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THE DETERMINANTS OF CITY SIZES AND INDUSTRY CONCENTRATION

BY

∅ 09/12/98

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1. Introduction

The spatial distribution of production is a traditional issue of economic geography that recently has catched the attention of standard economic analysis. One of the most striking features of the spatial organization of production is the high concentration of producers and consumers in urban clusters, and the wide range of variation in sizes of these urban units.

Spatial agglomeration has been most frequently explained by the presence of different kinds of economies of scale from concentrating in space both production and consumption. In the first group we have technological externalities of varying reaches (intra-industry or intra-city), risk sharing among firms by pooling labor markets, or lower cost of provision of some input produced under (internal) increasing returns to scale. The second group includes lower costs of provision of public goods and accessibility to a greater variety of consumption goods. Although each one of the spatial agglomeration incentives cited above certainly has its place in accounting for the observed patterns of spatial concentration of production, here we take the minimalist view that only costs to transactions across space together with internal increasing returns to scale in production are enough. Increasing returns basically make firms concentrate production in a limited number of locations. To minimize transport costs of goods produced, a firm will prefer a location with a large demand; but local demand is large precisely where most firms are settled. This circularity generates a self reinforcing incentive for industry concentration. On the other hand, because costs of moving in space also apply to workers accessing their own workplaces, if those costs increase with concentration there can be a trade off.

We develop a simple model of trade between two cities that includes transportation costs for goods and commuting costs for workers, and allows us to address the question of when and how far manufacturing will concentrate in some urban area, altogether with its respective

labor force. Two further elaborations of the basic model allow us respectively to evaluate the effects of differentiating goods either by cost to transport (or cost to trade if interpreted more generally)or by different price elasticities. The first case is a study of the effects of having nontraded goods available for consumption in each city over the results derived before. The purpose of the second differentiation is to assess the effects of variable relative prices in cities of different sizes on the demands for different types of goods. This turns out to reveal a demand driven motive that determines each industry's location decision, so that we are able to make predictions about what kind of industries will be more concentrated in each particular city.

A last question we approach here, that is more of an illustration of the nature of incentives for agglomeration postulated in this work, is related to the robustness of predictions of a two city model. An extension of the model to account for the interactions of three cities reveals a tendency for dispersion of equilibrium population distributions with respect to a comparable equilibrium with only two cities. The related literature include two basic sources: the standard intra-industry trade models (Krugman[80], Helpman and Krugman[90]) together with some recent work related to economic geography and manufacturing location (Krugman[91a],[91b]) form the first group; the second comes from urban economics models of commuting costs and land prices that use the preferences for variety and monopolistic competition with increasing returns framework (Dixit and Stiglitz[77]) to get agglomeration economies in cities. One illustrative example of the first group is Krugman[91] where the trade off for location is between proximity to manufactured goods markets on one hand and competition for agricultural goods (in fixed supply) on the other. We model trade in manufactured goods in a similar way but the additional urban land price structure completely changes results. Krugmann[80] develops the basic model of intra industry trade employed here and provides some results for industry concentration that depend on differences in tastes between cities. Among the second group are Abdel-Rahman[87] and

Rivera-Batiz[87]. In both works increasing returns and monopolistic competition in nontraded goods generate agglomeration economies stemming from the agents' preferences for variety. They both indicate that increasing land prices and greater availability of goods for consumption enter a trade off to determine city size. In both cases though trade aspects are neglected and relative wages determined exogenously. A recent work that pursues questions close to those in our Section 5 is Elizondo and Krugman [92].

The presentation is divided into six sections. Section 2 describes in detail the basic model and aims at the question of equilibrium city sizes. Section 3 adds nontraded goods as a second industry to the basic model and asks how this alters results. Section 4 extends the analysis of two different industries for the case when both produce traded goods, and develops implications for city sizes and industry concentrations. Section 5 treats a model of three cities and gives some insights about the characteristics of equilibria in larger systems. Finally, Section 6 closes with some comments and conclusions.

2. The Basic Model with Two Cities

In this section we present the model used to address the question of the coexistence of cities of different sizes in free trade.

This model assumes the existence of two spatially separated spots initially available for settling of productive activities. These sites may be predetermined by the physical characteristics of the landscape like climate or topography, or by the availability of a transportation network. A unspecified number of people freely choose between the two sites where they want to settle as dwellers and workers. Agents have utility from consumption of land and manufactured goods. We suppose that production takes no physical space and that occupation of land around each production site happens in the radial fashion of the monocentric city model (actually we simplify it to a one sided distribution of

population). Labor is the only productive factor and production happens under increasing returns to scale. Transportation costs are incurred to ship a good from one city to the other. Firms can costlessly differentiate their products and agents have utility on variety. Market equilibrium is one of monopolistic competition and a pattern of trade emerges such that one particular good will be manufactured by only one firm, and every agent consumes every good (unless transport costs are infinite). The exposition is divided into four subsections. In the first and second subsections respectively the goods market and the land market are analyzed; the third subsection describes overall market equilibria; finally in the fourth subsection the location choice problem is addressed.

Although agents can live at different locations into each city they are treated symmetrically otherwise, in particular they receive same wages, rents and eventual profits. Agents living in any city at location x, solve for (first subscript refers to one of two cities, second subscript refers to a specific good):

$$\max_{H,c_k} \quad H_i^{\alpha} \left\{ \sum_{k=1}^{n} c_{ik}^{(\sigma-1)/\sigma} \right\}^{(1-\alpha)\sigma/(\sigma-1)}$$

$$\text{s.t.} \quad Q_i(x)\, H_i + \sum_{k=1}^{n} p_{ik} c_{ik} + m_i(x) = W_i + R_i + \Pi_i = I_i$$

given $Q_i(x)$, p_{ik}, W_i, R_i, Π_i, $m_i(x)$, n
where,

H_i is living space.

c_{ik} is consumption of manufactured good k in city i.

$m_i(x)$ is commuting costs.

$Q_i(x)$ is the urban land price schedule.

p_{ik} is the price of good k faced by a citizen in city i; $p_{ik} \neq p_{jk}$ in general because of transport costs.

W_i, R_i, Π_i, I_i, are respectively wage rate, land rent ,profits and full income received in city i.

$n \equiv n_1 + n_2$ is the total number of goods produced in both cities.

p_{ik}, Π_i, and n are determined in the monopolisticaly competitive equilibrium of goods and labor markets in each city, as functions of wage rates and of the distribution of population.

$Q_i(x)$ and R_i are determined in the equilibrium of the urban land markets in each city, as functions of wage rates and the distribution of population.

W_i is determined in the goods trade equilibrium between cities, as a function of the distribution of population and transport costs.

2.1 Monopolistic Competition in Manufactured Goods.

Utility from consumption of manufactured goods is assumed to take the form:

$$C(c_1,\ldots,c_n) = [\ \textstyle\sum_{k=1}^{n} c_k^{(\sigma-1)/\sigma}\ ]^{\sigma/(\sigma-1)}$$

This specification embodies preference from variety, where n is the number of goods actually available in the market. All goods enter symmetrically, with σ being the elasticity of substitution between any two of them. Costs to transport manufactured goods between the cities are introduced in the model in Samuelson's "iceberg" fashion. We assume that, of each unit of a manufactured good shipped from one city to the other only a fraction $\tau<1$ arrives to be consumed. In this way transport costs are incurred in lost units of the goods themselves. If p is the price faced by a domestic consumer for some domestic good, then p/τ is that good's price faced by a foreign consumer.

The total demand c_{1k} for a good *k* manufactured in city one is the sum of the aggregate demands by city one (c_{1k}^1) and city two (c_{1k}^2):

$$c_{1k} = \frac{p_k^{-\sigma} E_1^a}{\sum_{j=1}^{n_1} p_j^{1-\sigma} + \sum_{j=n_1+1}^{n_1+n_2} (p_j/\tau)^{1-\sigma}} + \frac{(1/\tau)(p_j/\tau)^{-\sigma} E_2^a}{\sum_{j=1}^{n_1} (p_j/\tau)^{1-\sigma} + \sum_{j=n_1+1}^{n_1+n_2} p_j^{1-\sigma}}$$

where E_i^a is aggregate expenditures in manufactured goods by city i's inhabitants (to be determined ahead) and n_i is the number of goods manufactured in city i. The price elasticity of demand for good k manufactured in city one is:

$$\varepsilon_k = -\sigma - (1-\sigma)\{\ \delta(p_k c^1_{1k}/E^a_1) + (1-\delta)(p_k c^2_{1k}/E^a_2)\ \}$$

where δ is the fraction of total demand for good k coming from agents living in city one ($\delta = c^1_{1k}/c^1_{1k}+c^2_{1k}$). As the share of expenditures on good k in both cities get small, ε_k approaches $-\sigma$.

Labor alone is used in the production of each manufactured good. The technology is the same for every good, and involves constant fixed and variable cost components with the form:

$$X = \ell + \omega Y$$

where Y is output, X is the the cost in terms of labor and ℓ and ω are constants.

In this model firms can freely decide to produce a differentiated good, therefore avoiding to dispute markets directly. For this reason a particular good is produced only by one firm. This is also the reason why firms do not split production between the two cities, even if transport cost is high.

The profit maximizing pricing behavior of an individual firm is to set its price so as to equate marginal revenue to marginal cost. We suppose the number of goods available is sufficiently high to take the price elasticity ε as constant, that is, $\varepsilon=-\sigma$. If W is the wage rate then:

$$p = (\sigma/(\sigma-1))\omega W$$

where we require that $\sigma>1$, so that marginal revenue is always positive.

In the monopolistic competitive equilibrium, free entry of firms into manufacturing requires zero profits at the margin. This implies that

$$(p-\omega W)Y = \ell W$$

Then the levels of output and employment in equilibrium for any firm, are determined by:

$Y = \ell(\sigma-1)/\omega$

$X = \ell\sigma$

Finally, market clearing for labor pins down the number of goods being produced in equilibrium by:

$n = L/(\ell+\omega Y) = L/\ell\sigma$

When we let wage rates be different in the two cities, the relative price ratio of goods manufactured in cities 1 and 2 must equal the ratio of wages paid in cities 1 and 2

$p_1/p_2 = W_1/W_2$

Also, once the size of one firm is fixed, the number of goods manufactured in a city is proportional to the size of the work force located there. Thus,

$n_1/n_2 = L_1/L_2$

2.2 Land Prices in a Monocentric City Model.

Agents derive utility from living space and the aggregate consumption of manufactured goods in a Cobb-Douglas specification. Agents in each city are endowed with one unit of time which they share between working and commuting, and land ownership rights that entitles them to equal shares of the total rents paid in the city. Nominal labor and rental earnings are allocated to consumption and rental expenditures.

The urban land model is specified in the simplest way. Land is available as a strip with unit width. All jobs are located at one end of the strip, the Central Business District (CBD). Agents may decide to locate at any distance x from the CBD and incur commuting costs of m(x) units of time. Production is assumed to take no physical space.

From homotheticity of $C(c_1,...,c_n)$, the expenditure function for agents living in city i (facing prices p_{ik}) can be written as:

$e(p_{i1},...,p_{in},C(x)) = P(p_{i1},...,p_{in})C(x) \equiv P_i C(x)$

So that the first stage solution for the problem of an agent

living in any city brings up the following demand functions for land and consumption at each distance x from the CBD:

$$H(x) = \alpha(I - m(x))/Q(x)$$

$$C(x) = (1 - \alpha)(I-m(x))/P$$

The necessary condition for optimum location is:[1]

$$Q'(x)H(x) = -m'(x)$$

This condition states that movements along x induce income compensated price changes in demands for land and consumption. Substituting the demand for land into it and integrating, we get the land price schedule (with $m(0)=0$):

$$Q(x) = Q(0)[(I-m(x))/I]^{1/\alpha}$$

Substituting Q(x) into the demand for land we get:

$$H(x) = \alpha I^{1/\alpha}(I-m(x))^{(\alpha-1)/\alpha}/Q(0)$$

The price of land at the CBD (Q(0)) and the distance to the outer edge of the city (x^e) can be determined from the price of land at the edge (Q^e) and the size of population in the city (L) from the conditions:

$$Q(x^e) = Q^e \tag{1}$$

$$\int_0^{x^e} (1/H(x))dx = L$$

where $1/H(x)$ is the population density at x.

We assume that commuting costs are linear in the distance x, and consist entirely of time lost. We assume in addition that the average time spent to commute one unit of distance is a constant ξ. Therefore we make $m(x)=mx=\xi Wx$, where the number of commutes is normalized to one.

With this form of commuting costs in (1) we find:

$$Q(0) = mL + Q^e \tag{2}$$

$$x^e = (I/m)[1-(Q^e/mL+Q^e)^{\alpha}]$$

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[1]If $\upsilon(Q(x),P,I-m(x))$ is the indirect utility function, that follows from the FOC: $\partial\upsilon/\partial x=(\partial\upsilon/\partial Q)Q'(x)-(\partial\upsilon/\partial(I-m(x))m'(x)=0$ and from Roy's Identity.

We can now compute the city aggregates of interest. The aggregate amount of rents paid in the city (R^a) is given by:

$$R^a = \int_0^{x^e} Q(x)dx = (\alpha/(1+\alpha))[LI+Q^e x^e] \tag{3}$$

From the Cobb Douglas preferences, the total amount paid for rents (or total expenditure in land) is a fraction equal to α of the aggregate disposable income (aggregate income minus total commuting costs). So that aggregate expenditures in commuting (M^a) may be determined by:

$$\alpha(LI-M^a) = R^a \Rightarrow M^a = (1/(1+\alpha))[\alpha LI-Q^e x^e] \tag{4}$$

The sum of aggregate expenditures in land R^a in (3), in consumption E^a and commuting M^a in (4) must add to total income LI, that is:

$$R^a+M^a+E^a = LI \Rightarrow E^a = ((1-\alpha)/(1+\alpha))[LI+Q^e x^e] \tag{5}$$

Since rural land doesn't have any role in this model where all agents are urban dwellers, we arbitrarily set its price to zero, $Q^e=0$. Under this assumption, our solutions in (2) to (5) become:

$Q(0) = mL$

$x^e = I/m$

$R^a = M^a = (\alpha/(1+\alpha))LI$

$E^a = ((1-\alpha)/(1+\alpha))LI$

Each agent is entitled to an equal share of the aggregate rents paid in his city, so that we can determine R from

$$R^a=LR \Rightarrow R = (\alpha/(1+\alpha))(W+R) \Rightarrow R = \alpha W$$

Substituting back for $I = W+R = (1+\alpha)W$ we have finally:

$R^a = M^a = \alpha LW$

$E^a = (1-\alpha)LW$

2.3 The Determination of Wages.

The pattern of trade between cities that emerges is one in which each differentiated product will be produced in only one city. Market clearing for goods made in city one requires that:

$$\frac{p_1^{-\sigma}E_1^a}{n_1p_1^{1-\sigma}+ n_2(p_2/\tau)^{1-\sigma}} + \frac{(1/\tau)\ (p_1/\tau)^{-\sigma}E_2^a}{n_1(p_1/\tau)^{1-\sigma}+ n_2p_2^{1-\sigma}} = Y_1$$

On the left hand side E_1^a and E_2^a are respectively the aggregate expenditures on consumption in cities 1 and 2. The first term is the demand for a good made in city one by its own citizens; the second term is the demand for that good by city two's citizens. The analogous version for a good made in city two is:

$$\frac{(1/\tau)\ (p_2/\tau)^{-\sigma}E_1^a}{n_1p_1^{1-\sigma}+ n_2(p_2/\tau)^{1-\sigma}} + \frac{p_2^{-\sigma}E_2^a}{n_1(p_1/\tau)^{1-\sigma}+ n_2p_2^{1-\sigma}} = Y_2$$

Note that the quantities supplied Y_1 and Y_2 are equal if the technology is the same in both cities. The aggregate revenues from each city's manufactures must equate total wages paid since there's no profits in equilibrium; from (4):

$$n_1p_1Y_1 = W_1L_1 - M_1^a = (1-\alpha)L_1W_1$$
$$n_2p_2Y_2 = W_2L_2 - M_2^a = (1-\alpha)L_2W_2$$

Aggregate expenditures on consumption in each city are:

$$E_1^a = (1-\alpha)(L_1I_1 - M_1^a) = (1-\alpha)L_1W_1$$
$$E_2^a = (1-\alpha)(L_2I_2 - M_2^a) = (1-\alpha)L_2W_2$$

The market clearing equations can then be written as:

$$\frac{n_1p_1^{1-\sigma}}{n_1p_1^{1-\sigma}+ n_2(p_2/\tau)^{1-\sigma}}L_1W_1 + \frac{n_1(p_1/\tau)^{1-\sigma}}{n_1(p_1/\tau)^{1-\sigma}+ n_2p_2^{1-\sigma}}L_2W_2 = L_1W_1 \tag{6}$$
$$\frac{n_2(p_2/\tau)^{1-\sigma}}{n_1p_1^{1-\sigma}+ n_2(p_2/\tau)^{1-\sigma}}L_1W_1 + \frac{n_2p_2^{1-\sigma}}{n_1(p_1/\tau)^{1-\sigma}+ n_2p_2^{1-\sigma}}L_2W_2 = L_2W_2$$

Naturally one of the equations is superfluous by Walras law, and

any one of them is equivalent to trade balance equilibrium.

