the existence of potentially infinitely many firms. Each firm has a expectation of each type's choice of *action*, a_t^e , and take as given the asset price q . The firm then supplies an amount of insurance for each household in order to maximize profits. That is, when deciding how much to trade with an household of type t the firm chooses b in order to solve the problem

$$\max_b \pi_t(a_t^e) qb - \{1 - \pi_t(a_t^e)\} b$$

The firm's optimal supply is any point in the real line if

$$q = \frac{1 - \pi_t(a_t^e)}{\pi_t(a_t^e)}$$

The firm's problem has no solution if

$$q \neq \frac{1 - \pi_t(a_t^e)}{\pi_t(a_t^e)}$$

Suppose a firm trades with finitely many households of a certain type. Then, with positive probability the firm makes non-zero profits. We restrict the analysis to equilibria such that if a firm trades with a household of a certain type then the firm trade with countably many households of that type. This is equivalent to introducing Arrow securities contingent on the number of households of a certain type that are in the good state, and allowing the firms to trade those assets in order to insure themselves against the possibility of negative profits.

Given commodities prices p and asset prices q_t , a household of type t solves the following problem

$$\max_{a, x, b} U_t(x, a) \quad s.t. \quad \begin{cases} p(x^1 - e^1) + q_t b = 0 \\ p(x^2 - e^2) - b = 0 \end{cases}$$

Definition: An equilibrium in the price-taking model is a collection $((x_t, a_t, b_t, q_t)_{t=1}^T, p)$ such that:

- i) (x_t, a_t, b_t) solves a household t 's problem, given prices (p, q_t) ;
- ii) b_t solves the firm's problem when trading with a household of type t , given asset prices q_t and that firms expect a household of type t to take action $a_t^e = a_t$;
- iii) commodity markets clear.

Proposition 3.1: Let (H1)–(H2) hold. There is an equilibrium in the price-taking model. Moreover, at any equilibrium $a_t = 0$ for every t .

Proof: Suppose there is an equilibrium. In particular we must have $a_t^e = a_t$ and

$$q_t = \frac{1 - \pi_t(a_t)}{\pi_t(a_t)}$$

By the Kuhn-Tucker theorem, the solutions to the household's problem is completely characterized by the following system of equations

$$\begin{bmatrix} D_a \pi_t(a_t) \{u_t(x_t^1) - u_t(x_t^2)\} - D_a v_t(a_t) \\ \pi_t(a_t) Du_t(x_t^1) - \lambda_t^1 p \\ (1 - \pi_t(a_t)) Du_t(x_t^2) - \lambda_t^2 p \\ -\lambda_t^1 q_t + \lambda_t^2 \\ p(x_t^1 - e_t^1) + q_t b_t \\ p(x_t^2 - e_t^2) - b_t \end{bmatrix} = 0$$

Therefore, we get

$$\frac{\lambda_t^2}{\lambda_t^1} = q_t \quad \text{and} \quad x_t^1 = x_t^2$$

But then the first equation gives $Dv_t(a) = 0$. Thus, by assumption (H1) we must have $a = 0$, which is the desired result.

In order to show existence of an equilibrium it is sufficient to show the existence of a solution to the following system of equations

$$\begin{bmatrix} \vdots \\ \pi_t(0) Du_t(x_t^1) - \lambda_t^1 p \\ (1 - \pi_t(0)) Du_t(x_t^2) - \lambda_t^2 p \\ \lambda_t^1 \frac{1 - \pi_t(0)}{\pi_t(0)} - \lambda_t^2 \\ p(x_t^1 - e_t^1) + \frac{1 - \pi_t(0)}{\pi_t(0)} b_t \\ p(x_t^2 - e_t^2) - b_t \\ \vdots \\ \sum_t [\pi_t(0)(x_t^1 - e_t^1) + [1 - \pi_t(0)](x_t^2 - e_t^2)] \end{bmatrix} = 0$$

which can be alternatively written as

$$\begin{bmatrix} \vdots \\ Du_t(x_t) - \lambda_t p \\ p(x_t - \bar{e}_t) \\ \vdots \\ \sum_t (x_t - \bar{e}_t) \end{bmatrix} = 0$$

where $\bar{e}_t := \pi_t(0)e_t^1 + (1 - \pi_t(0))e_t^2$. But then we have reduced the problem to showing existence of an equilibrium in a smooth Walrasian economy, where equilibrium always exist.¹⁴ \square

The properties of equilibrium in the price-taking behavior model contrasts with the properties of equilibrium when *actions* are observable. In this case, for each t , the asset price can be taken as a function of the actual *action* level chosen by the household

$$q_t(a_t) = \frac{1 - \pi_t(a_t)}{\pi_t(a_t)}$$

Proposition 3.2: *Let (H1) – (H2) hold and suppose actions are observable. Then, at every competitive equilibrium we have*

$$x_t^1 = x_t^2 \quad \text{and} \quad a_t > 0 \quad \text{for all } t$$

¹⁴ A existence argument for the smooth Walrasian model can be found in Balasko (1988), Mas-Colell (1985) or Smale (1974)

This proposition follows immediately from the Kuhn-Tucker equations characterizing the solution of the household's problem. It shows, in particular, that the equilibrium in the price-taking behavior model does not satisfy Pareto optimality. This should not come as a surprise since in this model households' *action* cannot be observed. A natural extension to the notion of Pareto optimality would allow a Planner to allocate commodities to households, but not to choose households' *actions*, in order to maximize households welfare. Following Diamond (1967), Stiglitz (1981) and Geanakoplos and Polemarchakis (1986), a weaker notion of optimality will be used in this paper. I will show that equilibrium does not satisfy this weaker notion of optimality, and hence it does not satisfy the natural generalization of Pareto optimality.

Fix $\{p, (b_t, q_t)\}$ be a given set of commodity prices p and for each type t an asset price q_t and an asset holding b_t . Let a household of type t solve the following problem

$$\max_{a, x} U_t(x, a) \text{ s.t. } \begin{cases} p(x^1 - e^1) + q_t b = 0 \\ p(x^2 - e^2) - b = 0 \end{cases}$$

Notice that asset holding is **not** a type t household's choice variable. Let

$$\{x_t(p, b_t, q_t), a_t(p, b_t, q_t)\}$$

be the household's optimal choice, given $\{p, (b_t, q_t)\}$.

Definition: An equilibrium $\{(x_t, a_t, b_t)_t, p\}$ is **constrained optimal** if there is no asset holding for every household, commodity prices and asset prices $\{p', (b'_t, q'_t)\}$ such that:

i) if households optimize in the second period, given the asset holding b'_t , commodity prices p , and asset prices q_t then commodity and asset markets clear

ii) at least one household gets strictly better off, and no household gets worse off.

An equilibrium is **constrained sub-optimal** if it is not constrained optimal.

I will prove below that for "most" of the economies, a price-taking behavior model is constrained sub-optimal. In order to make this statement precise, I need to introduce the space of economies. Fix a profile of utility functions $U := (U_1, \dots, U_T)$ and let

$$E := \left\{ e \in \mathfrak{R}_{++}^{2CT} / e_t^1 \gg e_t^2 \text{ for every } t \text{ and } (e, U) \text{ satisfies } (H1) - (H2) \right\}$$

It is simple to verify that E is an open subset of \mathfrak{R}_{++}^{2CT} . A property holds for a **generic** set of economies if there is an open, full measure set $E^* \subset E$ such that the property holds for every economy (e, U) , $e \in E^*$.

Proposition 3.3: *There is a generic set $E^* \subset E$ such that for any economy (e, U) , with $e \in E^*$, every equilibrium is constrained sub-optimal.*

This proposition should not come as a surprise. Recall that I have assumed $Dv_t(0) = 0$ and $D\pi_t(0) > 0$. Moreover, at equilibrium we must have $a_t = 0$ for every t . Therefore, there is a Pareto improvement if each type chooses $a_t > 0$ small enough. The problem is to provide incentives so that each type's optimal choice is precisely $a_t > 0$. This can be achieved by providing partial insurance in the second period. A formal proof is presented in the appendix.

