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"A MODEL OF PRIVATE BANK-NOTE ISSUE"

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A model of private bank-note issue*

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Abstract

A random-matching model (of money) is formulated in which there is complete public knowledge of the trading histories of a subset of the population, called the banking sector, and no public knowledge of the trading histories of the complement of that subset, called the non bank sector. Each person, whether a banker or a non banker, is assumed to have the technological capability to create indivisible and durable objects called notes. If outside money is indivisible and sufficiently scarce, then the optimal mechanism is shown to have note issue and note destruction (redemption) by members of the banking sector.

1 Introduction

This paper presents a new model of a particular form of inside money. The term inside money refers to a kind of money which is inside the private sector in the sense that it disappears with sufficient netting of balance sheets. (For example, netting across banks and the rest of the private sector makes privately owned checking deposits disappear.) In other words, inside money is someone's liability and someone else's asset, a form of credit. It is money because it is used as a tangible medium of exchange by those for whom it is an asset. The challenge, therefore, in modeling inside money is to simultaneously

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have a role for both credit and tangible media of exchange. One reason this is challenging is because if credit functions too well, then tangible media of exchange are unnecessary or, to use Frank Hahn's term, inessential. Indeed, it is well-known that tangible media of exchange are inessential in the complete markets world of the Arrow-Debreu model.

Given the well-known need to inhibit the role of credit in order to get a role for tangible media of exchange, it is not surprising that the first nice models of tangible media of exchange — the random-matching models of Diamond (1984) and Kiyotaki and Wright (1989) — were models of outside money and were models that implicitly made extreme assumptions to rule out credit. There are two such assumptions (see Aiyagari and Wallace 1991, Huggett and Krasa 1996, and Kocherlakota 1996). First, people cannot commit to future actions. Second, each person's trading history is private information (some version of which is needed because the models are infinite horizon models).

Once the role of those assumptions is recognized, the obvious way to put some form of credit into the models is to weaken one or both of those assumptions.[1] The first paper to do that in the context of a random matching model was Kocherlakota and Wallace (1996). They study a setting with partial public knowledge of individual trading histories, one in which the public record of everyone's trading history is updated with a lag. They show that if the lag is finite and sufficiently large, then the optimal mechanism has a role for transfers of resources in exchange for outside money (a role for tangible media of exchange) and a role for other transfers of resources (a role for credit, mutual charity, or insurance). We follow that paper closely, but adopt a different version of partial public knowledge of individual trading histories. We assume that there is complete public knowledge of the trading histories of a subset of the population and no public knowledge of the trading histories of the complement of that subset.[2] The former, which we label the banking sector, will turn out to be the issuers of inside money, while the

[1]Perhaps because those assumptions were left implicit in the first random matching models of money, some of the early attempts to add credit to those models did not explicitly weaken those assumptions (see Diamond 1990 and Shi 1996). That is one reason why those formulation seem contrived. For example, in Diamond (1990), two people who reach a credit agreement stay together, an option not available to others. Cavalcanti et. al. (1996) is another recent attempt to model inside money, but it does not proceed by weakening one or both of the above assumptions in an explicit way.

[2]Rao Aiyagari suggested this assumption in conversations several years ago.

latter, which we label the non-bank sector, will turn out to be the users of inside money.

The kind of inside money that is in our model does not resemble what we see today. It resembles private bank notes, which were prominent in the first half of the 19th century and earlier — at a time when outside money was commodity money, predominantly, gold and silver. Our paper is about a social role for bank notes and the rationale for such a role that we adopt is one suggested by several economic historians. Hanson (1979) and Glassman and Redish (1988) argue that commodity outside money was significantly indivisible. Because its quantity was naturally limited, if it was indivisible, then it was scarce.[3] That scarcity is our rationale for inside money that takes the form of bank notes.[4]

The rest of the paper spells out the details. A complete description of the environment we study appears in section 2. In section 3, we study a special case with no banking sector and with an exogenous amount of indivisible outside money. That special case serves mainly as a simple introduction to the kind of mechanism design problem we study. In section 4, we turn to our main model, the model with a positive measure of people whose histories are public information, our banking sector. In order to be able to interpret banking as private while providing a public service, our optimum problem is to maximize the steady state expected utility of a representative non banker subject to making a representative banker at least as well off as the non banker. We show that any optimum has positive note issue by bankers and has bankers redeeming or destroying bank notes in a way that is interpretable as resulting from regulation. Also, for both a very small and a very large banking sector, we give additional description of the optimum. In section 5, we complicate the model by adding a productivity shock, the realization of which for each person is private information to the person: with probability p, a person is unable to produce at a date and with probability $1-p$, the

[3]Without attributing the scarcity to the indivisibility of commodity money, many observers commented on the scarcity of commodity money and some advocated note issue as a remedy. See, for example, Law (1705) and the remarks made in the early 19th century by a newly appointed Governor General of Australia cited in Butlin (1968).

[4]Different rationales are conceivable. One is that paper bank notes are easier to carry around than gold and silver, because they weigh less. A rationale like that could be given for a social role for the very modern version of bank notes — namely, stored value (smart) cards. Such cards permit one to avoid having to mess around with wads of paper money and handfuls of coins. Recognizability (the avoidance of counterfeits) is another conceivable rationale for both historically observed bank notes and stored valued cards.

person is able to produce. For the case of a sufficiently small banking sector, a sufficiently high discount factor, and a sufficiently small p, we show that there exists a simple scheme that approximates the optimum. This scheme has bankers whose most recent relevant productivity shock did not allow them to produce (and, therefore, did not allow them to redeem a note) consuming almost nothing and has those whose most recent relevant productivity shock allowed them to produce (and, therefore, allowed them to redeem a note) consuming what non bankers get for a note when they issue a note. We conclude in section 6 with a discussion of our assumptions. As part of that discussion, we point out some of the ways in which banks in the model differ from historically observed note-issuing banks. All the proofs appear in the appendix.