Defining the new variables ℓ and w for the ratios of population and wage rates between the cities, $\ell \equiv L_1/L_2$, $w \equiv W_1/W_2$, we can rewrite the market clearing more conveniently to get the relative wage as an implicit function of a given distribution of workers between the cities and the other parameters $\sigma>1$, $0<\tau<1$:

$$\frac{-(\tau^{1-\sigma}-w^{-\sigma})\tau^{\sigma-1}}{(\tau^{\sigma-1}-w^{-\sigma})w^{1-\sigma}} = \ell \qquad ;\ell \geq 0 \tag{7}$$

<u>Proposition 1:</u>

The wage ratio is a continuously differentiable, strictly increasing function $w(\ell)$ of the populations ratio s.t.

$w: \mathbb{R}_+ \to \mathbb{R}_+$

$$w(\ell) \equiv \mathrm{inv}(l(w)); \quad \text{where } l(w) = \frac{-(\tau^{1-\sigma}-w^{-\sigma})\tau^{\sigma-1}}{(\tau^{\sigma-1}-w^{-\sigma})w^{1-\sigma}}$$

$w(\ell) \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$

(proof in Appendix 1)

Note that the equilibrium w is independent of α or m. The additional structure of land markets and commuting costs does not interfere in the goods trade equilibrium. Also $w = w(\ell) \Leftrightarrow 1/w = w(1/\ell)$, from the symmetric treatment of cities in the model. Note that the upper bound for the equilibrium wage ratios depends positively on the elasticity σ and negatively on τ.

If there are no transport costs ($\tau=1$), equal wages ($w=1$) is an equilibrium for any relative city size ℓ, for any value of the elasticity σ. When transport costs are present though ($0<\tau<1$), relative wages are monotonically increasing with relative city sizes, a result well supported by empirical observations.

In this model transportation cost is the factor behind the rising ratio of wages with the ratio of populations (see Figure 1). From a

comparative statics standpoint, when the distribution of population changes at some initial equilibrium prices, there will be an excess demand for the goods manufactured in the city that gets bigger. Elaborating on this, suppose that one firm moves from the small to the big city. From the perspective of the moving labor there is an increase from p_2 to p_2/τ in the price of goods manufactured in the small city, and a corresponding decrease from p_1/τ to p_1 for goods made in the big city. From the perspective of the people already living in the big city, the price of the good manufactured by the moving firm decreases from p_2/τ to p_1 (since $p_1/p_2<\tau^{(1-\sigma)/\sigma}$ then $p_2/\tau< p_1$). Finally, from the perspective of the small city citizens that stayed, the price of the good manufactured by the moving firm increases from p_2 to p_1/τ. All the effects above summed, the aggregate demand for goods made in the big city must increase, while for goods made in the small city, decrease. When the wage ratio is allowed to shift, it needs to increase in order to clear the markets.

2.4 Real Incomes and the Location Choice.

Until now we have been taking the populations ratio ℓ as being exogenous and computed equilibrium prices as functnions of ℓ. Now we turn ℓ itself into an endogenous variable, letting workers freely decide to migrate between the cities. We assume that agents are indifferent about which city to live in if and only if real incomes are the same in both cities. We call equilibria the population ratios ℓ for which real incomes are the same in both cities. Among the equilibria, we call stable the ones at which an eventual migrant that is initially indifferent between cities makes it unworthy for other migrants to follow up.

City bigness contributes to real incomes by three distinct channels. Equilibrium nominal wages are relatively higher in the bigger city. Also, big city dwellers pay transport costs on a relatively

smaller fraction of all the goods they consume. On the other hand, land rents in the bigger city are relatively higher at any distance from the CBD. We construct a relative utility index as a function of the ratio ℓ that reflects the above trade off, making explicit the role of parameters α, σ, and τ.

The true price index of goods consumption is, in city i:

$P_i = [n_i p_i^{1-\sigma} + n_j(p_j/\tau)^{1-\sigma}]^{1/(1-\sigma)}$; $j = 1, 2.$

The cost of living index (including land consumption) in city i for an agent living at distance x from the CBD is:

$\Pi_i(x) = Q_i(x)^{\alpha} P_i^{\,1-\alpha} / \alpha^{\alpha}(1-\alpha)^{1-\alpha}$

Real income (indirect utility) at site x in city i can be measured by:

$U_i(x) = (I_i - m_i x)/\Pi_i(x)$

Since utility must be the same at any x, we pick x=0 to construct the relative utility index u:

$$u = U(0)/U(0) = \frac{w^{1-\alpha}\tau^{\alpha-1}}{\ell^{\alpha}}\left(\frac{\ell w^{1-\sigma}+\tau^{1-\sigma}}{\ell w^{1-\sigma}+\tau^{\sigma-1}}\right)^{(1-\alpha)/(1-\sigma)}$$

Substituting for ℓ in terms of w, we obtain the relative utility index as a function of equilibrium relative wages w:

$u(w): \mathbb{R}_+ \to \mathbb{R}_+$

$$u(w) = w^{(1-\sigma[2-\alpha])/(1-\sigma)}\left(\frac{\tau^{\sigma-1}-w^{-\sigma}}{\tau^{\sigma-1}-w^{\sigma}}\right)^{\alpha} \tag{8}$$

The function $u(\ell) \equiv u(w(\ell))$ defined jointly by (7) and (8) gives the relative utilities in the two cities for any given distribution of population between them. The equilibria are those ratios ℓ such that $u(\ell) = 1$, and the stable ones are those for which in addition, $u'(\ell)>0$. We proceed with some analysis of the function $u(\ell)$ and characterize the equilibria. Notice that if $u=u(\ell)$ then $1/u=u(1/\ell)$, as one should expect from the symmetry of the model. It is straightforward to check that, as long as $\alpha > 0$,

$\lim_{\ell\to\infty} u(\ell) = 0;$ $\qquad \lim_{\ell\to 0} u(\ell) = +\infty;$

As ℓ grows without bounds, relative utility in city one approaches

zero, what means that complete concentration of population in one city is never an equilibrium. Higher land prices sooner or later make life in the big city too expensive. It is also immediate that cities of the same size offer the same real income, $u(1)=1$. Cities with equal populations are then always in equilibrium, although it might be unstable. For $\ell>1$, $u(\ell)$ is either always less than one, or bigger than one at small ℓ, equal to one at a unique stable equilibrium, and finally less than one for all ℓ thereafter. The bottom line is:

Proposition 2:

a) There exist a unique stable long run equilibrium $\ell^*>1$ if and only if:

$$\tau < \left(\frac{(1-\alpha\sigma)(2\sigma-1)}{\alpha(\sigma-1)+(1-\alpha)(2\sigma-1)}\right)^{1/(\sigma-1)}$$

b) The symmetric equilibrium $\ell=1$ (that always exists) is stable only if there is no asymmetric equilibria.

(proof in Appendix 1)

The inequality in (a) describes the subset of parameter values leading to stable long run equilibria for cities with different population sizes. It prescribes high enough transport costs for given α and σ, or a low enough share of expenditures in housing for given τ and σ. A necessary condition for (a) to hold is that the r.h.s be strictly positive, what translates to $\alpha < 1/\sigma$, so that the share of expenditures on land can not be too high (see Figures 2 and 3). We can also show that:

Proposition 3:

If τ satisfies (a) in Proposition 1 then the equilibrium populations ratio is a strictly decreasing function of τ.

(proof in Appendix 1)

What Proposition 3 says essentially is that given preference parameters such that $\alpha<1/\sigma$ (the share of expenditures in land is not too big) then transport cost alone can explain all equilibrium relative sizes of cities (see Figure 4).

At this point we would like to add some qualification to the specific assumption we have made about the land ownership rights. If agents are restricted to own land only in the city he is currently living in, then we do not have a very realistic setting to interpret migration decisions from a dynamic standpoint. One natural alternative is to let agents be equal shareholders of the total rents paid in both cities instead. We can show though that this modification does not alter the qualitative results obtained with the first assumption, at a cost of a sensible loss in tractability (see Appendix 2).

3. Nontraded Goods and City Size

Here we develop an extension of the model in section 2 in order to illustrate how the presence of nontraded goods alter the equilibria. We include in the analysis a second industry that produces nontraded goods in each city. The notation distinguishes variables and parameters pertaining to each industry by means of "σ" or "ρ" subscripts depending on if the elasticity of substitution between goods manufactured by that industry is respectively σ or ρ. We now define as the consumption aggregate:

$$C(C_\sigma, C_\rho) = C_\sigma^{\phi} C_\rho^{1-\phi}$$

where

$$C_\sigma = [\ \textstyle\sum_{k=1}^{n_\sigma} \ c_k^{(\sigma-1)/\sigma}\ ]^{\sigma/(\sigma-1)} \quad ; \quad C_\rho = [\ \textstyle\sum_{k=1}^{n_\rho} \ c_k^{(\rho-1)/\rho}\ ]^{\rho/(\rho-1)}$$

The Cobb-Douglas preferences imply that a fraction equal to ϕ of expenditures on consumption goods is allocated to goods manufactured by industry sigma. The technological parameters are allowed to vary between industries so that the cost functions for all firms producing in each industry are:

$$X_\sigma = \ell_\sigma + \omega_\sigma Y_\sigma$$
$$X_\rho = \ell_\rho + \omega_\rho Y_\rho$$

Profit maximization then results in goods prices that are proportional to the wage rate in the city where they are manufactured:

$p_{i\sigma} = \omega_\sigma W_i \sigma/(\sigma-1)$

$p_{i\sigma} = \omega_\rho W_i \rho/(\rho-1)$

The zero profits condition determines output and employment for each firm in each industry:

$Y_\sigma = \ell_\sigma(\sigma-1)/\omega_\sigma$; $\quad X_\sigma = \ell_\sigma \sigma$;

$Y_\rho = \ell_\rho(\rho-1)/\omega_\rho$; $\quad X_\rho = \ell_\rho \rho$;

The full employment condition can then be used to determine the number of firms (goods) established (produced) in each city/industry:

$n_{1\sigma} = L_{1\sigma}/\ell_\sigma \sigma$; $\quad n_{1\rho} = L_{1\rho}/\ell_\rho \rho$;

$n_{2\sigma} = L_{2\sigma}/\ell_\sigma \sigma$; $\quad n_{2\rho} = L_{2\rho}/\ell_\rho \rho$;

Markets clear for both goods sigma and goods rho. Total demands for goods sigma and rho manufactured in cities one and two must equal their respective supplies. Then if industry sigma manufactures the traded goods and industry rho manufactures the nontraded goods, the market clearing conditions are (number subscripts refer to a specific city):

$$\frac{p_{1\sigma}^{-\sigma}\,\phi E_1^a}{n_{1\sigma}p_{1\sigma}^{1-\sigma} + n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}} + \frac{(1/\tau)\ (p_{1\sigma}/\tau)^{-\sigma}\phi E_2^a}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma} + n_{2\sigma}p_{2\sigma}^{1-\sigma}} = Y_{1\sigma} \tag{9}$$

$$\frac{(1/\tau)\ (p_{2\sigma}/\tau)^{-\sigma}\phi E_1^a}{n_{1\sigma}p_{2\sigma}^{1-\sigma} + n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}} + \frac{p_{2\sigma}^{-\sigma}\phi E_2^a}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma} + n_{2\sigma}p_{2\sigma}^{1-\sigma}} = Y_{2\sigma}$$

$$\frac{(1-\phi)E_1^a}{n_{1\rho}p_{1\rho}} = Y_{1\rho} \tag{10}$$

$$\frac{(1-\phi)E_2^a}{n_{2\rho}p_{2\rho}} = Y_{2\rho}$$

Bringing in land consumption and commuting costs, we can write for

the aggregate nominal disposable incomes in each of the two cities: [2]

$$E_1^a = (1-\alpha)W_1L_1; \qquad E_2^a = (1-\alpha)W_2L_2; \tag{11}$$

Aggregate city/industry revenues can be written as:

$$n_{1\rho}p_{1\rho}Y_{1\rho} = (1-\alpha)W_1L_{1\rho}\ ; \qquad n_{1\sigma}p_{1\sigma}Y_{1\sigma} = (1-\alpha)W_1L_{1\sigma} \tag{12}$$
$$n_{2\rho}p_{2\rho}Y_{2\rho} = (1-\alpha)W_2L_{2\rho}\ ; \qquad n_{2\sigma}p_{2\sigma}Y_{2\sigma} = (1-\alpha)W_2L_{2\sigma}$$

Substituting from (11) and (12) above into the traded goods market equilibrium conditions (9), we get:

$$\frac{n_{1\sigma}p_{1\sigma}^{1-\sigma}\phi L_1W_1}{n_{1\sigma}p_{1\sigma}^{1-\sigma}+n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}} + \frac{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}\phi L_2W_2}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}+n_{2\sigma}p_{2\sigma}^{1-\sigma}} = L_{1\sigma}W_1$$

$$\frac{n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}\phi L_1W_1}{n_{1\sigma}p_{1\sigma}^{1-\sigma}+n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}} + \frac{n_{2\sigma}p_{2\sigma}^{1-\sigma}\phi L_2W_2}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}+n_{2\sigma}p_{2\sigma}^{1-\sigma}} = L_{2\sigma}W_2$$

Substituting from (11) and (12) into the nontraded goods market equilibrium conditions (10) though we get that:

$$L_{1\rho}/L_1 = L_{2\rho}/L_2 = 1-\phi \ \Rightarrow\ L_{1\sigma}/L_1 = L_{2\sigma}/L_2 = \phi$$

So that the nontraded goods industry labor share in each city is a constant, what evidently implies the same for the traded goods. But then, since $n_{1\sigma}/n_{2\sigma} = L_1/L_2$, the market equilibrium conditions for the traded goods simplify to the form they had in (6) in section 2, what means that the relative wage ratio is still determined as a function of the ratio of city sizes as in (7).

