Bertrand, Cournot and Monopolistically Competitive Equilibria

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Abstract. Using an example, we study the analogs, for the differentiated product case, of the Cournot and Bertrand equilibria. These equilibria can be shown to exist and be unique if we impose a simple and natural restriction on the elasticities of the demand functions for the differentiated products. Our characterizations of these equilibria make it possible to compare them and to determine how they are affected by the size of the market and the number of firms. We are also able to prove the existence of Cournot free-entry equilibria in which the number of firms is determined endogenously. In addition, we are able to prove that, in a large market, the Cournot free-entry equilibria approximate the Dixit-Stiglitz monopolistically competitive equilibria. The free-entry equilibrium concept we study is an analog of the one studied by Novshek for the case of firms selling products that are perfect substitutes. Our results are extensions of Novshek's. While we were unable to establish a general existence result for Bertrand free-entry equilibria, we were able to prove that, when these equilibria exist, they are unique and that in large markets they also approximate the Dixit-Stiglitz equilibria.

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1. INTRODUCTION

The aim of this paper is to study a market for differentiated products. In the model employed, the demands for the differentiated goods are derived using the same approach as that taken by Dixit and Stiglitz in their analysis [2] of Chamberlin's monopolistically competitive equilibrium [1]. Our purpose, however, is to study these markets when the number of firms is finite and may be small. Thus, we are interested not only in the Chamberlinian equilibrium, but also in the analogs, for the differentiated product case, of the Cournot and Bertrand equilibria. These equilibria can be shown to exist and be unique if we impose a simple and natural restriction on the elasticities of the demand functions for the differentiated products. Under this demand function restriction, it is also easy to compare the Bertrand and Cournot equilibria and to show that the Bertrand prices and profits are lower and the Bertrand supplies are higher than those arising in the Cournot equilibria. The demand restriction imposed to obtain these results is simply that the elasticity of demand in the Dixit-Stiglitz monopolistically competitive equilibrium is higher than in the monopoly case in which only one firm produces. This turns out to be a condition that Chamberlin and Dixit-Stiglitz also imposed. In particular, Dixit and Stiglitz refer to "Chamberlin's DD and dd curves." The dd curve is the Dixit-Stiglitz demand curve faced by each firm in the monopolistically competitive equilibrium. The DD curve is the demand curve each firm would face if the firms colluded to charge the same price. Following Chamberlin, Dixit and Stiglitz assume that the dd curve is more elastic than the DD curve, a condition that is easily seen to be identical to the one we impose.

For the purpose of comparing the per firm profits obtained in the Bertrand and Cournot equilibria it is convenient to compare them both to the collusive outcome that arises when firms choose a common price and maximize per firm profits which are computed using the fact that, at any common price, each firm's sales will be given by the DD curve. As expected, the collusive prices and profits are higher and the collusive output levels are lower than those arising in the Cournot and Bertrand equilibria.

We also study Cournot and Bertrand analogs of Novshek's version [3] of the Cournot equilibrium in which the number of firms is determined by free entry. Novshek studied the case in which a typical potential firm produced with a technology that gave rise to a "U-shaped" average cost curve. While simply assuming that the demand curve sloped down, he was able to prove that if a typical potential firm's average cost minimizing output level is "small" relative to market demand when price equals minimum average cost, then the total amount supplied in the free entry Cournot equilibrium is approximately the "competitive output" and the price is, therefore, approximately the "competitive price;" i.e., the price is approximately equal to minimum average cost. Novshek's result demonstrated that even if firms don't take prices

as given, entry will drive the noncompetitive Cournot equilibrium price to the competitive price if the typical firm's efficient scale is small relative to the size of the market.

Like the competitive equilibrium, the monopolistically competitive equilibrium, is commonly viewed as an approximation to a noncompetitive equilibrium in which entry drives price down to average cost. Dixit and Stiglitz also argue that, in their formalization of the monopolistically competitive model, the presence of a large number of firms drives the elasticity of demand to a finite limiting value that they derive. Our aim is follow Novshek's approach and to provide a formalization of a natural noncompetitive equilibrium in which entry does, indeed, drive price to average cost and the demand elasticity to its Dixit-Stiglitz limit. For this purpose we define, for the differentiated products case, Cournot and Bertrand analog's of Novshek's free-entry equilibrium. Under the above mentioned restriction on the demand elasticity, we are then able to establish analogs of Novshek's result for the case of monopolistic competition. In particular, we can demonstrate that, if the market demand is large and if the Cournot and Bertrand free-entry equilibria exist, then in both of these equilibria, there will be a large number of firms producing differentiated products at approximately the Chamberlinian output levels and supplying them at approximately the Chamberlinian price. We are also able to demonstrate that, if the market is large enough to guarantee profits when only two firms produce at their Cournot duopoly output levels, then there will always exist a Cournot free-entry equilibrium. We are not able to demonstrate that this equilibrium is unique, however. We are also unable to demonstrate that a Bertrand free-entry equilibrium always exists even if the market is large. It seems that for some market sizes a Bertrand free-entry equilibrium will exist and for some it will not. We do demonstrate, however, that if a Bertrand free-entry equilibrium does exist, it is unique. We can also show that when a Bertrand free-entry equilibrium exists, there is always a Cournot free-entry equilibrium in which the number of firms producing is at least as large as in the Bertrand free-entry equilibrium.

While we do not extensively analyze the Cournot and Bertrand equilibria for the case in which our demand elasticity condition fails, we do show that, in the borderline case in which the Dixit-Stiglitz elasticity equals the monopoly elasticity, entry has no effect on firm profits. In that case, profits may or may not be positive depending on the size of the market. If profits are positive they remain positive even if there are many firms. As a consequence, the Dixit-Stiglitz monopolistically competitive equilibrium cannot be approximated by either a Cournot or Bertrand free entry equilibria because these equilibria must fail to exist. In this border line case, there is no reason to expect market forces, in particular, the possibility of entry, to lead to equilibria that approximate monopolistic competition.

Finally, it should be mentioned that our analysis is restricted to an example. It

is specifically the case in which the monopoly elasticity of demand is constant. The constancy of the monopoly elasticity of demand complements the constancy the Dixit-Stiglitz elasticity of demand to yield not only a tractable example but also a canonical one. Because the example is canonical in this sense, there is hope for supposing that the arguments developed in the analysis of this example can be extended. In the summary, we briefly discuss the possibility of obtaining such extensions.

2. Outline of the Paper

The demand functions and the inverse demand functions faced by the sellers of differentiated products in the Bertrand and Cournot equilibria are described in Section 2.1. The inverse demand functions are introduced in the subsection on "The Cournot Case" and the demand functions in the subsection on "The Bertrand Case." A series of remarks describing some features of the demand and inverse demand functions follows their introductions. These remarks also relate the demand and inverse demand functions to the Chamberlinian dd and DD curves mentioned in the introduction. In addition, the remarks form the basis for the discussion in the subsection that follows in which the parameters are interpreted and the elasticity restriction mentioned in the introduction is discussed formally. Next is a subsection in which we discuss the derivation of the demand and inverse demand functions from utility maximization. The final subsection of section 2.1 derives expressions for the elasticities of demand from both the demand and inverse demand functions. These expressions are discussed and interpreted in another series of remarks. The cost assumptions are described in Section 2.2.

The Dixit-Stiglitz equilibrium is defined and characterized in Section 3. Since we make quite specific assumptions about demand we are able to go farther than Dixit and Stiglitz were in describing their equilibrium. In particular we are able to determine the equilibrium number of firms and relate the number of firms to the size of the market. That relationship is discussed in Section 3.2. Section 3.3 presents two remarks that contain a comparative static analysis describing the relationship between the Dixit-Stiglitz equilibrium and the Dixit Stiglitz demand elasticity. Section 3.3 represents a detour from a our main purpose but is included since the relationship between the equilibrium number of firms and the Dixit Stiglitz demand elasticity is somewhat surprising.

For a fixed number of firms, the Cournot equilibrium is defined and characterized in the beginning of Section 4. The symmetric Cournot equilibrium is described in Proposition 1 of Section 4.1. In Proposition 2 of Section 4.2, the symmetric equilibrium is shown to be the unique Cournot equilibrium. Proposition 3 of Section 4.3 presents comparative static results relating the Cournot equilibrium price and output levels to the size of the market and the number of firms. Corollary 4 to Proposition 3, which begins Section 4.4, describes the effect of market size and the number of

firms on the Cournot profits. Proposition 5 demonstrates that, for any number of firms, if the market is large enough, the Cournot profits will be positive. Proposition 6, which concludes Section 4.4 shows that, for any market size, Cournot profits will be negative if there are many firms.

A Cournot free-entry equilibrium is defined at the beginning of Section 5, and Proposition 8 demonstrates that for virtually any market size, such an equilibrium exists. Proposition 9 demonstrates, that when the market is large the Cournot free-entry equilibrium approximates the Dixit-Stiglitz equilibrium.

The Bertrand equilibrium is defined, for a fixed number of firms, in the beginning of Section 6. A general characterization follows in Remark 39. Proposition 10 of Section 6.1 describes the symmetric Bertrand equilibrium which is shown to be the unique Bertrand equilibrium in Proposition 11 of Section 6.2.

Proposition 12 of Section 6.3 is the Bertrand analog of Proposition 3. It presents the comparative static results relating the Bertrand price and output levels to the size of the market and the number of firms. Corollary 13 to Proposition 12 which begins Section 6.4 shows that Bertrand profits increase with the size of the market. Proposition 15 demonstrates how entry and the number of firms affect the Bertrand profits. Proposition 16 is the Bertrand analog of Proposition 5. It demonstrates that, for any number of firms, if the market is large enough, the Bertrand profits will be positive.

The discussion of the Bertrand free-entry equilibrium, is facilitated by a comparison of the Cournot and Bertrand equilibria, and in making this comparison it is useful to compare each of these equilibria with the collusive outcome discussed in the introduction. The collusive outcome is defined in Section 7 and characterized in Proposition 17 of that section. Proposition 18 of Section 8 presents the comparison between the collusive outcome and the Cournot and Bertrand equilibria. For the discussion of the Bertrand free-entry equilibrium, the most important result is that the Bertrand profits are lower than the Cournot profits. This gives us Corollary 19 to Propositions 6 and 18 which asserts that, for any market size, Bertrand profits will negative if there are many firms.

A Bertrand free-entry equilibrium is defined at the beginning of Section 9. Although we are unable to show that such an equilibrium exists we are able to demonstrate that, if it exists, it is unique. This is done in Proposition 20. Proposition 21 demonstrates, that when the market is large and a Bertrand free-entry equilibrium exists it approximates the Dixit-Stiglitz equilibrium.

Section 10 discusses a case in which our equilibrium condition fails and shows that, in that case, we will typically fail to have existence of either a Cournot or Bertrand free-entry equilibrium in which firms produce.

3. Introduction to the Model

3.1. Demand.

The Cournot Case. Analysis of the Cournot Equilibrium begins with a description of the inverse demand function faced by each firm. We assume that when there are n firms, the **inverse demand function faced by firm** s is

$$\begin{aligned} p_s(x_1, \ldots, x_n) &= m x_s^{-\alpha} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{1}{1-\alpha}\right)[\gamma-(1-\alpha)]} \\ &= m x_s^{-\alpha} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha}\right)-1} \end{aligned} \quad (1)$$

where

$$x_t = \text{ the supply of firm } t,$$

$$p_s = \text{ the price the market will pay for good } s,$$

$$m > 0$$

and

$$0 < \gamma < 1 - \alpha < 1. \quad (2)$$

Remark 1. *It is straightforward to verify that*

$$\frac{\partial p_s(x_1, \ldots, x_n)}{\partial x_s} = -\frac{p_s(x_1, \ldots, x_n)}{x_s} \left[\alpha \left(1 - \frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} \right) + (1-\gamma) \left(\frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} \right) \right].$$

This expression is negative if both α and $1-\gamma$ are positive, and this is true even if (2) fails.

However, when $t \neq s$,

$$\frac{\partial p_s(x_1, \ldots, x_n)}{\partial x_t} = m[\gamma - (1-\alpha)] x_t^{-\alpha} x_s^{-\alpha} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha}\right)-2}$$

is negative if (2) holds but positive if

$$\gamma > 1 - \alpha.$$

When

$$\gamma = 1 - \alpha,$$

the price received by each firm s is unaffected by the amount supplied by any other firm, since in that case, (1) reduces to

$$p_s(x_1, \ldots, x_n) = m x_s^{-\alpha}$$

Remark 2. *It is useful to have an expression for the price received by each firm when all firms supply the same amount. If we evaluate the inverse demand function (1) at* $(x_1, \ldots, x_n) = (x, \ldots, x)$ *the result is*

$$p(x) \equiv p_s(x, \ldots, x) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-1} x^{\gamma-1}.$$

This is the inverse of what Dixit and Stiglitz call "the Chamberlinian DD curve." The demand elasticity of this inverse demand curve is $\frac{1}{1-\gamma}$.

Remark 3. *The monopoly case occurs when there is only one firm. In this case, the monopolist faces the inverse demand curve*

$$p_1(x_1) = m x_1^{\gamma-1},$$

which is the special case of (1) obtained when $n = 1$. *The demand elasticity of this inverse demand curve is also* $\frac{1}{1-\gamma}$. *Thus, the elasticity of the monopoly inverse demand curve is the same as the elasticity of the inverse of the Chamberlinian DD curve mentioned in Remark 2.*

Remark 4. *If we define*

$$H \equiv \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{1}{1-\alpha}\right)[\gamma-(1-\alpha)]}, \tag{3}$$

then we can rewrite (1) as

$$p_s(x_s) = m x_s^{-\alpha} H. \tag{4}$$

By following Dixit and Stiglitz and assuming that, because n *is large, the supply* x_s *chosen by firm* s *has virtually no effect on* H, *we can treat* H *as a constant. We will call the inverse demand curve* $p_s(x_s)$ *described in (4) the Dixit-Stiglitz inverse demand curve. It is, in fact, the inverse of what Dixit and Stiglitz refer to as "the dd curve" of "Chamberlinian terminology." The elasticity of the Dixit-Stiglitz inverse demand curve (4) is* $\frac{1}{\alpha}$ *when* H *is independent of* x_s.

The Bertrand Case:. For the purpose of investigating the Bertrand equilibrium it is necessary to describe the demand function. Fortunately, it is straightforward to invert the inverse demand functions given in (1) to obtain the corresponding **demand function faced by firm** s. The result is

$$\begin{aligned} x_s(p_1, \ldots, p_n) &= m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}} \left(\sum_{t=1}^{n} p_t^{-\frac{1-\alpha}{\alpha}} \right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)} - \frac{1}{\alpha}\right]} \\ &= m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}} \left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}} \right)^{\left[\frac{\left(1-\frac{1}{(1-\gamma)}\right)}{\left(1-\frac{1}{\alpha}\right)} - 1\right]}. \end{aligned} \tag{5}$$

Remark 5. *Note that*

$$\frac{\partial x_s(p_1,\ldots,p_n)}{\partial p_s}$$
$$= -\left[\frac{x_s(p_1,\ldots,p_n)}{p_s}\right]\left[\frac{1}{\alpha}\left(1-\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)}\right)+\frac{1}{(1-\gamma)}\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)}\right]$$

is negative if both α and $1-\gamma$ are positive. This is true even if (2) fails.

However, when $t \neq s$,

$$\frac{\partial x_s(p_1,\ldots,p_n)}{\partial p_t} = m^{\frac{1}{(1-\gamma)}}\left(\frac{1}{\alpha}-\frac{1}{(1-\gamma)}\right)p_t^{-\frac{1}{\alpha}}p_s^{-\frac{1}{\alpha}}\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)^{\left[\frac{\left(1-\frac{1}{(1-\gamma)}\right)}{\left(1-\frac{1}{\alpha}\right)}-2\right]}$$

is positive if (2) holds but negative if

$$\gamma > 1-\alpha.$$

When

$$\gamma = 1-\alpha,$$

the amount sold by each firm s is unaffected by the price charged by any other firm.

Remark 6. *Again, it will be useful in what follows to have an expression for the demand of each firm when all firms make the same choice. In this case, that means all firms charge the same price. If we evaluate the demand function (5) at $(p_1,\ldots,p_n) = (p,\ldots,p)$ the result is*

$$x(p) \equiv x_s(p,\ldots,p) = m^{\frac{1}{(1-\gamma)}}p^{-\frac{1}{(1-\gamma)}}n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}.$$

The demand function $x(p)$ is, of course, the inverse of $p(x)$ defined in Remark 2; it is the "Chamberlinian DD curve." As asserted in Remark 2, the elasticity of this demand curve is $\frac{1}{1-\gamma}$.

Remark 7. *When there is only one firm, that firm is a monopolist who faces the demand curve*

$$x_1(p_1) = m^{\frac{1}{(1-\gamma)}}p_1^{-\frac{1}{(1-\gamma)}}$$

which is easily seen to be the inverse of $p_1(x_1)$, the inverse demand curve described in Remark 3. As noted in that Remark, the elasticity of this demand curve is $\frac{1}{1-\gamma}$ and is the same as the elasticity of the Chamberlinian DD curve mentioned in Remarks 2 and 6.

Remark 8. *Let's define*

$$K \equiv \left(\sum_{t=1}^{n} p_t^{-\frac{1-\alpha}{\alpha}} \right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)} - \frac{1}{\alpha}\right]}, \tag{6}$$

and rewrite (5) as

$$x_s(p_s) = m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}} K. \tag{7}$$

We will again follow Dixit and Stiglitz and treat K as a constant. We, thereby, effectively assume that, because n is large, the price p_s chosen by firm s has virtually no effect on K. The elasticity of the demand curve (7), that we refer to as the Dixit-Stiglitz demand curve, is $\frac{1}{\alpha}$ because K is independent of p_s. The Dixit-Stiglitz demand curve is, of course, the inverse of $p_s(x_s)$, of Remark 4. It is, therefore, the dd curve of Chamberlinian terminology.

Interpreting the Parameters. As noted in the introduction, we will, at certain points in the analysis, consider cases in which market demand is "large." Note that, for all s and for each price vector $(p_1, \ldots, p_n)$, an increase in the parameter m increases the demand of firm s in (5). Also as m becomes large the demand of firm s becomes large. For this reason we use m as **the parameter that measures the size of market demand.** Thus, the case in which market demand is large is the case in which m is large.