2 The environment

Time is discrete and the horizon is infinite. There are N distinct divisible and perishable types of goods at each date and there is a $[0,1]$ continuum of each of N types of people, where $N \geq 3$. Each type is specialized in consumption and production: a type n person consumes good n and produces good $n+1$ (modulo N), for $n = 1, 2, ..., N$. Each type n person maximizes expected discounted utility with discount factor $\beta \in (0,1)$. Utility in a period is given by $u(x) - y$, where x is the amount consumed and y is the amount produced (see figure 1). The function u is defined on $[0, \infty)$, is increasing, twice differentiable and bounded, and satisfies $u(0) = 0$, $u'' < 0$, and $u'(0) = \infty$.[5]

In each period, people are randomly matched in pairs. Meetings are of two sorts: single-coincidence meetings, those between a type n person (the producer) and a type $n+1$ person (the consumer) for some n; and no-coincidence meetings, those in which neither person produces what the other consumes. (Because the number of types, N, exceeds two, there are no double-coincidence meetings.) We assume that people in a meeting know each other's type. We also assume that people cannot precommit to future

[5] Boundedness of u can be interpreted as coming from a bound on production. Suppose that the utility of consuming is r, which satisfies all the assumptions made about u except boundedness, and suppose that d is the disutility of production, where $d : [0, Z) \to \mathbb{R}_+$ is strictly convex, with $d(z)$ and $d'(z) \to \infty$ as $z \to Z$. This is equivalent to our specification if we let $x \equiv d(z)$ and let $u(x) \equiv r[d^{-1}(x)]$.

actions, so that those who produce in a single-coincidence meeting have to get a future reward for doing so.[6] In section 5, we also assume that at the same time as being matched, each person receives a productivity shock, the realization of which is private information. With probability p, the person is unable to produce, while with probability $1-p$ the person is able to produce.

The society is able to keep a public record of the actions of a fraction bN of each type of person, where $b \in [0, \frac{1}{N}]$. It has no public record for any one else. If b is equal to $\frac{1}{N}$, then there is a complete public record of every person's actions. If b is equal to zero, then there is no public record of any person's actions. For both of those special cases, the model reduces to special cases of Kocherlakota and Wallace (1997). As we will see, a person whose history is known can be induced to produce without receiving something tangible in exchange, because the person can be rewarded and punished in the future for actions they take currently; in contrast, a person whose history is not known must receive something tangible in order to produce.

We also assume that each person has a technology that permits the person to create indivisible and perfectly durable objects, called notes. The notes issued by a single person are uniform, but distinguishable from those issued by any one else, so that counterfeiting is not a problem. We also assume that any person can always destroy or freely dispose of notes that the person has acquired. We will see that an optimum does not require note issue by those whose histories are private information, the non bankers, and does require note issue and destruction (redemption) by those whose histories are known, the bankers.

To keep the model simple, we assume that each person can carry from one date to the next at most one unit of notes issued by others or one unit of outside money. As we explain in the conclusion, we believe that the beneficial role of notes is robust to weakening that assumption. We also assume that each trader in a meeting is able to see the trading partner's note or outside money holdings.

[6]The above specification is much like that in Shi (1995) and Trejos and Wright (1995).

3 Outside money, no banking sector, and no productivity shock

Here we assume that $b = 0$, that there is some outside money, and that there is no productivity shock, $p = 0$. We assume a symmetric initial distribution of outside money and, therefore, let m_i denote the fraction of each type with i units of money divided by N, so that $m_0 + m_1 = \frac{1}{N}$. We also assume that m_0 is positive so that not everyone is at capacity when all the outside money is held. We begin by specifying the set of incentive feasible allocations and do this exactly as in Kocherlakota and Wallace. We first describe a trading mechanism in which people in meetings play a coordination game: two people in a meeting simultaneously propose a net trade; if they name the same net trade, then that trade occurs; if they name different net trades, then each person leaves the meeting and proceeds to the next date with what the person owned prior to the meeting. Then we define symmetric and stationary equilibrium outcomes of this mechanism.[7] An allocation is incentive feasible if it is one of these equilibria.

To simplify the exposition, we anticipate some of the consequences of symmetry in our description of the strategies in meetings. One symmetry we impose is that all notes issued by non bankers are treated the same. This implies that a non banker will never produce in order to acquire a note issued by a non banker because the non banker could always hand out his or her own note instead. In other words, such symmetry assures that notes issued by non bankers do not circulate among non bankers. It follows that such notes are superfluous.[8] We use that fact and ignore note issue by non bankers in what follows. We also anticipate, as we will explain, some of the incentive constraints.

Meetings are either single-coincidence meetings or no-coincidence meetings. When two non bankers meet, trade can occur only if the meeting is a single-coincidence meeting and if the producer does not have a note and the consumer does — a consequence of the fact that trade among non bankers

[7] Kocherlakota and Wallace show that the set of symmetric and stationary equilibrium outcomes of this coordination mechanism is the same as those of a somewhat general trading mechanism that permits each person to leave a meeting without trading. Their first proposition applies to this setting.

[8] Even without symmetry, note issue by a subset of non bankers is not consistent with a steady state because no one would retire or destroy such notes and no individual banker in in the subset issuing notes would refrain from issuing another.

must be *quid pro quo* trade. In such a meeting, each non banker names a level of production, x_n, to be produced by the producer in the meeting and consumed by the consumer in the meeting. If they name the same positive amount, then they behave accordingly and the consumer transfers his or her note to the producer; otherwise, nothing happens in the meeting and they go on to the next date.

We denote by s coordinated actions in the above game that are symmetric across agent types and stationary. Symmetry and stationarity include the requirement that the action taken in a meeting depends only on the kind of meeting the people are in. Thus, s is simply some $x_n \in \mathbb{R}_+$, and s is incentive feasible if no one wants to deviate from the x_n specified.[9]

In order to express the no-defection conditions, we first describe in terms of x_n the expected discounted utilities of non bankers as of the beginning of a period before a meeting. We let v_i be that of a non banker with i units of money. For $x_n > 0$, we have

$$v_1 = \beta v_1 + m_0[u(x_n) + \beta(v_0 - v_1)] \tag{1}$$

and

$$v_0 = \beta v_0 + m_1[-x_n + \beta(v_1 - v_0)]. \tag{2}$$

These expressions are written under the presumption that when x_n is produced and consumed, a unit of outside money changes hands. Notice that m_i is the probability of meeting a particular type with i units of money. In the expression for v_1, m_0 is the probability of meeting a producer without money, while in the expression for v_0, m_1 is the probability of meeting a consumer with a unit of money. The no-defection conditions for non bankers assume that defection on the part of non bankers goes unpunished because such defection does not become part of a public record. Thus, the conditions are simply that the action in s is weakly preferred to leaving the meeting with what was brought into the meeting. There are two such conditions, one for the consumer and one for the producer:

$$u(x_n) + \beta(v_0 - v_1) \geq 0 \tag{3}$$

[9] Kocherlakota and Wallace include in s production levels by the consumer and production levels in no-coincidence meetings and show that those should be zero in the case under consideration.

and

$$-x_n + \beta(v_1 - v_0) \geq 0. \tag{4}$$

These conditions, non negative gains from trade, give rise to the following definition of incentive feasible allocations.