The true price index of consumption in city i is:

$$P_i = P_{i\sigma}^{\phi}P_{i\rho}^{1-\phi}$$

---

[2]We suppose that at any distance x from the CBD the distribution of workers for the two industries is the same, so that the average time spent commuting for the workers of each industry coincides with the city average.

where,

$$P_{i\sigma} = [n_{i\sigma}p_{i\sigma}^{1-\sigma} + n_{j\sigma}(p_{j\sigma}/\tau)^{1-\sigma}]^{1/1-\sigma}$$
$$P_{i\rho} = [n_{i\rho}p_{i\rho}^{1-\rho}]^{1/1-\rho}$$

The relative utility index is defined as the ratio of utilities attained at the CBD in both cities:

$$u \equiv U_1(0)/U_2(0) = (I_1/I_2)(Q_2(0)/Q_1(0))^{\alpha}(P_2/P_1)^{1-\alpha}$$

This gives:

$$u = w^{\phi(1-\sigma[2-\alpha])/(1-\sigma)}\left(\frac{\tau^{\sigma-1}-w^{-\sigma}}{\tau^{\sigma-1}-w^{\sigma}}\right)^{\phi\alpha}\ell^{(1-\phi)(1-\alpha\rho)/(\rho-1)}$$

Notice that the ratio of wages (w) is determined in the traded goods market alone. Then if $u(\ell)$ is the function studied in Section 2 that gives the ratio of utilities at any ratio of populations (ℓ), we can define an analogous function $u_n(\ell)$ incorporating nontraded goods, with the form:

$$u_n(\ell): \mathbb{R}_+ \rightarrow \mathbb{R}_+$$
$$u_n(\ell) = u(\ell)^{\phi}\,\ell^{(1-\phi)(1-\alpha\rho)/(\rho-1)}$$

It is easy to see that when all goods are nontraded (what amounts to make $\phi=0$ above), then $\ell=1$ is the only possible equilibrium if $\alpha\sigma\neq1$. Moreover, it is stable if $\alpha\sigma>1$ and unstable if $\alpha\sigma<1$ (when a small deviation leads to the disappearance of one city). For $0<\phi<1$ the following proposition describes how the equilibria compares with the case when only traded goods are present.

<u>Proposition 4:</u>

Supposing that $\alpha\sigma<1$ (so that asymmetric equilibria may exist with only traded goods), if also $\alpha\rho<1$ ($\alpha\rho>1$) then an equilibrium with both traded and nontraded goods ($\phi<1$) has a maximum populations ratio greater (smaller) or equal than that of an equilibrium with only traded goods ($\phi=1$).

(proof in Appendix 1)

Proposition 4 states that if the share of expenditures on land (α) is small enough, then the presence of nontraded goods tends to

exacerbate the concentration of population in larger cities. In particular this is the case if the price elasticity of demand for nontraded goods is smaller or equal than that for traded goods (i.e., $\rho\leq\sigma$). It is also important to note that the presence of nontraded goods brings up the possibility of complete concentration of population in one city. We can see this by rewriting $u_n(\ell)$ as:

$$u_n(\ell)=w^{(1-\alpha)(2\sigma-1)/(\sigma-1)}\ell^{-\alpha+((1-\phi)(1-\alpha\rho)/\phi(\rho-1))}$$

Now, since $w(\ell)$ is bounded, then

$$\lim_{\ell\to+\infty} u_n(\ell) >1 \iff \alpha< (1-\phi)/(\rho-\phi)$$

Thus we can have complete concentration of population if the share of expenditures on land (α) is too small or the elasticity of demand for nontraded goods is too small or finally if the share of expenditures on nontraded goods is too big.

4. Industrial Concentration and City Sizes

In this section we show that an extension of the model allowing for two industries that manufacture traded goods brings important insights about the way in which labor allocates between the industries in each city as a function of the population distribution between them. Here again the two industries sigma and rho differ by the different elasticities of demand (in trade equilibrium) for the goods each one manufactures. We may also let technological parameters vary but those will not influence the results. The notation is the same as in Section 3, where "σ" and "ρ" subscripts distinguish between different industries and between goods manufactured by each one. By convention we make, without loss of generality, $\sigma>\rho$.

Preferences and technologies are the same as described in Section 3. Consequently, the implications for prices and number of firms in each city and industry are also the same as before. The difference here is that goods manufactured by both industries are traded, so that the market clearing conditions in their nominal version (by substituting

(18) and (19) into the real versions) are changed to the following:

$$\frac{\phi\, n_{1\sigma}p_{1\sigma}^{1-\sigma}\, L_1W_1}{n_{1\sigma}p_{1\sigma}^{1-\sigma}+n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}}+\frac{\phi\, n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}\, L_2W_2}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}+n_{2\sigma}p_{2\sigma}^{1-\sigma}}=L_{1\sigma}W_1 \tag{13}$$

$$\frac{(1-\phi)\, n_{1\rho}p_{1\rho}^{1-\rho}\, L_1W_1}{n_{1\rho}p_{1\rho}^{1-\rho}+n_{2\rho}(p_{2\rho}/\tau)^{1-\rho}}+\frac{(1-\phi)\, n_{1\rho}(p_{1\rho}/\tau)^{1-\rho}\, L_2W_2}{n_{1\rho}(p_{1\rho}/\tau)^{1-\rho}+n_{2\rho}p_{2\rho}^{1-\rho}}=L_{1\rho}W_1$$

$$\frac{\phi\, n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}\, L_1W_1}{n_{1\sigma}p_{1\sigma}^{1-\sigma}+n_{2\sigma}(p_{2\sigma}/\tau)^{1-\sigma}}+\frac{\phi\, n_{2\sigma}p_{2\sigma}^{1-\sigma}\, L_2W_2}{n_{1\sigma}(p_{1\sigma}/\tau)^{1-\sigma}+n_{2\sigma}p_{2\sigma}^{1-\sigma}}=L_{2\sigma}W_2$$

$$\frac{(1-\phi)\, n_{2\rho}(p_{2\rho}/\tau)^{1-\rho}\, L_1W_1}{n_{1\rho}p_{1\rho}^{1-\rho}+n_{2\rho}(p_{2\rho}/\tau)^{1-\rho}}+\frac{(1-\phi)\, n_{2\rho}p_{2\rho}^{1-\rho}\, L_2W_2}{n_{1\rho}(p_{1\rho}/\tau)^{1-\rho}+n_{2\rho}p_{2\rho}^{1-\rho}}=L_{2\rho}W_2$$

We are interested in predictions regarding the values for the shares of labor employed in each industry and in each city, defined by:

$\ell_{1\sigma} \equiv L_{1\sigma}/L_1$; $\ell_{1\rho} \equiv L_{1\rho}/L_1$;
$\ell_{2\sigma} \equiv L_{2\sigma}/L_2$; $\ell_{2\rho} \equiv L_{2\rho}/L_2$;

After some steps of algebra we can get the following relations from the market clearing equations:

$$\ell_{1\sigma} = \phi[\tau^{1-\sigma}(\tau^{1-\sigma}-w^{-\sigma})^{-1}+\tau^{\sigma-1}(w\ell)^{-1}(\tau^{\sigma-1}-w^{-\sigma})^{-1}]$$
$$\ell_{1\rho} = (1-\phi)[\tau^{1-\rho}(\tau^{1-\rho}-w^{-\rho})^{-1}+\tau^{\rho-1}(w\ell)^{-1}(\tau^{\rho-1}-w^{-\rho})^{-1}]$$
$$\ell_{2\sigma} = -\phi w^{-\sigma}[(\tau^{\sigma-1}-w^{-\sigma})^{-1}+w\ell(\tau^{1-\sigma}-w^{-\sigma})^{-1}]$$
$$\ell_{2\rho} = -(1-\phi)w^{-\rho}[(\tau^{\rho-1}-w^{-\rho})^{-1}+w\ell(\tau^{1-\rho}-w^{-\rho})^{-1}]$$

Notice that the technological parameters have no influence whatsoever in the determination of the industry labor shares. By construction we have that $\ell_{1\sigma}+\ell_{1\rho}=1$ and $\ell_{2\sigma}+\ell_{2\rho}=1$. Either one of those can give us the trade equilibrium relative wage ratio.

$$- \frac{\phi(\tau^{\sigma-1}-w^{-\sigma})^{-1}\tau^{\sigma-1} + (1-\phi)(\tau^{\rho-1}-w^{-\rho})^{-1}\tau^{\rho-1}}{\phi(\tau^{1-\sigma}-w^{-\sigma})^{-1}w^{1-\sigma} + (1-\phi)(\tau^{1-\rho}-w^{-\rho})^{-1}w^{1-\rho}} = \ell \tag{14}$$

This relation is the analog version of (7) in section 2. It gives the ratio of wages w implicitly as a function of the ratio of populations ℓ and the other parameters ϕ, σ, ρ, τ.

Proposition 5:

The equilibrium ratio of wages is a continuously differentiable, strictly increasing function $w(\ell)$ of the populations ratio s.t.:

$w: \mathbb{R}_+ \to \mathbb{R}_+$

$w(\ell) \equiv \mathrm{inv}(l(w))$;

where,

$$l(w) = - \frac{\phi(\tau^{\sigma-1}-w^{-\sigma})^{-1}\tau^{\sigma-1} + (1-\phi)(\tau^{\rho-1}-w^{-\rho})^{-1}\tau^{\rho-1}}{\phi(\tau^{1-\sigma}-w^{-\sigma})^{-1}w^{1-\sigma} + (1-\phi)(\tau^{1-\rho}-w^{-\rho})^{-1}w^{1-\rho}}$$

$w(\ell) \in (\tau^{(\rho-1)/\rho}, \tau^{(1-\rho)/\rho})$

(proof in Appendix 1)

Notice that the range for equilibrium wage ratios is now determined by the smaller of the elasticities of substitution ρ. The values $w=w(\ell)$ may be substituted back into the expressions for $\ell_{1\sigma}$, $\ell_{1\rho}$, $\ell_{2\sigma}$, $\ell_{2\rho}$ to make them functions of any exogenous ratio of populations. Then we can define the functions:

$\ell_{1\sigma} = l_{1\sigma}(\ell)$; $\quad \ell_{1\rho} = l_{1\rho}(\ell)$; $\quad \ell_{2\sigma} = l_{2\sigma}(\ell)$; $\quad \ell_{2\rho} = l_{2\rho}(\ell)$;

We can use these functions to derive restrictions for the equilibrium ratios of populations. Note that because the industry labor shares add to 1 in any city and the model treats both cities symmetrically, the behavior of any one ratio immediately determines that of the others. We choose to focus on the analysis of $l_{2\sigma}(\ell)$.

Imposing nonnegativity on the industry labor shares result in the following restriction for the distribution of population:

Proposition 6:

There is a maximum (minimum) ratio of populations ℓ^* $(1/\ell^*)$ that can be an equilibrium.
(proof in Appendix 1)

From Proposition 6 we learn that, contrary to the one industry case, trade balance with two industries imposes bounds on the range of possible equilibrium ratios of populations.

The next proposition gives some additional characterization for the industry labor shares as functions of the ratio of populations and of the parameters ϕ, σ, ρ, and τ. If the cities are of equal size then the industry labor shares coincide with the industry shares of expenditures in consumption, that is:

$$l_{1\sigma}(1) = l_{2\sigma}(1) = \phi$$
$$l_{1\rho}(1) = l_{2\rho}(1) = 1-\phi$$

For asymmetric distributions of population we can say the following:

Proposition 7:

a) $l_{2\sigma}(\ell)=0$ for some ℓ sufficiently big.

b) For $0<\hat{\tau}<1$ such that $\sigma/\rho=(1-\hat{\tau}^{\sigma-1})/(1-\hat{\tau}^{\rho-1})$ then:

$\tau\geq\hat{\tau} \Rightarrow l_{2\sigma}(\ell) < \phi$ for all $\ell>1$

$\tau<\hat{\tau} \Rightarrow \exists!\ \ell^*$ s.t.: $\ell<\ell^* \Rightarrow l_{2\sigma}(\ell)>\phi$; $l_{2\sigma}(\ell^*)=\phi$; $\ell>\ell^* \Rightarrow l_{2\sigma}(\ell)<\phi$.

(proof in Appendix 1)

Item (a) of proposition 7 says that if the small city is smaller enough it will specialize completely in the industry rho. Item (b) says that if transport costs are small enough then the industry sigma labor share is less than ϕ in the small city, no matter how much smaller it is. Otherwise, for higher transport costs, industry sigma labor share in the small city will be bigger than ϕ while it is not too much smaller, and finally less than ϕ for bigger ratios of population.

Our intuitive explanation for these results is based on the different responses of the relative demands between cities for goods of different industries, as the price ratio between cities change. We resort again to a comparative statics reasoning. When at some initial trade equilibrium people move from the small to the big city, from their perspective the relative price between city one and city two for both industries decreases by a factor of τ^2 (from $p_{1i}/\tau p_{2i}$ to $\tau p_{1i}/p_{2i}$ for industries $i=\sigma,\rho$). Since the elasticities of substitution differ ($\sigma>\rho$), the excess demand for goods manufactured in the big city is greater for industry sigma than for industry rho, after other effects also generated by the migration (price changes from the perspective of people other than the migrants) are accounted for. When the relative wage between the cities is allowed to move, this is not enough in general to clear the markets for both industries, so that a shift of labor from one industry to another into each city may be necessary. The direction and extent of these reallocations of labor between industries must be a function of the size of the change in the wage ratio. Since goods sigma respond faster to prices, a large increase in relative wages can bring an excess supply for the industry sigma (and a respective excess demand for industry rho) in the big city and a corresponding excess demand for industry sigma (and a respective excess supply for industry rho) in the small city before labor shifts between industries in each city, so that industry sigma has to shrink in the big city and expand in the small city, and vice-versa for industry rho. If the relative wage change is small though, then industry rho will be in excess supply in the big city and in excess demand in the small city before labor moves, so that industry sigma expands in the big city and shrinks in the small.

Both the initial relative size of the cities and how high are transportation cost can influence the size of the change in the wage ratio that ensues migration. The greater is the initial population ratio ℓ the smaller is the shift in relative wages for a unit increase in it (if ℓ is big then it takes only a few migrants to increase it by one unit), so that eventually industry sigma disappears from the smaller

city. Also the higher are transportation costs (the smaller τ), the greater is the excess demand for (all) the goods manufactured in the big city (and the corresponding excess supply of goods manufactured in the small city) at the initial price ratio, and thus the greater must be the change in the wage ratio.

Propositions 6 and 7 do not completely characterize the behavior of the industries shares of labor though. Other important restrictions for the range of possible equilibria may arise that are illustrated by some simulations ahead.

As we did before in Section 2, we now add the assumption of free mobility of labor between cities and try to determine those population distributions that make people indifferent between living in either city. Price indexes of consumption for each industry in city i are:

$$P_{i\sigma} = [n_{i\sigma}p_{i\sigma}^{1-\sigma} + n_{j\sigma}(p_{j\sigma}/\tau)^{1-\sigma}]^{1/(1-\sigma)}$$
$$P_{i\rho} = [n_{i\rho}p_{i\rho}^{1-\rho} + n_{j\rho}(p_{j\rho}/\tau)^{1-\rho}]^{1/(1-\rho)}$$

The combined price index of consumption of all manufactured goods then is, in city i:

$$P_i = P_{i\sigma}^{\phi} P_{i\rho}^{1-\phi}$$

The relative utility index is defined as the ratio of utilities attained at the CBD for both cities:

$$u \equiv U_1(0)/U_2(0) = (I_1/I_2)(Q_2(0)/Q_1(0))^{\alpha}(P_2/P_1)^{1-\alpha}$$

Substituting from above, and having in mind that $w=w(\ell)$:

$$u = u(\ell) = w\tau^{\alpha-1}(w\ell)^{-\eta}\left(\frac{\tau^{1-\sigma}-w^{-\sigma}}{w^{-\sigma}-\tau^{\sigma-1}}\right)^{\frac{(1-\alpha)\phi}{(1-\sigma)}}\left(\frac{\tau^{1-\rho}-w^{-\rho}}{w^{-\rho}-\tau^{\rho-1}}\right)^{\frac{(1-\alpha)(1-\phi)}{(1-\rho)}}$$

where $\eta = \alpha+(1-\alpha)[\phi/(1-\sigma)+(1-\phi)/(1-\rho)]$.