We also noted in the introduction that we would be required to impose a restriction on the elasticity of demand. Specifically, we asserted that we would assume that the elasticity of demand in the Dixit-Stiglitz monopolistically competitive equilibrium is higher than in the monopoly case in which only one firm produces. We discussed the monopoly case in Remarks 3 and 7 and the Dixit-Stiglitz case in Remarks 4 and 8. In those Remarks we observed that **the elasticity of demand in the monopoly case is $\frac{1}{1-\gamma}$** and that **the elasticity of demand in the Dixit-Stiglitz monopolistically competitive equilibrium is $\frac{1}{\alpha}$**. Clearly, condition (2) implies that

$$\frac{1}{\alpha} > \frac{1}{1-\gamma}; \tag{8}$$

Thus, the case in which condition (2) holds is exactly the case in which the elasticity of demand in the Dixit-Stiglitz monopolistically competitive equilibrium is greater than the elasticity of demand in the monopoly case. As noted in Remarks 2-4 and Remarks 6-8, the monopoly elasticity is the same as the elasticity of the Chamberlinian DD curve and the Dixit-Stiglitz demand curve is the Chamberlinian dd curve. As a consequence, **condition (8) can also be interpreted as imposing what Dixit and Stiglitz call the "conventional" Chamberlinian condition that the dd curve is more elastic than the DD curve.**

Utility Maximization and Demand. It is straightforward to observe that, when $\gamma < 1$, the demand function we propose to study is the utility maximizing demand function of a representative consumer who consumes one other good in addition to the n differentiated products. In making this interpretation, we let $(x_1, \ldots, x_n)$ denote the vector of amounts consumed of the n differentiated products and we use y to denote the amount consumed of the one other good. When the representative consumer faces the price vector $(p_1, \ldots, p_n)$, has income I and maximizes the utility function

$$u(y, x_1, \ldots, x_n) = \left[y + \frac{m}{\gamma} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha}\right)} \right] \tag{9}$$

subject to the budget constraint

$$y + \sum_{t=1}^{n} p_t x_t = I,$$

his demand function is given by (5).

The condition $\gamma < 1$ is imposed to insure that the utility function (9) is quasi-concave. It is easy to verify that, whenever $\gamma < 1$,

$$\frac{m}{\gamma} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha}\right)}$$

is a strictly concave function of $(x_1, \ldots, x_n)$. This is then easily seen to imply that the utility function (9) is, indeed, a quasi-concave function of $(y, x_1, \ldots, x_n)$. Before leaving this point, it should be emphasized that condition (2) is not required to guarantee the quasi-concavity of the utility function (9), all that is required is $\gamma < 1$.

Remark 9. *The utility function (9) is, of course, is not quite a special case of the general utility function*

$$u(y, x_1, \ldots, x_n) = v\left(y, \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{1}{1-\alpha}\right)} \right)$$

assumed by Dixit and Stiglitz. The difference arises because they assumed that $v(\cdot, \cdot)$ was homothetic, a condition not satisfied by the function

$$v(y, z) = y + m\frac{z^{\gamma}}{\gamma}$$

used to define the utility function (9). Nevertheless, the interpretation of α is the same here as in the Dixit-Stiglitz paper; viz, In addition to being the elasticity of the demand curve faced by each firm in monopolistic competition, $\frac{1}{\alpha}$ ***measures the elasticity of substitution between goods s and t.***

Remark 10. *Dixit and Stiglitz imposed a condition analogous to condition (2) by effectively assuming that the elasticity of substitution between y and z was smaller than $\frac{1}{\alpha}$, the elasticity of substitution between the differentiated goods. The interpretation of that assumption in their analysis is the same as the interpretation of (2) in ours. In Chamberlinian terms, each of these assumptions guarantee that the dd curve is more elastic than the DD curve.*

Demand Elasticity. We can obtain the elasticity of demand by using either the demand function (5) or the inverse demand function (1). Let's begin by using the demand function (5) and the expression for

$$\frac{\partial x_s\left(p_1,\ldots,p_n\right)}{\partial p_s}$$

derived in Remark 5. The resulting expression for the elasticity is

$$\begin{aligned}\varepsilon_s^b\left(p_1,\ldots,p_n\right) &\equiv -\frac{p_s\frac{\partial x_s(p_1,\ldots,p_n)}{\partial p_s}}{x_s\left(p_1,\ldots,p_n\right)} \qquad (10)\\ &= \left[\frac{1}{\alpha}\left(1-\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)}\right)+\frac{1}{(1-\gamma)}\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)}\right].\end{aligned}$$

Using the inverse demand function (1) and the expression for

$$\frac{\partial p_s\left(x_1,\ldots,x_n\right)}{\partial x_s}$$

given in Remark 1, we obtain another expression for the elasticity of demand. Specifically,

$$\begin{aligned}\varepsilon_s^c\left(x_1,\ldots,x_n\right) &\equiv -\frac{p_s\left(x_1,\ldots,x_n\right)}{x_s\frac{\partial p_s(x_1,\ldots,x_n)}{\partial x_s}} \qquad (11)\\ &= \frac{1}{\left[\alpha\left(1-\frac{x_s^{1-\alpha}}{\sum_{t=1}^n x_t^{1-\alpha}}\right)+(1-\gamma)\left(\frac{x_s^{1-\alpha}}{\sum_{t=1}^n x_t^{1-\alpha}}\right)\right]}.\end{aligned}$$

Interpreting The Expressions for the Demand Elasticities.

The Bertrand Case. The important features of the expression for the demand elasticity $\varepsilon_s^b(p_1, \ldots, p_n)$ given in (10) are described in the following series of remarks.

Remark 11. *We have already noted in Remarks 3 and 7 that when there is only one firm, the elasticity of the demand curve faced by that firm is* $\frac{1}{1-\gamma}$*. The same result follows from (10), since, when* $n = 1$*,*

$$\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)} = 1$$

and (10) reduces to

$$\varepsilon_1^b(p_1) = \frac{1}{(1-\gamma)}.$$

Remark 12. *When* $n > 1$*, expression (10) tells us that the elasticity of demand faced by a typical firm* s*,* $\varepsilon_s^b(p_1, \ldots, p_n)$*, is a weighted average of* $\frac{1}{\alpha}$ *and* $\frac{1}{(1-\gamma)}$ *in which*

$$0 < \frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)} < 1$$

is the weight on the monopoly elasticity, $\frac{1}{(1-\gamma)}$*, and*

$$0 < 1 - \frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)} < 1$$

is the weight on the monopolistically competitive elasticity, $\frac{1}{\alpha}$*.*

The subsequent remarks are simple consequences of the one just made.

Remark 13. *When* $n > 1$ *and condition (2) holds,*

$$1 < \frac{1}{(1-\gamma)} < \varepsilon_s^b(p_1, \ldots, p_n) < \frac{1}{\alpha}, \tag{12}$$

i.e., the elasticity of a typical firm s*,* $\varepsilon_s^b(p_1, \ldots, p_n)$*, is less than the monopolistically competitive elasticity but above the monopoly elasticity.*

Remark 14. *Note that for*

$$\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)$$

fixed,

$$\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)} = \left[\frac{p_s}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{1}{\alpha}}}}\right]^{1-\frac{1}{\alpha}}.$$

is a decreasing function of p_s. Thus, as firm s raises its price, p_s, relative to the geometric average of all firms' prices

$$\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{1}{\alpha}}},$$

it's elasticity of demand, $\varepsilon_s^b(p_1,\ldots,p_n)$, rises and is closer to the monopolistically competitive elasticity $\frac{1}{\alpha}$.

Remark 15. *Note also that even if we allow*

$$\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)$$

to vary as p_s increases

$$\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)} = \frac{1}{\left(1 + p_s^{\frac{1}{\alpha}-1} \sum_{t\neq s} p_t^{1-\frac{1}{\alpha}}\right)},$$

is clearly a decreasing function of p_s. Thus,

$$\frac{\partial e_s^b\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s} < 0$$

and as firm s raises its price, p_s, it's elasticity of demand, $\varepsilon_s^b(p_1,\ldots,p_n)$, rises and is closer to the monopolistically competitive elasticity $\frac{1}{\alpha}$.

Remark 16. *In the special case that arises when all firms charge the same price, p, $\varepsilon_s^b(p_1,\ldots,p_n)$ reduces to*

$$\varepsilon^b(n) \equiv \varepsilon_s^b(p,\ldots,p) = \left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right) + \frac{1}{(1-\gamma)}\frac{1}{n}\right].$$

Remark 17. *When there are many firms and*

$$\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{1}{\alpha}}}$$

is very large relative to p_s, *then*

$$\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)} \simeq 0$$

and (10) tells us that

$$\varepsilon_s^b\left(p_1, \ldots, p_n\right) \simeq \frac{1}{\alpha}.$$

This means that the demand curve facing each firm has an elasticity near that faced by firms in Dixit and Stiglitz's model of monopolist competition. In particular,

$$\varepsilon^b(n) = \left[\frac{1}{\alpha}\left(1 - \frac{1}{n}\right) + \frac{1}{(1-\gamma)}\frac{1}{n}\right] \simeq \frac{1}{\alpha}$$

when n *is large and all firms charge the same price.*

Note that the comments made in this Remark are true whether (2) holds or not.

The Cournot Case. Once again, we list the important features of the expression for the demand elasticity $\varepsilon_s^c\left(x_1, \ldots, x_n\right)$ given in (11) in a series of remarks.

Remark 18. *We already know from Remarks 3, 7 and 11 that the expression for* $\varepsilon_1^c\left(x_1\right)$ *given in (11) must reduce to*

$$\varepsilon_1^c\left(x_1\right) = \frac{1}{(1-\gamma)}.$$

when $n = 1$. *This, also follows from (11) since*

$$\frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} = 1,$$

when $n = 1$.

Remark 19. *When* $n > 1$, *expression (11) tells us that*

$$\frac{1}{\varepsilon_s^c\left(x_1, \ldots, x_n\right)},$$

the inverse of the elasticity of demand faced by a typical firm s, is a weighted average of the inverse elasticities α and $(1-\gamma)$ in which

$$0 < \frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} < 1$$

is the weight on, $(1-\gamma)$, the inverse of the monopoly elasticity, $\frac{1}{(1-\gamma)}$, and

$$0 < 1 - \frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} < 1$$

is the weight on, α, the inverse of the monopolistically competitive elasticity, $\frac{1}{\alpha}$.

The following remarks are simple consequences of the one just made.

Remark 20. *When $n > 1$, and condition (2) holds,*

$$\alpha < \frac{1}{\varepsilon_s^c(x_1, \ldots, x_n)} < 1 - \gamma < 1$$

and

$$1 < \frac{1}{1-\gamma} < \varepsilon_s^c(x_1, \ldots, x_n) < \frac{1}{\alpha}, \tag{13}$$

i.e., the elasticity of a typical firm s, $\varepsilon_s^c(x_1, \ldots, x_n)$ is less than the monopolistically competitive elasticity but above the monopoly elasticity.

Remark 21. *Note that for*

$$\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)$$

fixed,

$$\frac{x_s^{1-\alpha}}{\sum_{t=1}^{n} x_t^{1-\alpha}} = \left[\frac{x_s}{\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}}}\right]^{1-\alpha}.$$

is an increasing function of x_s. Thus, as firm s raises its supply, x_s, relative to the geometric average of all firms' supplies

$$\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}},$$

it's elasticity of demand $\varepsilon_s^c(x_1,\ldots,x_n)$ *falls and is closer to the monopoly elasticity* $\frac{1}{1-\gamma}$. *Alternatively we could observe that, if the competition faced by firm* s *increases in the sense that* x_s *falls relative to*

$$\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}},$$

then $\varepsilon_s^c(x_1,\ldots,x_n)$ *increases and is closer to monopolistically competitive lower bound* $\frac{1}{\alpha}$.

Remark 22. *In the special case that arises when all firms supply the same amount, (11) reduces to*

$$\varepsilon^c(n) \equiv \varepsilon_s^c(x,\ldots,x) = \frac{1}{\left[\alpha\left(1-\frac{1}{n}\right)+(1-\gamma)\frac{1}{n}\right]}.$$

Remark 23. *When there are many firms and*

$$\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

is very large relative to x_s, *then*

$$\frac{x_s^{1-\alpha}}{\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)} \simeq 0$$

and (11) tells us that

$$\varepsilon_s^c(x_1,\ldots,x_n) \simeq \frac{1}{\alpha}.$$

This means that the demand curve facing each firm has an elasticity near that faced by firms in Dixit and Stiglitz's model of monopolist competition. In Particular,

$$\varepsilon^c(n) = \frac{1}{\left[\alpha\left(1-\frac{1}{n}\right)+(1-\gamma)\frac{1}{n}\right]} \simeq 0$$

when n *is large all firms supply the same amount.*

Again, the comments made in this Remark are true whether (2) holds or not.

Comparing the Bertrand, Cournot and Dixit-Stiglitz Elasticities.

Remark 24. *When $n = 1$,*

$$\varepsilon_1^c(x_1) = \varepsilon^c(1) = \frac{1}{(1-\gamma)} = \varepsilon^b(1) = \varepsilon_1^b(p_1).$$

When $n > 1$ and all firms supply the same amount and charge the same price,

$$\varepsilon^c(n) < \varepsilon^b(n).$$

These results are true whether or not condition (2) holds.

Proof: The case $n = 1$ was discussed in Remarks 3, 7, 11 and 18.
Since

$$f(x) = \frac{1}{x}$$

is convex, Jensen's inequality implies that

$$\begin{aligned}
\varepsilon^c(n) &= \frac{1}{\left[\alpha\left(1-\frac{1}{n}\right)+(1-\gamma)\frac{1}{n}\right]} \\
&= f\left(\alpha\left(1-\frac{1}{n}\right)+(1-\gamma)\frac{1}{n}\right) \\
&< \left(1-\frac{1}{n}\right)f(\alpha)+\frac{1}{n}f(1-\gamma) \\
&= \left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right] \\
&= \varepsilon^b(n). \;\|
\end{aligned}$$

Remark 25. *Clearly, if (2) failed and we instead had*

$$0 < 1-\alpha < \gamma < 1, \tag{14}$$

then $\frac{1}{\alpha}$, the elasticity of the demand curve faced by firms in monopolistic competition, would actually be less than the elasticity of the demand curve faced by a monopolist who faced no such competition. Condition (14) would also imply that when firms faced "competition" from other firms producing "similar" but differentiated products the elasticity of the demand curve (5) they faced would be lower than that faced by a monopolist. Furthermore, if (14) holds, then $\varepsilon_s^b(p_1,\ldots,p_n)$, the demand elasticity faced by firm s, would actually decrease and move closer to $\frac{1}{\alpha}$ when the "competition" faced by firm s increased in the sense that p_s rose relative to

$$\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{1}{\alpha}}}.$$

In addition, (14) implies that $\varepsilon_s^c(x_1, \ldots, x_n)$, *also decreases and moves closer to* $\frac{1}{\alpha}$ *when the "competition" faced by firm s increases in the sense that* x_s *falls relative to*

$$\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

The restriction (2) on the parameters α *and* γ *obviously rules out such cases.*

Remark 26. *Finally, note that if (2) fails and we instead have*

$$1-\alpha=\gamma,$$

then

$$\varepsilon_s^b(p_1, \ldots, p_n) = \varepsilon_s^c(x_1, \ldots, x_n) = \frac{1}{\alpha} = \frac{1}{1-\gamma}.$$

We will discuss this borderline case at some length in Section 10 at the end of the paper.

3.2. Costs. All firms will be assumed to face the same total cost function $C(x)$. We will consider two different cases: the case of increasing marginal cost and U-shaped average cost curves and the Dixit-Stiglitz case in which the marginal cost is constant and there is a fixed cost. For the most part the analysis of these two cases is the same, but there are some points at which it is useful to distinguish the cases. Also it is often possible to be more specific about the Dixit-Stiglitz case. In particular, the Monopolistically Competitive, Cournot and Bertrand equilibria can all be explicitly computed for this case.

Cost Case 1: The Case of U-Shaped Average Cost. In this case, the marginal cost function

$$MC(x) = C'(x)$$

will be assumed to be increasing and the average cost function

$$AC(x) = \frac{C(x)}{x}$$

will be assumed to U-shaped. We will also assume that $C''(x)$ exists and is continuous and that

$$\lim_{x\to 0} AC(x) = \infty.$$

In this case, as in Cost Case 2, there are both fixed and variable costs and the marginal costs are the marginal variable costs.

Cost Case 2: The Dixit-Stiglitz Case. Dixit-Stiglitz assumed that

$$C(x) = F + cx$$

where $F > 0$ and $c > 0$. In this case,

$$MC(x) = C'(x) = c$$

and

$$AC(x) = \frac{C(x)}{x} = \frac{F}{x} + c.$$

So marginal cost is constant and average cost is always declining and larger than marginal cost.

4. THE CHAMBERLIN-DIXIT-STIGLITZ EQUILIBRIUM

This section simply summarizes Dixit and Stiglitz's results as they relate to the present model. Let's begin by recalling the discussion in Remark 8 and in the subsection on *Utility Maximization and Demand.* As a result of that discussion we know that, when demand is derived from the utility function (9), the demand curve faced by each firm in Dixit and Stiglitz's model of monopolistic competition is

$$x_s(p_s) = m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}} K$$

where K, which is in fact related to p_s by (6), is treated as independent of p_s. Equivalently, we can, as noted in Remark 4, derive the Chamberlin-Dixit-Stiglitz equilibrium, using the inverse demand curve

$$p_s(x_s) = m x_s^{-\alpha} H$$

and treat H, which is, in fact, related to x_s by (3), as independent of x_s. Except for the special form of the constant mH multiplying $x_s^{-\alpha}$, this is exactly the demand function derived by Dixit and Stiglitz.

Definition 1. *In the Dixit Stiglitz equilibrium each firm produces*

$$x(\alpha) = \arg\max_{x_s} [R_s(x_s) - C(x_s)]$$

where (4) implies that the revenue function of firm s is

$$R_s(x_s) = p_s(x_s) x_s = m x_s^{1-\alpha} H$$

and H is treated as being independent of x_s. Since the price received by each firm is

$$p(\alpha) \equiv p_s(x(\alpha)) = AC(x(\alpha)), \tag{15}$$

no firm makes a profit. When each of the $n(\alpha)$ firms produces $x(\alpha)$, (3) becomes

$$\begin{aligned} H &= \left(n(\alpha)\, x(\alpha)^{1-\alpha}\right)^{\left(\frac{1}{1-\alpha}\right)[\gamma-(1-\alpha)]} \\ &= n(\alpha)^{\left(\frac{\gamma}{1-\alpha}-1\right)}\, x(\alpha)^{[\gamma-(1-\alpha)]}. \end{aligned}$$

and

$$p(\alpha) = mx(\alpha)^{-\alpha} H$$

becomes

$$p(\alpha) = mx(\alpha)^{\gamma-1}\, n(\alpha)^{\left(\frac{\gamma}{1-\alpha}-1\right)}, \tag{16}$$

which is the condition that determines the number of firms, $n(\alpha)$. The Dixit-Stiglitz equilibrium is completely described by the output, $x(\alpha)$, of each firm, the price, $p(\alpha)$, received by each firm and the number of firms, $n(\alpha)$.