4. Nonlinear Pricing Model

In the nonlinear pricing model there are $I > 1$ firms in the insurance market. The choice variables for each firm is the pair $(b_t^i, q_t^i) \in \mathbb{R}_+^2$, corresponding to the quantity b_t^i and the price q_t^i of the asset firm i is going to offer to a household of type t . A household may accept or reject the offer. However, a household cannot accept more than one offer. I assume that each type's utility function and initial endowments are common knowledge. Each household has a belief about second period commodities prices, p . Given this belief if a household of type t decides to accept the offer of firm i then she solves the problem:

$$\max_{(x_t, a_t)} U_t(x_t, a_t) \quad s.t. \quad \begin{cases} p(x_t^1 - e_t^1) + q_t^i b_t^i = 0 \\ p(x_t^2 - e_t^2) - b_t^i = 0 \end{cases}$$

In this case I write the solution of the household's problem as

$$U_t(p, b_t^i, q_t^i)$$

Given every firm offers $\{(b_t^i, q_t^i)_{i=1}^I\}$ a household of type t chooses the offer that solves the problem

$$\max_{\{(b_t^i, q_t^i)_{i=1}^I\}} U_t(p, b_t^i, q_t^i)$$

Let $\{x_t(p, (b_t^i, q_t^i)_i), a_t(p, (b_t^i, q_t^i)_i)\}$ denote a solution to the problem of a household of type t .

Each firm i has a belief about each other firm offer $\{(b_t^{i'}, q_t^{i'})_{i' \neq i}\}$ and a common expected second period commodity prices, p . I assume that if a household of type t is indifferent between the offers of two firms then a fair random device is used to determine which firm is going to supply insurance to the household. Given these beliefs, firm i

expected profits in trading with a household of type t are given by

$$\Pi_i \left\{ (b_t^i, q_t^i), (b_t^{i'}, q_t^{i'})_{i' \neq i}, p \right\} = \frac{1}{\#I'} \left(\pi_t \left[a_t (b_t^i, q_t^i, p) \right] q_t^i b_t^i - \left\{ 1 - \pi_t \left[a_t (b_t^i, q_t^i, p) \right] \right\} b_t^i \right)$$

provided that

$$U_t (p, b_t^i, q_t^i) \geq U_t (p, b_t^{i'}, q_t^{i'}) \text{ for all } i' \neq i$$

where $I' := \{i' / U_t (p, b_t^i, q_t^i) = U_t (p, b_t^{i'}, q_t^{i'})\}$. In the case

$$U_t (p, b_t^i, q_t^i) < U_t (p, b_t^{i'}, q_t^{i'}) \text{ for some } i' \neq i$$

then

$$\Pi_i \left\{ (b_t^i, q_t^i), (b_t^{i'}, q_t^{i'})_{i' \neq i}, p \right\} = 0$$

A collection of strategies $\{(b_t^i, q_t^i)_{i=1}^I\}$ is a **Nash equilibrium in the supply of asset for households of type t given commodity prices p** if for every i the pair (b_t^i, q_t^i) solves each firm i problem, given that i expects the remaining firms to charge $\{(b_t^{i'}, q_t^{i'})_{i' \neq i}\}$, and a household of type t to solve her problem given commodity prices p .

Definition: *An equilibrium in the nonlinear pricing model is a collection*

$$\left((x_t, a_t)_{t=1}^T, (b_t^i, q_t^i)_{i=1}^I, p \right)$$

such that:

- i) (x_t, a_t) solves the household of type t problem, given $\{p, (b_t^i, q_t^i)_{i=1}^I\}$;
- ii) $\{(b_t^i, q_t^i)_{i=1}^I\}$ is a Nash equilibrium in the supply of asset for type t , given commodity prices p ;
- iii) Commodity markets clear.

In order to analyze an equilibrium in the nonlinear pricing model, I consider an alternative (and artificial) model, which I refer to as *auxiliary-model*. This model rely strongly on the next lemma, and requires an additional assumption. Let

$$X := \left\{ \begin{array}{l} x \in \mathbb{R}_{++}^{2CT} / U_t (x_t^1, x_t^2, a_t) \geq U_t (e_t^1, e_t^2, a_t) \text{ and} \\ \sum_t \left[\pi_t (a_t) (x_t^1 - e_t^1)^\backslash + [1 - \pi_t (a_t)] (x_t^2 - e_t^2)^\backslash \right] = 0 \text{ for some } a_t \text{ for all } t \end{array} \right\}$$

Let X_t^1 be the projection of X into type t first period commodities, $p \in P$ and

$$b_t^M(p, a) := \pi_t(a) p (e_t^1 - e_t^2)$$

Let $x_t^1(p, y_1)$ be a type t household's first state optimal choice when facing relative prices p and income y_1 in the first state of nature.

(H3) For every t and $p \in P$, we have

$$\sup_{x_t^1 \in X_t^1, a, y_1 \in (pe_t^1, pe_t^1)} \left\{ (D_a \pi_t(a))^2 Du_t^1 D_{y_1} x_t^1 b_t^M(p, a) - \pi_t^3(a) D_a^2 v_t(a) \right\} < 0$$

Lemma 4.1: Let (H1) – (H3) hold. Fix $p \in P$, t and $b \geq 0$. There is a unique q_t such that

$$q_t = \frac{1 - \pi_t(a_t(p, b, q_t))}{\pi_t(a_t(p, b, q_t))}$$

Moreover, the function $q_t : P \times \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ defined implicitly by the above equation is C^∞ for every pair (p, b) such that $b \in [0, b_t^M(p)]$, where

$$b_t^M(p) := \pi_t(0)p(e_t^1 - e_t^2)$$

The lemma is proved in the appendix.

Given commodities prices p and the asset price function $q_t : P \times \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$, a household of type t in the auxiliary model solves the following problem

$$\max_{a, x, b} U_t(x, a) \quad \text{s.t.} \quad \begin{cases} p(x^1 - e^1) + q_t(p, b)b = 0 \\ p(x^2 - e^2) - b = 0 \end{cases}$$

Definition: An equilibrium in the auxiliary model is a collection $((x_t, a_t, b_t)_{t=1}^T, p)$ such that:

- i) (x_t, a_t, b_t) solves a type t household's problem, given prices $(p, q_t(\cdot))$;
- ii) markets clear.

Lemma 4.2: Fix an economy (e, U) satisfying (H1) – (H3) and a vector of relative prices p .

i) Suppose a collection (x_t, b_t, a_t) is a solution to a type t household's problem in the auxiliary model. Let $(q_t^i, b_t^i) := (q_t(p, b_t), b_t)$ for every i . Then the collection (x_t, a_t) is a solution of a type t household's problem in the nonlinear pricing model and $\{(b_t^i, q_t^i)_{i=1}^I\}$ is a Nash equilibrium in the supply of asset for type t , given commodity prices p .

ii) Suppose the collection (x_t, a_t) is a solution of a type t household's problem in the nonlinear pricing model and $\{(b_t^i, q_t^i)_{i=1}^I\}$ is a Nash equilibrium in the supply of asset

for type t , given commodity prices p . Let

$$(b_t, q_t) := \arg \max_{\{(b_t^i, q_t^i)_{i=1}^I\}} U_t(p, b_t^i, q_t^i)$$

Then the collection (x_t, b_t, a_t) is a solution to a type t household' problem in the auxiliary model.

Lemma 4.2 (ii) should be obvious. If any firm that is actually selling insurance for a household of type t charge a price q_t larger than the fair price

$$\frac{1 - \pi_t(a_t)}{\pi_t(a_t)}$$

then the firm must make positive profits. As in the standard Bertrand duopoly model, any firm different from i then can make higher profits by charging an asset price slightly smaller. This show that at equilibrium at the offer accepted by a household of type t the asset price must be the fair price. Given that households trade at the fair price, Bertrand competition once more ensure that firms should offer the contract that maximizes the consumer welfare. Lemma 4.2 (i) is not as immediate as Lemma (ii), and in fact it exploits basic properties of the household's problem. A formal proof is given in the appendix.

Proposition 4.1: *For every economy (e, U) satisfying $(H1) - (H3)$ there is a nonlinear pricing equilibrium.*

The proof of proposition 4.1 rely strongly on lemma 4.2. I use the equivalence between the auxiliary model and the nonlinear pricing model to show that the household's problem is well defined and it has a unique solution for each vector of commodity prices. Notice that the household's problem in the auxiliary model may not be convex. However, in the proof it is shown that in the nonlinear pricing model there is at most one solution to a household's problem. Therefore, also in the auxiliary model the household's problem can have at most one solution. The argument then concludes by showing the existence of an equilibrium in the auxiliary model. This essentially follows from a variation of the standard proof of existence of a competitive equilibrium using excess demand functions. A detailed argument is provided in the appendix.

The next proposition establish the constrained optimality of a nonlinear pricing equilibrium, provided that there is a unique commodity in the second period. Once more, the proof exploits the equivalence between the auxiliary-model and the nonlinear pricing model.