It should be stressed that we are only interested in computing $u(\ell)$ for those ratios ℓ that generate meaningful equilibria, that is, those that generate nonnegative industry labor shares.

We want to characterize the population ratios ℓ such that $0 \le l_{2\sigma}(\ell) \le 1$ and $u(\ell)=1$. Of course $\ell=1$ is such an equilibrium. We also know that $u(\ell) \le 1$ for ℓ big enough such that $0 \le l_{2\sigma}(\ell) \le 1$. Nevertheless we

can not be sure anymore that one can find such a ℓ, so that if $u'(1)>0$ it is possible that one of the cities will concentrate all the population, since the ratio ℓ at which utilities would be equalized in the two cities may be one at which some industry labor share is necessarily negative.

The analytical characterization of the boundaries for the sets of parameters that generate meaningful equilibria seems to be a difficult task though, so that we resort to some illustrative simulations. The simulations reveal that the nonnegativity restrictions on the industry labor shares may reduce dramatically the set of population distributions.

Two benchmarks are shown in Figure 5 and Figure 6, for low and high elasticities. In the case of high elasticities it is possible that $l_{2\sigma}(\ell)$ shoots over 1 (see Figure 6), bringing additional restrictions for the range of possible equilibrium population ratios. Figures 7 and 8 illustrate, respectively for the low and high values of elasticities used in Figures 5 and 6, how the equilibrium ratios of population and the concentrations (at those equilibria) of the industry sigma in city two vary for different values of the transportation cost parameter (τ). In the Figures 7 and 8 industry concentration is defined as the ratio between the industry share of the city's labor force and that same industry share of the total labor force.

Since the implications for industrial concentrations depend on preference parameters, one could ask the question of how to identify the industries by observable technological parameters. In equilibrium though, the ratio of average cost to marginal cost for a firm in industry sigma (rho) must equal $\sigma/(\sigma-1)$ $((\rho/(\rho-1))$. Thus observed measures of elasticities of scale may be used to test the implications of the model.

5. A Model with Three Cities

In this section we include in the analysis a third city in order to illustrate the workings of the intervening forces in favor or against agglomeration as the number of cities increases.

The previous discussion stressed transportation costs both intra city (commuting costs) and inter city (of goods) as the key elements in determining city sizes. Commuting costs generate increasing land prices with city size, thus pushing for dispersion of population over smaller cities. Costs to transport goods make consumption relatively cheaper in larger cities, where a relatively larger share of the marketed consumption goods are produced at home, thus leading to concentration of population in only one large city.

In the framework of previous sections these forces balance out in a equilibrium distribution of population between two cities. As the number of cities increases though, we should expect a tendency for dissipation of the incentive for concentration while the forces for dispersion should not change much. This is because the differences in the cities shares of the total number of goods available for consumption tend to diminish as the number of cities increase. The increase in the number of cities then makes the effect of the ratio of populations in any two cities to lose importance in the determination of the relative price of consumption between the cities, while its effect on the determination of the relative price of land should be pretty much unchanged (there should be some change in relative wages as well).

In this section we try to illustrate the effects alluded to above by exploring an extension of our previous framework that includes a third city. We show that if transportation costs among the three cities are the same, then there is no equilibrium in which the three cities are of different sizes. Also we show that equilibria in a three city system have always a smaller or equal maximum ratio of populations than comparable equilibria (given the same values of the parameters) in a two city system.

We suppose that preferences, technologies and land use are the same as described in Section 2 for the basic model. We keep restricting people to be equal shareholders of land in their own city only.

In a three city model, the market clearing condition for any good made in city i is (i, j, k, for cities):

$$\frac{p_i^{\ \sigma} E_i^a}{n_i p_i^{1-\sigma} + n_j (p_j/\tau_{ij})^{1-\sigma} + n_k (p_k/\tau_{ik})^{1-\sigma}} +$$

$$+ \frac{(1/\tau_{ij}) \quad (p_i/\tau_{ij})^{-\sigma} E_j^a}{n_i (p_i/\tau_{ij})^{1-\sigma} + n_j p_j^{1-\sigma} + n_k (p_k/\tau_{jk})^{1-\sigma}} +$$

$$+ \frac{(1/\tau_{ik}) \quad (p_i/\tau_{ik})^{-\sigma} E_k^a}{n_i (p_i/\tau_{ik})^{1-\sigma} + n_j (p_j/\tau_{jk})^{1-\sigma} + n_k p_k^{1-\sigma}} = Y_i$$

where τ_{ij} is transportation cost between cities i and j, p_i is price in city i, n_i is the number of goods manufactured in city i, E_i^a is the aggregate expenditures in consumption in city i, and Y_i is the supply of a good manufactured in city i (we suppose that technology is the same in both cities so that Y_i=Y_j). After some steps of tedious algebra we are able to get to the expressions linking the relative populations and relative wage rates in the cities (see Appendix 3). In the case of equal transportation costs among the cities those are:

$$\frac{\{(1+ \tau^{1-\sigma}) w_1^{\sigma} - w_2^{\sigma} - 1\} w_1^{\sigma-1}}{(1+ \tau^{1-\sigma}) - w_1^{\sigma} - w_2^{\sigma}} = \ell \tag{15}$$

$$\frac{\{(1+ \tau^{1-\sigma}) w_2^{\sigma} - w_1^{\sigma} - 1\} w_2^{\sigma-1}}{(1+ \tau^{1-\sigma}) - w_1^{\sigma} - w_2^{\sigma}} = \ell$$

where the number subscripts refer to ratios of variables between the city indicated and city three, as defined by:

$w_i \equiv W_i/W_3$, $\ell_i \equiv L_i/L_3$; $i=1,2$.

The relative utility index is also defined in terms of the indirect utility in city three.

$u_i = U_i/U_3 = (I_i/I_3)(Q_3(0)/Q_i(0))^{\alpha}(P_3/P_i)^{1-\alpha}$; $i=1,2$

The above can be written as:

$u_i = \ell_i^{-\alpha} w_i^{(1-\alpha)(2\sigma-1)/(\sigma-1)}$; $i=1,2$

So that when there are no incentives to migrate ($u_i=1$) we must have that:

$w_1 = \ell_1^{\beta}$

$w_2 = \ell_2^{\beta}$

where $\beta \equiv \alpha(\sigma-1)/(1-\alpha)(2\sigma-1)$

Substituting those back into the wage equations (15), we get two correspondences that must be simultaneously satisfied by the equilibrium ratios of population. Those can be rearranged to read as:

$$\ell_1 = \left\{ \frac{(1+\tau^{1-\sigma})(\ell_2^{\beta\sigma} - \ell_2^{1+\beta(1-\sigma)}) + \ell_2^{1+\beta} - 1}{(1-\ell_2^{1+\beta(1-\sigma)})} \right\}^{1/\beta\sigma} ;\text{for } \ell_2 \neq 1 \qquad (16)$$

$\ell_1 = \mathbb{R}$;for $\ell_2=1$

and,

$$\ell_2 = \left\{ \frac{(1+\tau^{1-\sigma})(\ell_1^{\beta\sigma} - \ell_1^{1+\beta(1-\sigma)}) + \ell_1^{1+\beta} - 1}{(1-\ell_1^{1+\beta(1-\sigma)})} \right\}^{1/\beta\sigma} ;\text{for } \ell_1 \neq 1$$

$\ell_2 = \mathbb{R}$;for $\ell_1=1$

Note the symmetry of the expressions above. Obviously $\ell_1=\ell_2=1$ is a solution. The study of the correspondences above bring up the following characterization of equilibria:

<u>Proposition 8:</u>

a) There is no equilibrium in which the three cities have different sizes.

b) A necessary and sufficient condition for exactly 2 asymmetric equilibria to exist, the first with two large and one small city, the

other with one large and two small cities is that:

$$\tau < \left\{\frac{1-\beta(2\sigma-1)}{1+\beta(\sigma+1)}\right\}^{1/(\sigma-1)}$$

c) If the condition in (b) is not attended then only equilibria with two small and one big city may survive, if τ is not too large. If τ is large enough though, then there is no equilibrium other than the trivial completely symmetric one $\ell_1=\ell_2=1$.

(proof in Appendix 1)

This Proposition says basically two things. One is that, as in the case of the two city model, high enough transportation costs are necessary for any asymmetric equilibrium at all to exist. The second is that the presence of a third city dilutes the incentive for concentration as much as to preclude complete diversity of city sizes in equilibrium. The next proposition gives another illustration.

Proposition 9:

For given values of the parameters α, σ, and τ, the maximum equilibrium ratio of populations in the three city model is smaller or equal than the corresponding maximum equilibrium ratio of populations in the two city model.

(proof in Appendix 1)

This confirms the sense in which the presence of a third city influences spatial concentration. Whatever be the total absolute populations, in a three city system the ratio between the populations of the larger and the smaller cities in equilibrium can not exceed that same ratio in a two city system.

The results described lead us to speculate that, if transportation costs are independent of geographical distance or any city specific factors, then the spatial agglomeration incentives as modeled here would imply an increasingly homogeneous distribution of population as the number of cities increases. Richer results can nevertheless be obtained

by letting transportation costs vary between different pairs of cities, object of future work.

6. Conclusion and Comments

The framework we employed to study urban concentration of economic activity embodies two fundamental antagonic incentives for spatial location of labor as the only freely mobile factor of production. On one side, costs to transport goods make them cheaper at their production site, on the other, spatial congestion makes living at close quarters more expensive. Variations of the specific nature of trade as it interacts with the balance of those forces generate all the results.

The basic model we explore in Section 2 leads us to two main conclusions: first, if all goods are traded, complete concentration of population in one city is not an equilibrium if people spend any amount at all on land; second, that there is much latitude for equilibria with unequal populations, and in this case regional divergence increases with transport costs.

The treatment of nontraded goods in Section 3 reveals some room for regional divergence (or concentration of population in only one city) as nontraded goods get more "important" in the consumption basket. In addition we are able to state conditions under which the presence of nontraded goods can either exacerbate or diminish spatial concentration.

The two industry model of Section 4 where all goods are traded and demand elasticities are different for different industries explores a consequence of the increasing wage ratio in the ratio of populations. It adds results in mainly two directions. First it indicates other possibilities of regional divergence, when the only combinations of parameters that equate utilities in the cities can not guarantee market clearing. We also learn that the industry labor shares can vary sensibly between cities, especially if the equilibrium distribution of population is markedly uneven. It is found that for low transportation costs the

industry with the high elasticity of demand tends to vanish from the small city as it gets smaller. A possible criticism of the analysis is that the results are strictly demand driven, any technological differentiation between industries being innocuous in the characterization of equilibria. Finally, from the analysis in Section 5 of a model with three cities we get indications that incentives for regional concentration get diluted as the number of potential sites for location increases. We can conclude that equilibrium population distributions present less disparities of city sizes with three than with two cities. The results also indicate that this framework of analysis can not explain much variation in city sizes as the number of cities increases, unless transportation costs can vary between different pairs of cities.

APPENDIX 1

Proofs of Propositions

Proof of Proposition 1:

$l(w) > 0 \Leftrightarrow w \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$

$w \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma}) \Rightarrow l'(w) > 0$, so that a strictly increasing inverse of $l(w)$ exists. Furthermore:

$$\lim_{w \to \tau^{(\sigma-1)/\sigma}} l(w) = 0$$

$$\lim_{w \to \tau^{(1-\sigma)/\sigma}} l(w) = +\infty$$

□

Proof of Proposition 2:

Since $w(\ell)$ is strictly monotonic, the qualitative behavior of $u(\ell)$ can be inferred by that of $u(w) \equiv u(l(w))$. Thus , we are particularly interested in the occurrence of multiple roots for $u(w)=1$. The function $u(w)$ is continuously differentiable on the domain $(\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$. Furthermore, $u(1)=1$ and

$$\lim_{w \to \tau_{+}^{(\sigma-1)/\sigma}} u(w) = +\infty$$

$$\lim_{w \to \tau_{-}^{(1-\sigma)/\sigma}} u(w) = 0$$

A necessary condition for the existence of multiple roots for $u(w)=1$ is that $u'(w)=0$ for some $w \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$.

$$u'(w) = 0 \Leftrightarrow w^{\sigma} + w^{-\sigma} = \frac{\sigma[2-\alpha(2\sigma-1)]-1}{\sigma(2-\alpha\sigma)-1}\tau^{1-\sigma} + \frac{\sigma(2-\alpha)-1}{\sigma(2-\alpha\sigma)-1}\tau^{\sigma-1}$$

The function $z(w) \equiv w^{\sigma}+w^{-\sigma}$, defined for $w \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$, $\sigma>1$, is strictly convex and has a minimum at $w=1$ where $z(1)=2$. Also,

$$z(\tau^{(\sigma-1)/\sigma}) = z(\tau^{(1-\sigma)/\sigma}) = \tau^{\sigma-1}+\tau^{1-\sigma}$$

Thus we can find exactly two solutions, $w^{*} \neq 1$ and $1/w^{*}$, for $u'(w)=0$, $w \in (\tau^{(\sigma-1)/\sigma}, \tau^{(1-\sigma)/\sigma})$ iff

$$2 < \frac{\sigma[2-\alpha(2\sigma-1)]-1}{\sigma(2-\alpha\sigma)-1}\tau + \frac{\sigma(2-\alpha)}{\sigma(2-\alpha\sigma)-1}\tau < \tau^{1-\sigma} + \tau^{\sigma-1}$$

Noting that the coefficients of τ in the middle term add up to 2, the above can be written as:

$$\tau^{(\sigma-1)} < \frac{\sigma[2-\alpha(2\sigma-1)]-1}{\sigma(2-\alpha)-1} < 1$$

The second inequality is automatically satisfied since $\sigma>1$. A sufficient condition for multiple roots of $u(w)=1$ is that $u'(1)>0$:

$$u'(1) > 0 \Leftrightarrow \tau^{(\sigma-1)} < \frac{\sigma[2-\alpha(2\sigma-1)]-1}{\sigma(2-\alpha)-1}$$

But this coincides with the necessary condition stated before. Now since there is only one w^*, there is a unique root $w^{**}\in(\tau^{(\sigma-1)/\sigma},\tau^{(1-\sigma)/\sigma})$ such that $u(w^{**})=1$

Note: the last term in the r.h.s. above can also be written as:

$$\frac{(1-\alpha\sigma)(2\sigma-1)}{\alpha(\sigma-1)+(1-\alpha)(2\sigma-1)}$$

□

Proof of Proposition 3:

In equilibrium, $u(w,\alpha,\sigma,\tau)=1$.

$\partial u/\partial\tau|_{u=1} = \alpha(\sigma-1)((1-\tau^{1-\sigma}w^{-\sigma})^{-1}+ (1-\tau^{1-\sigma}w^{\sigma})^{-1}) < 0$

$\partial u/\partial w|_{u=1} < 0$ (see Proposition 1).

Then $\partial w/\partial\tau|_{u=1}< 0$; but $w(\ell)$ is a strictly increasing function.