Remark 27. *In both Cost Cases 1 and 2, $x(\alpha)$ is determined by*

$$C'(x(\alpha)) = (1-\alpha)\, AC(x(\alpha)). \tag{17}$$

So that

$$p(\alpha) = \frac{C'(x(\alpha))}{(1-\alpha)} \tag{18}$$

The number of firms is

$$n(\alpha) = \left[\frac{mx(\alpha)^{\gamma-1}}{AC(x(\alpha))}\right]^{\frac{1}{\left(1-\frac{\gamma}{1-\alpha}\right)}}. \tag{19}$$

Proof: When, as is true in both cases one and two,

$$C''(x_s) \geq 0,$$

the maximand

$$[R_s(x_s) - C(x_s)]$$

is strictly concave and $x(\alpha)$ is obtained as the solution to

$$MR_s(x(\alpha)) = MC(x(\alpha)) \tag{20}$$

where the marginal revenue is

$$MR_s(x_s) = (1-\alpha)\, mx_s^{-\alpha} H = (1-\alpha)\, p_s(x_s). \tag{21}$$

Together (15), (20) and (21) imply (17). Equation (18) follows from (15) and (17). Substituting (15) in (16) and solving for $n(\alpha)$ we get (19).||

Remark 28. *Since*

$$\alpha = \frac{1}{\textit{demand elasticity}}$$

the relationship between $p(\alpha)$ *and the marginal cost in (18) is a standard result which asserts that*

$$\textit{price} = \frac{\textit{marginal cost}}{\left[1 - \frac{1}{\textit{demand elasticity}}\right]}.$$

Remark 29. *Note that in Cost Case 1,*

$$x(0) = \underset{x}{argmin} AC(x)$$

is the competitive output of each firm and the competitive outcome is the Chamberlinian outcome. When $\alpha > 0$, *(17) implies that*

$$MC(x(\alpha)) < AC(x(\alpha))$$

so that

$$x(\alpha) < x(0);$$

i.e., the monopolistically competitive output of each firm is less than the average cost minimizing competitive level $x(0)$.

The equilibrium is described graphically by Figure 1 which is familiar from textbook expositions of Chamberlin's model. Figure 1 is obviously drawn for Cost Case 1.

4.1. When Marginal Cost is Constant.

Remark 30. *In this case, we can explicitly compute the Dixit-Stiglitz monopolistically competitive equilibrium. In particular, (17) and (18) imply that, in equilibrium, each firm supplies*

$$x(\alpha) = \left(\frac{1}{\alpha} - 1\right)\frac{F}{c}$$

and charges

$$p(\alpha) = \frac{c}{(1-\alpha)}.$$

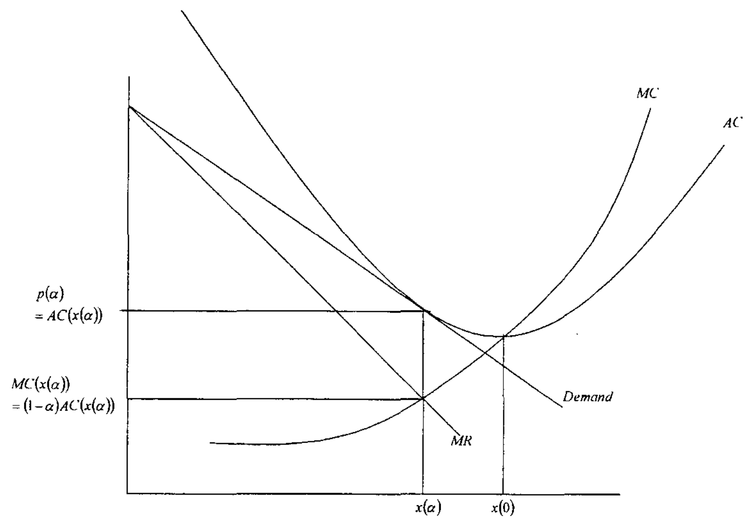

Figure 1:

Using (19), we observe that the number of firms is

$$
\begin{aligned}
n(\alpha) &= \left[\frac{mx(\alpha)^{\gamma-1}}{AC(x(\alpha))}\right]^{\frac{1}{\left(1-\frac{\gamma}{1-\alpha}\right)}} && (22)\\
&= \left[(1-\alpha)\frac{m\left[\left(\frac{(1-\alpha)}{\alpha}\right)\frac{F}{c}\right]^{\gamma-1}}{c}\right]^{\frac{1}{\left(1-\frac{\gamma}{1-\alpha}\right)}}\\
&= \left[m\left[\frac{(1-\alpha)}{c}\right]^{\gamma}\left[\frac{F}{\alpha}\right]^{\gamma-1}\right]^{\frac{1}{\left(1-\frac{\gamma}{1-\alpha}\right)}}.
\end{aligned}
$$

4.2. The Size of the Market, Price, Output Per Firm and the Number of Firms.

Remark 31. *Note that, since $x(\alpha)$ is simply determined by condition (17) which only involves the cost function and the demand parameter α, $x(\alpha)$ is independent of other aspects of demand and, in particular, of the size of the market parameter m. Since condition (15) asserts that the price is*

$$p(\alpha) = AC(x(\alpha)),$$

it also is independent of the size of the market.

Remark 32. *The expression for the equilibrium number of firms given in (19) implies that the number of firms is affected by the size of demand. It is also clear from expressions (19) and (22) that the equilibrium number of firms, $n(\alpha)$, grows with the market size parameter m when and only when condition (2) holds so that*

$$\frac{1}{\alpha} > \frac{1}{1-\gamma}$$

and the elasticity of demand in monopolistic competition is higher than in the monopoly case.

If this elasticity assumption fails and we have

$$\frac{1}{\alpha} < \frac{1}{1-\gamma},$$

then $n(\alpha)$ is a decreasing function of m. In the case where

$$\frac{1}{\alpha} = \frac{1}{1-\gamma},$$

we cannot solve (19) to get (22). In that case,

$$H = K = 1$$

and (16) becomes

$$p(\alpha) = mx(\alpha)^{\gamma-1} = AC(x(\alpha))$$

which is independent of $n(\alpha)$*. In this particular case, the number of firms in monopolistic competition is indeterminate.*

How the Firm's Price Affects Demand when the Market is Large. Let's ask, in particular, what the size of the market means for the Dixit-Stiglitz assumption that K and H are independent of the actions taken by any firm. Let's first suppose that firm s charges p_s while all other firms all charge $p(\alpha)$. Substituting in (6) we get

$$K = (n(\alpha)-1)^{\left[\frac{1}{(1-\alpha)}\right]\left[\frac{\alpha}{(1-\gamma)}-1\right]}\left(\frac{p_s^{-\frac{1-\alpha}{\alpha}}}{n(\alpha)-1} + p(\alpha)^{-\frac{1-\alpha}{\alpha}}\right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}.$$

Alternatively, let's suppose that firm s supplies x_s while all other firms supply $x(\alpha)$. Then (3) becomes

$$H = (n(\alpha)-1)^{\frac{\gamma}{1-\alpha}-1}\left(\frac{x_s^{1-\alpha}}{n(\alpha)-1} + x(\alpha)^{1-\alpha}\right)^{\left(\frac{1}{1-\alpha}\right)[\gamma-(1-\alpha)]}.$$

If condition (2) holds then $n(\alpha)$ is large when m is large and

$$K \simeq (n(\alpha)-1)^{\left[\frac{1}{(1-\alpha)}\right]\left[\frac{\alpha}{(1-\gamma)}-1\right]} p(\alpha)^{\left[\frac{1}{\alpha}-\frac{1}{(1-\gamma)}\right]} \tag{23}$$

while

$$H \simeq (n(\alpha)-1)^{\frac{\gamma}{1-\alpha}-1} x(\alpha)^{[\gamma-(1-\alpha)]}. \tag{24}$$

This means that in a large market K is, indeed, approximately independent of p_s and H is approximately independent of x_s as assumed by Dixit and Stiglitz. **But this argument applies only to the case in which the demand elasticity is higher in monopolistic competition than in monopoly.** If condition (2) fails and $n(\alpha)$ is a decreasing function of m because

$$\frac{1}{\alpha} < \frac{1}{1-\gamma}$$

then the number of firms will be small when the market size parameter is large. In that case, there is no reason for (23) or (24) to hold in a large market.

The case in which

$$\frac{1}{\alpha} = \frac{1}{1-\gamma},$$

is special. We noted above that, in that case,

$$K = H = 1$$

and the number of firms in monopolistic competition is indeterminate. Thus, in that case, the Dixit-Stiglitz assumptions that K is approximately independent of p_s and H is approximately independent of x_s hold in the strongest possible sense. In that case, K is completely independent of p_s, and H is completely independent of x_s. This is true whether the market is large or small.

4.3. Comparative Statics of $x(\alpha)$, $p(\alpha)$ and $n(\alpha)$.

Remark 33. *As $\frac{1}{\alpha}$, the elasticity of demand, rises the output, $x(\alpha)$, produced by each firm in the Dixit Stiglitz equilibrium rises and the price, $p(\alpha)$ falls. These results are illustrated in Figure 2. In Figure 2, $\alpha < \alpha'$, so that $\frac{1}{\alpha'} < \frac{1}{\alpha}$ and*

$$\frac{AC(x(\alpha))}{MC(x(\alpha))} = \frac{1}{1-\alpha} < \frac{1}{1-\alpha'} = \frac{AC(x(\alpha'))}{MC(x(\alpha'))}.$$

Proof: Implicitly differentiating (17) we get

$$x'(\alpha) = -\frac{AC(x(\alpha))}{[C''(x(\alpha)) - (1-\alpha)AC'(x(\alpha))]} \tag{25}$$

Since

$$AC'(x) = \frac{C'(x) - \frac{C(x)}{x}}{x},$$

(17) implies that

$$AC'(x(\alpha)) = \frac{-\alpha AC(x(\alpha))}{x(\alpha)} < 0, \tag{26}$$

and the expression for $x'(\alpha)$ in (25) is negative.

Note that (15) implies that

$$p'(\alpha) = AC'(x(\alpha))\,x'(\alpha). \tag{27}$$

Since we have shown that $x'(\alpha)$ and $AC'(x(\alpha))$ are negative, the expression for $p'(\alpha)$ in (27) is positive.

In Cost Case 2, the result follows immediately from the expressions

$$x(\alpha) = \left(\frac{1}{\alpha} - 1\right)\frac{F}{c}$$

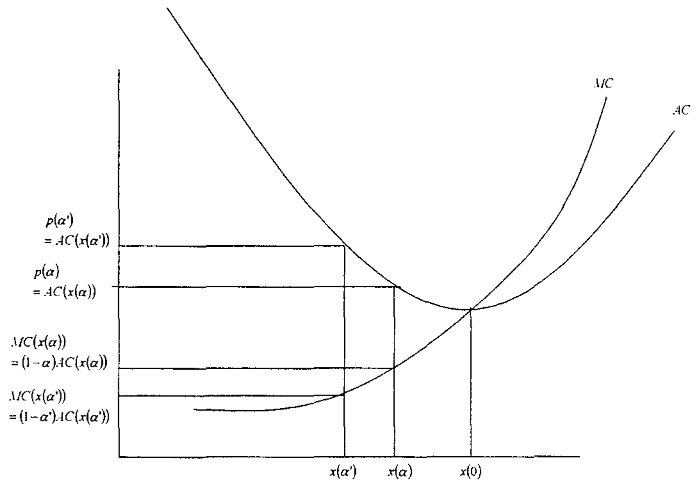
MC
AC
$p(\alpha')$
$= AC(x(\alpha'))$
$p(\alpha)$
$= AC(x(\alpha))$
$MC(x(\alpha))$
$= (1-\alpha)AC(x(\alpha))$
$MC(x(\alpha'))$
$= (1-\alpha')AC(x(\alpha'))$
$x(\alpha')$
$x(\alpha)$
$x(0)$

Figure 2:

and

$$p(\alpha) = \frac{c}{(1-\alpha)}$$

obtained in Remark 30. These expressions imply that

$$x'(\alpha) = -\frac{1}{\alpha^2}\frac{F}{c} < 0$$

and

$$p'(\alpha) = \frac{c}{(1-\alpha)^2}. \;\|$$

Remark 34. *Assume that* m *and, hence,*

$$n(\alpha) = \left[\frac{m}{AC(\alpha)\,x(\alpha)^{1-\gamma}}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}$$

are large. In that case, increases in $\frac{1}{\alpha}$, *the elasticity of demand, lead to a reduction in* $n(\alpha)$ *the number of firms that produce in the Dixit-Stiglitz equilibrium. This is a surprising result, since we expect the effect of an elasticity increase to be an increase the number firms.*

Proof: We rewrite the expression for $n(\alpha)$ in (19) as

$$n(\alpha) = \left[\frac{m}{\lambda(\alpha)}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}$$

where

$$\lambda(\alpha) = AC(\alpha)\,x(\alpha)^{1-\gamma}.$$

Differentiating, we get

$$\begin{aligned} n'(\alpha) &= \left[\frac{\lambda'(\alpha)}{\left(\frac{\gamma}{1-\alpha}\right)-1}\right] m^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}} \lambda(\alpha)^{\frac{1}{\left(\frac{\gamma}{1-\alpha}\right)-1}-1} \\ &\quad + \frac{\frac{\gamma}{(1-\alpha)^2}}{\left[1-\left(\frac{\gamma}{1-\alpha}\right)\right]^2} log\left(\frac{m}{\lambda(\alpha)}\right)\left[\frac{m}{\lambda(\alpha)}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}} \\ &= \Theta\left[\frac{1}{\left(\frac{\gamma}{1-\alpha}\right)-1}\right]\left[\frac{m}{\lambda(\alpha)}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}. \end{aligned} \tag{28}$$

where

$$\Theta \equiv \left[\frac{\lambda'(\alpha)}{\lambda(\alpha)} - \frac{\frac{\gamma}{(1-\alpha)^2}}{\left[1-\left(\frac{\gamma}{1-\alpha}\right)\right]} log\left(\frac{m}{\lambda(\alpha)}\right)\right].$$

When condition (2) holds, we have

$$\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}>0$$

and

$$n\left(\alpha\right)=\left[\frac{m}{\lambda\left(\alpha\right)}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}>1$$

implies that

$$\frac{m}{\lambda\left(\alpha\right)}>1$$

so that

$$log\left(\frac{m}{\lambda\left(\alpha\right)}\right)>0.$$

Thus,

$$-\frac{\frac{\gamma}{(1-\alpha)^2}}{\left[1-\left(\frac{\gamma}{1-\alpha}\right)\right]}log\left(\frac{m}{\lambda\left(\alpha\right)}\right),$$

the second term in the expression for Θ, is negative. The term

$$\lambda'\left(\alpha\right)=p'\left(\alpha\right)x\left(\alpha\right)^{1-\gamma}+\left(1-\gamma\right)p\left(\alpha\right)x\left(\alpha\right)^{-\gamma}$$

is positive, however. This means that the sign of the term Θ and of $n'\left(\alpha\right)$ is ambiguous. However, when m and, hence

$$n\left(\alpha\right)=\left[\frac{m}{\lambda\left(\alpha\right)}\right]^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}$$

are large enough Θ is negative so that the expression for $n'\left(\alpha\right)$ in (28) is positive.||

5. COURNOT EQUILIBRIUM

At this point we fix the number of firms at n.

Definition 2. *In a* **Cournot equilibrium,** *firm s faces the inverse demand function (1) and supplies the profit maximizing output level x_s^c which is defined formally as*

$$x_s^c=\underset{x_s}{argmax}\left[R_s\left(x_1^c,\ldots,x_s,\ldots,x_n^c\right)-C\left(x_s\right)\right]$$

where (1) implies that

$$\begin{aligned}R_s\left(x_1,\ldots,x_n\right)&=p_s\left(x_1,\ldots,x_n\right)x_s\\&=mx_s^{1-\alpha}\left(\sum_{t=1}^{n}x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}\right)-1}.\end{aligned}$$

We identify the **Cournot equilibrium** *with the vector,* $(x_1^c, \ldots, x_n^c)$ *specifying the profit maximizing amount produced by each of the* n *firms.*

Remark 35. *When condition (2) holds, the Cournot equilibrium* $(x_1^c, \ldots, x_n^c)$ *is characterized by the* n *equations*

$$p_s\left(x_1^c, \ldots, x_n^c\right) = \frac{C'\left(x_s^c\right)}{\left[(1-\alpha)\left(1 - \frac{(x_s^c)^{1-\alpha}}{\sum_{t=1}^n (x_t^c)^{1-\alpha}}\right) + \gamma\left(\frac{(x_s^c)^{1-\alpha}}{\sum_{t=1}^n (x_t^c)^{1-\alpha}}\right)\right]} \tag{29}$$

which, recalling expression (11) for the elasticity $\varepsilon_s^c\left(x_1^c, \ldots, x_n^c\right)$, *(29) can also be written as*

$$p_s\left(x_1^c, \ldots, x_n^c\right) = \frac{C'\left(x_s^c\right)}{\left[1 - \frac{1}{\varepsilon_s^c\left(x_1^c, \ldots, x_n^c\right)}\right]}. \tag{30}$$

Proof: The first order condition satisfied at x_s^c is

$$\frac{\partial R_s\left(x_1^c, \ldots, x_n^c\right)}{\partial x_s} = C'\left(x_s^c\right) \tag{31}$$

where

$$\begin{aligned} & \frac{\partial R_s\left(x_1, \ldots, x_n\right)}{\partial x_s} \\ = \; & m x_s^{-\alpha}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-2\right)}\left[(1-\alpha)\left(\sum_{t=1}^n x_t^{1-\alpha} - x_s^{1-\alpha}\right) + \gamma x_s^{1-\alpha}\right] \\ = \; & p_s\left(x_1, \ldots, x_n\right)\left[(1-\alpha)\left(1 - \frac{x_s^{1-\alpha}}{\sum_{t=1}^n x_t^{1-\alpha}}\right) + \gamma\left(\frac{x_s^{1-\alpha}}{\sum_{t=1}^n x_t^{1-\alpha}}\right)\right]. \end{aligned} \tag{32}$$

Substituting (32) in the first order conditions (31) yields (29) which, using the definition of $\varepsilon_s^c\left(x_1^c, \ldots, x_n^c\right)$ in (11) can also be written as (30).