Proposition 4.2: *Every nonlinear pricing equilibrium is constrained optimal, provided $C = 1$.*

Proof: The proof proceeds by contradiction, and it is an immediate variation of the standard proof of the first welfare theorem. Suppose there is a equilibrium (x^*, b^*, a^*) in the auxiliary model which is not constrained optimal. Therefore, there is an allocation of assets holding for each type in each individual state of nature, (b_t^1, b_t^2) , such that if $x_t(b_t^1, b_t^2)$ and $a_t(b_t^1, b_t^2)$ are the optimal solution of the consumer problem given (b_t^1, b_t^2) then markets clear

$$\sum_t \left[\pi_t(a_t(b_t^1, b_t^2)) (x_t^1(b_t^1, b_t^2) - e_t^1) + [1 - \pi_t(a_t(b_t^1, b_t^2))] (x_t^2(b_t^1, b_t^2) - e_t^2) \right] = 0$$

$$\pi_t(a_t(b_t^1, b_t^2)) b_t^1 = (1 - \pi_t(a_t(b_t^1, b_t^2))) b_t^2$$

no household gets worse off and some household gets strictly better off

$$\begin{pmatrix} U_1(x_1(b_1^1, b_1^2), a_1(b_1^1, b_1^2)) \\ \vdots \\ U_T(x_T(b_T^1, b_T^2), a_T(b_T^1, b_T^2)) \end{pmatrix} > \begin{pmatrix} U_1(x_1^*, a_1^*) \\ \vdots \\ U_T(x_T^*, a_T^*) \end{pmatrix} \quad (*)$$

Let

$$q_t := \frac{(1 - \pi_t(a_t(b_t^1, b_t^2)))}{\pi_t(a_t(b_t^1, b_t^2))}, \quad b_t := \frac{b_t^1}{q_t} \Rightarrow b_t^2 = b_t$$

Inequality (*) implies that the pair $(x_t(b_t^1, b_t^2), a_t(b_t^1, b_t^2))$ cannot be a feasible solution for a type $t = t'$. Since the type faces no restriction on choosing a_t , we must have

$$\begin{aligned} (x_t^1(b_t^1, b_t^2) - e_t^1) + q_t(b_t) (x_t^2(b_t^1, b_t^2) - e_t^2) &\geq 0 \text{ for all } t \\ (x_{t'}^1(b_{t'}^1, b_{t'}^2) - e_{t'}^1) + q_t(b_{t'}) (x_{t'}^2(b_{t'}^1, b_{t'}^2) - e_{t'}^2) &> 0 \text{ for } t' \end{aligned}$$

By assumption

$$q_t(b) = \frac{1 - \pi_t(a_t(q_t(b), b))}{\pi_t(a_t(q_t(b), b))} \text{ for every } b \text{ and } q_t(b_t) = q_t$$

Therefore

$$\sum_t \left[\pi_t(a_t(q_t(b_t), b_t)) (x_t^1(b_t^1, b_t^2) - e_t^1) + [1 - \pi_t(a_t(q_t(b_t), b_t))] (x_t^2(b_t^1, b_t^2) - e_t^2) \right] > 0$$

which contradicts the feasibility of the Pareto improving equilibrium, and completes the proof. \square

The previous proof of constrained optimality rely strongly on the existence of a single commodity. A closer investigation of the proof shows in fact that the argument requires only that at the candidate Pareto improving allocation households face the same commodity prices as in the original equilibrium. If relative prices change, however, as they may in the many commodities case, then the argument does not hold. In order to show this point, suppose we try to generalize the above argument to the many commodities case.

Let (x^*, a^*, b^*, p^*) be an equilibrium in the nonlinear pricing model and suppose there is an allocation of assets holding for each type in each individual state of nature, (b_t^1, b_t^2) , and commodity prices p such that if $x_t(b_t^1, b_t^2, p)$ and $a_t(b_t^1, b_t^2, p)$ are the optimal solution of the consumer problem given (b_t^1, b_t^2, p) then markets clear, no household gets worse off and some household gets strictly better off. Following the same arguments used in the proof of proposition 4.2, we get

$$\begin{aligned} p^* \left(x_t^1(b_t^1, b_t^2) - e_t^1 \right) + q_t(p, b_t) p^* \left(x_t^2(b_t^1, b_t^2, p) - e_t^2 \right) &\geq 0 \text{ for all } t \\ p^* \left(x_{t'}^1(b_{t'}^1, b_{t'}^2) - e_{t'}^1 \right) + q_t(p, b_{t'}) p^* \left(x_{t'}^2(b_{t'}^1, b_{t'}^2, p) - e_{t'}^2 \right) &> 0 \text{ for some } t' \end{aligned}$$

By assumption

$$q_t(p, b) = \frac{1 - \pi_t(a_t(q_t(p, b), b, p))}{\pi_t(a_t(q_t(p, b), b, p))} \text{ for every } b \text{ and } p$$

and

$$q_t(p, b_t) = q_t := \frac{(1 - \pi_t(a_t(b_t^1, b_t^2)))}{\pi_t(a_t(b_t^1, b_t^2))}, \text{ where } b_t := \frac{b_t^1}{q_t}$$

Therefore

$$\sum_t [\pi_t(a'_t) p^* (x_t^1(b_t^1, b_t^2, p) - e_t^1) + [1 - \pi_t(a'_t)] p^* (x_t^2(b_t^1, b_t^2, p) - e_t^2)] > 0$$

where

$$a'_t := a_t(q_t(p, b_t), b_t, p)$$

The above inequality, however, does not violate feasibility since, at relative prices p^* households in general would **not** choose $a_t(q_t(p, b_t), b_t, p)$. A trivial case in which households choose this action level is when $p = p^*$. This equality must hold in the one commodity case, but not in the general case. Therefore, it is not possible to generalize to the proof of the constrained optimality of a nonlinear pricing equilibrium to the multi-commodity case. In fact, I will show that for “most” of economies with at least two commodities every nonlinear pricing equilibrium is constrained sub-optimal. This result, however, requires a the space of economies to include the specification of utility functions as well. Let

$$\mathcal{U}_t := \left\{ U_t : \mathfrak{R}_{++}^C \times [0, 1) \rightarrow \mathfrak{R} \in C^\infty / U_t \text{ satisfies } (H1) \right\}$$

I endow this set with the compact-open topology.¹⁵ Let $\mathcal{U} := \times_t \mathcal{U}_t$ be the space of utility functions, endowed with the product topology. Finally, the space of economies is defined with

$$E \times \mathcal{U} := \left\{ (e, U) \in E \times \mathcal{U} / e_t^1 \gg e_t^2, (e, U) \text{ satisfies } (H2) - (H3) \right\}$$

once more endowed with the product topology.

Proposition 4.3: *There is an open and dense set of economies $(E \times \mathcal{U})^* \subset E \times \mathcal{U}$ such that for all economies in this set every nonlinear pricing equilibrium is constrained sub-optimal, provided $C > T$.*

Proposition 4.3 says that for a large set of economies every constrained optimal allocation cannot be supported as a nonlinear pricing equilibrium in the multi-commodity case. Why does the introduction of many commodities matter? In the multi-commodity case neither households nor firms take into account the effect of their optimal choice on the equilibrium commodity prices. The vector of equilibrium prices is not affected by a single household's behavior and thus each household takes prices as given when choosing among the firms' offers. Competition among firms then lead to the offering of the contract that maximizes household utility for a given vector of relative prices. Since all households of a same type choose the same contract, there might be another contract, and the proposition says that in general there will be, whose introduction leads to a change in the vector of relative prices and in the household's optimal choice such that markets clear and ever household gets strictly better off. This contract, however, do not maximize firms' profits. Therefore, it is a lack of coordination, and not of rationality, that leads to the constrained optimality property of equilibrium in the nonlinear pricing model.

5. Appendix

5.1. Proof of Proposition 3.3

For every t , let $\pi_t^\varepsilon : (\varepsilon, 1) \rightarrow [0, 1] \in C^\infty$ and $v_t^\varepsilon : (\varepsilon, 1) \rightarrow \mathbb{R} \in C^\infty$ such that

$$\begin{aligned} D^s \pi_t^\varepsilon(a) &= D^s \pi_t(a) \text{ for all } s \text{ and } a \geq 0 \\ D \pi_t^\varepsilon(a) &> 0 \text{ for all } a \in (\varepsilon, 1), \pi_t^\varepsilon(\varepsilon) \in (0, 1) \\ D^s v_t^\varepsilon(a) &= D^s v_t(a) \text{ for all } s \text{ and } a \geq 0 \\ D v_t^\varepsilon(a) &> 0 \text{ for all } a \neq 0 \\ D^2 v_t^\varepsilon(a) &< 0 \text{ for all } a \end{aligned}$$

¹⁵See Mas-Colell (1985).

and $\varepsilon < 0$ is large enough so that

$$U_t^e(x, a) : \pi_t^e(a) u_t(x^1) + (1 - \pi_t^e(a)) u_t(x^2) - v_t^e(a)$$

satisfies (H2).