□

Proof of Proposition 4:

Suppose $\ell^* >1$ is an equilibrium with only traded goods so that $u(\ell^*)=1$, $u(\ell)>1$ if $1<\ell<\ell^*$ and $u(\ell)<1$ if $\ell^*<\ell$. If $\alpha\rho<1$ then $u_n(\ell)>1$ for all $0<\ell\leq\ell^*$, so that for any ℓ_n^* s.t. $u_n(\ell_n^*)=1$, by continuity of u we must have that $\ell^*<\ell_n^*$. If $\alpha\rho>1$ then $u_n(\ell)<1$ for all $\ell\geq\ell^*$, so that for any ℓ_n^* s.t. $u_n(\ell_n^*)=1$, by continuity of u we must have $\ell_n^*<\ell^*$.

If $\ell^*=1$ is the only equilibrium with only traded goods then $0<u(\ell)<1$ for all $\ell>1$, but still, for all $\ell>1$, if $\alpha\rho<1$ then $u_n(\ell)>u(\ell)$ and if $\alpha\rho>1$ then $u_n(\ell)<u(\ell)$. □

Proof of Proposition 5:

$l(w) > 0 \Leftrightarrow w \in \mathbb{B}$, where $\mathbb{B}$ is defined as:

$\mathbb{B} \equiv \{w \in \mathbb{R}_+ : w \in (\tau^{(\sigma-1)/\sigma}, 1/v) \cup (\tau^{(\rho-1)/\rho}, \tau^{(1-\rho)/\rho}) \cup (v, \tau^{(1-\sigma)/\sigma})\}$

and $v \in (\tau^{(1-\rho)/\rho}, \tau^{(1-\sigma)/\sigma})$ is the solution to:

$$\phi/(1-\phi) = -(1-\tau^{(1-\sigma)}v^{-\sigma})/(1-\tau^{(1-\rho)}v^{-\rho})$$

But,

$w \in (v, \tau^{(1-\sigma)/\sigma}) \Rightarrow l_{2\rho}(w) < 0$

$w \in (\tau^{(\sigma-1)/\sigma}, 1/v) \Rightarrow l_{1\rho}(w) < 0$, so that the only interesting domain left is the interval:

$w \in (\tau^{(\rho-1)/\rho}, \tau^{(1-\rho)/\rho})$

Now, $l'(w) > 0$ for $w \in (\tau^{(\rho-1)/\rho}, \tau^{(1-\rho)/\rho})$, so that a strictly increasing inverse exists. Furthermore,

$$\lim_{w \to \tau^{(\sigma-1)/\sigma}} l(w) = 0$$

$$\lim_{w \to \tau^{(1-\sigma)/\sigma}} l(w) = +\infty$$

□

Proof of Proposition 6:

Given that $l_{2\sigma}(1) = \phi$ and $\lim_{\ell \to +\infty} l_{2\sigma}(\ell) = -\infty$, then since $l_{2\sigma}(\ell)$ is continuous there exists ℓ^{**} such that $l_{2\sigma}(\ell^{**}) = 0$. Take ℓ^{*} to be the sup of the set $\{\ell^{**} : l(\ell^{**}) = 0\}$

□

Proof of Proposition 7:

Item (a) is a direct consequence of Proposition 6. We prove (b):

$l_{2\sigma}(l(w)) = 0 \Leftrightarrow (1-\tau^{1-\sigma}w^{-\sigma})(1-\tau^{1-\rho}w^{\rho}) = (1-\tau^{1-\sigma}w^{\sigma})(1-\tau^{1-\rho}w^{-\rho}) \Leftrightarrow P(w) = 0$

where,

$$P(w) \equiv \tau^{1-\sigma}w^{2\sigma} - \tau^{1-\rho}w^{\sigma+\rho} - \tau^{2-\sigma-\rho}w^{2\sigma-\rho} + \tau^{2-\sigma-\rho}w^{\rho} + \tau^{1-\rho}w^{\sigma-\rho} - \tau^{1-\sigma}$$

The Descartes' rule of signs[3] says that the polynomial above has either 1 or 3 positive roots. Now, one of the roots must be $\ell=1$;

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[3] See Appendix 4.

furthermore, if $\ell^*\neq 1$ is a root then $1/\ell^*$ is the third one. Then since $\lim_{\ell\to+\infty} l_{2\sigma}(\ell)=-\infty$, $\lim_{\ell\to 0} l_{2\sigma}(\ell)=-\phi/(\tau^{(\sigma-1)+\sigma(\rho-1)/\rho}-1)>0$ and $l_{2\sigma}(\ell)$ is continuous, then $\ell^*\neq 1$ is a root iff $\hat{l}_{2\sigma}(1)>0$. But $l_{2\sigma}(1)>0 \Leftrightarrow \sigma/\rho>(1-\tau^{\sigma-1})/(1-\tau^{\rho-1})$. Finally, a threshold $\hat{\tau}$ can always be found since: $\lim_{\tau\to 0}(1-\tau^{\sigma-1})/(1-\tau^{\rho-1})=1$; $\lim_{\tau\to 1}(1-\tau^{\sigma-1})/(1-\tau^{\rho-1})=(\sigma-1)/(\rho-1)>\sigma/\rho$, and $(1-\tau^{\sigma-1})/(1-\tau^{\rho-1})$ is strictly increasing in $\tau\in(0,1)$.

□

Proof of Proposition 8:

The system of equations (16) can be represented, for $\ell_1,\ell_2\neq 1$, as:

$\ell_1=f(\ell_2)$

$\ell_2=f(\ell_1)$

where the function f is defined as:

$$f(\ell) \equiv \left\{\frac{(1+\tau^{1-\sigma})(\ell^{\beta\sigma}-\ell^{1+\beta(1-\sigma)})+\ell^{1+\beta}-1}{(1-\ell^{1+\beta(1-\sigma)})}\right\}^{1/\beta\sigma}$$

We begin by giving some characterizations for the function $f(\ell)$.

Because of the symmetry of the system (16), if there is no solution for $f(\ell)=\ell$ then there is no equilibrium other than the obvious one $\ell_1=\ell_2=1$. Solutions different than unity of $f(\ell)=\ell$ coincide with solutions different than unity of $p(\ell)=0$ for $p(\ell)$ as defined below:

$$p(\ell)\equiv 2\ell^{1+\beta}-(1+\tau^{1-\sigma})\ell^{1+\beta(1-\sigma)}+\tau^{1-\sigma}\ell^{\beta\sigma}-1;\quad \ell\in\mathbb{R}$$

$$\forall\ \ell\neq 1,\ p(\ell)=0 \Leftrightarrow f(\ell)=\ell$$

The ranking of the powers in $p(\ell)$ is important in determining the number of roots of $p(\ell)=0$ by the Descartes' rule of sign. The a priori restrictions $1+\beta>0$, $1+\beta>1+\beta(1-\sigma)$, and $\beta\sigma>0$ though leave us with only one ranking possibility, namely: $1+\beta > 1+\beta(1-\sigma) > \beta\sigma$, such that there is more than just one change of sign in the coefficients of $p(\ell)$, what would indicate that no solution different than unity for $p(\ell)=0$ existed, since $\ell=1$ is always one of the solutions. So we conclude that the condition $1+\beta(1-\sigma) > \beta\sigma$ is necessary for any asymmetric equilibria to exist. It is worth noting also that the inequality $1+\beta(1-\sigma) > \beta\sigma$ is equivalent to $\alpha < 1/\sigma$ after substituting for β, the same necessary

condition for the existence of an asymmetric equilibrium that we found for the model with two cities.

We suppose from now on that $\alpha< 1/\sigma$, so that $p(\ell)=0$ has either two or no solutions different than unity.

Solutions different than unity for $f(\ell)=1$ coincide with solutions different than unity for $q(\ell)=0$ for $q(\ell)$ defined by:

$$q(\ell)\equiv \ell^{1+\beta}-\tau^{1-\sigma}\ell^{1+\beta(1-\sigma)}+(1+\tau^{1-\sigma})\ell^{\beta\sigma}-2; \quad \ell \in \mathbb{R}$$

$$\forall\ \ell\neq 1,\ q(\ell)=0 \Leftrightarrow f(\ell)=1$$

The Descartes' rule of signs tells us that $q(\ell)=0$ also has either one or three positive roots. Since one of them is unity then we end up again with either two or no roots different than unity. Note in addition that ℓ^* is a solution to $p(\ell)=0$ if and only if $1/\ell^*$ is a solution of $q(\ell)=0$.

Finally we want to know about $f(\ell)$ at $f(\ell)=0$. First note that:

$$\lim_{\ell\to 0} f(\ell)=-1$$

$$\lim_{\ell\to \infty} f(\ell)=-\infty$$

Furthermore, solutions different than unity for $f(\ell)=0$ coincide with solutions different than unity for $r(\ell)=0$ as defined by:

$$r(\ell) \equiv \ell^{1+\beta}-(1+\tau^{1-\sigma})\ell^{1+\beta(1-\sigma)}+(1+\tau^{1-\sigma})\ell^{\beta\sigma}-1; \quad \ell \in \mathbb{R}$$

$$\forall\ \ell\neq 1,\ r(\ell)=0 \Leftrightarrow f(\ell)=0$$

Again, $q(\ell)=0$ has either two or no roots different than unity. Furthermore, if ℓ is a root of $q(\ell)=0$ then $1/\ell$ is too. The case of no roots is clearly not interesting since then there is no ℓ that makes $f(\ell)$ positive.

Lastly, we want to know about the number of local maxima and minima of $f(\ell)$:

$$f'(\ell) = \{ f(\ell)^{1-\beta\sigma}/\beta\sigma\ell(1-\ell^{1+\beta(1-\sigma)}) \}$$
$$\{ [(1+ \tau^{1-\sigma})(\beta\sigma\ell^{\beta\sigma}- (1+\beta(1-\sigma))\ell^{1+\beta(1-\sigma)})+ (1+\beta)\ell^{1+\beta}] [1-\ell^{1+\beta(1-\sigma)}] +$$
$$+ [(1+ \tau^{1-\sigma})(\ell^{\beta\sigma}- \ell^{1+\beta(1-\sigma)})+ \ell^{1+\beta}- 1] (1+\beta(1-\sigma))\ell^{1+\beta(1-\sigma)} \}$$

The roots different than unity of $f'(\ell)=0$ are either the (two) solutions of $f(\ell)=0$ or they must coincide with solutions for:

$$-\beta\sigma\ell^{2+\beta(2-\sigma)}+ (1+\tau^{1-\sigma})(1+\beta(1-2\sigma))\ell^{1+\beta}+ (1+\beta)\ell^{1+\beta}-$$
$$-(2+\tau^{1-\sigma})(1+\beta(1-\sigma))\ell^{1+\beta(1-\sigma)}+ (1+\tau^{1-\sigma})\beta\sigma\ell^{\beta\sigma}= 0$$

It is easy to check that this expression has at most three roots, one being zero and the other unity. This means that there is at most one root different than unity of $f'(\ell)$ that is not simultaneously a root of $f(\ell)=0$.

The facts we highlighted above about $f(\ell)$ are enough to characterize the solutions of the system (16) depending on the value of the limit of $f(\ell)$ at $\ell=1$. This can be computed using L'Hospital's rule to be:

$$\lim_{\ell\to 1} f(\ell)= \frac{(1+\tau^{1-\sigma})(1-\beta(2\sigma-1))+1+\beta}{1-\beta(\sigma-1)}$$

Taking in account that $\alpha<1/\sigma$ we can see that:

$$\lim_{\ell\to 1} f(\ell) >1 \Leftrightarrow \tau^{\sigma-1} < \frac{1-\beta(2\sigma-1)}{1+\beta(\sigma+1)}$$

First we show items (b) and (c) in the proposition and then we come back to item (a).

If $\lim_{\ell\to 1} f(\ell) > 1$ then $f(\ell)=\ell$ at two points at least, one strictly greater than unity (that corresponds to the equilibrium with two large and one small city) and the other strictly less than unity (corresponding to the equilibrium with one large and two small cities), but two is the maximum number of roots for $f(\ell)=\ell$.

If $\lim_{\ell\to 1} f(\ell) \leq 1$ and some $\ell>1$ existed such that $f(\ell)=\ell$ (corresponding to an equilibrium with two large and one small city), then necessarily $f(1/\ell)=1$, but this would imply either that more than two roots different than unity for $f(\ell)=\ell$ existed or that more than one local maximum different than unity existed (in case $\lim_{\ell\to 1} f(\ell)=1$), but both are impossibilities. We can not dismiss though the existence of some $\ell<1$ such that $f(\ell)=\ell$ (corresponding to equilibria with one large and two small cities), but it is easy to see that $\partial f/\partial\tau <0$, so that for τ high enough, $f(\ell)<1$ for all $\ell>1$, and consequently, $f(\ell)<\ell$ for all $\ell<1$ too, what means that no equilibrium other than $\ell_1=\ell_2=1$ survives.

Rests to be shown that there is no equilibrium with three cities of different sizes. Again we look separately at the two cases, when

$\lim_{\ell\to 1} f(\ell)>1$ and when $\lim_{\ell\to 1} f(\ell)\le 1$.

If $\lim_{\ell\to 1} f(\ell)>1$ then there exists at most one $\ell^*<1$ such that $f(\ell^*)=\ell^*$. Furthermore, for all $\ell<1$, $f(\ell)<\ell$ if $\ell<\ell^*$, and $f(\ell)>\ell$ if $\ell>\ell^*$. The set of solutions of (16) is given in all generality by the set of solutions of $\ell=f(f(\ell))$. But then, for all $\ell<1$:

if $\ell>\ell^* \Rightarrow f(\ell) > \ell \Rightarrow f(\ell) > \ell^* \Rightarrow f(f(\ell)) > f(\ell) \Rightarrow f(f(\ell)) > \ell$

if $\ell<\ell^* \Rightarrow f(\ell) < \ell \Rightarrow f(\ell) < \ell^* \Rightarrow f(f(\ell)) < f(\ell) \Rightarrow f(f(\ell)) < \ell$

what proves that ℓ^* is the only possible equilibrium for $\ell<1$. But from the symmetry of treatment of the cities in the model, if there is no equilibrium with $1>\ell_1\neq\ell_2<1$ then there is no equilibrium at all such that $1\neq\ell_1\neq\ell_2\neq 1$.

If $\lim_{\ell\to 1} f(\ell)\le 1$ then $f(\ell)<\ell$ for all $\ell>1$, what obviously means that no equilibrium with three cities of different sizes can happen. Now, again because of the symmetry of the model, if there is no equilibrium for $1<\ell_1\neq\ell_2>1$ then there is no equilibrium at all such that $1\neq\ell_1\neq\ell_2\neq 1$.

□

Proof of Proposition 9:

We saw in Proposition 8 that the kind of asymmetric equilibria that survives in the three city model for the largest set of parameters (given α and σ, for the widest range of τ) are those with two small and one big city. When there is multiple equilibria, i.e., when equilibria with one small and two big cities also exist, then the first are always larger, comparing equilibria bigger than unity and respectively smaller, comparing equilibria less than unity. We can see this by noting that with multiple equilibria there is a unique ℓ^* and a unique ℓ^{**} such that $f(\ell^*)=\ell^*$ and $f(\ell^{**})=1$ and $\ell^{**} > \ell^*$. The maximum equilibrium ratio of populations in the three city model then corresponds to the ratio greater than unity (population of big divided by population of the small city) in an equilibrium with two small and one big city. We show that for any such equilibrium there corresponds an equilibrium in the model of two cities for which the maximum ratio of populations is bigger.