Definition 1 *An allocation $s = x_n$ is incentive feasible if there exists (v_1, v_0) such that (1)-(4) hold.*

Let $v = N(m_1 v_1 + m_0 v_0)$ define the expected utility of non bankers, an expectation taken with respect to the stationary (and initial) distribution of money holdings for non bankers. Then our optimum problem is the following.

Problem 1. *Maximize v by choice of an incentive feasible s (an x_n that satisfies definition 1).*

This problem is easily solved. By (1) and (2),

$$\frac{1-\beta}{N} v = m_0 m_1 [u(x_n) - x_n] \tag{5}$$

and

$$(\rho + \frac{1}{N})\beta(v_1 - v_0) = m_0 u(x_n) + m_1 x_n, \tag{6}$$

where $\rho = (1-\beta)/\beta$. Using (6), inequality (4) is equivalent to

$$(1 + \frac{\rho}{m_0}) x_n \leq u(x_n), \tag{7}$$

while inequality (3) is equivalent to $[m_1/(m_1+\rho)]x_n \leq u(x_n)$. It follows that satisfaction of (7) implies satisfaction of (3). Therefore, by (5), problem 1 is equivalent to choosing x_n to maximize $u(x_n) - x_n$ subject to (7). Let x' be the unique solution to $\frac{du(x)}{dx} = 1$ and let x'' be the unique positive x_n satisfying (7) at equality. Then the solution to problem 1 is $x_n = x^*$, where $x^* = \min\{x', x''\}$ (see figure 1). Because $m_0 > 0$, $x^* > 0$.

We now describe how the solution depends on the amount of money, m_1. First, the constraint set, (7), is decreasing in the amount of money. And because $m_0 m_1$ is a maximum at $m_0 = m_1 = \frac{1}{2N}$, it follows that welfare is strictly decreasing in the amount of money for $m_1 \geq \frac{1}{2N}$. For $m_1 \leq \frac{1}{2N}$,

welfare depends on whether constraint (7) is binding. If $x^* < x'$ at $m_1 = \frac{1}{2N}$, then welfare reaches a maximum at some $m_1 \in (0, \frac{1}{2N})$; otherwise, it reaches a maximum at $m_1 = \frac{1}{2N}$. Notice that the price of the good in terms of money, which is also the price level in this economy, is $\frac{1}{x^*}$. If there is some $m_0 < \frac{1}{N}$, say m_0', such that x' satisfies (7) with equality at $m_0 = m_0'$, then the solution to problem 1 is $x_n = x'$ for all $m_1 \leq \frac{1}{N} - m_0'$. Therefore, for all such low amounts of money, the solution is a price that does not depend on the amount of money.[10] For all $m_1 \geq \frac{1}{N} - m_0'$, the solution is a price that is increasing in the amount of money. Alternatively, in terms of bargaining or the division of the gains from trade, for all $m_1 < \frac{1}{N} - m_0'$, the gains are divided among the consumer and the producer with more going to the consumer the larger is m_1. For all $m_1 \geq \frac{1}{N} - m_0'$, all the gains go to the consumer and $v_0 = 0$.

As just noted, we can interpret the solution to problem 1 as a suggested price. In the coordination game, a potential defector sees as options either trading at the suggested price or not trading at all. However, the solution to problem 1 is not only Nash given those options; it is also robust to cooperative deviation by a pair in a meeting. Because the game in a meeting resembles a fixed-sum game (the producer would like to give up as little as possible for a unit of money and the consumer would like to get as much as possible for a unit of money), there is no cooperative and favorable deviation from the problem 1 solution for a pair in a meeting.

4 A banking sector, no outside money, and no productivity shock

We now assume that there is a banking sector; that is, $b \in (0, \frac{1}{N}]$. We are interpreting b as both the exogenous maximum size of the banking sector and the actual size. For b sufficiently close to zero, which we consider the relevant assumption historically, proposition 2 implies that full utilization of the capacity of the banking sector is optimal. And to keep matters relatively simple, we now assume that there is no outside money, and we continue to assume that $p = 0$. We again follow Kocherlakota and Wallace in defining incentive feasible allocations. We first describe the appropriate coordination

[10] In this case, the solution resembles a customary price that over a range does not depend on the amount of money.

game in which two people in a meeting simultaneously propose a net trade. As above, the net trade is carried out if they propose the same net trade; otherwise each person leaves the meeting with what the person brought into the meeting. Then we define symmetric and stationary equilibrium outcomes of this game.[11] An allocation is incentive feasible if it is one of these equilibria. As we will see, symmetry and stationarity have important consequences here.

As in section 3, we anticipate some of the consequences of symmetry in our description of the strategies in meetings. We now assume that all notes issued by non bankers are treated the same and that all notes issued by (nondefecting) bankers are treated the same. The first implies, as above, that a non banker does not produce in order to acquire a note issued by a non banker. It follows, as above, that such notes are superfluous.[12] In particular, they are not needed to induce bankers to produce, because the trading histories of bankers are known. Therefore, we continue to ignore note issue by non bankers and, henceforth, use the term notes to refer to notes issued by bankers. The assumption that all notes issued by bankers are treated the same implies that they are superfluous in trades between bankers. We also anticipate, as we will explain, some of the incentive constraints.

We describe, in turn, meetings between two bankers, between two non bankers, and between a banker and a non banker. Meetings are either single-coincidence meetings or no-coincidence meetings. When two bankers meet, trade occurs only if the meeting is a single-coincidence meeting. In such a meeting, each names a level of production, y, to be produced by the producer in the meeting and consumed by the consumer in the meeting. When two non bankers meet, trade can occur only if the meeting is a single-coincidence meeting and if the producer does not have a note and the consumer does. In such a meeting, each names a level of production, x_n, to be produced by the producer in the meeting and consumed by the consumer in the meeting. If they name the same positive amount, then they behave accordingly and the consumer transfers his or her note to the producer; otherwise, nothing happens in the meeting and they go on to the next date.

[11]The Kocherlakota-Wallace demonstration that the set of symmetric and stationary equilibrium outcomes of this coordination mechanism is the same as those of a somewhat general trading mechanism that permits each person to leave a meeting without trading can be adapted to the setting in this section.

[12]The symmetry is crucial for this result. Without it, the following possibility arises: a subset of nonbankers issues notes which are used by the other non bankers and are retired by bankers. We have not studied whether such non banker note issue can be optimal.