Let $\Xi := ((\varepsilon, 1) \times \mathfrak{R}_{++}^{2C \times 3})^T \times P$. If ξ is a generic element of Ξ then I also write

$$\xi = (\dots, (a_t, x_t^1, x_t^2, b_t, \lambda_t^1, \lambda_t^2), \dots, p^1)$$

By the Kuhn-Tucker theorem, an equilibrium must satisfy $F(\xi, e, U) = 0$, where $F : \Xi \times E \times \mathcal{U} \rightarrow (\mathfrak{R}^{2C \times 4})^T \times \mathfrak{R}^{C-1}$ is defined by

$$F(\xi, e, U) = \begin{bmatrix} \vdots \\ D_a \pi_t^e(a_t) \{u_t(x_t^1) - u_t(x_t^2)\} - D_a v_t^e(a_t) \\ \pi_t^e(a_t) D u_t(x_t^1) - \lambda_h^1 p \\ (1 - \pi_t^e(a_t)) D u_t(x_t^2) - \lambda_h^2 p \\ -\lambda^1 q_t(0) + \lambda^2 \\ -p(x_t^1 - e_t^1) - b_t q_t(0) \\ -p(x_t^1 - e_t^1) + b_t \\ \vdots \\ \sum_t [\pi_t(a_t)(x_t^1 - e_t^1) + [1 - \pi_t(a_t)](x_t^2 - e_t^2)] \end{bmatrix}$$

where

$$q_t(a_t) := \frac{1 - \pi_t^e(a_t)}{\pi_t^e(a_t)}$$

An economy is said to be **regular** if $D_\xi F$ is non-singular. Suppose we have a regular equilibrium (ξ^*, e, U) . constrained optimality is equivalent to showing the existence of $(p', (b'_t, q'_t)_{t=1}^T)$ such that if households solve the problem

$$\max_{a, x} U_t(x, a) \text{ s.t. } \begin{cases} p(x^1 - e^1) + q_t b = 0 \\ p(x^2 - e^2) - b = 0 \end{cases}$$

then markets clear and households get strictly better off. Therefore, it is sufficient to show the existence of ξ with $a_t \geq 0$ for all t such that

$$\begin{aligned} F^P(\xi, e, U) &= 0 \\ G(\xi, e, U) - G(\xi^*, e, U) &\gg 0 \end{aligned}$$

where

$$G(\xi, e, U) = (U_1(x_1, a_1), \dots, U_T(x_T, a_T))$$

$$F^P(\xi, e, U) = \begin{bmatrix} \vdots \\ D_a \pi_t^e(a_t) \{u_t(x_t^1) - u_t(x_t^2)\} - D_a v_t^e(a_t) \\ \pi_t^e(a_t) Du_t(x_t^1) - \lambda_h^1 p \\ (1 - \pi_t^e(a_t)) Du_t(x_t^2) - \lambda_h^2 p \\ -p(x_t^1 - e_t^1) - b_t q_t(a) \\ -p(x_t^2 - e_t^1) + b_t \\ \vdots \\ \sum_t [\pi_t(a_t)(x_t^1 - e_t^1) + [1 - \pi_t(a_t)](x_t^2 - e_t^2)] \end{bmatrix}$$

It is simple to verify that at any solution of the above problem we must have $a_t \geq 0$ for every t . In fact, by Kuhn-Tucker sufficient conditions, any solution to the above system of equations must solve in particular

$$\max_{a, x} U_t^e(a, x) \text{ s.t. } \begin{cases} p(x_t^1 - e_t^1) = -b q_t(b) \\ p(x_t^2 - e_t^1) = b_t \end{cases}$$

By the Kuhn-Tucker equations we must have $u_t(x_t^2) < u_t(x_t^1)$. If $a_t < 0$ then the household can improve her total utility by choosing $a_t' = 0$, which is an absurd. For notational simplicity, I will drop the superscript "e" in the rest of this proof.

Following the same procedure used in Lisboa (1995), we have to show that the following system of equations

$$\begin{aligned} F(\xi, e, U) &= 0 \\ d_F^T D_\xi F^P + d_G^T D_\xi G &= 0 \\ d^T d - 1 &= 0 \end{aligned}$$

does not have a solution for a generic set of economies.¹⁶ It is a standard argument to show that the set of economies that does not satisfy this system is open.¹⁷ The proof that this set has full measure will be done in two steps.

Step 1: *There is a full measure set of economies $E^* \subset E$ such that if $e \in E^*$ and $F(\xi, e, U) = 0$ then $D_\xi F$ is non-singular.*

Fix an economy (e, U) . I claim $D_{\xi, e} F$ is surjective at equilibrium. Computing $D_{\xi, e} F$

¹⁶This approach to analyze optimality of equilibrium was proposed by Smale (1974) and developed by Geanakoplos and Polemarkachis (1986), Cass and Citanna (1994) and Citanna, Kajii and Villanacci (1994). I follow Cass and Citanna (1994) and Citanna, Kajii and Villanacci (1994) approaches more closely.

¹⁷Lisboa (1995) presents this argument in detail.

gives

$$\begin{bmatrix} a_t & x_t^1 & x_t^2 & b_t & \lambda_t^1 & \lambda_t^2 & e_t^1 & e_t^2 & p^\backslash \\ D^2\pi_t\Delta u_t - D^2v_t^e & D\pi_t Du_t^1 & -D\pi_t Du_t^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ D\pi_t (Du_t^1)^T & \pi_t D^2 u_t^1 & 0 & 0 & -p^T & 0 & 0 & 0 & -\lambda_t^1 \begin{bmatrix} 0 \\ I \end{bmatrix} \\ -D\pi_t (Du_t^2)^T & 0 & \pi_t D^2 u_t^2 & 0 & 0 & -p^T & 0 & 0 & -\lambda_t^2 \begin{bmatrix} 0 \\ I \end{bmatrix} \\ 0 & 0 & 0 & 0 & -q_t & 1 & 0 & 0 & \\ 0 & -p & 0 & -q_t & 0 & 0 & p & 0 & Z_t^{1\backslash} \\ 0 & 0 & -p & 1 & 0 & 0 & 0 & p & Z_t^{2\backslash} \\ D\pi_t (z_t^1 - z_t^2)^\backslash & I_t^1 & I_t^2 & 0 & 0 & 0 & -I_t^1 & -I_t^2 & 0 \end{bmatrix}$$

where

$$I_t^s := \pi_t^s [0 \ I], \quad Z_t^{s\backslash} := (z_t^{s2}, \dots, z_t^{sC}), \quad \Delta u_t := \{u_t^1 - u_t^2\}, \quad u_t^s := u_t(x_t^s), \quad \pi_t^2 := 1 - \pi_t$$

Pre-multiplying this matrix by a vector

$$d := (\dots, (da_t, dx_t^1, dx_t^2, db_t, d\lambda_t^1, d\lambda_t^2), \dots, dp^\backslash) \in \mathfrak{R}^{(4+2C)T+C-1}$$

and using the fact that at equilibrium $a_t = 0$ and $x_t^1 = x_t^2$ for every t gives

$$(D^2\pi_t\Delta u_t - D^2v_t^e) da_t + D\pi_t Du_t dx_t^1 - D\pi_t Du_t dx_t^2 = 0 \quad (t.1)$$

$$D\pi_t Du_t da_t + \pi_t D^2 u_t dx_t^1 - p^T d\lambda_t^1 + \pi_t \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} = 0 \quad (t.2)$$

$$-D\pi_t Du_t da_t + (1 - \pi_t) D^2 u_t dx_t^2 - p^T d\lambda_t^2 + (1 - \pi_t) \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} = 0 \quad (t.3)$$

$$-q_t d\lambda_t^1 + d\lambda_t^2 = 0 \quad (t.4)$$

$$-p dx_t^1 - q_t db_t = 0 \quad (t.5)$$

$$p dx_t^2 + db_t = 0 \quad (t.6)$$

$$p^T d\lambda_t^1 - \pi_t \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} = 0 \quad (t.7)$$

$$p^T d\lambda_t^2 - (1 - \pi_t) \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} = 0 \quad (t.8)$$

$$\sum_t \left(\lambda_t^1 dx_t^{1\backslash} + \lambda_t^2 dx_t^{2\backslash} + d\lambda_t^1 z_t^{1\backslash} + d\lambda_t^2 z_t^{2\backslash} \right) = 0 \quad (M)$$

From (t.7) – (t.8) we get $d\lambda_t^1 = d\lambda_t^2 = 0$ and $dp^\backslash = 0$. From the differentiability strict concavity of U_t , $(da_t, dx_t) = 0$ for every t and from (t.6), $db_t = 0$. Therefore, $D_{\xi,e}F$ is surjective.