It is straightforward to verify that the maximand $\left[R_s\left(x_1^c,\ldots,x_s,\ldots,x_n^c\right)-C\left(x_s\right)\right]$ is a strictly concave function of x_s when condition (2) holds by demonstrating that

$$\begin{aligned}
&\frac{\partial^2 R_s\left(x_1,\ldots,x_n\right)}{\partial x_s^2} \qquad (33)\\
= \;& -\alpha m x_s^{-(\alpha+1)}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-2\right)}\left[(1-\alpha)\left(\sum_{t=1}^n x_t^{1-\alpha}-x_s^{1-\alpha}\right)+\gamma x_s^{1-\alpha}\right]\\
&+\left(\gamma-2(1-\alpha)\right) m x_s^{-2\alpha}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-3\right)}\left[(1-\alpha)\left(\sum_{t=1}^n x_t^{1-\alpha}-x_s^{1-\alpha}\right)+\gamma x_s^{1-\alpha}\right]\\
&+(1-\alpha)\gamma m x_s^{-2\alpha}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-2\right)}
\end{aligned}$$

is negative. To do this, we simply rewrite (33) by combining the last two terms to get

$$\begin{aligned}
&\frac{\partial^2 R_s\left(x_1,\ldots,x_n\right)}{\partial x_s^2}\\
= \;& -\alpha m x_s^{-(\alpha+1)}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-2\right)}\left[(1-\alpha)\left(\sum_{t=1}^n x_t^{1-\alpha}-x_s^{1-\alpha}\right)+\gamma x_s^{1-\alpha}\right]\\
&-m\gamma(1-\alpha)x_s^{-2\alpha}\left(\sum_{t=1}^n x_t^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}-2\right)}\\
&\times\left(\left(2-\frac{\gamma}{1-\alpha}\right)\frac{\left[\frac{(1-\alpha)}{\gamma}\left(\sum_{t=1}^n x_t^{1-\alpha}-x_s^{1-\alpha}\right)+x_s^{1-\alpha}\right]}{\sum_{t=1}^n x_t^{1-\alpha}}-1\right).
\end{aligned}$$

The first term in this expression is clearly negative. The second term is also negative because $\frac{\gamma}{(1-\alpha)}<1$ implies that both

$$\left(2-\frac{\gamma}{1-\alpha}\right)$$

and

$$\frac{\left[\frac{(1-\alpha)}{\gamma}\left(\sum_{t=1}^n x_t^{1-\alpha}-x_s^{1-\alpha}\right)+x_s^{1-\alpha}\right]}{\sum_{t=1}^n x_t^{1-\alpha}}$$

exceed one.||

In the next section we will describe the symmetric Cournot equilibrium in which all firms supply the same amount. The following section demonstrates that the symmetric equilibrium is the only Cournot equilibrium.

5.1. Symmetric Cournot Equilibrium.

Proposition 1. *When condition (2) holds, there exists a symmetric Cournot equilibrium in which each firm s produces*

$$x_s^c = x^c$$

and charges

$$p^c = p\left(x^c\right) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^c\right)^{\gamma-1} \tag{34}$$

where x^c is the solution to

$$\begin{aligned} mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^c\right)^{\gamma-1} &= \frac{C'\left(x^c\right)}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}. \\ &= \frac{C'\left(x^c\right)}{1-\frac{1}{\varepsilon^c(n)}} \end{aligned} \tag{35}$$

The Cournot price p^c can also be obtained as the solution to

$$\begin{aligned} p^c &= \frac{C'\left(x\left(p^c\right)\right)}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}. \\ &= \frac{C'\left(x\left(p^c\right)\right)}{1-\frac{1}{\varepsilon^c(n)}} \end{aligned} \tag{36}$$

where because of Remark 6

$$x^c = x\left(p^c\right) \equiv m^{\frac{1}{(1-\gamma)}}\left(p^c\right)^{-\frac{1}{(1-\gamma)}} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}. \tag{37}$$

Proof: As noted in Remarks 2 and 22, when all n firms supply the same amount x^c, they all charge (34) and the expression for the elasticity $\varepsilon_s^c\left(x_1^c, \ldots, x_n^c\right)$ becomes

$$\varepsilon_s^c\left(x^c, \ldots, x^c\right) = \varepsilon^c\left(n\right) = \frac{1}{\left[\alpha\left(1-\frac{1}{n}\right)+(1-\gamma)\left(\frac{1}{n}\right)\right]}. \tag{38}$$

Substituting (34) and (38) in (30) we observe that x^c is the solution to (35). A solution clearly exists since the price

$$p\left(x^c\right) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^c\right)^{\gamma-1},$$

on the left side of equation (35) grows without bound as x^c approaches zero and approaches zero as x^c becomes large. The solution is unique since the $p(x^c)$ is a decreasing function of x^c, while marginal cost $C'(x^c)$ is either increasing in x^c (as in Cost Case 1) or independent of x^c (as in Cost Case 2).

Using the observation in Remark 6 we can invert $p(x)$ to get (37) and rewrite (35) as in (36).||

At this point we will not consider the question of whether the firms make positive profits in the Symmetric Cournot equilibrium. We will presently discuss this issue at length, however.

Remark 36. *Note that the relationship between the Cournot price, p^c, and the marginal cost in (36) is*

$$\text{price} = \frac{\text{marginal cost}}{\left[1 - \frac{1}{\text{demand elasticity}}\right]}.$$

As we noted in Remark 28 this standard result also holds in the Dixit-Stiglitz case.

Remark 37. *It is clear that x^c and p^c depend on m, n and α. In the subsequent discussion, it will often be useful to emphasize the dependence of x^c and p^c on m and n by using the notation $x^c(m,n)$ and $p^c(m,n)$ to denote the Cournot equilibrium output and price when there are n firms and the market size parameter is m. When there is no danger of confusion we will simply denote $x^c(m,n)$ and $p^c(m,n)$ by x^c and p^c.*

The Case when Marginal Cost is Constant.

Remark 38. *When marginal cost is constant and equal to c,*

$$\begin{aligned} x^c &= \left[\frac{cn^{1-\left(\frac{\gamma}{1-\alpha}\right)}}{m\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}\right]^{\frac{1}{\gamma-1}} \\ &= \left[\frac{cn^{1-\left(\frac{\gamma}{1-\alpha}\right)}}{m\left[1-\frac{1}{\varepsilon^c(n)}\right]}\right]^{\frac{1}{\gamma-1}}. \end{aligned} \tag{39}$$

and

$$\begin{aligned} p^c &= \frac{c}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]} = \frac{c}{1-\frac{1}{\varepsilon^c(n)}} \\ &> \frac{c}{(1-\alpha)} = p(\alpha). \end{aligned} \tag{40}$$

In this case, the profit per unit can be easily calculated as

$$\begin{aligned} p^c - c &= c\left[\frac{1}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}-1\right] \\ &= c\left[\frac{\frac{1}{\varepsilon^c(n)}}{1-\frac{1}{\varepsilon^c(n)}}\right] \\ &= c\left[\frac{1}{\varepsilon^c(n)-1}\right] \end{aligned}$$

Proof: In this case, (35) becomes

$$mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^c\right)^{\gamma-1} = \frac{c}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}$$

an equation in which the right side is independent of $x.^c$ The solution is given in (39) Also (36) becomes (40).||

5.2. Uniqueness of the Cournot Equilibrium. In the previous section we characterized the symmetric Cournot Equilibrium in which all firms produce the same amount. In this section we prove that this is the only Cournot equilibrium by establishing the following Proposition.

Proposition 2. *Assume that condition (2) holds. In a Cournot equilibrium all firms must produce the same amount.*

Proof: Using the expression (1) for the inverse demand function we can rewrite condition (29) as

$$m\left(x_s^c\right)^{-\alpha}\left(\sum_{t=1}^{n}\left(x_t^c\right)^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}\right)-1} = \frac{C'\left(x_s^c\right)}{\left[(1-\alpha)\left(1-\frac{\left(x_s^c\right)^{1-\alpha}}{\sum_{t=1}^{n}\left(x_t^c\right)^{1-\alpha}}\right)+\gamma\left(\frac{\left(x_s^c\right)^{1-\alpha}}{\sum_{t=1}^{n}\left(x_t^c\right)^{1-\alpha}}\right)\right]}$$

which implies that for all s, x_s^c is the solution to

$$\begin{aligned} & m(1-\alpha)\left(\sum_{t=1}^{n}\left(x_t^c\right)^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}\right)-1} \qquad (41) \\ = \;& C'\left(x_s^c\right)\left(x_s^c\right)^{\alpha}+\left(x_s^c\right)^{1-\alpha}\left[(1-\alpha)-\gamma\right]\left(\sum_{t=1}^{n}\left(x_t^c\right)^{1-\alpha}\right)^{\left(\frac{\gamma}{1-\alpha}\right)-2} \end{aligned}$$

If we can prove that, for

$$\sum_{t=1}^{n} (x_t^c)^{1-\alpha}$$

fixed, there is a unique x_s^c that solves (41), then all firms must supply that amount in the Cournot equilibrium. When

$$\sum_{t=1}^{n} (x_t^c)^{1-\alpha}$$

is fixed, the left hand side of equation (41) is independent of s as is

$$\mu \equiv [(1-\alpha)-\gamma] \left(\sum_{t=1}^{n} (x_t^c)^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha}\right)-2},$$

the positive coefficient of $(x_s^c)^{1-\alpha}$ on the right hand side of equation (41). Since, when

$$\sum_{t=1}^{n} (x_t^c)^{1-\alpha}$$

is fixed,

$$\begin{aligned} & \frac{\partial \left[C'(x_s^c)(x_s^c)^{\alpha} + (x_s^c)^{1-\alpha}\mu \right]}{\partial x_s^c} \\ = \; & C''(x_s^c)(x_s^c)^{\alpha} + \alpha C'(x_s^c)(x_s^c)^{\alpha-1} + (1-\alpha)(x_s^c)^{-\alpha}\mu > 0, \end{aligned}$$

the right hand side of equation (41) is an increasing function of x_s^c. As a result the x_s^c that solves equation (41) is, indeed, unique and the same for all firms s. Thus, all firms must supply the same amount. ||

5.3. Price and Supply Related to Market Size and the Number of Firms. The proposition established in this section shows that either an increase in the size of the market or a decrease in the number of firms raise both the Cournot equilibrium price and the per firm output. It should be emphasized that these results depend crucially on the assumption that the demand elasticity in monopolistic competition exceeds that of monopoly.

Proposition 3. *Assume that condition (2) holds. In that case, $x^c(m,n)$ and $p^c(m,n)$ are increasing functions of the market size parameter m. Also, $p^c(m,n)$ is a decreasing function of the number of firms, n. If, in addition, there are more than two firms, then $x^c(m,n)$ is a decreasing function of the number of firms, n.*

Proof: We will proceed by treating n as well as m as continuous variables and implicitly differentiating the equations (35) and (36) that determine $x^c(m,n)$ and $p^c(m,n)$ respectively. We will then be able to determine the signs of the resulting derivatives $\frac{\partial x^c}{\partial m}$, $\frac{\partial p^c}{\partial m}$, $\frac{\partial x^c}{\partial n}$ and $\frac{\partial p^c}{\partial n}$. The general arguments we give will apply to both Cost Cases 1 and 2. Since, in Cost Case 2, where marginal cost is constant, we have derived exact expressions for $x^c(m,n)$ and $p^c(m,n)$ we could also verify that the proposition holds in that Case by directly computing $\frac{\partial x^c}{\partial m}$, $\frac{\partial p^c}{\partial m}$, $\frac{\partial x^c}{\partial n}$ and $\frac{\partial p^c}{\partial n}$.

Let's define

$$\xi(m,n) \equiv \zeta(m,n)\,\delta(n)$$

where

$$\zeta(m,n) \equiv mn^{\left(\frac{\gamma}{1-\alpha}-1\right)}$$

and

$$\delta(n) \equiv 1 - \frac{1}{\varepsilon^c(n)} = \left[(1-\alpha)\left(1-\frac{1}{n}\right) + \gamma\left(\frac{1}{n}\right)\right].$$

If we also define the function $F(x,m,n)$ by

$$F(x,m,n) \equiv \xi(m,n)\,x^{\gamma-1} - C'(x),$$

then we can rewrite the condition (35) satisfied at $x^c(m,n)$ as

$$F(x^c,m,n) \equiv \xi(m,n)\,(x^c)^{\gamma-1} - C'(x^c) = 0. \tag{42}$$

Implicitly differentiating (42) we get

$$\begin{aligned}\frac{\partial x^c}{\partial n} &= -\frac{F_n(x^c,m,n)}{F_x(x^c,m,n)} \\ &= -\frac{\xi_n(m,n)\,(x^c)^{\gamma-1}}{\xi(m,n)\,(\gamma-1)\,(x^c)^{\gamma-2} - C''(x^c)}\end{aligned} \tag{43}$$

and

$$\begin{aligned}\frac{\partial x^c}{\partial m} &= -\frac{F_m(x^c,m,n)}{F_x(x^c,m,n)} \\ &= -\frac{\zeta_m(m,n)\,\delta(n)\,(x^c)^{\gamma-1}}{\xi(m,n)\,(\gamma-1)\,(x^c)^{\gamma-2} - C''(x^c)}.\end{aligned}$$

Since

$$F_x(x^c,m,n) = \xi(m,n)\,(\gamma-1)\,(x^c)^{\gamma-2} - C''(x^c) < 0, \tag{44}$$

and

$$F_m(x^c,m,n) = \zeta_m(m,n)\,(x^c)^{\gamma-1} = n^{\left(\frac{\gamma}{1-\alpha}-1\right)}\,(x^c)^{\gamma-1} > 0,$$

$$\frac{\partial x^c}{\partial m} > 0.$$

Combining (34) and (35) we get

$$p^c(m,n) = \frac{C'(x^c(m,n))}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]}. \tag{45}$$

Thus,

$$\frac{\partial p^c}{\partial m} = \frac{C''(x^c(m,n))\frac{\partial x^c}{\partial m}}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]} > 0.$$

Expression (43) for $\frac{\partial x^c}{\partial n}$, expression (42) for $F(x^c, m, n)$ and condition (44) combine to imply that $F_n(x^c, m, n)$ and, hence,

$$\frac{\partial x^c}{\partial n}$$

have the same sign as $\xi_n(m,n)$. For the purpose of computing $\xi_n(m,n)$ let's observe that $\xi(m,n)$ can be written as

$$\xi(m,n) = m(1-\alpha)\left(n^{\left(\frac{\gamma}{1-\alpha}-1\right)} + n^{\left(\frac{\gamma}{1-\alpha}-2\right)}\left[\frac{\gamma}{(1-\alpha)}-1\right]\right).$$

Thus,

$$\begin{aligned}
\xi_n(m,n) &= m(1-\alpha)\left[\left(\frac{\gamma}{1-\alpha}-1\right)n^{\left(\frac{\gamma}{1-\alpha}-2\right)} + \left(\frac{\gamma}{1-\alpha}-2\right)n^{\left(\frac{\gamma}{1-\alpha}-3\right)}\left(\frac{\gamma}{1-\alpha}-1\right)\right] \\
&= -m(1-\alpha)\left(1-\frac{\gamma}{1-\alpha}\right)n^{\left(\frac{\gamma}{1-\alpha}-2\right)}\left[1+\left(\frac{\gamma}{1-\alpha}-2\right)\frac{1}{n}\right].
\end{aligned}$$

Condition (2) implies that $\xi_n(m,n)$ has the same sign as

$$\varphi(n) \equiv -\left(1+\left(\frac{\gamma}{1-\alpha}-2\right)\left(\frac{1}{n}\right)\right).$$

Clearly,

$$\varphi(1) = 1-\frac{\gamma}{1-\alpha} > 0$$

but

$$\varphi(2) = -\frac{1}{2}\left(\frac{\gamma}{1-\alpha}\right) < 0$$

and

$$\varphi'(n) \equiv \left(\frac{\gamma}{1-\alpha}-2\right)\left(\frac{1}{n^2}\right) < 0.$$

So

$$\varphi(n) < \varphi(2) < 0$$

and

$$\frac{\partial x^c}{\partial n} < 0$$

if there are more than 2 firms.

Differentiating (45) we get

$$\begin{aligned}\frac{\partial p^c}{\partial n} &= \frac{C''(x^c(m,n))\frac{\partial x^c}{\partial n}}{\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]} \\ &\quad+\frac{C'(x^c)\left[\gamma-(1-\alpha)\right]}{n^2\left[(1-\alpha)\left(1-\frac{1}{n}\right)+\gamma\left(\frac{1}{n}\right)\right]^2}.\end{aligned}$$

The first term is negative when $n \geq 2$, and condition (2) implies the second term is negative. Thus, we know that p^c decreases with n, when $n \geq 2$. In fact, we can also show that this holds for $n \geq 1$, by the following argument.