There are four nonvacuous kinds of meetings between a banker and a non banker. If a non banker has a note, then the non banker cannot be induced to produce. Therefore, it is only necessary to consider a meeting between such a non banker who is a consumer in a single-coincidence meeting with a banker. They each name a level of production, y_1, and a note transfer variable $\delta \in \{0,1\}$, where 1 (0) means that the non banker surrenders (does not surrender) his or her note. If a non banker has no note, then there are three kinds of meetings to consider. If the meeting is a no-coincidence meeting, then each names a note transfer variable $\gamma \in \{0,1\}$. If the banker is the producer in a single-coincidence meeting, then each names a level of production, y_0, and a note transfer variable $\theta \in \{0,1\}$. Finally, if the non banker is the producer, they each name a level of production, x_b, and a note transfer variable $\alpha \in \{0,1\}$. For γ, θ, and α, 1 (0) means that the banker gives (does not give) the non banker a note.

We now denote by s coordinated actions in the above game that are symmetric across agent types and stationary. Symmetry and stationarity include the requirement that the action taken in a meeting depends only on which of the above six kinds of meetings the people are in. Thus, s is now the vector, $(x_n, x_b, y, y_0, y_1, \alpha, \gamma, \delta, \theta) \in \mathbb{R}^9_+$, where, as indicated above, the first two components are production levels by non bankers, the next three are production levels by bankers, and the remaining four variables, each an element of $\{0,1\}$, describe transfers of notes. A vector s is an incentive feasible allocation if two conditions are satisfied: s is consistent with an unchanging stock of notes held by non bankers and that stock and s are such that no one wants to deviate from the actions in s.

In order to express the constant stock-of-notes condition, we now let $m_i N$ denote the fraction of each type who are non bankers and who start with i notes ($i = 0, 1$) and let bN denote the fraction of each type who are bankers, so that

$$m_0 + m_1 + b = \frac{1}{N}. \tag{8}$$

In terms of this notation, (m_0, m_1) is constant if (8) holds and if

$$m_0 b\alpha + m_0 b(N-2)\gamma + m_0 b\theta = m_1 b\delta, \tag{9}$$

where the left side of (9) is the inflow into note holdings of non bankers and the right side is the outflow. If $\alpha = \gamma = \theta = \delta = 0$ (bankers neither issue notes

nor accept them), then any (m_0, m_1) satisfying (8) also satisfies (9). However, if $\alpha = \gamma = \theta = \delta = 0$, then no stock of outstanding notes can be approached from nearby stocks. Because it makes sense to require that any positive stock be approachable from nearby stocks, we associate $\alpha = \gamma = \theta = \delta = 0$ uniquely with $m_1 = 0$. Therefore, we add as a requirement,

$$m_1 > 0 \quad \text{implies} \quad (\alpha, \gamma, \theta, \delta) \neq (0, 0, 0, 0). \tag{10}$$

Our specification implies that there are only a few steady state magnitudes of m_1, notes held by non bankers. We record what they are and a fact about stability in the following lemma.

Lemma 1 *There are seven possible steady state magnitudes of m_1 consistent with (8)-(10): 0, m_0, $2m_0$, $(N-2)m_0$, $(N-1)m_0$, Nm_0, and $\frac{1}{N}-b$. Moreover, if $(\alpha, \gamma, \delta, \theta)$ is constant at its steady state magnitude, then each interior steady state (one satisfying $m_1 \in (0, \frac{1}{N} - b)$) is stable.*

Notice that all the interior steady states satisfy $m_1 > m_0$. Therefore, from our results in the last section, a reasonable surmise is that a steady state with $m_1 = m_0$ has some desirable properties.

We now turn to no-defection conditions. In order to express them, we first describe in terms of the components of s the expected discounted utilities of non bankers and that of bankers, each as of the beginning of a period before a meeting. We now let v_i be that of a non banker with i notes and let w be that of a banker. We have

$$v_1 = \beta v_1 + m_0[u(x_n) + \beta(v_0 - v_1)] + b[u(y_1) + \delta\beta(v_0 - v_1)], \tag{11}$$

$$\begin{aligned} v_0 \;=\; & \beta v_0 + m_1[-x_n + \beta(v_1 - v_0)] + b[u(y_0) + \theta\beta(v_1 - v_0)] \\ & + b\alpha[-x_b + \beta(v_1 - v_0)] + b(N-2)\gamma\beta(v_1 - v_0) \end{aligned} \tag{12}$$

and

$$w = \beta w + b[u(y) - y] + m_0 \alpha u(x_b) - m_0 y_0 - m_1 y_1. \tag{13}$$

The expressions for v_1 and v_0 are written as a sum of the product of the probability of a particular kind of meeting and the net gain from that meeting implied by s. For example, the expression for v_1 uses the fact that a non

banker with a note does not produce. With probability m_0, such a non banker is in a single-coincidence meeting with a non-banker producer who does not have a note, while with probability b such a non banker is in a single-coincidence meeting with a banker producer. In the former meeting, the non banker surrenders his or her note, while in the latter meeting the non banker does if $\delta = 1$. The expression for v_0 is analogous. The expression for w reflects the fact that there is no individual state variable for bankers.

The following no-defection conditions assume again that defection on the part of non bankers goes unpunished. In contrast, defection by a banker is punished by having the defecting bank get the payoff from autarky, which is zero. (Below, we offer two interpretations of how the autarky punishment of a defecting bank is accomplished.) The no-defection conditions are now

$$u(x_n) + \beta(v_0 - v_1) \geq 0, \tag{14}$$

$$u(y_1) + \beta(v_0 - v_1) \geq 0 \;\text{ if } \delta > 0, \tag{15}$$

$$-x_n + \beta(v_1 - v_0) \geq 0, \tag{16}$$

$$-x_b + \beta(v_1 - v_0) \geq 0 \;\text{ if }\; \alpha > 0, \tag{17}$$

$$v_1 \geq v_0 \geq 0 \;\text{ and }\; w \geq 0, \tag{18}$$

and

$$-\max\{y,\, y_0,\, y_1\} + \beta w \geq 0. \tag{19}$$

Inequalities (14)-(17) express the requirement that a non banker receive nonnegative gains from trade. Inequalities (18) express the free disposal possibility and the fact that autarky is always an option for any person. Inequality (19) expresses the requirement that the payoff to bankers when they produce cannot be worse than autarky.