If an equilibrium with two small and one big city exists, then the respective maximum ratio of populations is some $\ell_*>1$ that solves $f(\ell)=1$ or equivalently, $q(\ell)=0$ defined as before by:

$$q(\ell)\equiv \ell^{1+\beta}-\tau^{1-\sigma}\ell^{1+\beta(1-\sigma)}+(1+\tau^{1-\sigma})\ell^{\beta\sigma}-2$$

Asymmetric equilibria in the two city model correspond to solutions different than unity of $g(\ell)=0$ where:

$$g(\ell) \equiv \ell^{1+\beta}-\tau^{1-\sigma}\ell^{1+\beta(1-\sigma)}+\tau^{1-\sigma}\ell^{\beta\sigma}-1$$

As we have seen in Section 2, $g(\ell)=0$ has either one or three solutions, unity bee in one of them. Moreover, if ℓ is one root so is $1/\ell$, and since $\lim_{\ell\to 0}g(\ell)=-1$ and $\lim_{\ell\to+\infty}g(\ell)=+\infty$. When unity is the only root, $g(\ell)>0$ for $\ell>1$ and $g(\ell)<0$ for $\ell<1$. When there are three roots, ℓ_{**} being one of them,

$$g(\ell)\begin{cases} <0; & \ell<1/\ell \\ >0; & \ell\in(1/\ell_{**},1) \\ <0; & \ell\in(1,\ell_{**}) \\ >0; & \ell>\ell_{**} \end{cases}$$

Now, $q(\ell)=0$ may be rewritten as:

$$\ell^{1+\beta}-\tau^{1-\sigma}\ell^{1+\beta(1-\sigma)}+\tau^{1-\sigma}\ell^{\beta\sigma}-1 = 1-\ell^{\beta\sigma}$$

But the left hand side coincides with $g(\ell)$, so that if $\ell_*>1$ solves $q(\ell)=0$, then $g(\ell_*)<0$ and thus we can conclude that some $\ell_{**}>\ell_*$ solves $g(\ell)=0$. □

APPENDIX 2

An Alternative Assumption about Land Ownership Rights

When land ownership rights are the same for everyone in both cities, the rent income received everywhere is: $R \equiv (R_1^a+R_2^a)/L_1+L_2$. Substituting for R_1^a and R_2^a we get:

$$R = \alpha(L_1W_1+L_2W_2)/L_1+L_2$$

so that R is a populations weighted average of the wage rates in both cities.

Plugging this into the market clearing equations (6) and many steps of algebra later, we can get the following expression for the trade balance equilibrium:

$$\begin{aligned}&\ell^2[(1-\alpha^2)w^{\sigma+1}\tau^{\sigma-1}+\alpha^2w^2-w]+\\&\quad+\ell[(1-\alpha^2)w^{2\sigma}+(w-1)w^{\sigma}(\alpha\tau^{1-\sigma}+(1-\alpha(1-\alpha))\tau^{\sigma-1})-(1-\alpha^2)w]-\\&\quad-[(1-\alpha^2)w^{-\sigma-1}\tau^{\sigma-1}+\alpha^2w^{-2}-w^{-1}]w^{2\sigma+1}=0\end{aligned}$$

This is a relation that gives us the endogenous ratio of wages in cities 1 and 2 as a function of the parameters α, σ, τ, and a given relative size of the cities populations. Since we can't write w explicitly as a function of the parameters and of ℓ, we solve the inverse function that gives ℓ as the roots of the second degree polynomial whose coefficients are functions of the parameters α, σ, τ and the wage ratio (w).

$$\ell^2 + B(w)\ell + C(w) = 0$$

$$B(w) \equiv \frac{(1-\alpha^2)w^{2\sigma-1}+(w-1)w^{\sigma-1}(\alpha\tau^{1-\sigma}+(1-\alpha(1-\alpha))\tau^{\sigma-1})-(1-\alpha^2)}{(1-\alpha^2)w^{\sigma}\tau^{\sigma-1}+\alpha^2w-1}$$

$$C(w) \equiv -w^{2\sigma-1}\frac{(1-\alpha^2)w^{-\sigma}\tau^{\sigma-1}+\alpha^2w^{-1}-1}{(1-\alpha^2)w^{\sigma}\tau^{\sigma-1}+\alpha^2w-1}$$

where we are omitting α, σ, τ as parameters of B and C to simplify the notation. We can then show that:

Proposition:

The wage ratio is a continuously differentiable, strictly increasing function $w(\ell)$ of the populations ratio s.t.:

$w:\ \mathbb{R}_+ \to \mathbb{R}_+$

$w(\ell) \equiv \mathrm{inv}(l(w))$; where $l(w) = (-B(w)+[B(w)^2-4C(w)]^{1/2})/2$

$w(\ell) \in (1/w^*, w^*)$, where w^* solves $(1-\alpha^2)w^{*\sigma}\tau^{\sigma-1}+\alpha^2w^*-1 = 0$

proof:

The proof is by analysis of the functions B(w) and C(w). Our first worry should be the range for w that generates meaningful roots for the expression above. First thing to notice is that, as we should expect from the symmetry of the model, if ℓ is a solution for w then $1/\ell$ is a solution for 1/w. That allows us to restrict attention just to $w\geq1$ without loss of generality. One can check that, for $w\geq1$, $0\leq\alpha<1$, $\sigma>1$, $0<\tau<1$, we have $B'(w)<0$ and $C'(w)<0$. The expression in the denominator of B(w) and C(w) is monotonically increasing in w. It has a unique root $w^*>1$ such that

$$(1-\alpha^2)w^{*\sigma}t^{\sigma-1}+\alpha^2w^*-1 = 0$$

The expression in the numerator of C(w) is positive for all $w>1/w^*$. Thus, for $w\geq1$, C(w) is always decreasing in w, negative valued in $[0,w^*)$, positive valued in $(w^*,+\infty)$, with a discontinuity at w equal to w^* where:

$$\lim_{w \to w^*_-} C(w) = -\infty \ ; \qquad \lim_{w \to w^*_+} C(w) = +\infty \ ;$$

The expression in the numerator of B(w) is zero at w=1, and is monotonically increasing for all $w\geq1$. The expression in the denominator again generates a discontinuity at $w=w^*$, making B(w) negative on $[1,w^*)$, positive on $(w^*,+\infty)$ and:

$$\lim_{w \to w^*_-} B(w) = -\infty \ ; \qquad \lim_{w \to w^*_+} B(w) = +\infty \ ;$$

Now, since the product of the roots of the binomium must equal C(w) and the sum of its roots must equal -B(w), we can be sure that no positive root exists in $[w^*,+\infty)$ and that only one positive root exists in $[1,w^*)$. We are only interested in the positive roots, that are given by:

$$l(w) = \{ -B(w)+[B(w)^2-4C(w)]^{1/2} \}/2; \qquad 1\leq w<w^*$$

With some more derivatives and the prior results on B(w) and C(w) we can establish that $l'(w) >0$ for $w\geq1$. □

The ratio of indirect utilities in city one to city two can be written as:

$$u(\ell) = \frac{w(\ell+1)+\alpha(w\ell+1)}{(\ell+1)+\alpha(w\ell+1)} \; \frac{1}{(w\ell)^{\alpha}} \left(\frac{\ell w^{1-\sigma}\tau^{\sigma-1} + 1}{\ell w^{1-\sigma} + \tau^{\sigma-1}} \right)^{(1-\alpha)/(1-\sigma)}$$

Notice that, as before, if $u=u(\ell)$ then $1/u=u(1/\ell)$, and also $u(1)=1$, so that cities of equal size are always in equilibrium, although this is not necessarily stable. One can check that:

$\lim_{\ell\to\infty} u(\ell) = 0;$ $\qquad \lim_{\ell\to 0} u(\ell) = +\infty$

Since $u(\ell)$ is continuous for $0<\ell<+\infty$, we have at least one stable long-run equilibrium (where $u(\ell)$ crosses $u=1$ with a negative slope).

A sufficient subset of parameters for this type of asymmetrical equilibria to occur is the one that satisfies $u'(1) > 0$. This set is the analogous to the one defined by inequality (a) in Proposition 2, and is given by:

$$\begin{aligned} S = \{ \alpha,\sigma,\tau : \; & \alpha^3[(2\sigma^2-7\sigma+3)\tau^{2(\sigma-1)}+(4\sigma^2-4\sigma+5)\tau^{\sigma-1}+(2\sigma^2-5\sigma)]+ \\ & + \alpha^2[(5\sigma-5)\tau^{2(\sigma-1)}+(-3\sigma)\tau^{\sigma-1}+(-\sigma+6)-(1+\sigma)\tau^{1-\sigma}]+ \\ & + \alpha[(-2\sigma^2+7\sigma+3)\tau^{2(\sigma-1)}+(-4\sigma^2-5)\tau^{\sigma-1}+(-2\sigma^2+\sigma)+2\tau^{1-\sigma}]+ \\ & + [(-6\sigma)\tau^{2(\sigma-1)}+(4\sigma+3)\tau^{\sigma-1}+(2\sigma-3)] > 0 \} \end{aligned}$$

It is easy to show that S is nonempty.

APPENDIX 3

Derivation of Equilibrium Wages for the Model with Three Cities

In this appendix we indicate the steps in derivation of the expressions giving the relative wages in the three city model. We let transportation costs be different between different pairs of cities, but we assume that any trade between two cities can only be done directly, or in other words, that there is no way to avoid transportation costs between two cities by using the longer route through the third city. In the case of equal transportation costs though the issue is immaterial, since the direct trade is always the cheaper one.

Using again the accounting relations developed in Section 2, i.e., $E_i^a=(1-\alpha)L_iW_i$ and $n_ip_iY_i=(1-\alpha)L_iW_i$ we can write the market clearing conditions for goods made in city one and 2 respectively as:

$$w_1^{\sigma}\tau_{23}^{\sigma-1}= \frac{(\tau_{12}\tau_{23})^{1-\sigma}\, w_1\ell_1}{\ell_1(w_1\tau_{12})^{1-\sigma}+\ell_2 w_2^{1-\sigma}+(\tau_{12}/\tau_{13})^{1-\sigma}} + \frac{\tau_{23}^{\sigma-1}\, w_2\ell_2}{\ell_1 w_1^{1-\sigma}+\ell_2(w_2\tau_{12})^{1-\sigma}+(\tau_{12}/\tau_{23})^{1-\sigma}} + \frac{1}{\ell_1(w_1\tau_{23})^{1-\sigma}+\ell_2(w_2\tau_{13})^{1-\sigma}+(\tau_{13}\tau_{23})^{1-\sigma}}$$

$$w_2^{\sigma}\tau_{13}^{\sigma-1}= \frac{\tau_{13}^{\sigma-1}\, w_1\ell_1}{\ell_1(w_1\tau_{12})^{1-\sigma}+\ell_2 w_2^{1-\sigma}+(\tau_{12}/\tau_{13})^{1-\sigma}} + \frac{(\tau_{12}/\tau_{13})^{1-\sigma}\, w_2\ell_2}{\ell_1 w_1^{1-\sigma}+\ell_2(w_2\tau_{12})^{1-\sigma}+(\tau_{12}/\tau_{23})^{1-\sigma}} + \frac{1}{\ell_1(w_1\tau_{23})^{1-\sigma}+\ell_2(w_2\tau_{13})^{1-\sigma}+(\tau_{13}\tau_{23})^{1-\sigma}}$$

Isolating and equating the last terms in each expression we can get ℓ_1 or ℓ_2 as an explicit function of the other variables:

$$\ell_2 w_2 = \frac{\ell_1 w_1(\tau_{12}^{\sigma-1} - w_1^{-\sigma} w_2^{\sigma}) + w_1^{\sigma}\tau_{23}^{\sigma-1} - w_2^{\sigma}\tau_{13}^{\sigma-1}}{(\tau_{12}^{\sigma-1} - w_1^{\sigma} w_2^{-\sigma})}$$

Substituting those back into the versions we developed of the market clearing equations, we can finally get to:

$$\ell_1 = \frac{\{ (\tau_{23}^{2(\sigma-1)}-1)w_1^{\sigma} - (\tau_{23}^{\sigma-1}\tau_{13}^{\sigma-1}-\tau_{12}^{\sigma-1})w_2^{\sigma} - (\tau_{12}^{\sigma-1}\tau_{23}^{\sigma-1}-\tau_{13}^{\sigma-1}) \} w_1^{\sigma-1}}{(\tau_{12}^{2(\sigma-1)}-1) - (\tau_{12}^{\sigma-1}\tau_{23}^{\sigma-1}-\tau_{13}^{\sigma-1})w_1^{\sigma} - (\tau_{12}^{\sigma-1}\tau_{13}^{\sigma-1}-\tau_{23}^{\sigma-1})w_2^{\sigma}}$$

$$\ell_2 = \frac{\{ (\tau_{13}^{2(\sigma-1)}-1)w_2^{\sigma} - (\tau_{13}^{\sigma-1}\tau_{23}^{\sigma-1}-\tau_{12}^{\sigma-1})w_1^{\sigma} - (\tau_{12}^{\sigma-1}\tau_{13}^{\sigma-1}-\tau_{23}^{\sigma-1}) \} w_2^{\sigma-1}}{(\tau_{12}^{2(\sigma-1)}-1) - (\tau_{12}^{\sigma-1}\tau_{23}^{\sigma-1}-\tau_{13}^{\sigma-1})w_1^{\sigma} - (\tau_{12}^{\sigma-1}\tau_{13}^{\sigma-1}-\tau_{23}^{\sigma-1})w_2^{\sigma}}$$

where $\ell_i=L_i/L_3$ and $w_i=W_i/W_3$, for $i=1,2$.

Making $\tau_{ij}=\tau$ above we get the expressions displayed in the text.

APPENDIX 4

The Descarte's Rule of Signs

The Descartes' rule of signs is originally stated for polynomials with real coefficients. If we write down the polynomial in decreasing order of the exponents and count the number of sign changes (v) of the nonvanishing coefficients, the rule is stated as follows:

The number of positive real roots, p, of a polynomial equation with real coefficients does not exceed v, the number of sign variations of the coefficients; moreover, $v-p$ is a nonnegative even integer.