Clearly the demand function $x(p)$ derived in Remark 6 depends on the parameters m and n as well as on price. Let's make that dependence explicit by writing

$$x(p,m,n) = m^{\frac{1}{(1-\gamma)}} p^{-\frac{1}{(1-\gamma)}} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}. \tag{46}$$

Observe that

$$\frac{\partial x(p,m,n)}{\partial p} = \left(\frac{1}{\gamma-1}\right) m^{\left(\frac{1}{1-\gamma}\right)} n^{\left(\frac{1}{\gamma-1}\right)\left[1-\left(\frac{\gamma}{1-\alpha}\right)\right]} p^{\frac{1}{\gamma-1}-1} < 0 \tag{47}$$

and

$$\frac{\partial x(p,m,n)}{\partial n} = \left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right] m^{\left(\frac{1}{1-\gamma}\right)} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]-1} p^{\frac{1}{\gamma-1}} < 0 \tag{48}$$

if condition (2) holds. If we use the notation introduced in (46) to rewrite (36), the condition that determines $p^c(m,n)$, it becomes

$$p^c(m,n) = \frac{C'(x(p^c(m,n),m,n))}{\left((1-\alpha)+\left(\frac{1}{n}\right)\left[\gamma-(1-\alpha)\right]\right)}$$

which we choose to rewrite as

$$\Psi(p^c(m,n),m,n) = 0 \tag{49}$$

where

$$\Psi(p,m,n) \equiv \frac{C'(x(p,m,n))}{p} - \left((1-\alpha) + \left(\frac{1}{n}\right)[\gamma - (1-\alpha)]\right). \tag{50}$$

Implicitly differentiating (49) we get

$$\frac{\partial p^c}{\partial n} = -\frac{\Psi_n(p^c,m,n)}{\Psi_p(p^c,m,n)} \tag{51}$$

where (50) implies that

$$\begin{aligned}\Psi_p(p,m,n) &= \frac{C''(x(p,m,n))\frac{\partial x(p,m,n)}{\partial p}}{p} \\ &\quad -\frac{C'(x(p,m,n))}{p^2}\end{aligned}$$

and

$$\begin{aligned}\Psi_n(p,n) &= \frac{C''(x(p,m,n))\frac{\partial x(p,m,n)}{\partial n}}{p} \\ &\quad +\frac{1}{n^2}[\gamma-(1-\alpha)].\end{aligned}$$

These expressions together with (47), (48), (51) and condition (2) imply that

$$\Psi_p(p,n) < 0$$

$$\Psi_n(p,n) < 0$$

and

$$\frac{\partial p^c}{\partial n} < 0. \parallel$$

5.4. Profitability, Market Size and the Number of Firms. In this Section we establish three results. The first, which we state as a Corollary to Proposition 3, demonstrates that, when the Cournot per firm output level, $x^c(m,n)$, is less than the average cost minimizing output level, $x(0)$, both per unit firm profits and firm profits increase with increases in the market size parameter, m, and decrease when n, the number of firms, increases. The next result, Proposition 5, demonstrates, that, for any n, all firms earn a profit in the n-firm Cournot equilibrium if the market size parameter, m, is large enough. The final result obtained in this section, Proposition 6, demonstrates that, for any m, all firms lose money in the n-firm Cournot equilibrium if n is sufficiently large.

Corollary 4. *Assume that condition (2) holds. The per unit profit,*

$$p^c(m,n) - AC(x^c(m,n)),$$

and profits per firm,

$$p^c(m,n)\, x^c(m,n) - C(x^c(m,n)),$$

are both increasing in m if

$$x^c(m,n) < x(0).$$

The per unit profit,

$$p^c(m,n) - AC(x^c(m,n)),$$

and profits per firm,

$$p^c(m,n)\, x^c(m,n) - C(x^c(m,n)),$$

are both decreasing in n, if there are more than two firms and

$$x^c(m,n) < x(0).$$

In Cost Case 2 where average cost is always falling, the hypothesis

$$x^c(m,n) < x(0)$$

is unnecessary.

Proof: When

$$x^c(m,n) < x(0),$$

the increase in $x^c(m,n)$ caused by an increase in m causes $AC(x^c(m,n))$ to fall and the Proposition 3 asserts that $p^c(m,n)$ is increasing in m. Since per unit profits,

$$p^c(m,n) - AC(x^c(m,n)),$$

and $x^c(m,n)$ both increase with m, profits per firm,

$$p^c(m,n)\, x^c(m,n) - C(x^c(m,n)) = [p^c(m,n) - AC(x^c(m,n))]\, x^c(m,n),$$

must also increase with m.

Similarly, when there are more than two firms and

$$x^c(m,n) < x(0),$$

the decrease in $x^c(m,n)$ caused by an increase in n causes $AC(x^c(m,n))$ to rise and the proposition asserts that $p^c(m,n)$ is decreasing in n. Since per unit profits,

$$p^c(m,n) - AC(x^c(m,n)),$$

and $x^c(m,n)$ both decrease with n, profits per firm,

$$p^c(m,n)\,x^c(m,n) - C(x^c(m,n)) = [p^c(m,n) - AC(x^c(m,n))]\,x^c(m,n),$$

must also decrease with n. ||

Let's fix the number of firms at n. We now show that we can always make m large enough so that we can have n firms producing profitably at the Cournot outcome for the given n. In particular, we will show that the Cournot equilibrium is profitable for all firms if the market size parameter, m, exceeds or equals

$$m^*(n,\alpha) = n^{\left(1-\frac{\gamma}{1-\alpha}\right)} \frac{C'(x(\alpha))}{x(\alpha)^{\gamma-1}\left[1-\frac{1}{\varepsilon^c(n)}\right]}.$$

Proposition 5. *Assume that condition (2) holds. If $m = m^*(n,\alpha)$ and there are n firms, then the Cournot equilibrium output is*

$$x^c(m^*(n,\alpha),n) = x(\alpha),$$

the Cournot price charged by all firms is

$$p^c(m^*(n,\alpha),n) = m^*(n,\alpha)\,n^{\left(\frac{\gamma}{1-\alpha}-1\right)}x(\alpha)^{\gamma-1} = \frac{C'(x(\alpha))}{\left[1-\frac{1}{\varepsilon^c(n)}\right]}. \tag{52}$$

and this price exceeds average cost so that all firms earn a profit. This equilibrium is described in Figure 3.

If $m > m^*(n,\alpha)$ and there are n firms, then the Cournot equilibrium output is

$$x^c(m,n) \geq x(\alpha),$$

the Cournot price charged by all firms is

$$p^c(m,n) \geq \frac{C'(x(\alpha))}{\left[1-\frac{1}{\varepsilon^c(n)}\right]}$$

and this price exceeds average cost so that all firms earn a profit.

If $m = m^*(n,\alpha)$ and there are $2 \leq n' < n$ firms, then the Cournot equilibrium output is

$$x^c(m,n') \geq x(\alpha),$$

the Cournot price charged by all firms is

$$p^c(m^*(n,\alpha),n') \geq \frac{C'(x(\alpha))}{\left[1-\frac{1}{\varepsilon^c(n)}\right]}$$

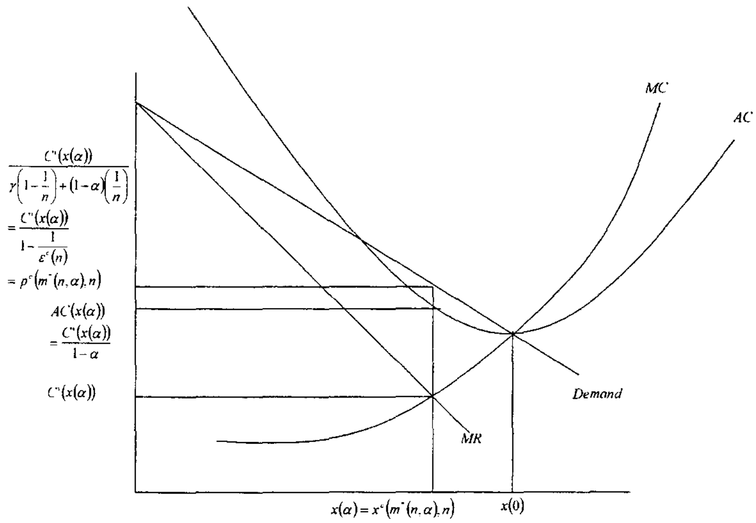

Figure 3:

and this price exceeds average cost so that all firms earn a profit.

If $m = m^*(n, \alpha)$ *and there are* $n' > n \geq 2$ *firms, then the Cournot equilibrium output is*

$$x^c(m^*(n, \alpha), n') \leq x(\alpha),$$

the Cournot price charged by all firms is

$$p^c(m^*(n, \alpha), n') \leq \frac{C'(x(\alpha))}{\left[1 - \frac{1}{\varepsilon^c(n)}\right]}.$$

Proof: It is immediate to verify that $m^*(n, \alpha)$ has been chosen so that $x(\alpha)$ satisfies condition (35) so that

$$x^c(m^*(n, \alpha), n) = x(\alpha).$$

The Cournot equilibrium price in (52) is obtained by substituting $x(\alpha)$ for x^c in (35) and combining (34) and (35). Since

$$C'(x(\alpha)) = (1 - \alpha) AC(x(\alpha))$$

(52) implies

$$\begin{aligned} p^c\left(m^*\left(n,\alpha\right),n\right) &= \frac{(1-\alpha)\,AC\left(x\left(\alpha\right)\right)}{\left[1-\frac{1}{\varepsilon^c(n)}\right]} \\ &> AC\left(x\left(\alpha\right)\right). \end{aligned} \tag{53}$$

The last inequality follows from the observation made in Remark 20 which implies that

$$\frac{(1-\alpha)}{\left[1-\frac{1}{\varepsilon^c(n)}\right]} > 1.$$

If $m > m^*(n,\alpha)$ and there are n firms, Proposition 3 implies that the Cournot equilibrium output, $x^c(m,n)$, exceeds $x(\alpha)$ and the Cournot price, $p^c(m,n)$, charged by all firms exceeds that given in (52). The fact that all firms make a profit is a consequence of Corollary 4 if

$$x^c(m,n) < x(0).$$

It is possible, however, that m is so large that $x^c(m,n)$ exceeds $x(0)$. In that case,

$$p^c(m,n) > C'\left(x^c(m,n)\right) \geq AC\left(x^c(m,n)\right)$$

which means that all firms make a profit.

If $m = m^*(n,\alpha)$ and there are $2 \leq n' < n$ firms, Proposition 3 again implies that the Cournot equilibrium output, $x^c\left(m^*(n,\alpha),n'\right)$, exceeds $x(\alpha)$ and the Cournot price, $p^c\left(m^*(n,\alpha),n'\right)$, charged by all firms exceeds that given in (52). The fact that all firms make a profit is a also a consequence of the Corollary 4 if

$$x^c\left(m^*(n,\alpha),n'\right) < x(0).$$

Again the possibility of

$$x^c\left(m^*(n,\alpha),n'\right) \geq x(0)$$

exists and again firms operate profitably in that case because then

$$p^c\left(m^*(n,\alpha),n'\right) > C'\left(m^*(n,\alpha),n'\right) \geq AC\left(m^*(n,\alpha),n'\right).$$

Finally, if $m = m^*(n,\alpha)$ and there are $n' > n$ firms, the results that

$$x^c\left(m^*(n,\alpha),n'\right) \leq x(\alpha)$$

and that the Cournot price, $p^c\left(m^*(n,\alpha),n'\right)$, charged by all firms is less than that given in (52) are corollaries of Proposition 3.||

Now let's fix the market size parameter m. The next proposition demonstrates that, when is n large enough, all firms lose money in the Cournot equilibrium.

Proposition 6. *For every m, there exists an $\bar{n}$ such that, when the market size is m and there are $n \geq \bar{n}$, firms*

$$p^c(m,n) < AC(x^c(m,n))$$

so that all firms lose money in the Cournot equilibrium.

Proof: For m and x, fixed let

$$n^*(m,x) = \left(\frac{m\left[1-\frac{1}{\varepsilon^c(n)}\right]}{x^{1-\gamma}C'(x)}\right)^{\frac{1}{1-\left(\frac{\gamma}{1-\alpha}\right)}}$$

Note that

$$x^c(m,n^*(m,x)) = x$$

and

$$p^c(m,n^*(m,x)) = m\left[n^*(m,x)\right]^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma-1} = \frac{C'(x)}{\left[1-\frac{1}{\varepsilon^c(n)}\right]}.$$

These equations hold even if $n^*(m,x)$ is not an integer which it need not always be. But if $n^*(m,x)$ is an integer, and if there are $n^*(m,x)$ firms, then each firm produces

$$x^c(m,n^*(m,x)) = x$$

and charges $p^c(m,n^*(m,x))$ in the Cournot equilibrium.

Also note that

$$\lim_{x\to 0} n^*(m,x) = \infty$$

and that $n^*(m,x)$ is an increasing function of m and a decreasing function of x in both Cost Cases 1 and 2. Finally, note also that

$$m^*(n^*(m,x(\alpha)),\alpha) = m.$$

Let's fix

$$\bar{x} < x(\alpha)$$

small enough so that

$$n^*(m,\bar{x}) \geq 2.$$

Then Proposition 3 applies to guarantee that

$$x^c(m,n) \leq \bar{x} < x(\alpha)$$

when

$$n \geq n^* (m, \bar{x}).$$

Recall that, in Cost Case 1, marginal cost, $C' (x)$, is always increasing while average cost, $AC (x)$, is decreasing when output is below

$$x (\alpha) < x (0)$$

and that, in Cost Case 2, marginal cost, $C' (x)$, is constant and that average cost, $AC (x)$, is always decreasing. Thus, not only is

$$x^c (m, n) \leq \bar{x} < x (\alpha)$$

when

$$n \geq n^* (m, \bar{x})$$

but we also have

$$\begin{aligned} p^c (m, n) &= \frac{C' (x^c (m, n))}{\left[1 - \frac{1}{\varepsilon^c(n)}\right]} \\ &\leq \frac{C' (x (\alpha))}{\left[1 - \frac{1}{\varepsilon^c(n)}\right]} \end{aligned} \tag{54}$$

and

$$AC (x^c (m, n)) > AC (\bar{x}) > AC (x (\alpha)). \tag{55}$$

Now let

$$\epsilon = \frac{AC (\bar{x}) - AC (x (\alpha))}{2}$$

and choose

$$\bar{n} > n^* (m, \bar{x})$$

sufficiently large so that

$$n \geq \bar{n}$$

implies

$$\begin{aligned} &\frac{C' (x (\alpha))}{\left[1 - \frac{1}{\varepsilon^c(n)}\right]} \\ &< \frac{C' (x (\alpha))}{(1 - \alpha)} + \epsilon. \end{aligned} \tag{56}$$

When

$$n \geq \bar{n},$$

inequality (56) combines with the definition of ϵ and condition (17) that determines $x(\alpha)$ to imply

$$\begin{aligned}
& \frac{C'(x(\alpha))}{\left[1-\frac{1}{\varepsilon^c(n)}\right]} \\
< \ & \frac{C'(x(\alpha))}{(1-\alpha)}+\epsilon \\
= \ & AC(x(\alpha))+\epsilon \\
= \ & AC(x(\alpha))+\frac{AC(\bar{x})-AC(x(\alpha))}{2} \\
= \ & \frac{AC(\bar{x})+AC(x(\alpha))}{2}.
\end{aligned} \tag{57}$$

Since $\bar{n}$ has been chosen to exceed $n^*(m,\bar{x})$, (54), (55) and (57) hold when

$$n \geq \bar{n}.$$

Since (55) implies

$$\frac{AC(\bar{x})+AC(x(\alpha))}{2} < AC(\bar{x}),$$

(54), (55) and (57) combine to imply that

$$p^c(m,n) < AC(x^c(m,n))$$

when

$$n \geq \bar{n}. \ \|$$

When $\bar{n}$ is as defined in the proof of Proposition 6, the Cournot equilibrium price, $p^c(m,n)$ and output, $x^c(m,n)$ for the case,

$$n \geq \bar{n}$$

are illustrated in Figure 4.

6. COURNOT-FREE ENTRY EQUILIBRIUM

Now we assume that there is an infinity of potential firms all of which possess the same cost function satisfying the conditions of either Cost Case 1 or Cost Case 2.

Definition 3. *In a* ***Cournot free-entry equilibrium*** *there are n^c firms that produce. The vector of amounts produced by these firms, $(x_1^c,\ldots,x_{n^c}^c)$, is a Cournot*

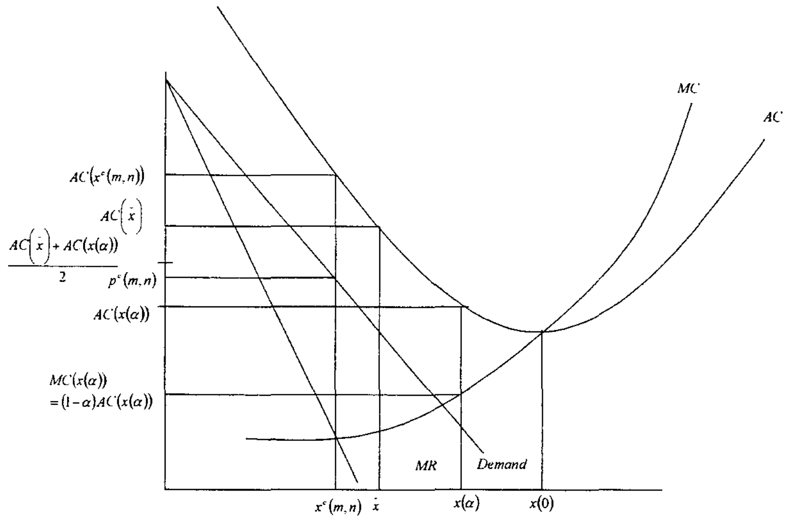
MC
AC
AC(x^c(m,n))
AC(x̄)
AC(x̄) + AC(x(α))
2
p^c(m,n)
AC(x(α))
MC(x(α))
= (1 − α)AC(x(α))
MR
Demand
x^c(m,n)
x̄
x(α)
x(0)

Figure 4:

equilibrium and each of these producing firms makes a nonnegative (but possibly zero) profit. Thus, for each s,

$$\begin{aligned} & \max_{x_s} \left[R_s \left(x_1^c, \ldots, x_s, \ldots, x_{n^c}^c \right) - C \left(x_s \right) \right] \\ = & \left[R_s \left(x_1^c, \ldots, x_{n^c}^c \right) - C \left(x_s^c \right) \right] \geq 0 \end{aligned}$$

where (1) implies that

$$\begin{aligned} R_s \left(x_1, \ldots, x_n \right) &= p_s \left(x_1, \ldots, x_n \right) x_s \\ &= m x_s^{1-\alpha} \left(\sum_{t=1}^{n} x_t^{1-\alpha} \right)^{\left(\frac{\gamma}{1-\alpha} \right) - 1}. \end{aligned}$$

In addition, none of the non-producing firms can make a profit. Thus, the final condition satisfied in a Cournot-Free Entry equilibrium is

$$\max_{x} \left[R_s \left(x_1^c, \ldots, x_{n^c}^c, x \right) - C \left(x \right) \right] \leq 0.$$

In analyzing the Cournot free-entry equilibria, we make use of fact established in Proposition 2 that, for each n, the unique Cournot equilibrium is the symmetric Cournot Equilibrium shown to exist and described in Proposition 1. Of course, in that symmetric equilibrium, all firms produce the same amount $x^c(m, n)$.

The proof that a Cournot free-entry equilibrium exists makes use of the following Lemma which applies to a firm that faces n rivals each producing $\hat{x}$. Note that (1) implies that the price received by such a firm will be

$$p_{n+1} \left(\hat{x}, \ldots, \hat{x}, x \right) = m x^{-\alpha} \left(x^{1-\alpha} + n \hat{x}^{1-\alpha} \right)^{\left(\frac{1}{1-\alpha} \right) [\gamma - (1-\alpha)]}$$

if it supplies x units.