These conditions give rise to the following definition of incentive feasible allocations.

Definition 2 *An allocation $s = (x_n, x_b, y, y_0, y_1, \alpha, \gamma, \delta, \theta)$ is incentive feasible if there exists (m_0, m_1) and (v_1, v_0, w) such that (8)-(19) hold.*

Our specification is consistent with two interpretations of how the autarky punishment of a defecting bank is accomplished. One interpretation is that the response to defection by a banker triggers autarkic play, $s \equiv 0$, by every person in every subsequent meeting. The other is that it triggers autarkic play only by the defecting banker and by all trading partners of the defecting banker. The virtue of the first, relative to the second, is that it is robust to cooperative defections by any pair in a meeting. If it is believed that $s \equiv 0$ will be played by everyone in the future, then there is no beneficial cooperative defection from $s \equiv 0$ by any pair in a meeting. The second interpretation is not robust to cooperative defections by pairs in a meeting. The critical meeting is between a non banker without a note who produces what the defecting bank consumes. The non banker is not willing to defect by accepting a note issued by the defecting bank; the note is identifiable and if it will not be accepted by others in the future, then the non banker does not produce currently to obtain it. However, the non banker is willing to defect by accepting from the defecting bank a note issued by another bank, because, under the second alternative, such a note will be accepted by others in the future.

With $v = (m_1 v_1 + m_0 v_0)/(m_1 + m_0)$, the expected utility of non bankers, an expectation taken with respect to the stationary distribution of note holdings for non bankers, our optimum problem is the following.

Problem 2. *Maximize v by choice of an incentive feasible s (one that satisfies definition 2) subject to $w \geq v$.*

We first note that a solution to problem 2 exists and is continuous.

Lemma 2 *A solution to problem 2 exists. If $v^*(b)$ denotes the maximized value of v in problem 2 and if $G(b)$ denotes the set of maximizers, then $v^*(b)$ is a continuous function and G is uhc in b.*

The proof is an application of the Theorem of the Maximum. The boundedness of u implies that the v_i's and w are bounded and those bounds and the incentive constraints involving the production components of s imply bounds on production. That fact is used in the proof of lemma 2. The fact that those bounds do not depend on b is used in the proof of proposition 2.

We now turn to some description of the solution to problem 2.

Proposition 1 *If $b \in (0, \frac{1}{N})$, then any solution to problem 2 has bankers both creating and destroying notes.*

The proof shows that any allocation without notes being created and destroyed is not optimal. Creation means $\max\{\alpha, \gamma, \theta\} = 1$ and destruction means $\delta = 1$. If both do not occur, then, by (8)-(10), either $m_1 = 0$ and $\max\{\alpha, \gamma, \theta\} = 0$ (no non banker has a note and none are being created), or $m_0 = 0$ and $\delta = 0$ (every non banker has a note and none are being destroyed). In either case, non bankers do not produce so that v and w depend only on the y's; in particular, non banker welfare depends only on y_0 and y_1, which in this case can be thought of as gifts from bankers. Starting with such a gift-only incentive feasible allocation, we construct an incentive feasible alternative with $m_1 \in (0, \frac{1}{N} - b)$ and show that it increases both v and w. The alternative we construct has $m_1 = m_0$ and $\alpha = 1$, note issue only in exchange for production.

Proposition 1 says that any optimum has note issue by bankers and, therefore, has note issue and destruction ($\delta = 1$) — destruction in the sense that whenever a banker meets a non banker who has a note and who consumes what the banker produces, the banker produces enough to get the non banker to surrender the note. It is reasonable to interpret such action to be the result of regulation, because a banker who did not face punishment for defecting would not produce in order to acquire an outstanding note. Also, proposition 2 does not depend on the particular criterion we use. Its conclusion and that of lemma 2 also hold for any welfare function that is increasing and continuous in v and w.

We next turn to a further description of the optimum. A particular kind of allocation, which we call a *no-gift allocation*, plays an important role in that description and in a subsequent result. We, therefore, define such allocations, without regard to whether or not they are incentive feasible.

Definition 3 *A no-gift allocation s satisfies $\alpha = \delta = 1$, $\gamma = \theta = 0$ (and, hence, $m_1 = m_0$), $y_0 = 0$, and $x_n = x_b = y = y_1 = x$ for some x.*

Notice that in a no-gift allocation, notes are created only when the non banker produces as much as he or she would to acquire a note from a non banker, and that bankers do not produce for non bankers except to acquire (and destroy) a note and, in that circumstance, produce as much as non bankers do. The following lemma describes the solution to problem 2 from among no-gift allocations. For ease of exposition, we also refer to a no-gift allocation s as the x associated to s in the above definition.

Lemma 3 *Let x' denote the solution to $\frac{du(x)}{dx} = 1$, let $x''(b)$ denote the unique positive solution for x to $(1 + \frac{2\rho}{\frac{1}{N}+b})x = u(x)$, and let $\hat{x}(b) = \min\{x', x''(b)\}$. Then $\hat{x}(b)$ is the x that solves problem 2 from among no-gift allocations. Moreover, $\hat{x}(b)$ is weakly increasing in b and if $\hat{v}(b)$ and $\hat{w}(b)$ denote the expected discounted utilities of non bankers and bankers attained by $\hat{x}(b)$, then*

$$\hat{v}(b) = \frac{\frac{1}{N} + b}{4(1-\beta)}[u(\hat{x}(b)) - \hat{x}(b)] \tag{20}$$

and

$$\hat{w}(b) = 2\hat{v}(b). \tag{21}$$

The proof is straightforward. For any no-gift allocation, x, the implied expected discounted utility for non bankers is the right-side of (20) evaluated at x. That follows directly from (11) and (12) evaluated at a no-gift allocation. It also follows from (13) that $w = 2v$ for any such allocation. (An intuitive explanation is that in a no-gift allocation bankers produce and consume twice as frequently as do non bankers because there is no state variable for bankers that helps determine what they do.) That, in turn, implies that the constraint $w \geq v$ in problem 2 is not binding for any no-gift allocation satisfying $u(x) \geq x$. Also, $2v = v_1 + v_0$, (a consequence of $m_0 = m_1$) and $w = 2v$ imply that satisfaction of the non banker production incentive constraint (14) implies satisfaction of the banker incentive constraint (19). Consequently, solving problem 2 by choice of a no-gift allocation is very simple; it amounts to choosing an x to maximize v subject only to (16) (and $x_n = x$), because such a solution necessarily satisfies (14) and $u(x) \geq x$. Finally, it is easily shown that (16) becomes $(1 + \frac{2\rho}{\frac{1}{N}+b})x \leq u(x)$ for no-gift allocations, which accounts for our definition of $x''(b)$. That $x''(b)$ is increasing in b is immediate from its definition. That, in turn, implies that $\hat{x}(b)$ is weakly increasing.