We offer here a proof that the rule also holds for nonnegative rational exponents. Take some function:

$$f(x) = a_0 x^{p_n/q_n} + a_1 x^{p_{n-1}/q_{n-1}} + \ldots + a_{n-1} x^{p_1/q_1} + a_n$$

with $p_n, \ldots, p_1$ and $q_n, \ldots, q_1$ positive integers and

$$p_n/q_n > p_{n-1}/q_{n-1} > \ldots > p_1/q_1 > 0$$

Write the exponents with a common denominator:

$$p_{n-i}/q_{n-i} = (q_n q_{n-1} \ldots p_{n-i} \ldots q_1)/(q_n q_{n-1} \ldots q_{n-i} \quad q_1); \; i=0, \ldots, n-1$$

Make the change of variables $z = x^{1/(q_n q_{n-1} \ldots q_1)}$ so that,

$$f(z) = a_0 z^{p_n q_{n-1} \ldots q_1} + a_1 z^{q_n p_{n-1} \ldots q_1} + \ldots + a_{n-1} z^{q_n q_{n-1} \ldots p_1} + a_n$$

Since the rule applies to f(z) and x is a strictly increasing function of z then f(z) and f(x) have the same number of roots. □

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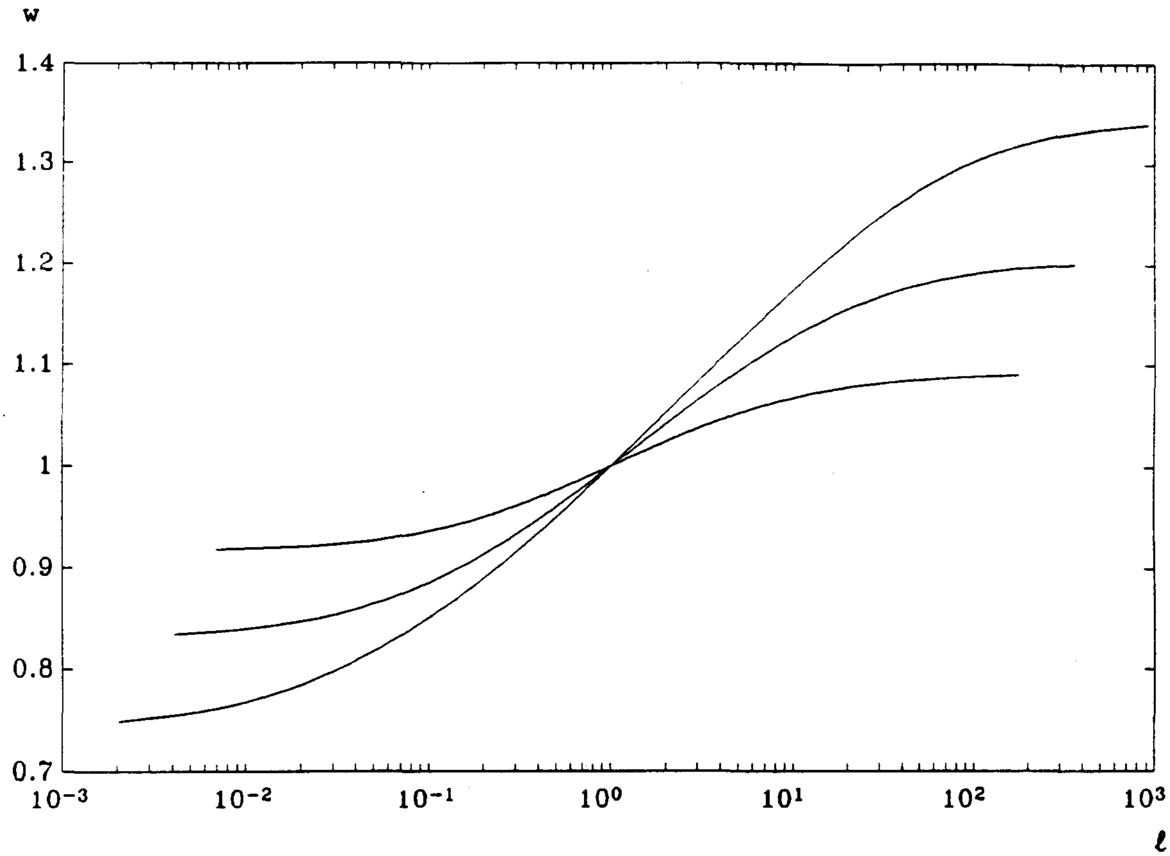
FIGURE 1. Equilibrium wage ratios.

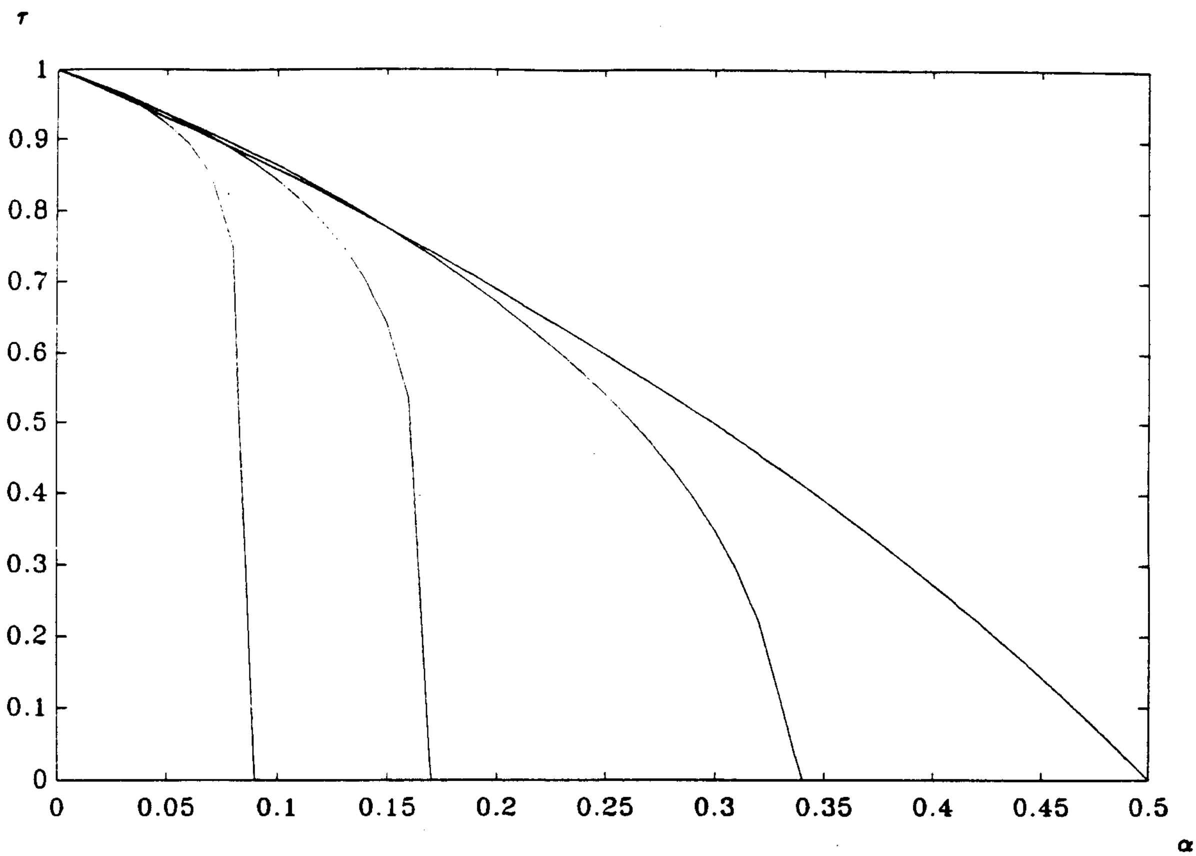


w - ratio of wages (W_1/W_2) in equilibrium

ℓ - ratio of populations (L_1/L_2)

Figure 1 depicts the function $w(\ell,\sigma,\tau)$ for σ=6 and three different values for τ. For ℓ>1, the curve at the top is $w(\ell,6,0.7)$, the one in the middle is $w(\ell,6,0.8)$ and the one at the bottom is $w(\ell,6,0.9)$. Note that w gets steeper for higher transportation costs (lower τ).

FIGURE 2. Range of parameters for asymmetric equilibria.



τ - transportation cost

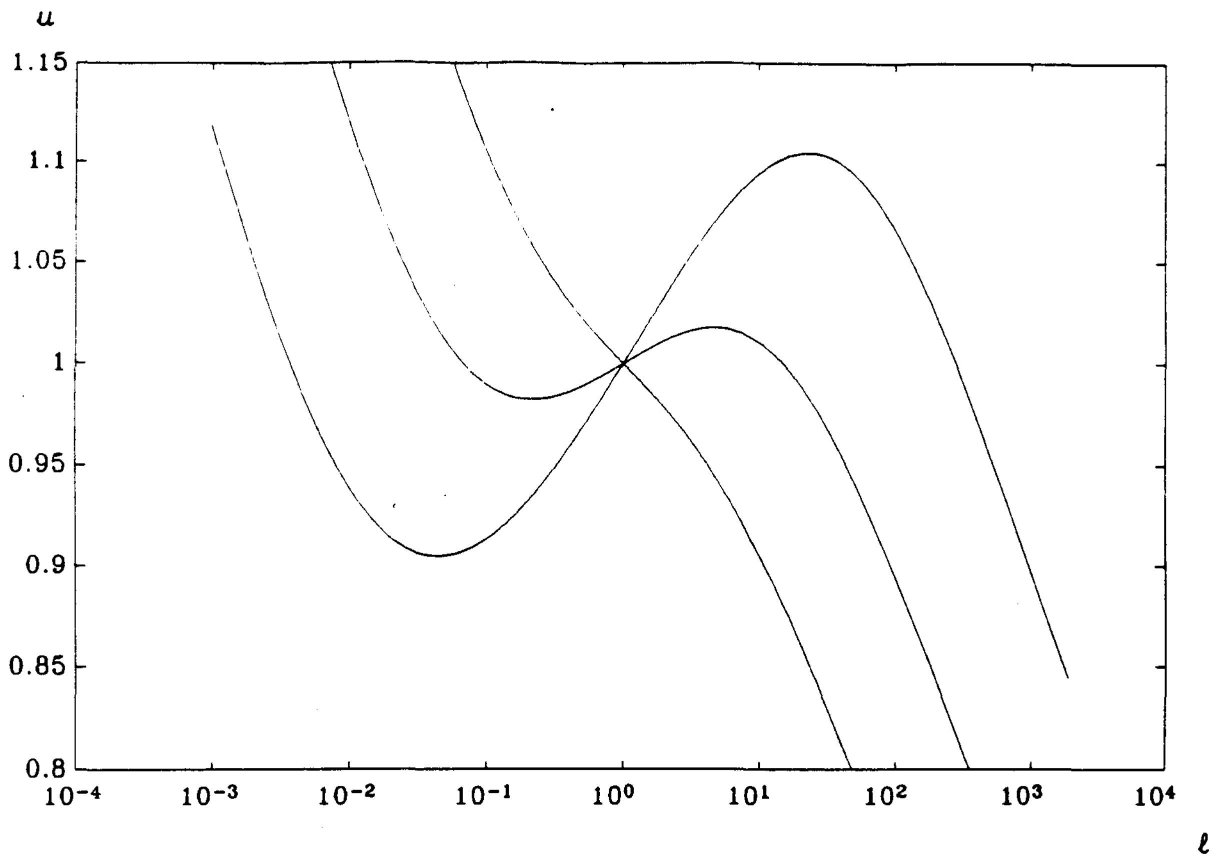
α - share of expenditures on land

Figure 2 depicts the function:

$$T(\alpha,\sigma)= \left(\frac{(1-\alpha\sigma)(2\sigma-1)}{\alpha(\sigma-1)+(1-\alpha)(2\sigma-1)}\right)^{1/(\sigma-1)}$$

for four different values of σ. From the l.h.s. in the picture, $T(\alpha,12)$ is the first to hit the x-axis, $T(\alpha,6)$ is the second, $T(\alpha,3)$ the third and $T(\alpha,2)$ is the last, at $\alpha=0.5$.

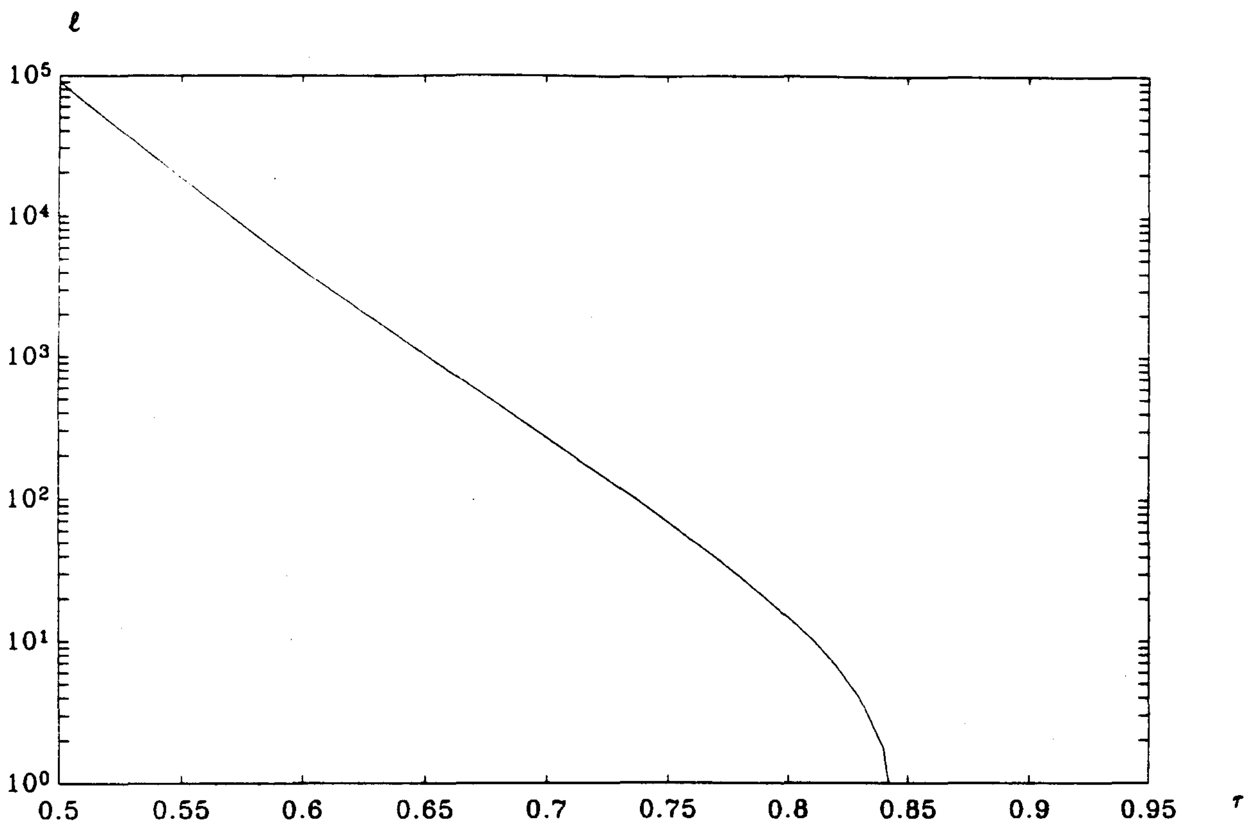
FIGURE 3. Indirect utility ratios



u - ratio of indirect utilities (U_1/U_2)
ℓ - ratio of populations (L_1/L_2)

Figure 3 depicts the function $u(\ell,\alpha,\sigma,\tau)$ for α=0.10, σ=6, and three different values for τ. For ℓ>1, the curve at the top is $u(\ell,0.10,6,0.7)$, the one in the middle is $u(\ell,0.10,6,0.8)$ and the one at the bottom is $u(\ell,0.10,6,0.9)$.