Lemma 7. *The profits of a firm who faces n rivals, each producing $\hat{x}$, is a decreasing function of the amount, $\hat{x}$. Formally,*

$$\max_{x} \left[p_{n+1} \left(\hat{x}, \ldots, \hat{x}, x \right) x - C \left(x \right) \right]$$

is a decreasing function of x.

Proof: The lemma follow immediately from the fact that $p_{n+1}(\hat{x}, \ldots, \hat{x}, x)$ is a decreasing function of $\hat{x}$. ||

Proposition 8. *Assume that* $\tilde{n} > 1$. *If* $m \geq m^* (\tilde{n}, \alpha)$, *there exists a Cournot free-entry equilibrium with* $n^c > \tilde{n}$ *firms in which all firms produce*

$$x^c (m, n^c) < x (\alpha)$$

and charge a price

$$p^c (m, n^c) > AC (x (\alpha)).$$

.

Proof: From Proposition 5 we know that if $m \geq m^* (\tilde{n}, \alpha)$, then all firms make a profit in the n firm Cournot equilibrium if $2 < n \leq \tilde{n}$. In fact when $m > m^* (\tilde{n}, \alpha)$, Proposition 5 implies that all firms make a profit in the n firm Cournot equilibrium if $2 < n \leq n^* (m, x (\alpha))$, where $n^* (m, x)$ is the function defined in the Proof of Proposition 6. We have already noted that

$$m^* (n^* (m, x (\alpha)), \alpha) = m$$

which also implies that

$$\tilde{n} = n^* (m^* (\tilde{n}, \alpha), x (\alpha))$$

Since $n^* (m, x)$ is increasing in m, $m > m^* (\tilde{n}, \alpha)$ implies

$$n^* (m, x (\alpha)) > \tilde{n}.$$

Proposition 6 tells us that for every m, there is some $\bar{n}$, such that profits are negative if there are more than $\bar{n}$ firms. There must therefore exist some $n^c \in (n^* (m, x (\alpha)), \bar{n})$ for which

$$p^c (m, n^c) \geq AC (x^c (m, n^c))$$

and

$$p^c (m, n^c + 1) < AC (x^c (m, n^c + 1)).$$

We can demonstrate that the n^c firm Cournot equilibrium is, in fact, a Cournot free-entry equilibrium by demonstrating that

$$\max_x [R_{n^c+1} (x^c (m, n^c), \ldots, x^c (m, n^c), x) - C (x)] < 0.$$

But this follows immediately from Lemma 7 and Proposition 3 which tells us that because $n^c > n^* (m, x (\alpha)) \geq \tilde{n} \geq 2$,

$$x^c (m, n^c) > x^c (m, n^c + 1).$$

Finally note that, since $n^c > n^* (m, x (\alpha))$,

$$x^c (m, n^c) < x (\alpha)$$

and

$$p^c (m, n^c) \geq AC (x^c (m, n^c)) > AC (x (\alpha)). \;\|$$

Proposition 9. *When the market size parameter m is sufficiently large, the amount supplied by each of the n^c firms in the Cournot free-entry equilibrium is*

$$x^c(m,n^c) \simeq x(\alpha)$$

and the price charged by each of these firms is

$$p^c(m,n^c) \simeq AC(x(\alpha)).$$

Proof: In equilibrium,

$$\frac{C'(x^c(m,n^c))}{\left[(1-\alpha)\left(1-\frac{1}{n^c}\right)+\gamma\left(\frac{1}{n^c}\right)\right]} = p^c(m,n^c) \geq AC(x^c(m,n^c)).$$

Thus,

$$\frac{1}{\left[(1-\alpha)\left(1-\frac{1}{n^c}\right)+\gamma\left(\frac{1}{n^c}\right)\right]} \geq \frac{AC(x^c(m,n^c))}{C'(x^c(m,n^c))}. \tag{58}$$

Clearly

$$\lim_{m\to\infty} n^*(m,x(\alpha)) = \infty.$$

So if we choose m large enough, then $n^*(m,x(\alpha))$ and $n^c > n^*(m,x(\alpha))$ will also be large. By choosing m large enough we can, therefore, be sure that

$$\frac{AC(x(\alpha))}{C'(x^c(\alpha))} + \epsilon = \frac{1}{(1-\alpha)} + \epsilon > \frac{1}{\left[(1-\alpha)\left(1-\frac{1}{n^c}\right)+\gamma\left(\frac{1}{n^c}\right)\right]}$$

Combining this inequality with (58) yields

$$\epsilon > \frac{AC(x^c(m,n^c))}{C'(x^c(m,n^c))} - \frac{AC(x(\alpha))}{C'(x^c(\alpha))}.$$

Thus, for m sufficiently large

$$x^c(m,n^c) \simeq x(\alpha)$$

and

$$p^c(m,n^c) = \frac{C'(x^c(m,n^c))}{\left[(1-\alpha)\left(1-\frac{1}{n^c}\right)+\gamma\left(\frac{1}{n^c}\right)\right]} \simeq AC(x(\alpha)). \;\|$$

7. BERTRAND EQUILIBRIUM

We fix the number of firms at n.

Definition 4. *In a* **Bertrand equilibrium,** *firm s faces the demand function (5) and charges the profit maximizing price p_s^b which is defined formally as*

$$p_s^b = \underset{p_s}{argmax}\left[R_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right].$$

where (5) implies that

$$\begin{aligned} R_s\left(p_1,\ldots,p_n\right) &= x_s\left(p_1,\ldots,p_n\right)p_s \\ &= m^{\frac{1}{(1-\gamma)}} p_s^{1-\frac{1}{\alpha}}\left(\sum_{t=1}^{n} p_t^{-\frac{1-\alpha}{\alpha}}\right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}. \end{aligned} \tag{59}$$

We identify the **Bertrand equilibrium** *with the vector, $\left(p_1^b,\ldots,p_n^b\right)$ specifying the price charged by each of the n firms.*

Remark 39. *When condition (2) holds, the Bertrand equilibrium $\left(p_1^b,\ldots,p_n^b\right)$ is characterized by the n equations*

$$p_s = \frac{C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)}{\left(1-\left[\frac{1}{\alpha}\left(1-\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)}\right)+\frac{1}{(1-\gamma)}\frac{p_s^{1-\frac{1}{\alpha}}}{\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)}\right]^{-1}\right)}. \tag{60}$$

which, recalling the expression (10) for the elasticity $\varepsilon_s^b\left(p_1^b,\ldots,p_n^b\right)$, can also be written as

$$p_s = \frac{C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)}{\left[1-\frac{1}{\varepsilon_s^b(p_1,\ldots,p_n)}\right]} \tag{61}$$

These equations are the Bertrand analogs of (29) and (30).

Proof: The first order condition satisfied at p_s^b is

$$\begin{aligned} &\frac{\partial}{\partial p_s}\left[R_s\left(p_1^b,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)\right] \\ =\; &\frac{\partial R_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s} - C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s} = 0 \end{aligned} \tag{62}$$

where

$$\frac{\partial R_s\left(p_1,\ldots,p_n\right)}{\partial p_s} = p_s\frac{\partial x_s\left(p_1,\ldots,p_n\right)}{\partial p_s} + x_s\left(p_1,\ldots,p_n\right)$$

Substituting this expression in (62) and using the fact, observed in Remark 5, that

$$\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s} < 0 \tag{63}$$

we get

$$\begin{aligned}
&\frac{\partial}{\partial p_s}\left[R_s\left(p_1^b,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)\right] \qquad (64)\\
= &\left(p_s^b\left[1+\frac{x_s\left(p_1^b,\ldots,p_n^b\right)}{p_s^b\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s}}\right] - C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)\right)\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s}\\
= &\left(p_s^b\left[1-\frac{1}{e_s^b\left(p_1^b,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)\right)\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s}\\
= &\ 0.
\end{aligned}$$

Because of (63), (10) and (64) imply (60) and (61).

Note that, Remarks 13 and 15, (63) and the condition $C''(x) \geq 0$ imply

$$\begin{aligned}
&\frac{\partial}{\partial p_s}\left(p_s\left[1-\frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right)\\
= &\left[1-\frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] + p_s\left[\frac{\frac{\partial e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}{\partial p_s}}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)^2}\right]\\
&-C''\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\frac{\partial x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}{\partial p_s}\\
> &\ 0.
\end{aligned}$$

In addition, it is easy to verify that

$$p_s^b\left[1-\frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)$$

is negative when p_s is near zero and positive when p_s is large. Thus, there exists a unique solution, p_s, to

$$\frac{\partial}{\partial p_s}\left[R_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right] = 0. \tag{65}$$

In addition, at the price, p_s, for which (65) is satisfied,

$$\begin{aligned}
&\frac{\partial^2}{\partial p_s^2}\left[R_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right] \\
=& \left[\frac{\partial}{\partial p_s}\left(p_s\left[1 - \frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right)\right] \\
&\times \frac{\partial x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}{\partial p_s} \\
&+ \left(p_s\left[1 - \frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right)\frac{\partial^2 x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}{\partial p_s^2} \\
=& \left[\frac{\partial}{\partial p_s}\left(p_s\left[1 - \frac{1}{e_s^b\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)}\right] - C'\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right)\right]\frac{\partial x_s\left(p_1^b,\ldots,p_n^b\right)}{\partial p_s} \\
<& \ 0.
\end{aligned}$$

Thus, the price p_s at which (65) is satisfied must be

$$p_s^b = \underset{p_s}{argmax}\left[R_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right) - C\left(x_s\left(p_1^b,\ldots,p_s,\ldots,p_n^b\right)\right)\right]. \ \|$$

7.1. Symmetric Bertrand Equilibrium.

Proposition 10. *When condition (2) holds, there exists a symmetric Bertrand equilibrium in which each firm s charges*

$$p_s^b = p^b$$

and supplies

$$x^b \equiv x\left(p^b\right) = m^{\frac{1}{(1-\gamma)}}\left(p^b\right)^{-\frac{1}{(1-\gamma)}} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]} \tag{66}$$

where p^b is the solution to

$$\begin{aligned}
p^b &= \frac{C'\left(x\left(p^b\right)\right)}{\left(1 - \left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right) + \frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right)} \\
&= \frac{C'\left(m^{\frac{1}{(1-\gamma)}}\left(p^b\right)^{-\frac{1}{(1-\gamma)}} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}\right)}{\left(1-\frac{1}{\varepsilon^b(n)}\right)}.
\end{aligned} \tag{67}$$

The Bertrand output x^b can also be obtained as the solution to

$$\begin{aligned} p\left(x^b\right) &= mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^b\right)^{\gamma-1} = \frac{C'\left(x^b\right)}{\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right)} \qquad (68) \\ &= \frac{C'\left(x^b\right)}{\left(1-\frac{1}{\varepsilon^b(n)}\right)}. \end{aligned}$$

Proof: When all firms charge p^b, Remark 6 implies that the demand of each firm is as given in (66) and Remark 16 implies that.

$$\varepsilon_s^b\left(p,\ldots,p\right) = \varepsilon^b\left(n\right).$$

Equation (67) is obtained by substituting $x\left(p^b\right)$ and

$$\varepsilon^b\left(n\right) = \frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{\left(1-\gamma\right)}\frac{1}{n}$$

in (60) and (61).

Note that a solution to (67) clearly exists since the demand in (66) grows without bound as p^b approaches zero.

Finally, note that when all firms supply x^b, $p^b = p\left(x^b\right)$ where $p\left(x\right)$ is the function defined Remark 2. Thus, we can substitute $p\left(x^b\right)$ for p^b and x^b for $x\left(p^b\right)$ in equation (67) to get (68).||

Remark 40. *The relationship between the Bertrand equilibrium price, p^b, and the marginal cost in (67) is again the standard result*

$$\textit{price} = \frac{\textit{marginal cost}}{\left[1-\frac{1}{\textit{demand elasticity}}\right]}.$$

that as noted in Remarks 28 and 36 also holds in the Cournot and Dixit-Stiglitz equilibria.

Remark 41. *As is true with the symmetric Cournot equilibrium, it will often be useful to emphasize the dependence of x^b and p^b on m and n by using the notation $x^b\left(m,n\right)$ and $p^b\left(m,n\right)$ to denote the Cournot equilibrium output and price. When there is no danger of confusion we will simply denote $x^b\left(m,n\right)$ and $p^b\left(m,n\right)$ by x^b and p^b.*

The Case When Marginal Cost is Constant. In this case, (68) becomes

$$mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^b\right)^{\gamma-1}=\frac{c}{\left(1-\frac{1}{\varepsilon^b(n)}\right)}$$

so that

$$x^b=\left[\frac{cn^{1-\left(\frac{\gamma}{1-\alpha}\right)}}{m\left(1-\frac{1}{\varepsilon^b(n)}\right)}\right]^{\frac{1}{\gamma-1}}.$$

Also (67) becomes

$$p^b=\frac{c}{\left(1-\frac{1}{\varepsilon^b(n)}\right)}.$$

Finally note that

$$p^b-c=c\left[\frac{1}{\left(1-\frac{1}{\varepsilon^b(n)}\right)}-1\right].$$

7.2. Uniqueness of the Bertrand Equilibrium. In the previous section we characterized the symmetric Bertrand Equilibrium in which all firms charge the same price. In this section we prove that this is the only Bertrand equilibrium by establishing the following Proposition.

Proposition 11. *Assume that condition (2) holds. In a Bertrand equilibrium, all firms must charge the same price.*

Proof: We can demonstrate that, for $\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)$ fixed, there is one value of p_s that solves (61) by demonstrating that, for $\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)$ fixed,

$$\frac{C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)}{\left[1-\frac{1}{\varepsilon_s^b\left(p_1^b,\ldots,p_n^b\right)}\right]}$$

is decreasing in p_s. First, we note that the expression (5) for the demand function tells us that, for $\left(\sum_{t=1}^{n} p_t^{1-\frac{1}{\alpha}}\right)$ fixed, $x_s\left(p_1^b,\ldots,p_n^b\right)$ is decreasing in p_s. Since $C''(x)\geq 0$,

$$C'\left(x_s\left(p_1^b,\ldots,p_n^b\right)\right)$$

is nonincreasing in p_s if $\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)$ is fixed. Also Remark 14 implies that, for $\left(\sum_{t=1}^n p_t^{1-\frac{1}{\alpha}}\right)$ fixed, $\frac{1}{\varepsilon_s^b\left(p_1^b,\ldots,p_n^b\right)}$, and hence

$$\frac{1}{\left[1-\frac{1}{\varepsilon_s^b\left(p_1^b,\ldots,p_n^b\right)}\right]}$$

is decreasing in p_s. ||

7.3. Price and Supply Related to Market Size and the Number of Firms.

Proposition 12. *Assume that condition (2) holds. In that case, $x^b(m,n)$ and $p^b(m,n)$ are increasing functions of the market size parameter m. Also, $p^b(m,n)$ is a decreasing function of the number of firms, n. In addition, there exist an $\hat{n}$, independent of m, such that, if, $n>\hat{n}$, then $x^b(m,n)$ is a decreasing function of the number of firms, n.*

Proof: Let's define

$$\Phi(p,m,n)\equiv\frac{C'(x(p,m,n))}{p}-\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right) \tag{69}$$

where $x(p,m,n)$ is defined in (46). Equation (67) which determines $p^b(m,n)$ can be rewritten as

$$\Phi\left(p^b(m,n),m,n\right)=0. \tag{70}$$

Implicitly differentiating (70) we get

$$\frac{\partial p^b(m,n)}{\partial n}=-\frac{\Phi_n\left(p^b(m,n),m,n\right)}{\Phi_p\left(p^b(m,n),m,n\right)} \tag{71}$$

where (69) implies that

$$\Phi_p(p,m,n) = \frac{C''(x(p,m,n))\frac{\partial x}{\partial p}}{p} - \frac{C'(x(p,m,n))}{p^2} \tag{72}$$

and

$$\Phi_n(p,m,n) = \frac{C''(x(p,m,n))\frac{\partial x}{\partial n}}{p} + \frac{\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}{n^2\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^2}.$$

These expressions together with (2), (47), (48) and (71) imply that

$$\Phi_p(p,m,n) < 0, \tag{73}$$

$$\Phi_n(p,m,n) < 0$$

and

$$\frac{\partial p^b(m,n)}{\partial n} < 0.$$

Implicitly differentiating (70) once again we get

$$\frac{\partial p^b(m,n)}{\partial m} = -\frac{\Phi_m\left(p^b,m,n\right)}{\Phi_p\left(p^b,m,n\right)} \tag{74}$$

where (69) implies that

$$\Phi_m(p,m,n) = \frac{C''(x(p,m,n))\frac{\partial x}{\partial m}}{p} \tag{75}$$

and where (2) and (46) imply that

$$\frac{\partial x}{\partial m} = \left(\frac{1}{1-\gamma}\right) m^{\left(\frac{1}{1-\gamma}-1\right)} n^{\left(\frac{1}{\gamma-1}\right)\left[1-\left(\frac{\gamma}{1-\alpha}\right)\right]} p^{\frac{1}{\gamma-1}} > 0. \tag{76}$$

Equations (75) and (76) imply that

$$\Phi_m(p,m,n) > 0. \tag{77}$$

Combining (73), (74) and (77) we get

$$\frac{\partial p^b(m,n)}{\partial m} > 0.$$

For the purpose of computing $\frac{\partial x^b(m,n)}{\partial m}$ and $\frac{\partial x^b(m,n)}{\partial n}$, let's define

$$Q(x,m,n) \equiv mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)n}\frac{1}{n}\right]^{-1}\right)x^{\gamma-1} - C'(x)$$

and rewrite (68) as

$$Q\left(x^b(m,n),m,n\right) = 0. \tag{78}$$

Implicitly differentiating (78) we get

$$\frac{\partial x^b(m,n)}{\partial m} = -\frac{Q_m\left(x^b(m,n),m,n\right)}{Q_x\left(x^b(m,n),m,n\right)} \tag{79}$$

and

$$\frac{\partial x^b(m,n)}{\partial n} = -\frac{Q_n(x^b,m,n)}{Q_x(x^b,m,n)} \tag{80}$$

Since

$$\begin{aligned} & Q_x(x,m,n) \\ = \;& (\gamma-1)\,mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right)x^{\gamma-2}-C''(x) \\ < \;& 0 \end{aligned} \tag{81}$$

and

$$\begin{aligned} & Q_m(x,m,n) \\ = \;& n^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right)x^{\gamma-1}>0 \end{aligned}$$

the expression for $\frac{\partial x^b}{\partial m}$ in (78) is positive. When we compute $Q_n(x,m,n)$, the result is

$$Q_n(x,m,n) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-2}x^{\gamma-1}\Delta$$

where

$$\begin{aligned} \Delta \equiv\;& \left[\left(\frac{\gamma}{1-\alpha}\right)-1\right]\left(1-\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-1}\right) \\ & +\left(\frac{1}{n}\left[\frac{1}{\alpha}-\frac{1}{(1-\gamma)}\right]\left[\frac{1}{\alpha}\left(1-\frac{1}{n}\right)+\frac{1}{(1-\gamma)}\frac{1}{n}\right]^{-2}\right). \end{aligned}$$

has a negative first term and a positive second term. However, when n is sufficiently large the positive second term in this expression is near zero and

$$\begin{aligned} \Delta &\simeq \left[\left(\frac{\gamma}{1-\alpha}\right)-1\right](1-\alpha) \\ &= \alpha-(1-\gamma)<0. \end{aligned}$$

Thus, when n is sufficiently large, $Q_n(x^b,m,n)$ is negative and (80) and (81) imply that

$$\frac{\partial x^b(m,n)}{\partial n}<0. \;\|$$

.