Obviously $\hat{v}(b)$ and $\hat{w}(b)$ are each continuous and strictly increasing. Also, $\hat{v}(b)$ is a lower bound on $v^*(b)$, the value attained by any solution to problem 2 (from among all allocations). The following proposition says a little more about the relationship between $v^*(b)$ and $\hat{v}(b)$.

Proposition 2 *Let $s^*(b)$ be a solution to problem 2 and let $v^*(b)$ be the maximized objective in that problem. Then,*

(i) if b is sufficiently close to zero, then $s^*(b)$ satisfies $\alpha = \delta = 1$ and $\gamma = \theta = 0$;

(ii) $\lim_{b \to 0} v^*(b) = \hat{v}(0)$; *and*
(iii) $\lim_{b \to \frac{1}{N}} v^*(b) = \hat{w}(\frac{1}{N})$.

The proof of conclusion (i) uses the fact that if b sufficiently close to 0, then v depends mainly on what happens in trades between non bankers. In accord with what we saw in section 3, those trades are more beneficial with $m_1 = m_0$ than with m_1 discreetly higher than m_0, the only alternatives other than $m_1 = 0$. We first show that $m_1 = m_0$ is necessary for a maximum of v without regard to the constraint $w \geq v$. We then complete the proof of the first part of the proposition by showing that $\alpha = 1$ is necessary and sufficient for $w \geq v$. Necessity follows because with b close to zero, production by non bankers is the only source of positive utility for bankers. Sufficiency follows from the choice of a no-gift allocation and lemma 3. The proof of (ii) is a consequence of the fact that as $b \to 0$, the production levels in meetings with bankers do not matter because the frequency of such meetings is arbitrarily small. The proof of (iii) is a consequence of the fact that as $b \to \frac{1}{N}$, both v and w are determined by what happens in meetings with bankers. It follows that $y = \hat{x}(\frac{1}{N})$ (as defined in lemma 3) maximizes w at $b = \frac{1}{N}$ and implies $w = \hat{w}(\frac{1}{N})$.[13] Because v cannot exceed w, the proof amounts to showing that it is incentive feasible to have $v = \hat{w}(\frac{1}{N})$. One way to accomplish that is to have non bankers produce zero and receive from banker producers a gift which is positive and less than $\hat{x}(\frac{1}{N})$.

The proposition 2 facts about $v^*(b)$ are depicted in figure 2. As indicated there, we have not been able to show that $v^*(b)$ is bounded above by $\hat{w}(b)$ or that it is increasing throughout its domain. However, the continuity of $\hat{v}(b)$ and $v^*(b)$, the strict monotonicity of $\hat{v}(b)$, and conclusion (ii) of proposition 2 imply that $v^*(b)$ is increasing in the neighborhood of $b = 0$. Therefore, if the capacity to keep track of trading histories is sufficiently small, which we think is the historically plausible situation, then the society should use all of it.

Our analysis of steady states is locally robust in the following sense. If the initial stock of bank notes is sufficiently close to that implied by an incentive feasible allocation, then there exists a sequence of actions that is incentive feasible, converges to the incentive feasible steady state, and

[13] As noted above, when $b = \frac{1}{N}$, our model is a special case of Kocherlakota and Wallace. The result that $y = \hat{x}(\frac{1}{N})$ maximizes w depends on the assumption that y is assumed to be state independent. Kocherlakota and Wallace allowed for some state dependence and were not able to show that state independence achieves an optimum for w when $\hat{x}(\frac{1}{N}) < x'$.

implies convergence of the stock of notes to the steady state stock. As we now explain, this is a consequence of the convergence noted in lemma 1.

Given an incentive feasible s and an initial condition for m_1, the sequence of actions is constructed as follows. First, the note transfer variables and the production levels for which incentive constraints are slack at the steady state are fixed at their steady state values. A sequence for the production levels for which incentive constraints are binding at the steady state is chosen by imposing at equality dynamic versions of the corresponding incentive constraints. (The dynamic versions of the incentive constraints are (14)-(19) with production levels dated t and with v_1, v_0, and w dated $t+1$.) Therefore, if s_t denotes the date t term in the sequence of actions, s_t is a function of expected discounted utilities dated $t+1$. When that function is substituted into the dynamic versions of (11)-(13) (versions with the left side variables dated t , with actions on the right side dated t, and with the expected discounted utilities on the right side dated $t+1$), the result is three equations involving v_1, v_0, and w dated t and $t+1$ and m_1 and m_0 dated t. When augmented by the autonomous law of motion for m_1 and m_0 described in the proof of lemma 1, the result is a first-order difference equation system in (v_1, v_0, w, m_1, m_0). Because the initial conditions are m_1 and m_0 (or one of them if we use the fact that the sum of the two is constant for all t) and because the system is recursive in that the law of motion for the m's does not depend on the v's and w, the convergence in lemma 1 implies the existence of a convergent sequence for the v's and w and, hence, for the production components of s.[14]

The stationarity and symmetry we have imposed are responsible for, among other things, the conclusion that there is a finite set of steady state amounts of notes and that $m_1 > 0$ implies $m_1 \geq m_0$. That implication matters because, as we saw in section 3, the optimal amount of money may satisfy $0 < m_1 < m_0$. One way to expand the set of steady state stocks of notes is to adopt randomized strategies for bankers: each banker issues a note with some probability and destroys a note with some probability. We were reluctant to assume such randomness, because knowledge of trading histories would then have to include knowledge of the randomizing device being used.

[14] Provided the implicit function theorem can be invoked to allow us to solve the dynamic versions of (11)-(13) with the substitutions just described for the expected discounted utilities at $t+1$, the asserted result is an application of standard local convergence results for first-order difference equations (see, for example, Theorem 6.6 in Stokey and Lucas with Prescott 1989, p. 147).

Another way to expand the set of steady state stocks of notes is to abandon symmetry and our extreme form of stationarity. For example, because bankers are identified by something like an identifying number, we could have had some of the bankers issue notes only at some dates; or, because a banker's realized trading history is known, we could have made note issue and destruction depend on that history. Those seem to be straightforward extensions of what we have done.