By the transversality theorem (Hirsch 1975, chap. 3), for almost every e the derivative $D_\xi F$ is surjective at any solution of the system $F(\xi, e, U) = 0$, which is the desired result.

Step 2: If $F(\xi, e, U) = 0$ and $d_F^T DF^P + d_G^T DG = 0$ then $(d_F, d_G) = 0$.

Using the same notation as in the previous step and $d_G := (d\alpha_1, \dots, d\alpha_T)$, the system $d_F^T D_\xi F^P + d_G^T D_\xi G = 0$ gives

$$(D^2 \pi_t \Delta u_t - D_\alpha^2 v_t) da_t + D\pi_t Du_t (dx_t^1 - dx_t^2) + d\lambda_t^1 D_\alpha q_t = 0 \quad (t.1)$$

$$D\pi_t Du_t da_t + \pi_t D^2 u_t dx_t^1 - p^T d\lambda_t^1 + \pi_t \begin{bmatrix} 0 \\ dp \backslash \end{bmatrix} + d\alpha_t \pi_t Du_t = 0 \quad (t.2)$$

$$-D\pi_t Du_t da_t + (1 - \pi_t) D^2 u_t dx_t^2 - p^T d\lambda_t^2 + (1 - \pi_t) \begin{bmatrix} 0 \\ dp \backslash \end{bmatrix} + d\alpha_t (1 - \pi_t) Du_t = 0 \quad (t.3)$$

$$-q_t d\lambda_t^1 + d\lambda_t^2 = 0 \quad (t.4)$$

$$p dx_t^1 = 0 \quad (t.5)$$

$$p dx_t^2 = 0 \quad (t.6)$$

$$\sum_t (\lambda_t^1 dx_t^{1\backslash} + \lambda_t^2 dx_t^{2\backslash} + d\lambda_t^1 z_t^{1\backslash} + d\lambda_t^2 z_t^{2\backslash}) = 0 \quad (M)$$

From first order conditions and (t.5) – (t.6) we get $Du_t dx_t^s = 0$ for every t and $s = 1, 2$.

Dividing equation (t.2) by π_t , equation (t.3) by $-(1 - \pi_t)$, and adding them up gives

$$D^2 u_t (dx_t^1 - dx_t^2) + D\pi_t Du_t da_t \left(\frac{1}{\pi_t} + \frac{1}{1 + \pi_t} \right) = 0$$

and thus

$$(dx_t^1 - dx_t^2)^T D^2 u_t (dx_t^1 - dx_t^2) = 0 \Rightarrow (dx_t^1 - dx_t^2) = 0, da_t = 0$$

From (t.1) we get $d\lambda_t^1 = 0$ which immediately gives, $d\lambda_t^2 = 0$.

From equations (t.2) and (M) we get

$$dx_t^{1T} D^2 u_t dx_t^1 + dx_t^{1T} \begin{bmatrix} 0 \\ dp \backslash \end{bmatrix} = 0$$

$$\sum_t \lambda_t^1 (1 + q_t) dx_t^{1\backslash} = 0$$

which gives

$$\sum_t \lambda_t^1 (1 + q_t) dx_t^{1T} D^2 u_t dx_t^1 = 0 \Rightarrow dx_t^1 = 0 \text{ for all } t$$

since $D^2 u_t$ is negative definite and $\lambda_t^1 (1 + q_t) > 0$ for every t . This immediately gives $dx_t^2 = 0$ and $dp \backslash = 0$, which completes the proof. \square

5.2. Proof of Lemma 4.1

Fix $p \in P$, t and b . Let

$$q_t^* : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+ \text{ given by}$$

$$q_t^*(q_t) = \frac{\{1 - \pi_t(a_t(p, q_t, b))\}}{\pi_t(a_t(p, q_t, b))}$$

$$q_t^M : = \frac{1 - \pi_t(0)}{\pi_t(0)} \text{ and } q_t^m := \lim_{a \rightarrow 1} \frac{1 - \pi_t(a)}{\pi_t(a)}$$

Notice that

$$a_t(p, b, q_t^M) \geq 0 \Rightarrow \frac{1 - \pi_t(a_t(p, b, q_t^M))}{\pi_t(a_t(p, b, q_t^M))} \leq q_t^M$$

$$a_t(p, b, q_t^m) < 1 \Rightarrow \frac{1 - \pi_t(a_t(p, b, q_t^m))}{\pi_t(a_t(p, b, q_t^m))} \geq q_t^m$$

$$q_t^*(q_t) \in [q_t^m, q_t^M] \text{ for all } q_t$$

Solving a household type t problem facing asset holding b and prices p and q gives that $q_t^*(\cdot)$ is a non-decreasing function. In fact, the optimal solution of the household's problem is given by the Kuhn-Tucker equations

$$F_t(x^1, x^2, a, \lambda, p, q, b) := \begin{bmatrix} D\pi_t(a) \{u_{h,t}(x^1) - u_t(x^2)\} - D_a v(a) \\ \pi_t(a) Du_t(x^1) - \lambda_h^1 p \\ (1 - \pi_t(a)) Du_t(x^2) - \lambda_h^2 p \\ p(x^1 - e^1) + q_t b \\ p(x^2 - e^2) - b \end{bmatrix} = 0$$

A variation of the standard proof of differentiability of the demand of a Walrasian household shows that $D_{x,a,\lambda} F_t$ is non-singular. Therefore, by the implicit function theorem, the household's optimal choice, $x_t(\cdot)$ and $a_t(\cdot) \in C^\infty$. Applying the chain rule theorem to the identity

$$D_a \pi_t(a_t(p, q_t, b)) \{u_t(x_t^1(p, q_t, b)) - u_t(x_t^2(p, q_t, b))\} - D_a v_t(a_t(p, q_t, b)) = 0$$

gives

$$D_q a_t(p, q_t, b) = \frac{D_a \pi_t Du_t^1 D_{y_1} x_t^1 b}{D_a^2 \pi_t \{u_t(x_t^1) - u_t(x_t^2)\} - D_a^2 v_t} < 0 \Rightarrow D_q q_t^* = -\frac{D_a \pi_t D_q a_t}{\pi_t^2} > 0$$

Therefore, $q_t^*(\cdot)$ is an increasing function. Moreover, the previous computations show that under (H3) we have $0 < D_q q_t^* < 1$. Therefore, $q_t^*(q) - q$ is a decreasing function, positive at $q = q_t^m$ and negative at $q = q_t^M$. Therefore, there is q' such that $q_t^*(q') = q'$.