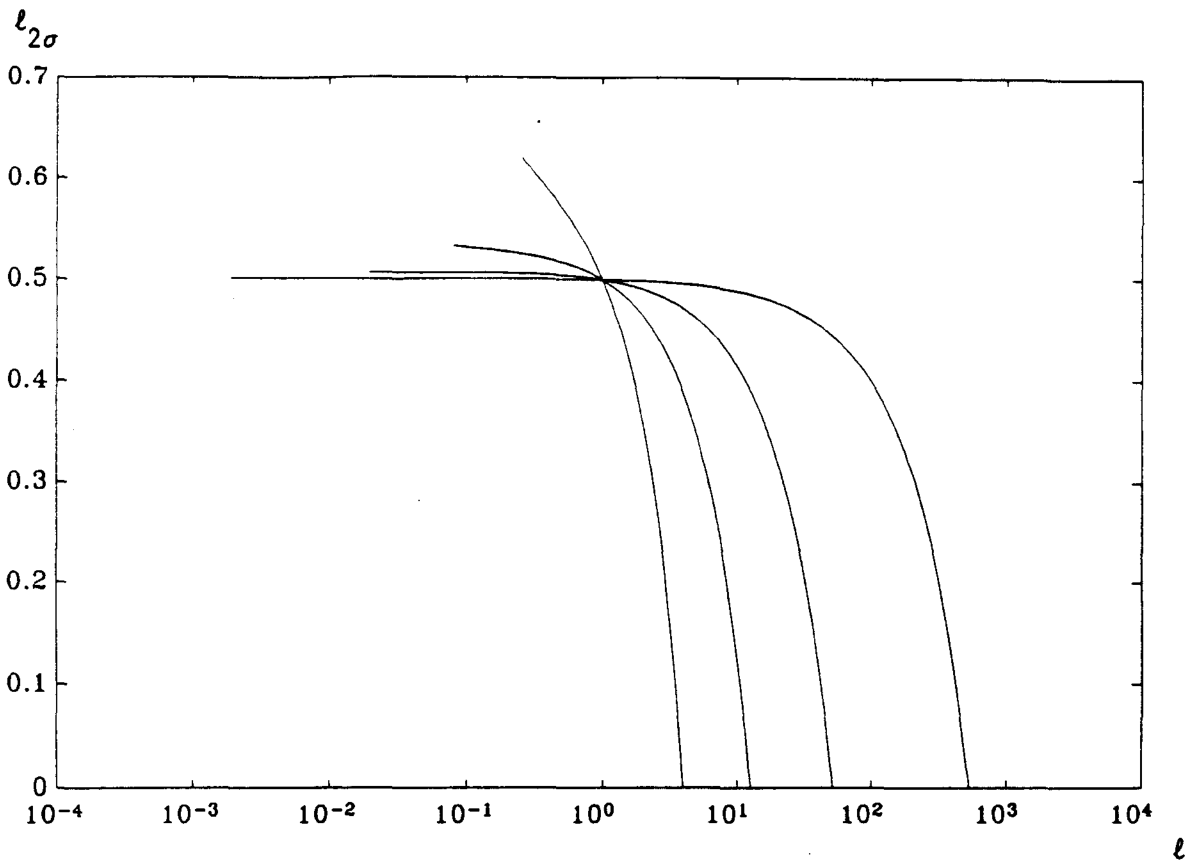
FIGURE 4. Equilibrium ratios of populations.



ℓ - ratio of populations (L_1/L_2)
τ - transportation cost

Figure 4 depicts the equilibrium ratios of populations for values of the parameters $\alpha=0.10$, $\sigma=6$, as a function of τ. Equilibrium ratios of populations are those ℓ s.t. $u(\ell,\alpha,\sigma,\tau)=1$. Note that $\ell=1$ is always an equilibrium for all τ, and that this is stable if and only if it is unique.

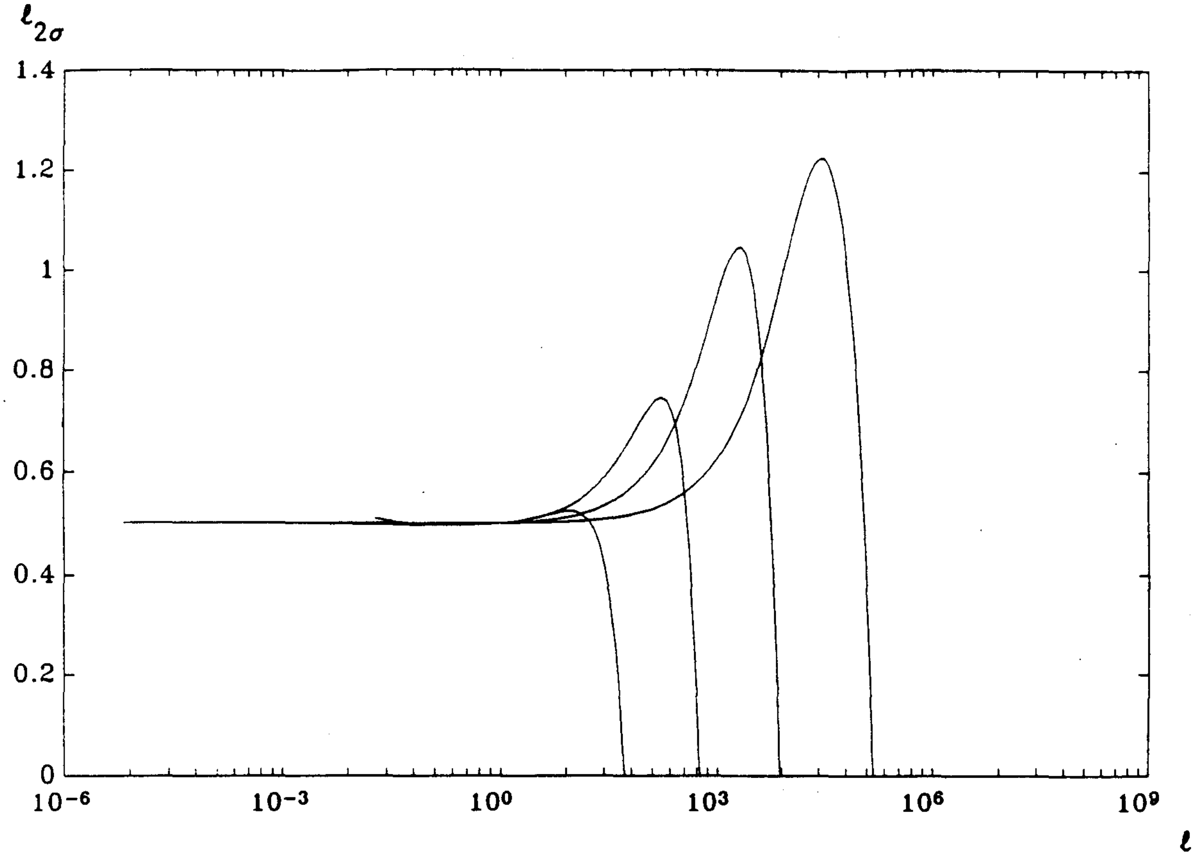
FIGURE 5. Industry sigma labor shares for small elasticities.



$\ell_{2\sigma}$ - industry sigma labor share in city two $(L_{2\sigma}/L_2)$
ℓ - ratio of populations (L_1/L_2)

Figure 5 depicts the function $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)$ for ϕ=0.5, σ=4, ρ=3, and four different values of τ. From the l.h.s., the first curve to hit the x-axis is $l_{2\sigma}(\ell,0.5,4,3,0.9)$, the second is $l_{2\sigma}(\ell,0.5,4,3,0.7)$, the third is $l_{2\sigma}(\ell,0.5,4,3,0.5)$, and the last is $l_{2\sigma}(\ell,0.5,4,3,0.3)$. The value of ℓ at which $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)$ hits the x-axis define an upper bound for the range of equilibrium ratios of populations. At such a bound the industry with high elasticity of demand vanishes from the small city.

FIGURE 6. Industry sigma labor shares for high elasticities.

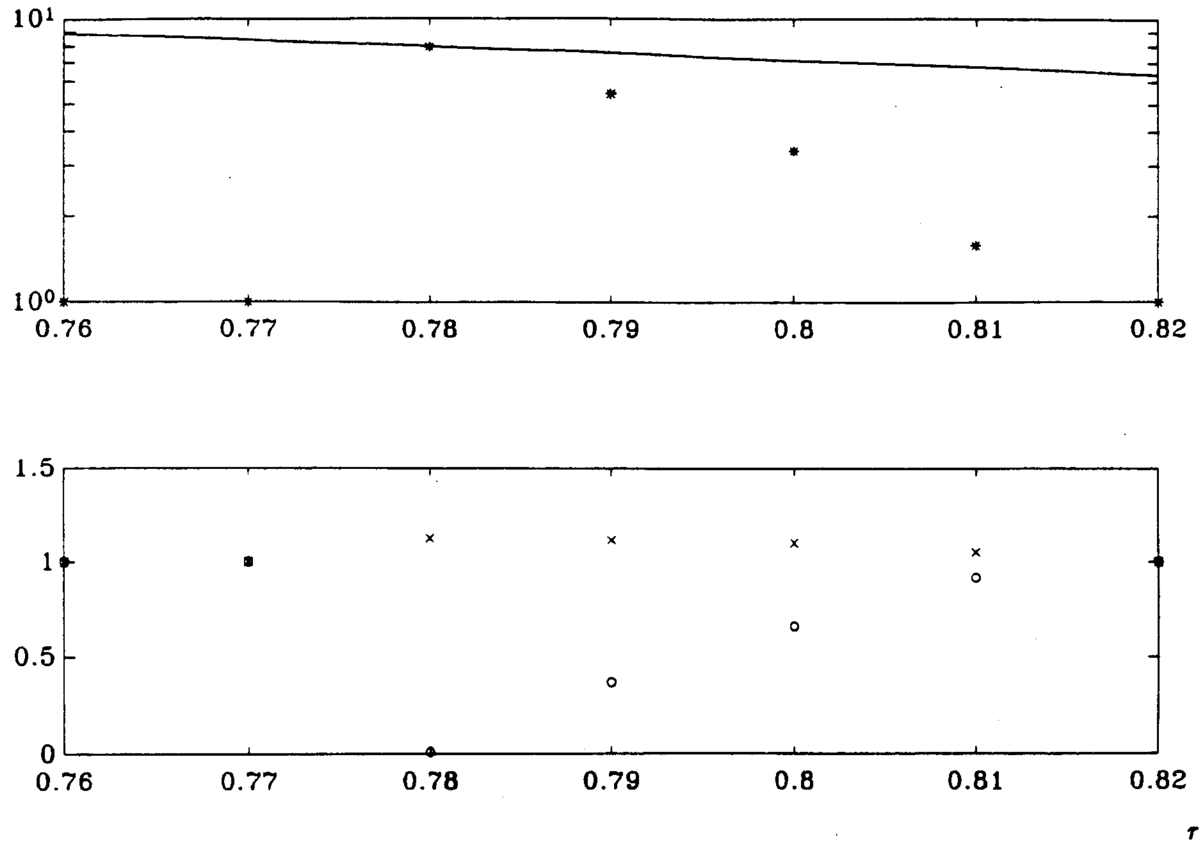


$\ell_{2\sigma}$ - industry sigma labor share in city two $(L_{2\sigma}/L_2)$

ℓ - ratio of populations (L_1/L_2)

Figure 6 depicts the function $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)$ for ϕ=0.5, σ=12, ρ=8, and four different values of τ. From the l.h.s., the first curve to hit the x-axis is $l_{2\sigma}(\ell,0.5,4,3,0.9)$, the second is $l_{2\sigma}(\ell,0.5,4,3,0.8)$, the third is $l_{2\sigma}(\ell,0.5,4,3,0.7)$, and the last is $l_{2\sigma}(\ell,0.5,4,3,0.6)$. In addition to the restriction on ℓ defined by $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)\geq 0$, here the restriction $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)\leq 1$ is also effective for τ=0.7 and τ=0.6. Note that here, in contrast with Figure 5, the industry sigma labor share increases for a while in city two as it gets smaller, before finally vanishing from there at ℓ s.t. $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)=0$.

FIGURE 7. Equilibrium population ratios and industry concentrations (I)



* equilibrium population ratios ℓ (ℓ s.t. $u(\ell,\alpha,\phi,\sigma,\rho,\tau)=1$)

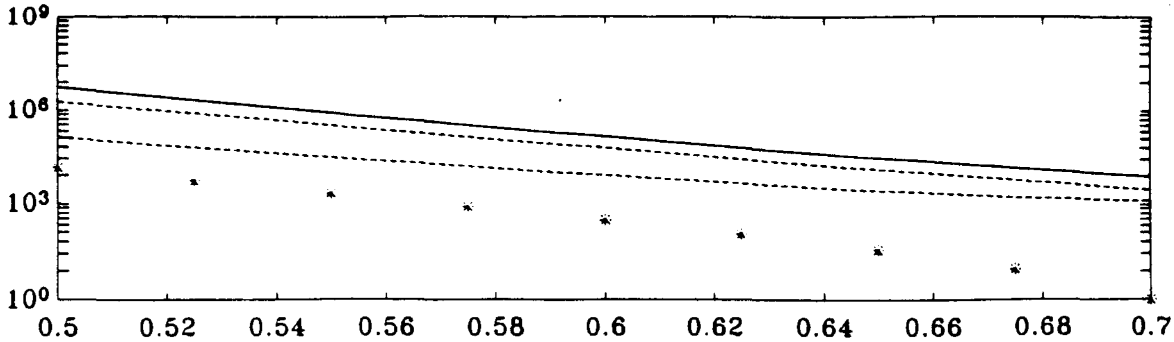
—— population ratios ℓ s.t. $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)=0$

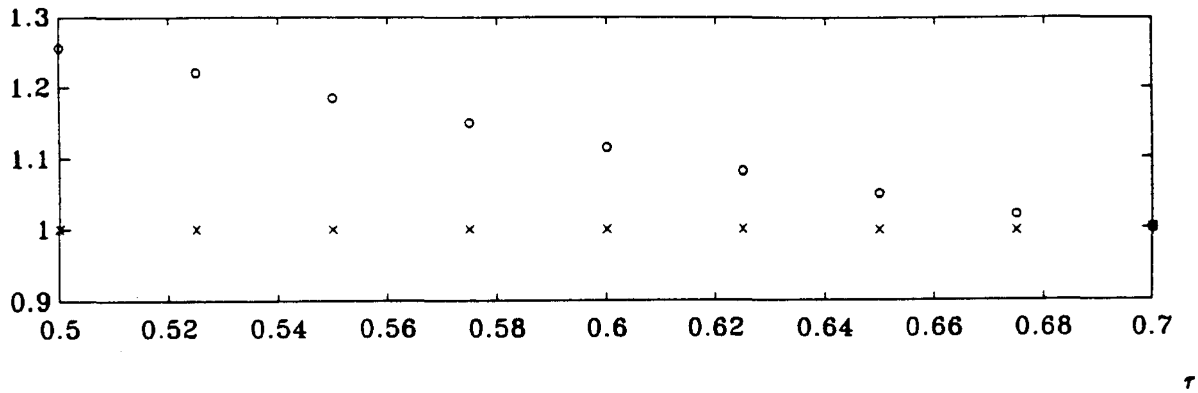
x concentration of industry sigma in city one $((L_{1\sigma}/L_1)/(L_\sigma/L))$

o concentration of industry sigma in city two $((L_{2\sigma}/L_2)/(L_\sigma/L))$

For values of the parameters $\alpha=0.15$, $\phi=0.5$, $\sigma=4$, $\rho=3$, Figure 7 illustrates how equilibrium population ratios increase as τ decreases, until it hits the upper bound. Note how the concentration of industry sigma in city two declines monotonically. Note also that if τ is too low there is a unique equilibrium populations ratio $\ell=1$, (for $\tau=0.77$ and $\tau=0.76$) and that it is unstable, since the utilities ratio is greater than one at the upper bound.

FIGURE 8. Equilibrium population ratios and industry concentrations (II)





*	equilibrium population ratios ℓ (ℓ s.t. $u(\ell,\alpha,\phi,\sigma,\rho,\tau)=1$)
——	population ratios ℓ s.t. $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)=0$
----	population ratios ℓ s.t. $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)=1$
x	concentration of industry sigma in city one $((L_{1\sigma}/L_1)/(L_\sigma/L))$
o	concentration of industry sigma in city two $((L_{2\sigma}/L_2)/(L_\sigma/L))$

For values of the parameters $\alpha=0.10$, $\phi=0.5$, $\sigma=12$, $\rho=8$, Figure 8 pictures progressively increasing equilibrium population ratios as τ decreases, as in Figure 7. Here though the concentration of industry sigma in city two at first increases as city two shrinks, because of the humped shape of $l_{2\sigma}(\ell,\phi,\sigma,\rho,\tau)$ for high elasticities (see Figure 6). Note that there can not be any equilibrium between the dotted lines.