5 Private-information productivity shocks

Here we introduce the productivity shock which determines whether or not a person can produce. The shock is such that a non banker has essentially no incentive to misrepresent its outcome. After all, a non banker always has the option of not producing, the consequence of which is only that the non banker does not acquire a note. A banker, however, has an incentive to misrepresent the outcome of the productivity shock unless the banker receives an explicit reward for producing. The only possible reward is to make the banker's subsequent consumption depend on the banker's announced productivity shock.

Rather than attempt to deal with this set up generally, we here focus on a special case, which we think is instructive. We study, as we did in part of proposition 2, a situation in which the banking sector is sufficiently small. When that is true, non banker and banker expected utilities depend hardly at all on what happens in meetings with bankers. Moreover, we here also assume that the discount factor is sufficiently high so that $m_1 = m_0$ is the optimal steady state stock of notes in the hands of non bankers when there are no bankers. And, finally, we assume that the probability of not being able to produce, p, is sufficiently small.

We begin with a somewhat general description of allocations and of the optimum problem as we now pose it. We assume that each banker prior to being paired and prior to receiving a productivity shock is characterized by a state variable denoted z, where $z \in Z$. A banker's state is public information. As noted above, the productivity shocks are realized simultaneously with being paired at random in meetings, and the realization of productivity shocks is an independent drawing from the same binomial distribution for each person at each date.

As above, we continue to assume that people in a meeting play a co-

ordination game. Now, however, in accord with the revelation principle, we assume that there is a prior stage to the game in which the *potential producer* in single-coincidence meetings announces his or her realized productivity shock. (Here and below *potential producer* means the type $n-1$ person in a meeting with a type n person and the term *potential consumer* means the type n in such a meeting, whether or not the type $n-1$ person has realized the good productivity shock and whether or not the type n person has a note.) The people in a meeting — and, in particular, people making productivity announcements — know the kind of meeting they are in (no-coincidence or single coincidence), the note holdings of any non banker in the meeting, and the state of any banker in the meeting. We denote by $\mathcal{I}$ the set of possible preannouncement information known by those in a meeting and we let the productivity announcement, if there is one, be either 1 or 0.

We can define an allocation to be the s of the previous section and a rule for augmenting the state of bankers, both of which are now made dependent on the objects in the set $\mathcal{I}$ and the productivity announcement. That is, $s : \mathcal{I} \times \{0,1\} \to \mathbb{R}^9_+$ and there is a function, denoted $z' : \mathcal{I} \times \{0,1\} \to Z$, where z' determines the state of the banker at the next date. With this change in the meaning of s, we define an incentive feasible allocation to be a pair (s, z') such that two conditions are met: (s, z') is consistent with an unchanging stock of notes in the hands of non bankers and an unchanging distribution of bankers over states in Z; and each person is willing to truthfully announce the person's realized productivity shock and does not want to deviate from the actions in s. The optimum problem is now

Problem 3. *Maximize v (as defined in the last section) by choice of an incentive feasible (s, z') subject to $Ew \geq v$, where Ew denotes the expectation of banker utility with respect to the stationary distribution of bankers over the set Z.*

Because we work in detail only with very special allocations, very special pairs (s, z'), we will not present an explicit definition of incentive feasibility for general pairs (s, z'). Instead, we describe a special class of allocations, called *two-state allocations*, and present an explicit definition of incentive feasibility for them. For such allocations, Z contains only two elements and we can dispense with the notation for elements of Z by letting w_i for $i = 0, 1$ denote the expected discounted utility of a banker in state i.

Definition 4 *A two-state allocation is an (s, z') that satisfies the following conditions:*

(i) A banker issues a note if and only if the banker is a potential consumer in a meeting with a non banker and the non banker produces $x_{b0} > 0$ when the banker is in state 0 and $x_{b1} > x_{b0}$ when the banker is in state 1.

(ii) If a banker is a potential producer and realizes the good productivity shock, then the banker produces y for a banker, produces nothing for a non banker without a note, and produces y_1 for a non banker with a note and destroys that note.

(iii) If the banker produced the amount called for in (ii) in the most recent meeting in which the banker was a potential producer (whether or not the banker realized the good productivity shock), then the banker is in state 1. If not, then the banker is in state 0.

According to (i) and (ii), only production by non bankers for bankers is dependent on the banker's state. Also, according to (iii), the banker's state is not dependent on the banker's consumption experience. It turns out that such two-state allocations are sufficient for our purposes. (We do not claim that any such allocation maximizes the utility of bankers.)

If people behave as described in definition 4 whenever their production realizations permit such behavior, then a unique steady state outstanding stock of notes and distribution of bankers over states is implied. This steady state is described in the following lemma.

Lemma 4 *Let b_i denote the fraction of each type who are bankers in state i divided by N, so that $b_0 + b_1 = b$. If people behave according to definition 4 whenever their production realizations permit such behavior, then there exists a unique constant distribution of notes in the hands of non bankers and of bankers over states given by $m_1 = m_0 = \frac{1}{2}(\frac{1}{N} - b)$ and $b_0 = pb$, $b_1 = (1-p)b$. Moreover, this constant distribution is locally stable.*

These conclusions follow directly from the law of motion for the stock notes in the hands of non bankers and the distribution of bankers over the two states that is implied by definition 4. The local stability is obtained easily because the law of motion is recursive; note holdings of non bankers at date $t + 1$ do not depend on the distribution of bankers over states at t.

We now express non banker and banker expected utilities in terms of the production levels in two-state allocations and the constant distributions

given in lemma 4 under the assumption that people behave in accord with their true productivities.

$$\begin{aligned} v_1 &= \beta v_1 + (1-p)m_0[u(x_n) + \beta(v_0 - v_1)] \\ &\quad + (1-p)b[u(y_1) + \beta(v_0 - v_1)], \end{aligned} \tag{22}$$

$$\begin{aligned} v_0 &= \beta v_0 + (1-p)m_1[-x_n + \beta(v_1 - v_0)] \\ &\quad + (1-p)\sum_{j=0}^{1} b_j[-x_{bj} + \beta(v_1 - v_0)], \end{aligned} \tag{23}$$

$$\begin{aligned} w_1 &= \beta w_1 + (1-p)b[u(y) - y] + (1-p)m_0 u(x_{b1}) \\ &\quad -(1-p)m_1 y_1 + p(b + m_1)\beta(w_0 - w_1) \end{aligned} \tag{24}$$

and

$$\begin{aligned} w_0 &= \beta w_0 + (1-p)b[u(y) - y] + (1-p)m_0 u(x_{b0}) \\ &\quad -(1-p)m_1 y_1 + (1-p)(b + m_1)\beta(w_1 - w_0). \end{aligned} \tag{25}$$

The expressions for v_1 and v_0 in (22) and (23) are straightforward amended versions of (11) and (12). The last term in the expressions for w_1 and w_0 involve the probability that a banker switches states: $p(b + m_1)$ is the probability of being called on to produce and not being able to, an event which triggers a switch from state 1 to state 0; while $(1-p)(b+m_1)$ is the probability of being called on to produce and being able to, an event which triggers a switch from state 0 to state 1.