Let $\xi : P \times \mathfrak{R}_+ \rightarrow \mathfrak{R}$ be given by

$$\xi(p, b) = \frac{1 - \pi_t(a_t(p, b, q_t))}{\pi_t(a_t(p, b, q_t))} - q_t$$

For any $b \in (0, b_t^M(p))$, $\xi \in C^\infty$. Taking derivatives at a solution of the system $\xi(p, b) = 0$ we get

$$D\xi = - \left[1 + \frac{D_a \pi_t D_q a_t}{\pi_t^2}, \frac{D_a \pi_t D_b a_t}{\pi_t^2}, \frac{D_a \pi_t D_p a_t}{\pi_t^2} \right]$$

where

$$\frac{D_a \pi_t D_q a_t}{\pi_t^2} < 1$$

by assumption (H3). Therefore, by the Implicit Function Theorem, we can locally write $q_t = \zeta_t(p, b) \in C^\infty$ where

$$\zeta_t(p, b) = \frac{1 - \pi_t(a_t(p, b, \zeta_t(p, b)))}{\pi_t(a_t(p, b, \zeta_t(p, b)))}$$

Since for each (p, b) there is a unique q_t such that $\xi(p, b) = 0$ and the domain of q_t is connected, the function ζ_t is globally well defined and C^∞ on $(0, b_t^M(p))$. \square

5.3. Proof of Lemma 4.2

Let (x_t, b_t, a_t) be a solution to a type t household's problem in the auxiliary model given prices q and $(b_t^i, q_t^i) := (b_t, q_t(b_t))$ for every i . It is sufficient to show that $\{(b_t^i, q_t^i)_{i=1}^I\}$ is a Nash equilibrium in the supply of asset for households of type t given commodity prices p . We prove it by contradiction. Suppose there is a profitable deviation (b'_t, q'_t) for some firm. It must be the case that

$$\pi_t[a_t(b'_t, q'_t, p)] q'_t b'_t - \{1 - \pi_t[a_t(b'_t, q'_t, p)]\} b'_t > 0 \quad (i.1)$$

and

$$U_t(p, b_t^i, q_t^i) \leq U_t(p, b'_t, q'_t) \quad (i.2)$$

The inequality (i.1) implies

$$q'_t > \frac{\{1 - \pi_t[a_t(p, q'_t, b'_t)]\}}{\pi_t[a_t(p, q'_t, b'_t)]}$$

Using the previous lemma's proof notation, recall that $q_t^*(q_t) - q_t$ is a decreasing function. Since

$$q_t^*(q_t(p, b'_t)) - q_t(p, b'_t) = 0$$

and $q'_t > q_t^*(q'_t)$ we must have $q'_t > q_t(p, b'_t)$ and hence

$$U_t(p, b'_t, q'_t) < U_t\left(p, b'_t, \frac{\{1 - \pi_t[a_t(p, q_t(p, b'_t))]\}}{\pi_t[a_t(p, q_t(p, b'_t))]\}\right) < U_t\left(p, b_t^i, \frac{\{1 - \pi_t[a_t(p, q_t(p, b_t))]\}}{\pi_t[a_t(p, q_t(p, b_t))]\}\right)$$

where the second inequality from the fact that $b_t^i = b_t$ solves the problem

$$\max U_t\left(p, b, \frac{\{1 - \pi_t[a_t(p, q_t(\cdot))]\}}{\pi_t[a_t(p, q_t(\cdot))]\}\right)$$

where $q_t(p, b_t) = q_t^i$. But this contradicts (i.2).

To prove the other direction, let (x_t, a_t) be a solution to a type household's problem in the nonlinear pricing model facing prices p and firms' offers $\{(b_t^i, q_t^i)_{i=1}^I\}$ and assume that this collection of strategies is a Nash equilibrium in the supply of asset for households type t given p . We have to show that if

$$(b_t^i, q_t^i) = \arg \max U_t(p, b_t^i, q_t^i)$$

then

$$q_t^i = \frac{\{1 - \pi_t [a_t(p, q_t^i, b_t^i)]\}}{\pi_t [a_t(p, q_t^i, b_t^i)]}$$

and b_t^i solves the problem

$$\max_b U_t \left(p, b, \frac{\{1 - \pi_t [a_t(p, q_t(b), b)]\}}{\pi_t [a_t(p, q_t(b), b)]} \right)$$

where

$$q_t(p, b) = \frac{\{1 - \pi_t [a_t(p, q_t, b)]\}}{\pi_t [a_t(p, q_t, b)]} \text{ for all } (p, b)$$

If this is not the case then either

$$q_t^i > \frac{\{1 - \pi_t [a_t(p, q_t^i, b_t^i)]\}}{\pi_t [a_t(p, q_t^i, b_t^i)]} \quad (e.1)$$

or there is b such that

$$U_t(p, b, q_t(p, b)) > U_t(p, b_t^i, q_t^i) \quad (e.2)$$

Suppose (e.1) happens. Then if firm $j \neq i$ charges q_t^j

$$q_t^i > q_t^j > \frac{\{1 - \pi_t [a_t(p, q_t^i, b_t^i)]\}}{\pi_t [a_t(p, q_t^i, b_t^i)]}$$

then by taking q_t^j as close to q_t^i as necessary firm j can make larger expected profits, which contradicts the definition of a Nash equilibrium in the supply of asset. Therefore, it must be the case that

$$q_t^i = \frac{\{1 - \pi_t [a_t(p, q_t^i, b_t^i)]\}}{\pi_t [a_t(p, q_t^i, b_t^i)]} \Rightarrow q_t^i = q_t(p, b_t^i)$$

and hence no firm is making profits in this Nash equilibrium.

If (e.2) happens there is $\varepsilon > 0$ small enough such that

$$U_t(p, b_t, q_t(p, b_t) + \varepsilon) > U_t(p, b_t^i, q_t^i)$$

But then by offering the pair $(b_t, q_t(p, b_t) + \varepsilon)$ a firm j can make larger profits, which contradicts the definition of Nash equilibrium in the supply of asset and completes the proof. \square

5.4. Proof of Proposition 4.1

By lemma 4.2, it is sufficient to show existence of equilibrium in the auxiliary model. For each $p \in P$, a type t household is solving the following problem in the auxiliary model

$$\max U_t(x_t, a_t) \quad s.t \quad \begin{cases} p(x_t^1 - e_t^1) + q_t^p(p, b_t)b_t = 0 \\ p(x_t^2 - e_t^2) - b_t = 0 \end{cases}$$

Standard arguments show that we can restrict the household's problem to the compact set

$$\{(x, b, a) / U_t(x, a) \geq U_t(e_t, 0), b \in [0, b_t^M(p)]\}$$

This show existence of a solution to the household's problem.

Suppose there are two solutions to the household's problem

$$\{(x_{t,i}(p), b_{t,i}(p), a_{t,i}(p))_{i=1}^2\}$$

By the lemma 4.2 both solutions $\{(b_{t,i}, q_{t,i}(b_{t,i}))_{i=1}^2\}$ must also solve the firm problem in the nonlinear pricing model. Let

$$b_{t,\alpha} = \alpha b_{t,1} + (1 - \alpha)b_{t,2}, \quad q_{t,\alpha} = \frac{\alpha b_{t,1} q_{t,1} + (1 - \alpha)b_{t,2} q_{t,2}}{b_{t,\alpha}}, \quad \alpha \in (0, 1)$$

By strict concavity of the utility function,

$$U_t(b_{t,\alpha}, q_{t,\alpha}) > \alpha U_t(b_{t,1}, q_{t,1}) + (1 - \alpha)U_t(b_{t,2}, q_{t,2})$$

Moreover, (H1) – (H3) implies that, using the Kuhn-Tucker equations,

$$a_t(b_{t,\alpha}, q_{t,\alpha}) \geq \alpha a_t(b_{t,1}, q_{t,1}) + (1 - \alpha)a_t(b_{t,2}, q_{t,2})$$

On the other hand

$$D_a^2 \left(\frac{1 - \pi_t(a)}{\pi_t(a)} \right) = \frac{-D_a^2 \pi_t(a) \pi_t^2(a_t) + (D_a \pi_t(a))^2 2\pi_t(a)}{\pi_t^4(a)} > 0$$

Which gives, since $q_{t,1} < q_{t,2} \Leftrightarrow b_{t,1} < b_{t,2}$,

$$\begin{aligned} \left(\frac{1 - \pi_t(a(b_{t,\alpha}, q_{t,\alpha}))}{\pi_t(a(b_{t,\alpha}, q_{t,\alpha}))} \right) &\leq \left(\frac{1 - \pi_t(\alpha a(b_{t,1}, q_{t,1}) + (1 - \alpha)a(b_{t,2}, q_{t,2}))}{\pi_t(\alpha a(b_{t,1}, q_{t,1}) + (1 - \alpha)a(b_{t,2}, q_{t,2}))} \right) \\ &< \alpha \left(\frac{1 - \pi_t(a(b_{t,1}, q_{t,1}))}{\pi_t(a(b_{t,1}, q_{t,1}))} \right) + (1 - \alpha) \left(\frac{1 - \pi_t(a(b_{t,2}, q_{t,2}))}{\pi_t(a(b_{t,2}, q_{t,2}))} \right) \\ &= \alpha q_{t,1} + (1 - \alpha)q_{t,2} < q_{t,\alpha} \end{aligned}$$

Therefore, if a firm set asset price and demand

$$\{q_{t,\alpha}, b_{t,\alpha}\}$$

then the household strictly prefer this contract and the firm makes positive profits, which contradicts $\{(x_{t,i}(p), b_{t,i}(p), a_{t,i}(p))_{i=1}^2\}$ being solutions to the nonlinear pricing model.