7.4. Profitability, Market Size and the Number of Firms. Using the results established in Proposition 12 relating the Bertrand price and per firm output to the size of the market and the number of firms we can now see how increases in the market's size and the number of firms affects per unit profits and total profits.

Corollary 13. *Assume that condition (2) holds. The per unit profit,*

$$p^b(m,n) - AC\left(x^b(m,n)\right),$$

and profits per firm,

$$p^b(m,n)\,x^b(m,n) - C\left(x^b(m,n)\right),$$

are both increasing in m, if

$$x^b(m,n) < x(0).$$

Proof: Proposition 12 asserts that an increase in m causes both $p^b(m,n)$ and $x^b(m,n)$ to increase. When

$$x^b(m,n) < x(0).$$

the increase in $x^b(m,n)$ causes $AC\left(x^b(m,n)\right)$ to fall and Thus, per unit profits,

$$p^b(m,n) - AC\left(x^b(m,n)\right),$$

and $x^b(m,n)$ both increase with m, and profits per firm,

$$p^b(m,n)\,x^b(m,n) - C\left(x^b(m,n)\right) = \left[p^b(m,n) - AC\left(x^b(m,n)\right)\right]x^b(m,n),$$

must also increase with m. ||

Profits of a New Entrant and the Relationship Between the Number of Firms and Bertrand Profits. We can now prove that the Bertrand profits are decreasing in n. This result is obtained by showing that an entrant who faces n firms each charging $p^b(m,n)$ will earn profits that are lower than the n firm Bertrand profits

$$p^b(m,n)\,x^b(m,n) - C\left(x^b(m,n)\right)$$

but are higher than the profits

$$p^b(m,n+1)\,x^b(m,n+1) - C\left(x^b(m,n+1)\right)$$

earned in the $n+1$ firm Bertrand equilibrium.

We begin by observing that (5) implies that, if we have n firms producing and selling at price $\tilde{p}$, and a new entrant charges p its sales will be

$$x_{n+1}(\tilde{p},\ldots,\tilde{p},p) = m^{\frac{1}{(1-\gamma)}} p^{-\frac{1}{\alpha}} \left(p^{-\frac{1-\alpha}{\alpha}} + n\tilde{p}^{-\frac{1-\alpha}{\alpha}}\right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}.$$

Considered as a function of p, $x_{n+1}(\tilde{p},\ldots,\tilde{p},p)$ is the demand function faced by the new entrant. Now let's define the new entrant's inverse demand function $p(x,\tilde{p},n)$ implicitly by

$$x_{n+1}(\tilde{p},\ldots,\tilde{p},p(x,\tilde{p},n)) = x.$$

Remark 42. *We are going to make use of the fact that the problem of choosing p to maximize*

$$[R_{n+1}(\tilde{p},\ldots,\tilde{p},p) - C(x_{n+1}(\tilde{p},\ldots,\tilde{p},p))]$$

can also be solved by choosing x to maximize

$$[p(x,\tilde{p},n)\,x - C(x)].$$

Formally,

$$\begin{aligned} & \max_p [R_{n+1}(\tilde{p},\ldots,\tilde{p},p) - C(x_{n+1}(\tilde{p},\ldots,\tilde{p},p))] \\ = & \max_x [p(x,\tilde{p},n)\,x - C(x)]. \end{aligned}$$

Lemma 14. *Assume that condition (2) holds. Then, $p(x,\tilde{p},n)$ is an increasing function of $\tilde{p}$ and a decreasing function of n. As a consequence, the function*

$$\begin{aligned} & \max_x [p(x,\tilde{p},n)\,x - C(x)] \\ = & \max_p [R_{n+1}(\tilde{p},\ldots,\tilde{p},p) - C(x_{n+1}(\tilde{p},\ldots,\tilde{p},p))] \end{aligned}$$

is also an increasing function $\tilde{p}$ and a decreasing function of n.

Proof: Note that $p(x,\tilde{p},n)$ is the solution to

$$\Pi(p(x,\tilde{p},n),x,\tilde{p},n) = 0,$$

where

$$\Pi(p,x,\tilde{p},n) \equiv x - m^{\frac{1}{(1-\gamma)}} p^{-\frac{1}{\alpha}} \left(p^{-\frac{1-\alpha}{\alpha}} + n\tilde{p}^{-\frac{1-\alpha}{\alpha}}\right)^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}.$$

Implicitly differentiating we get

$$p_{\tilde{p}}(x,\tilde{p},n) = -\frac{\Pi_{\tilde{p}}(p(x,\tilde{p},n),x,\tilde{p},n)}{\Pi_p(p(x,\tilde{p},n),x,\tilde{p},n)}.$$

and

$$p_n(x,\tilde{p},n) = -\frac{\Pi_n(p(x,\tilde{p},n),x,\tilde{p},n)}{\Pi_p(p(x,\tilde{p},n),x,\tilde{p},n)}.$$

Since (2) implies

$$\Pi_{\tilde{p}}(p, x, \tilde{p}, n) < 0$$

and since

$$\Pi_n(p, x, \tilde{p}, n) < 0,$$

and

$$\Pi_p(p, x, \tilde{p}, n) > 0,$$

we have

$$p_n(x, \tilde{p}, n) < 0$$

and

$$p_{\tilde{p}}(x, \tilde{p}, n) > 0.$$

Since $p(x, \tilde{p}, n)$ an increasing function $\tilde{p}$ and a decreasing function of n for each x, the same is true of

$$p(x, \tilde{p}, n)\, x - C(x).$$

This implies that

$$\max_x \left[p(x, \tilde{p}, n)\, x - C(x)\right]$$

is an increasing function $\tilde{p}$ and a decreasing function of n. ||

By applying Lemma 14 twice and using the fact that

$$p^b(m, n+1) < p^b(m, n)$$

we obtain the following proposition.

Proposition 15. *Assume that condition (2) holds. The profits earned by entrant who faces n firms each charging $p^b(m,n)$ exceed the $n+1$ firm Bertrand profits*

$$p^b(m, n+1)\, x^b(m, n+1) - C\left(x^b(m, n+1)\right)$$

but are lower than the n firm Bertrand profits

$$p^b(m, n)\, x^b(m, n) - C\left(x^b(m, n)\right).$$

Formally,

$$\begin{aligned}
& p^b(m,n)\, x^b(m,n) - C\left(x^b(m,n)\right) \\
= \; & \max_p \left[R_n\left(p^b(m,n), \ldots, p^b(m,n), p\right) - C\left(x_n\left(p^b(m,n), \ldots, p^b(m,n), p\right)\right)\right] \\
> \; & \max_p \left[R_{n+1}\left(p^b(m,n), \ldots, p^b(m,n), p\right) - C\left(x_{n+1}\left(p^b(m,n), \ldots, p^b(m,n), p\right)\right)\right] \\
> \; & \max_p [R_{n+1}\left(p^b(m,n+1), \ldots, p^b(m,n+1), p\right) \\
& - C\left(x_{n+1}\left(p^b(m,n+1), \ldots, p^b(m,n+1), p\right)\right)] \\
= \; & p^b(m,n+1)\, x^b(m,n+1) - C\left(x^b(m,n+1)\right).
\end{aligned}$$

Proof: Since
$$\max_x \left[p\left(x,\tilde{p},n\right)x - C\left(x\right)\right]$$
is a decreasing function of n,
$$\begin{aligned} & p^b\left(m,n\right)x^b\left(m,n\right) - C\left(x^b\left(m,n\right)\right) \\ = & \max_x \left[p\left(x,p^b\left(m,n\right),n-1\right)x - C\left(x\right)\right] \\ > & \max_x \left[p\left(x,p^b\left(m,n\right),n\right)x - C\left(x\right)\right] \\ = & \max_p \left[R_{n+1}\left(p^b\left(m,n\right),\ldots,p^b\left(m,n\right),p\right) - C\left(x_{n+1}\left(p^b\left(m,n\right),\ldots,p^b\left(m,n\right),p\right)\right)\right]. \end{aligned}$$
Also since
$$\max_x \left[p\left(x,\tilde{p},n\right)x - C\left(x\right)\right]$$
is an increasing function $\tilde{p}$ and since Proposition 12 implies that
$$p^b\left(m,n+1\right) < p^b\left(m,n\right),$$
$$\begin{aligned} & \max_p \left[R_{n+1}\left(p^b\left(m,n\right),\ldots,p^b\left(m,n\right),\ldots,\tilde{p},p\right) - C\left(x_{n+1}\left(p^b\left(m,n\right),\ldots,p^b\left(m,n\right),p\right)\right)\right] \\ & = \max_x \left[p\left(x,p^b\left(m,n\right),n\right)x - C\left(x\right)\right] \\ < & \max_x \left[p\left(x,p^b\left(m,n+1\right),n\right)x - C\left(x\right)\right] \\ = & p^b\left(m,n+1\right)x^b\left(m,n+1\right) - C\left(x^b\left(m,n+1\right)\right). \parallel \end{aligned}$$

When the Market is Large Enough to Ensure Bertrand Profits. Let's

fix the number of firms at n. We now show, by an argument analogous to that used to establish Proposition 5, that we can always make m large enough so that we can have n firms producing profitably at the Bertrand outcome. For the given n, we let
$$m^b\left(n,\alpha\right) \equiv n^{\left(1-\frac{\gamma}{1-\alpha}\right)}\frac{C'\left(x\left(\alpha\right)\right)}{x\left(\alpha\right)^{\gamma-1}\left(1-\frac{1}{e^b(n)}\right)},$$
the Bertrand analog of $m^*\left(n,\alpha\right)$. Since,
$$e^b\left(n\right) > e^c\left(n\right),$$
$$m^b\left(n,\alpha\right) > m^*\left(n,\alpha\right).$$

Proposition 16. *Assume that condition (2) holds. If $m = m^b(n, \alpha)$ and there are n firms, then the Bertrand equilibrium output is*

$$x^b(m, n) = x(\alpha),$$

the Bertrand price charged by all firms is

$$p^b(m, n) = m^b(n, \alpha)\, n^{\left(\frac{\gamma}{1-\alpha}-1\right)} x(\alpha)^{\gamma-1} = \frac{C'(x(\alpha))}{\left(1 - \frac{1}{e^b(n)}\right)}. \tag{82}$$

and this price exceeds average cost so that all firms earn a profit.

If $m > m^b(n, \alpha)$ and n is sufficiently large, then the Bertrand equilibrium output is

$$x^b(m, n) \geq x(\alpha),$$

the Bertrand price charged by all firms is

$$p^b(m, n) \geq \frac{C'(x(\alpha))}{\left(1 - \frac{1}{e^b(n)}\right)}$$

and this price exceeds average cost so that all firms earn a profit.

If $m = m^b(n, \alpha)$ and there are $n' < n$ firms, and n' is sufficiently large then the Bertrand equilibrium output is

$$x^b(m, n') \geq x(\alpha),$$

the Bertrand price charged by all firms is

$$p^b(m, n') \geq \frac{C'(x(\alpha))}{\left(1 - \frac{1}{e^b(n)}\right)}$$

and this price exceeds average cost so that all firms earn a profit.

If $m = m^b(n, \alpha)$ and there are $n' > n$ firms, and n is sufficiently large then the Bertrand equilibrium output is

$$x^b(m, n') \leq x(\alpha),$$

the Bertrand price charged by all firms is

$$p^b(m, n') \leq \frac{C'(x(\alpha))}{\left(1 - \frac{1}{e^b(n)}\right)}.$$

Proof: It is immediate to verify that $m^b(n,\alpha)$ has been chosen so that $x(\alpha)$ satisfies condition (68) and

$$x^b(m,n) = x(\alpha).$$

The Bertrand equilibrium price in (82) is obtained by substituting $x(\alpha)$ for $x(p^b)$in (67). Since

$$C'(x(\alpha)) = (1-\alpha)\,AC(x(\alpha))$$

(82) implies

$$\begin{aligned} p^b(m,n) &= \frac{(1-\alpha)\,AC(x(\alpha))}{\left(1-\frac{1}{e^b(n)}\right)} \\ &> AC(x(\alpha)). \end{aligned}$$

The last inequality follows from Remark 13.

If $m > m^*(n,\alpha)$ and there are n firms, the results that the Bertrand equilibrium output exceeds $x(\alpha)$ and the Bertrand price, $p^b(m,n)$, charged by all firms exceeds that given in (52) are corollaries of Proposition 12. The fact that all firms make a profit is a consequence of Corollary 13 if

$$x^b(m,n) < x(0).$$

If

$$x^b(m,n) \geq x(0)$$

the fact that

$$p^b(m,n) > C'\left(x^b(m,n)\right) \geq AC\left(x^b(m,n)\right)$$

implies that all firms make a profit.

The remainder of the proof follows immediately from Proposition 12 and Corollary 13.||

8. THE COLLUSIVE OUTCOME

Definition 5. *Suppose that there are n firms and that the market size parameter is m. In the* ***symmetric collusive outcome****, output per firm, $x^M(m,n)$, is chosen to maximize industry profits. Thus,*

$$x^M(m,n) = \underset{x}{argmax}\,[p(x)\,x - C(x)]$$

where

$$p(x) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma-1}$$

is the inverse demand function defined in Remark 2. The collusive price charged by each firm is

$$p^M(m,n) = p\left(x^M(m,n)\right)$$

Remark 43. *We could, of course, have proceeded more generally in defining the collusive outcome by not requiring that all firms produce the same output and charge the same price. It is easy to see, however, that, because of the symmetry of the demand function (5), industry profit maximization implies that all firms act alike.*

Remark 44. *The function*

$$\begin{aligned} & [p(x)x - C(x)] \\ = \;& mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma} - C(x) \end{aligned}$$

relating profits per firm to output per firm, x, is a strictly concave function.

Proof:

$$\frac{\partial \left[mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma} - C(x)\right]}{\partial x} = \gamma mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma-1} - C'(x) \tag{83}$$

and

$$\frac{\partial^2 \left[mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma} - C(x)\right]}{\partial x^2} = \gamma(\gamma-1)\,mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}x^{\gamma-1} - C''(x) < 0. \; \| \tag{84}$$

Proposition 17. *In the symmetric collusive outcome, each firm s produces x^M where x^M is the solution to*

$$mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^M\right)^{\gamma-1} = \frac{C'\left(x^M\right)}{\gamma}. \tag{85}$$

and charges

$$p^M = p\left(x^M\right) = mn^{\left(\frac{\gamma}{1-\alpha}\right)-1}\left(x^M\right)^{\gamma-1}$$

The Collusive price p^M can also be obtained as the solution to

$$p^M = \frac{C'\left(x\left(p^M\right)\right)}{\gamma}. \tag{86}$$

where, because of Remark 6,

$$x^M = x\left(p^M\right) \equiv m^{\frac{1}{(1-\gamma)}}\left(p^M\right)^{-\frac{1}{(1-\gamma)}} n^{\left[\frac{\alpha}{(1-\alpha)}\right]\left[\frac{1}{(1-\gamma)}-\frac{1}{\alpha}\right]}.$$

Proof: Since $[p(x)x - C(x)]$ is strictly concave, x^M is characterized by the first order condition obtained by setting the first derivative computed in (83) equal to zero. Equations (85) and (86) follow immediately from that first order condition.$\|$

Remark 45. *The relationship between the collusive price, p^M, and the marginal cost in (86) is once more the standard result*

$$\text{price} = \frac{\text{marginal cost}}{\left[1 - \frac{1}{\text{demand elasticity}}\right]}.$$

that as noted in Remarks 28, 36 and 40 also holds in the Dixit-Stiglitz, Cournot and Bertrand equilibria.

Remark 46. *Although $x^M(m,n)$ and $p^M(m,n)$ depend on m, n and α. we have already suppressed this dependence in our statement of Proposition 17. In the subsequent discussion, we will simply denote $x^M(m,n)$ and $p^M(m,n)$ by x^M and p^M.when there is no danger of confusion or when there is no need to emphasize the dependence of these outcomes on m and n.*

9. BERTRAND AND COURNOT EQUILIBRIA AND THE COLLUSIVE OUTCOME COMPARED

Proposition 18. *At every m and n, the Bertrand equilibrium price, $p^b(m,n)$, is lower than the Cournot equilibrium price, $p^c(m,n)$, which is, in turn, lower than the collusive price, $p^M(m,n)$. In addition, the Bertrand equilibrium supply, $x^b(m,n)$, exceeds the Cournot equilibrium supply, $x^c(m,n)$ which, in turn, exceeds the Collusive output, $x^M(m,n)$. Finally, The Bertrand equilibrium profits,*

$$p\left(x^b(m,n)\right)x^b(m,n) - C\left(x^b(m,n)\right),$$

are lower than the Cournot equilibrium profits,

$$p\left(x^c(m,n)\right)x^c(m,n) - C\left(x^c(m,n)\right)$$

which are, in turn, lower than the collusive profits

$$p\left(x^M(m,n)\right)x^M(m,n) - C\left(x^M(m,n)\right).$$

Proof: Recalling, Remarks 36, 40 and 45, let's rewrite (36), (67) and (86) as

$$\frac{p^c}{C'(x(p^c))} = \frac{1}{\left[1 - \frac{1}{\varepsilon^c(n)}\right]} \tag{87}$$

$$\frac{p^b}{C'(x(p^b))} = \frac{1}{\left[1 - \frac{1}{\varepsilon^b(n)}\right]} \tag{88}$$

and

$$\frac{p^M}{C'\left(x\left(p^M\right)\right)} = \frac{1}{\left[1 - \frac{1}{(1-\gamma)^{-1}}\right]}. \tag{89}$$

respectively. Remarks 13, 20 and 24 imply that

$$\varepsilon^b\left(n\right) > \varepsilon^c\left(n\right) > \frac{1}{1-\gamma}.$$

Thus, (87), (88) and (89) imply

$$\frac{p^b}{C'\left(x\left(p^b\right)\right)} < \frac{p^c}{C'\left(x\left(p^c\right)\right)} < \frac{p^M}{C'\left(x\left(p^M\right)\right)}. \tag{90}$$

Since

$$x'\left(p\right)$$

is decreasing in p and $C''\left(x\right) \geq 0$

$$\frac{p}{C'\left(x\left(p\right)\right)}$$

is increasing in p and (90) implies

$$p^b < p^c < p^M.$$

Since $x'\left(p\right) < 0$, these inequalities imply

$$x\left(p^b\right) > x\left(p^c\right) > x\left(p^M\right). \tag{91}$$

Because of Remark 44 and the fact that

$$\frac{\partial\left[p\left(x^M\right)x^M - C\left(x^M\right)\right]}{\partial x} = 0,$$

the function

$$p\left(x\right)x - C\left(x\right)$$

is as shown in Figure 5.