Let $x_t(p)$ denote the optimal choice of commodities for a household of type t facing commodity prices p . It is immediate that $x_t(p)$ satisfies boundary behavior and that the aggregate

excess demand function is bounded below. By the standard existence theorem, it is sufficient to show that $x_t(p)$ is continuous.

Let $p^n \rightarrow p \in P$, and (x^n, b^n, a^n) solve the household t problem at prices p^n and (x, b, a) solve the problem at prices p . Suppose (x^n, b^n) does not converge to (x, b) . Therefore, there is $\varepsilon > 0$ such that, taking a subsequence if necessary,

$$\| (x^n, b^n) - (x, b) \| > \varepsilon$$

for every n . Since $b^n \in [0, b_t^M(p^n)]$, and $q_t^n \in [q_t^m, q_t^M]$, taking a subsequence if necessary, there is (b^*, q^*) such that $(b^n, q^n) \rightarrow (b^*, q^*)$. Since $q_t^m > 0$, taking another subsequence if necessary, by the Kuhn-Tucker equations there is x^* such that $x^n \rightarrow x^*$ and $a^n \rightarrow a^*$ for some $a^* \in [0, 1]$. By continuity, (x^*, b^*) is a feasible solution at p . Therefore, we must have

$$U(x, b, a) > U_t(x^*, b^*, a^*)$$

There is (x', b, a) with $x' \ll x^*$ which still satisfies the inequality. But then for n large enough x' is feasible and strictly better than x^n , which is the desired contradiction. The proof is then complete. \square

5.5. Proof of Proposition 4.3

Let $q : E \times \mathcal{U} \times P \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^T$ be the mapping that satisfies

$$\begin{aligned} q(p, b, e, U) &= (\dots, q_t(p, b_t, e_t, U_t), \dots) \\ &= \left(\dots, \frac{1 - \pi_t(a_t(b_t, p, e_t, U_t, q_t(p, b_t, e_t, U_t)))}{\pi_t(a_t(b, p, e_t, U_t, q_t(p, b_t, e_t, U_t)))}, \dots \right) \end{aligned}$$

By lemma 4.1 we know that such a mapping is well defined and $q_t(\cdot) \in C^\infty$ for every t . Using the same notation as in proposition 3.3, since households restrictions satisfy the constraint qualification condition, an equilibrium must satisfy $F(\xi, e, U) = 0$, where

$$F(\xi, e, U) = \begin{bmatrix} \vdots \\ D_a \pi_t(a_t) \{u_t(x_t^1) - u_t(x_t^2)\} - D_a v_t(a_t) \\ \pi_t(a_t) D u_t(x_t^1) - \lambda_h^1 p \\ (1 - \pi_t(a_t)) D u_t(x_t^2) - \lambda_h^2 p \\ -\lambda^1 (q_t(p, b_t, e_t, U_t) + b D_b q_t) + \lambda^2 \\ -p(x_t^1 - e_t^1) - b_t q_t(p, b_t, e_t, U_t) \\ -p(x_t^1 - e_t^1) + b_t \\ \vdots \\ \sum_t [\pi_t(a_t)(x_t^1 - e_t^1) + [1 - \pi_t(a_t)](x_t^2 - e_t^2)] \end{bmatrix}$$

Suppose we have a regular equilibrium (ξ^*, e, U) . Following the same argument used in section 3, I have to show that the system of equations

$$\begin{aligned} F(\xi, e, U) &= 0 \\ d_F^T D_\xi F^P + d_G^T D_\xi G &= 0 \\ d^T d - 1 &= 0 \end{aligned}$$

does not have a solution for a generic set of economies. In fact, since the household t problem for a given choice of asset holding satisfy the Kuhn-Tucker sufficient conditions theorem, a solution of the Kuhn-Tucker equations is also a household's optimal choice. Openness is a standard argument. The proof that this set is dense will be done in four steps. By the price-taking behavior model it suffices to show that if $a := (a_1, a_2, \dots, a_T) > 0$ then a nonlinear equilibrium is constrained sub-optimal.

Step 1: *There is a dense set of economies $E^* \subset E$ such that if $e \in E^*$ and $F(\xi, e, U) = 0$ then $D_\xi F$ is non-singular.*

The argument follows from an immediate generalization of the first step in the proof of proposition 3.3.

Step 2: *For a dense set of economies the following matrix has full column rank if $a > 0$*

$$\left[\lambda_1^1 \left(z_1^{1\backslash} + b_1 D_p q_1 \right) + \lambda_1^2 z_1^{2\backslash}, \dots, \lambda_t^1 \left(z_T^{1\backslash} + b_T D_p q_T \right) + \lambda_T^2 z_T^{2\backslash} \right]$$

Fix a regular economy such that at any equilibrium we have $a > 0$. Consider the system by

$$\begin{aligned} F(\xi, e, u) &= 0 \\ \sum_{t=1}^{T-1} \alpha_t \left(\lambda_t^1 z_t^{1\backslash} + D_p q_t b_t + \lambda_t^2 z_t^{2\backslash} \right) &= 0 \quad (*) \\ \sum_{t=1}^{T-1} (\alpha_t^*)^2 - 1 &= 0 \end{aligned}$$

Computing derivatives gives

$$\begin{bmatrix} a_t & x_t^1 & x_t^2 & b_t & \lambda_t^1 & \lambda_t^2 & e_t^1 & e_t^2 & p^\backslash & \alpha \\ D^2 \pi_t \Delta u_t - D^2 v_t & D \pi_t D u_t^1 & -D \pi_t D u_t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D \pi_t (D u_t^1)^T & \pi_t D^2 u_t^1 & 0 & 0 & -p^T & 0 & 0 & 0 & -\lambda_t^1 \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 \\ -D \pi_t (D u_t^2)^T & 0 & \pi_t^2 D^2 u_t^2 & 0 & 0 & -p^T & 0 & 0 & -\lambda_t^2 \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 \\ 0 & 0 & 0 & \lambda_t^1 D_b Q_t & Q_t & -1 & 0 & 0 & * & 0 \\ 0 & -p & 0 & -Q_t & 0 & 0 & p & 0 & * & 0 \\ 0 & 0 & -p & 1 & 0 & 0 & 0 & p & Z_t^{2\backslash} & 0 \\ D \pi_t (z_t^1 - z_t^2)^\backslash & I_t^1 & I_t^2 & 0 & 0 & 0 & -I_t^1 & -I_t^2 & 0 & 0 \\ 0 & \alpha_t [0 \ I] & \alpha_t [0 \ I] & 0 & 0 & 0 & \alpha_t [0 \ I] & \alpha_t [0 \ I] & D p & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha \end{bmatrix}$$

where $Q_t := q_t + b_t Dq_t$, $Dp = \sum_t \alpha_t^1 (b_t D_{bp} q_t)$. Since there is t such that $\alpha_t \neq 0$, if $d\alpha_{t'} = 0$ for some t' it is simple to verify that the above matrix is surjective. Suppose $\alpha_t \neq 0$ for every t . Pre-multiply this matrix by a vector

$$d := \left(\dots, (da_t, x_t^1, dx_t^2, db_t, d\lambda_t^1, d\lambda_t^2), \dots, dp^1, dr^1, dl \right) \in \mathbb{R}^{(4+2C)T+C-1}$$

I have to show that $d = 0$. From the columns corresponding to (x_t, e_t) we get

$$d\lambda_t^s p - \pi_t^s \begin{bmatrix} 0 \\ dp^1 \end{bmatrix} + d\alpha_t \lambda_t^s \begin{bmatrix} 0 \\ dr^1 \end{bmatrix} = 0 \Rightarrow d\lambda_t^s = 0, \frac{1 - \pi_t}{\lambda_t^2} = \frac{\pi_t}{\lambda_t^1}$$

But, from the Kuhn-Tucker equations characterizing the solutions of the consumer problem, this implies $x_t^1 = x_t^2$, and hence $a_t = 0$ for every t , which is a contradiction. Therefore, the derivative of the system $(*)$ with respect to (ξ, α, e) is surjective. By the transversality theorem, for almost every $e \in E$, at any solution of the system $(*)$, the derivative with respect to (ξ, α) is also surjective, which is impossible since this matrix has more rows than columns. Therefore, for almost every e the system $(*)$ has no solution. Since $F(\xi, e) = 0$ always has a solution, this means that at any equilibrium the system

$$\begin{aligned} \sum_{t=1}^{T-1} \alpha_t \left(\lambda_t^1 z_t^1 + D_p q_t b_t + \lambda_t^2 z_t^2 \right) &= 0 \\ \sum_{t=1}^{T-1} (\alpha_t^s)^2 - 1 &= 0 \end{aligned}$$

does not have a solution, which is equivalent to the desired matrix having full column rank. The proof of this step is then complete.

Step 3: If $F(\xi, e, U) = 0$, $a > 0$, $d_F^T D F^P + D_G D G = 0$ and $(d_F, d_G) \neq 0$ then it must be the case that $dx_t^s \neq 0$ for every (t, s) .

Consider the system

$$\begin{aligned} F(\xi, e, U) &= 0 \\ d_F^T D F^P + D_G D G &= 0 \\ d^T d - 1 &= 0 \end{aligned}$$

Taking derivatives, the system $d_F D_\xi F + d_G D_\xi G = 0$, using the same notation as in proposition 3.3, gives

$$(D^2 \pi_t \Delta u_t - D_a^2 v_t) da_t + D \pi_t (z_t^1 - z_t^2)^1 dp^T = 0 \quad (t.1)$$

$$D \pi_t Du_t da_t + \pi_t D^2 u_t dx_t^1 - p^T d\lambda_t^1 + \pi_t \begin{bmatrix} 0 \\ dp^1 \end{bmatrix} + d\alpha_t \pi_t Du_t = 0 \quad (t.2)$$

$$-D\pi_t Du_t da_t + \pi_t^2 D^2 u_t dx_t^2 - p^T d\lambda_t^2 + \pi_t^2 \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} + d\alpha_t \pi_t^2 Du_t = 0 \quad (t.3)$$

$$(-q_t - b_t Dq_t) d\lambda_t^1 + d\lambda_t^2 = 0 \quad (t.4)$$

$$pdx_t^1 = 0 \quad (t.5)$$

$$pdx_t^2 = 0 \quad (t.6)$$

$$\sum_t \left(\lambda_t^1 dx_t^{1\backslash} + \lambda_t^2 dx_t^{2\backslash} + d\lambda_t^1 \left(z_t^{1\backslash} + b_t D_p q_t \right) + d\lambda_t^2 z_t^{2\backslash} \right) = 0 \quad (M)$$

Suppose $dx_t^1 = 0$ for some t . From equation (t.2) we get

$$D\pi_t Du_t^1 da_t - p d\lambda_t^1 + \pi_t \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} + d\alpha_t \pi_t Du_t^1 = 0$$

Using the first order conditions this equations gives

$$\pi_t \begin{bmatrix} 0 \\ dp^\backslash \end{bmatrix} + p \left[\frac{\lambda_t^1}{\pi_t} D\pi_t da_t + d\alpha_t \lambda_t^1 \right] = p d\lambda_t^1$$

and hence

$$\lambda_t^1 [D\pi_t da_t + d\alpha_t \pi_t] = \pi_t d\lambda_t^1 \quad (*)$$

But this implies $dp^\backslash = 0$. Using (t.3), (t.6) and once more the first order conditions we get

$$(1 - \pi_t) dx_t^2 D^2 u_t^2 dx_t^2 = 0 \Rightarrow dx_t^2 = 0$$

Following the last argument for every t we get $dx_t^1 = dx_t^2 = 0$ and, using (t.1), $da_t = 0$ for every t . From (*) we get $d\lambda_t^s = d\alpha_t \lambda_t^s$ for every t and s . But then equation (M) gives

$$\sum_t d\alpha_t \left(\lambda_t^1 \left(z_t^{1\backslash} + b_t D_p q_t \right) + \lambda_t^2 z_t^{2\backslash} \right) = 0$$

By step 2, it must be the case that $d\alpha_t = 0$ for every t , which immediately gives $d = 0$ and is the desired contradiction.

Let $\Psi(\xi, e, u) = 0$ denote the system of equations (t.1) – (t.6), (M) and

$$d\alpha dx_t^1 dx_t^2 = 0 \quad (t.7)$$

for every t , where $d\alpha := \sum_t d\alpha_t^2$.

Step 4: For a dense set of economies the following system has no solution

$$F(\xi, e, U) = 0$$

$$\Psi(\xi, e, U) = 0$$

Let (ξ^*, e, U) be a regular equilibrium satisfying the previous steps, t satisfy $a_t^* > 0$ and

$$V_t^s := \{x \in \mathfrak{R}_{++}^C / \|x_t - x_t^{s*}\| < \varepsilon\}$$

Choose $\varepsilon > 0$ small enough so that $dV_t^1(\varepsilon) \cap dV_t^2(\varepsilon) = \emptyset$. Let $\Phi_t^s : \mathfrak{R}_{++}^C \rightarrow [0, 1] \in C^\infty$ satisfy

$$\Phi_t^s(x) = \begin{cases} 1 & \text{if } x \in V_t^s(\varepsilon/2) \\ 0 & \text{if } x \notin V_t^s(\varepsilon) \end{cases}$$

Hirsch (1975, p.41) shows the existence of this function. Consider the following perturbation of the households utility functions

$$\begin{aligned} U_t(x_t, a_t, H_t^1, H_t^2, h_t) &: = \pi_t(a_t) \left(u_t(x_t^1) + \Phi_t^1(x_t) (x_t^1 - x_t^{1*})^T H_t^1 (x_t^1 - x_t^{1*}) \right) \\ &+ (1 - \pi_t(a_t)) \left(u_t(x_t^2) + \Phi_t^2(x_t) (x_t^2 - x_t^{2*})^T H_t^2 (x_t^2 - x_t^{2*}) \right) - v_t(a_t) \end{aligned}$$

It is simple to verify that for H_t^s and h_t small enough, $U_t(\cdot, H_t^1, H_t^2, h_t)$ satisfy (H1) – (H3). Notice that the household's Kuhn-Tucker conditions at $x_t = x_t^*$ are still satisfied for every H_t^s small enough.¹⁸

Pre-multiplying the matrix $D \begin{bmatrix} F(\xi, e, U) \\ \Psi(\xi, e, U) \end{bmatrix}$ by

$$\Delta := \left(\dots, \left(\Delta a_t, \Delta x_t^1, \Delta x_t^2, \Delta q_t, \Delta \lambda_t^1, \Delta \lambda_t^2, \Delta l_t \right), \dots, \Delta p^\backslash \right) \in \mathfrak{R}^{(5+2C)T+C-1}$$

gives

$$\begin{aligned} \Delta x_t^s dx_t^{s,c} &= 0 \text{ for all } c, dx_t^s \neq 0 \Rightarrow \Delta x_t^s = 0 \\ (D^2 \pi_t \Delta u_t - D^2 v_t) \Delta a_t &= 0 \Rightarrow \Delta a_t = 0 \\ \Delta l_t dx_t^{1T} dx_t^1 dx_t^{2T} dx_t^2 &= 0, dx_t^s \neq 0 \text{ for all } s \Rightarrow \Delta l_t = 0 \\ \Delta \lambda_t^s p^T + \lambda_t^s \begin{bmatrix} 0 \\ \Delta p^\backslash \end{bmatrix} &= 0 \Rightarrow \Delta \lambda_t^s = 0 \Rightarrow \Delta p^\backslash = 0 \\ \Delta q_t D q_t &= 0, D q_t > 0 \Rightarrow \Delta q_t = 0 \end{aligned}$$

This how that for every t such that $a_t > 0$ we must have

$$\left(\Delta a_t, \Delta x_t^1, \Delta x_t^2, \Delta q_t, \Delta \lambda_t^1, \Delta \lambda_t^2, \Delta l_t \right) = 0 \text{ and } \Delta p^\backslash = 0$$

The same computations used in step 2, proposition 3.3 show that for every t such that $a_t = 0$ we also must have

$$\left(\Delta a_t, \Delta x_t^1, \Delta x_t^2, \Delta q_t, \Delta \lambda_t^1, \Delta \lambda_t^2, \Delta l_t \right) = 0$$

¹⁸This computations are done in detail in Lisboa (1995).

Therefore, the derivative

$$D_{\xi,d,H} \begin{bmatrix} F \\ \Psi \end{bmatrix}$$

is surjective. By the Transversality Theorem, for a dense choice of H , if the system

$$\begin{aligned} F(\xi, e, u) &= 0 \\ \Psi(\xi, e, u) &= 0 \end{aligned}$$

has a solution then the derivative

$$D_{\xi,d} \begin{bmatrix} F \\ \Psi \end{bmatrix}$$

is also surjective, which is impossible since this matrix has more rows than columns. The proof is then complete. \square

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