As Figure 5 and Remark 44 imply,

$$p\left(x\right)x - C\left(x\right)$$

is a decreasing function above x^M, (91) implies

$$p\left(x^M\right)x^M - C\left(x^M\right) > p\left(x^c\right)x^c - C\left(x^c\right) > p\left(x^b\right)x^b - C\left(x^b\right). \;\|$$

Remark 47. *Proposition 18, of course, implies that the Bertrand profits are negative whenever the Cournot profits are negative and the Cournot profits are positive whenever the Bertrand profits are positive. As a consequence we have the following corollary of Propositions 6 and 18.*

Corollary 19. *For every m, there exists an $\bar{n}$ such that when the market size is m and there are $n \geq \bar{n}$ firms, all firms lose money in the Bertrand equilibrium.*

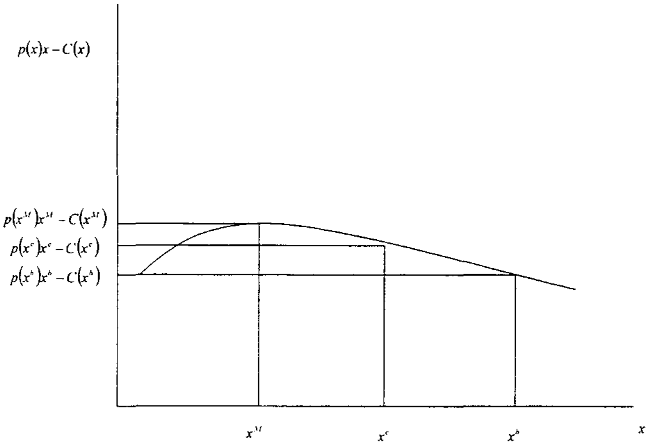

Figure 5:

10. BERTRAND-FREE ENTRY EQUILIBRIUM

As in the discussion of the Cournot free-entry equilibrium we assume that there is an infinity of potential firms all of which possess the same cost function satisfying the conditions of either Cost Case 1 or Cost Case 2.

Definition 6. *In a* ***Bertrand free-entry equilibrium*** *there are n^b firms that produce. The vector of prices charged by these firms, $\left(p_1^b, \ldots, p_{n^b}^b\right)$, is a Bertrand equilibrium and each of these producing firms makes a nonnegative (but possibly zero) profit. Thus, for each s,*

$$\begin{aligned} & \max_{p_s} R_s\left(p_1^b, \ldots, p_s, \ldots, p_{n^b}^b\right) \\ & -C\left(x_s\left(p_1^b, \ldots, p_s, \ldots, p_{n^b}^b\right)\right) \\ = \; & R_s\left(p_1^b, \ldots, p_{n^b}^b\right) \\ & -C\left(x_s\left(p_1^b, \ldots, p_{n^b}^b\right)\right) \end{aligned}$$

is nonnegative. In addition, none of the non-producing firms can make a profit. Thus, the final condition satisfied in a Bertrand free-entry equilibrium is that

$$\max_{p} R_s\left(p_1^b, \ldots, p_{n^b}^b, p\right) - C\left(x_{n+1}\left(p_1^b, \ldots, p_s, \ldots, p_{n^b}^b, p\right)\right)$$

is nonpositive.

In discussing the Bertrand free-entry equilibria, we make use of the fact established in Proposition 11 that, for each n, the unique Bertrand equilibrium is the symmetric Bertrand equilibrium shown to exist and described in Proposition 10. Of course, in that symmetric equilibrium, all firms charge the same price $p^b(m,n)$.

It appears that, in general, a Bertrand free-entry equilibrium may fail to exist. In any case, because of Proposition 15, it is not possible to use an argument that parallels the one used to establish the existence of a Cournot free-entry equilibrium. We can however, make some observations.

Remark 48. *Note that if we have a Bertrand free-entry equilibrium with n^b firms, then there will also be a Cournot free-entry equilibrium with n^c firms where*

$$n^c \geq n^b.$$

This is true since as we noted in Remark 47, the Cournot profits are positive whenever the Bertrand profits are positive .

Proposition 20. *If $m \geq \max_{n \leq \tilde{n}} m^b(n, \alpha)$, then there exists an $N > \tilde{n}$ such that all firms earn a nonnegative profit in the Bertrand equilibrium with n firms when*

$$n \leq N$$

and all firms suffer a loss in the Bertrand equilibrium with n firms when

$$n > N.$$

If, in addition, $\tilde{n}$ is sufficiently large,

$$x^b(m, N) < x(\alpha).$$

If there exists a Bertrand-free entry equilibrium then there are N firms in equilibrium.

Proof: From Proposition 16, we know note that if $m \geq \max_{n \leq \tilde{n}} m^b(n, \alpha)$, then firms make a profit in the n firm Bertrand equilibrium when $n \leq \tilde{n}$. From Corollary 19 we know that for every m, there is some $\bar{n}$, such that profits are negative if there are more than $\bar{n}$ firms. There must, therefore, exist some $N \in [\tilde{n}, \bar{n})$ for which all firms earn a nonnegative profit in the Bertrand equilibrium with n firms when

$$n \leq N$$

and all firms suffer a loss in the Bertrand equilibrium with n firms when

$$n > N.$$

Proposition 16 also implies that

$$x^b(m, N) < x(\alpha)$$

if $\tilde{n}$ and, therefore N, is sufficiently large.

Since firms make a nonnegative profit in the n firm Bertrand equilibrium when

$$n \leq N,$$

Proposition 15 implies that entry will be profitable when there are less than N firms producing in the Bertrand equilibrium. Thus, we can't have a Bertrand-free entry equilibrium with less than N firms. We can't have a Bertrand-free entry equilibrium with more than N firms because firms must suffer a loss in the Bertrand equilibrium when there are more than N firms. ||

Remark 49. *There will exist a Bertrand-free entry equilibrium if*

$$\max_{p}\left[R_{N+1}\left(p^{b}(m,N),\ldots,p^{b}(m,N),p\right)-C\left(x_{N+1}\left(p^{b}(m,N),\ldots,p^{b}(m,N),p\right)\right)\right]<0.$$

It appears that this condition can hold for some m. It also seems clear that if this condition holds for some m, the inequality can be reversed by simply raising m slightly.

Although we don't know whether a Bertrand-Free Entry Equilibrium always exists we do know that if one does exist in a large market, it will approximate the Dixit-Stiglitz equilibrium.

Proposition 21. *When the market size parameter m is sufficiently large, the amount produced by each of the n^b firms in a Bertrand free-entry equilibrium, if one exists, is*

$$x^{b}\left(m,n^{b}\right)\simeq x(\alpha)$$

and the price charged by each of these firms is

$$p^{b}\left(m,n^{b}\right)\simeq AC\left(x(\alpha)\right).$$

The proof of this proposition is analogous to that given for Proposition 9.

11. A Case When Our Elasticity Condition Fails

We will not explicitly consider the case in which

$$\frac{1}{\alpha}<\frac{1}{1-\gamma}$$

but we will consider the borderline case in which

$$\frac{1}{\alpha}=\frac{1}{1-\gamma}.$$

In this case(1) reduces to

$$p_s(x_1,\ldots,x_n)=mx_s^{-\alpha}$$

and (5) reduces to

$$x_s(p_1,\ldots,p_n)=m^{\frac{1}{(1-\gamma)}}p_s^{-\frac{1}{\alpha}}.$$

Remark 50. *When*

$$\frac{1}{\alpha}=\frac{1}{1-\gamma},$$

the Cournot and Bertrand equilibria and the collusive outome all coincide. In each of these equilibria all firms charge

$$p^M = p_s\left(x^M\right) = m\left(x^M\right)^{-\alpha} = \frac{C'\left(x^M\right)}{(1-\alpha)}, \tag{92}$$

and supply the amount x^M that solves (92).

In the case of constant marginal cost, the solution to (92) is

$$x^M = \left[\frac{m(1-\alpha)}{c}\right]^{\frac{1}{\alpha}}.$$

So all firms produce that amount and charge

$$p^M = \frac{c}{(1-\alpha)}.$$

which, as noted in Remark 30, is the Dixit-Stiglitz price $p(\alpha)$.

The equilibrium price and output are independent of the number of firms, n, but, when marginal cost is increasing, both increase with the size of the market as measured by m. When marginal cost is constant, the equilibrium price remains at $p(\alpha)$ whether the market is large or small, but x^M increases with the size of the market. The equilibrium coincides with the Dixit-Stiglitz equilibrium if $m = m(\alpha)$ where

$$m(\alpha) = \frac{C'(x(\alpha))}{x(\alpha)^{\alpha}(1-\alpha)}.$$

In that case, firms obviously earn no profits.

When $m < m(\alpha)$,

$$x^M < x(\alpha),$$

and firms earn negative profits. When $m < m(\alpha)$, and marginal cost is increasing, we also have

$$p^M = \frac{C'\left(x^M\right)}{(1-\alpha)} < p(\alpha).$$

The case in which $m < m(\alpha)$ and marginal cost is increasing is illustrated Figure 6. In that case,

$$AC\left(x^M\right) > p_s\left(x^M\right) = \frac{C'\left(x^M\right)}{(1-\alpha)}.$$

Figure 7 illustrates a case in which marginal cost is constant and $m < m(\alpha)$ so that

$$\left[\frac{m(1-\alpha)}{c}\right]^{\frac{1}{\alpha}} = x^M < x(\alpha) = \left(\frac{1}{\alpha} - 1\right)\frac{F}{c}.$$

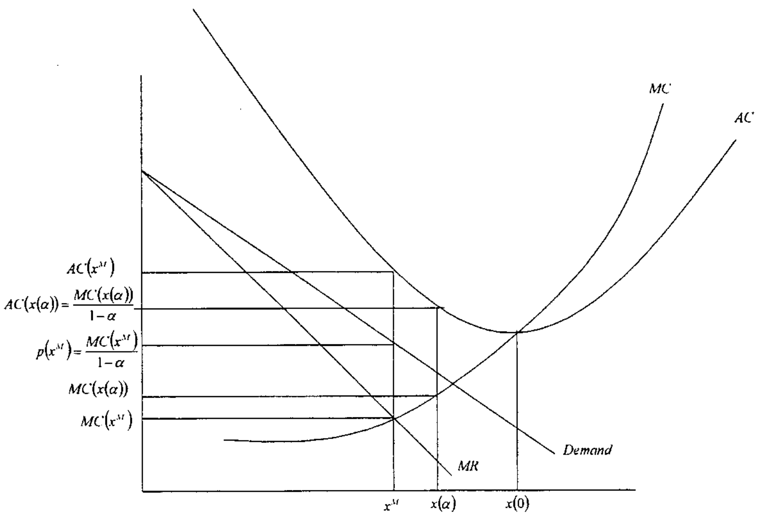
MC
AC
$AC(x^M)$
$AC(x(\alpha)) = \frac{MC(x(\alpha))}{1-\alpha}$
$p(x^M) = \frac{MC(x^M)}{1-\alpha}$
$MC(x(\alpha))$
$MC(x^M)$
Demand
MR
x^M
$x(\alpha)$
$x(0)$

Figure 6:

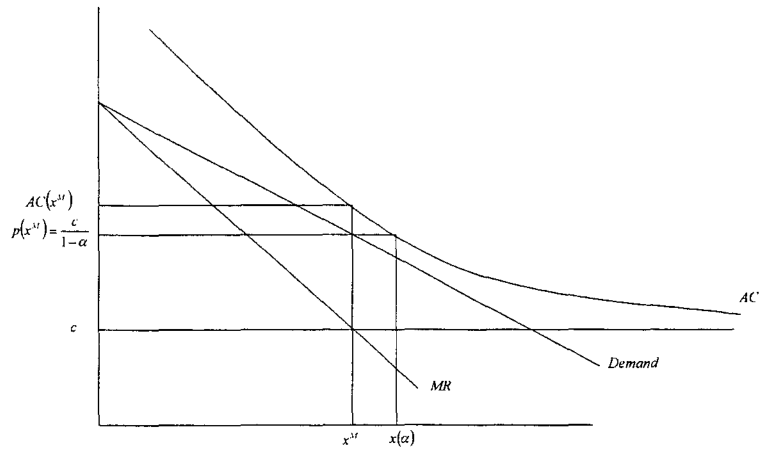

Figure 7:

In Figure 7, as in Figure 6, we also have

$$AC\left(x^M\right) > p^M = \frac{c}{(1-\alpha)}.$$

When $m > m(\alpha)$,

$$x^M > x(\alpha),$$

and firms earn positive profits. When $m > m(\alpha)$, *and marginal cost is increasing, we also have*

$$p\left(x^M\right) = \frac{C'\left(x^M\right)}{(1-\alpha)} > p(\alpha).$$

Finally, equilibrium profits,

$$m\left(x^M\right)^{1-\alpha} - C\left(x^M\right)$$

are independent of n. *Thus, if profits are positive because* $m > m(\alpha)$, *the entry of new firms has no impact on profits. For this reason, neither the Cournot nor Bertrand*

free-entry equilibria exist in this case. If $m = m(\alpha)$, there exist both Cournot and Bertrand free-entry equilibrium which coincides with the Dixit-Stiglitz equilibrium and any number of firms might produce in equilibrium. When $m < m(\alpha)$, there exist trivial Cournot and Bertrand free-entry equilibrium in which no firms produce.

Proof: When

$$\frac{1}{\alpha} = \frac{1}{1-\gamma},$$

choosing x_s to maximize

$$\begin{aligned} R_s\left(x_1^c, \ldots, x_s, \ldots, x_n^c\right) - C\left(x_s\right) &= m x_s^{1-\alpha} - C\left(x_s\right) \\ &= m x_s^{\gamma} - C\left(x_s\right) \end{aligned}$$

is equivalent to choosing p_s to maximize

$$\begin{aligned} & R_s\left(p_1^b, \ldots, p_s, \ldots, p_n^b\right) - C\left(x_s\left(p_1^b, \ldots, p_s, \ldots, p_n^b\right)\right) \\ = \;& m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}} - C\left(m^{\frac{1}{(1-\gamma)}} p_s^{-\frac{1}{\alpha}}\right). \end{aligned}$$

Thus, in the Cournot, Bertrand and Collusive cases, firms solve the same problem and $x^c = x^b = x^M$. Because $\alpha < 1$, and $C''(x_s) > 0$,

$$R_s\left(x_1^c, \ldots, x_s, \ldots, x_n^c\right) - C\left(x_s\right) = m x_s^{1-\alpha} - C\left(x_s\right)$$

is clearly a strictly concave function of x_s. Thus, $x^c = x^b = x^M$ is still characterized by (29) which now becomes (92), a single equation in the one unknown x^c. Note that, since the solution to (92) is unique and the same for all s, the only equilibrium is one in which all firms s supply the same amount. Note also that, because the equation that determines x^M is independent of n, the solution x^M is also independent of n. It is, furthermore, easy to check by implicitly differentiating (92) that x^M is an increasing function of m.

The fact that firms earn negative profits when

$$x^M < x(\alpha)$$

follows from the fact

$$\frac{AC\left(x^M\right)}{MC\left(x^M\right)}$$

rises as x^M falls below $x(\alpha)$, while

$$\frac{p^M}{MC\left(x^M\right)}$$

remains constant at $\frac{1}{1-\alpha}$ as x^M falls below $x(\alpha)$.

Conversely, as x^M rises above $x(\alpha)$,

$$\frac{AC(x^M)}{MC(x^M)}$$

falls while

$$\frac{p^M}{MC(x^M)}$$

remains constant at $\frac{1}{1-\alpha}$. Thus, firms earn positive profits when

$$x^M > x(\alpha). \|$$

12. SUMMARY:

For the special case of the inverse demand functions given in (1) and the demand functions given in (5) we have described the Cournot and Bertrand equilibria for a fixed number of firms. Our characterizations of these equilibria have made it possible to determine how they are affected by the size of the market and the number of firms. For the special cases we considered, we were also able to prove the existence of Cournot free-entry equilibria in which the number of firms is determined endogenously In addition, we were able to prove that, in a large market, the Cournot free-entry equilibria approximate the Dixit-Stiglitz monopolistically competitive equilibria. While we were unable to establish a general existence result for Bertrand free-entry equilibria, we were able to prove that, when these equilibria exist, they are unique and that in large markets they also approximate the Dixit-Stiglitz equilibria.

While the results that emerged were primarily as expected, the arguments required were more involved than those normally required in the case when firms sell goods that are perfect substitutes. It is natural to ask to what extent these arguments can be extended. In searching for extensions, it would seem to be necessary to restrict attention to cases in which an analog of our elasticity condition continues to hold. One such extension that appears promising is that in which the demand functions are derived from utility maximization of the utility function

$$u(y, x_1, \ldots, x_n) = \left[y + mh\left(\sum_{t=1}^{n} x_t^{1-\alpha}\right)\right]$$

where $h'(\cdot) > 0$, $h''(\cdot) < 0$ and

$$\alpha < -\frac{zh''(z)}{h'(z)} < 1. \tag{93}$$

We have considered the case of

$$h(z) = \frac{z^{\gamma}}{\gamma}$$

for which

$$-\frac{zh''(z)}{h'(\cdot)} = (1-\gamma).$$

The extension we propose here would be one in which

$$-\left[\frac{zh''(z)}{h'(z)}\right]^{-1}$$

would become the monopoly elasticity of demand and (93) would be the extension of (2). In this extension, the monopoly elasticity of demand would, of course, no longer be constant.

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