The no-defection conditions are now,

$$\min\{u(x_n), u(y_1)\} + \beta(v_0 - v_1) \geq 0, \tag{26}$$

$$-\max\{x_n, x_{b1}\} + \beta(v_1 - v_0) \geq 0, \tag{27}$$

$$v_1 \geq v_0 \geq 0 \text{ and } w_j \geq 0 \text{ for } j = 0, 1 \tag{28}$$

and

$$-\max\{y, y_1\} + \beta(w_1 - w_0) \geq 0. \tag{29}$$

Because the conditions for note issue and destruction are given, the incentive constraints for non bankers can be written more succinctly than in section 4. The production incentive constraint for bankers now requires that a banker, in either state, who is a potential producer and is able to produce be willing to do so. If the banker is in state 1 and chooses not to produce, then the banker switches to state 0 and, thereby, sacrifices $\beta(w_1 - w_0)$. If the banker is in state 0 and chooses not to produce, then the banker stays in state 0 and, thereby, sacrifices the gain from switching to state 1, $\beta(w_1 - w_0)$. Hence, (29) assures that a banker in either state is willing to truthfully announce an ability to produce.

We can now define incentive feasibility for two-state allocations.

Definition 5 *A two-state allocation, a 5-tuple of production levels $(x_n, x_{b0}, x_{b1}, y, y_1)$, is incentive feasible if there exist v_0, v_1, w_0, and w_1 such that (22)-(29) hold when (m_0, m_1, b_0, b_1) is equal to the steady state distribution of lemma 4.*

We next describe a two-state allocation that is an approximate solution to problem 3 for a region of the parameter space. The region has a sufficiently high discount factor, β, a sufficiently small probability of being unable to produce, p, and a size of the banking sector, b, that is arbitrarily close to zero.

Proposition 3 *Assume that a solution to problem 3 exists and that the maximized value of the objective, denoted $v^*(b)$, is a continuous function. Let p_0 denote the unique solution for p of $(1 - 2p)u(x') = x'$ and let ρ_0 denote the unique solution for ρ of $(1 - p_0)u(x') = (1 + 2N\rho)x'$, where x' solves $\frac{du(x)}{dx} = 1$. If $p < p_0$ and $\rho < \rho_0$, then there exists a two-state allocation with $x_n = x_{b1} = y = y_1 = x'$ that is feasible for problem 3 and that implies a magnitude of v, denoted $\tilde{v}(b)$, satisfying $\lim_{b \to 0}[v^*(b) - \tilde{v}(b)] = 0$.*

The assumption that a solution to problem 3 exists and is continuous follows directly from the Theorem of the Maximum if the set Z in problem 3 is finite. Notice that because $u(x') > x'$, the p_0 defined in the proposition

is unique and satisfies $p_0 \in (0, \frac{1}{2})$. The definition of p_0 also implies that $(1 - p_0)u(x') > x'$, which, in turn, implies that $\rho_0 > 0$.

In the proof, we demonstrate that

$$\lim_{b \to 0} v^*(b) \leq \frac{1-p}{4N(1-\beta)}[u(x') - x'] = \lim_{b \to 0} \tilde{v}(b). \tag{30}$$

Notice that $\frac{1-p}{4N(1-\beta)}[u(x') - x']$ is the product of $(1 - p)$ and $\hat{v}(0)$ when $\hat{x}(0) = x'$ (see lemma 3). That is an upper bound on $\lim_{b \to 0} v^*(b)$ because the best that can happen in the limit as $b \to 0$ is that x' is produced whenever the potential producer is able to produce (which accounts for the presence of the term $(1 - p)$) and that $m_1 = m_0$, which maximizes the probability that the potential producer does not have a note and the potential consumer does. The equality part of (30) holds by definition. Given (30), the limit conclusion follows if $v^*(b) \geq \tilde{v}(b)$. That, in turn, follows if $\tilde{v}(b)$ is feasible in problem 3. The feasibility of $\tilde{v}(b)$ in problem 3 depends on the assumptions about p and ρ. Its feasibility is related to the fact that the allocation is almost a no-gift allocation. It deviates from a no-gift allocation in having an x_{b0} that is sufficiently small. (Indeed, the only reason to have x_{b0} be positive, rather than zero, is to prevent non bankers who are unable to produce from claiming to be able to produce and, thereby, acquiring a note from a state-0 banker.) For such an allocation and for any b, as $p \to 0$, the implied v and Ew approach the payoffs from a no-gift allocation with $x = x'$. Therefore, with p sufficiently small, the constraint $Ew \geq v$ is satisfied. We assume that ρ is sufficiently small in order to insure that bankers and non bankers can be induced to produce x'. If that were not the case, then $m_1 = m_0$ would not give a solution that is close to $v^*(b)$ even for b close to zero.

Were we willing to assume that both b and p are arbitrarily close to zero, then we could have worked with a two-state allocation that is simpler and that resembles even better the situation of historically observed bankers than that given by definition 4. In that case, we could have had bankers in state 0 neither issuing notes nor consuming when they meet non banker producers. In the limit, as $p \to 0$, that would produce a steady state distribution of notes which would continue to have $m_1 = m_0$ and which, therefore, would give rise to the conclusion of proposition 3. Notice, also, that for such an allocation and for our two-state allocation, the results would be unaltered by amending the allocation to allow production of bankers for bankers to be dependent on the state of the consuming bank. Obviously, because proposition 3 assumes

that b is arbitrarily close to zero, production levels in meetings with bankers do not matter for either v or Ew.

Finally, notice that the convergence established in lemma 4 implies that our proposition 3 result is locally robust in the same sense as were the results in section 4. That is, for any (steady state) allocation satisfying definition 5, there exists a neighborhood of the steady state distribution of notes in the hands of non bankers and bankers over states such that if the initial distribution is in that neighborhood, then there exists a sequence of incentive feasible two-state allocations that converges to the (steady state) allocation. The construction of that sequence and the argument for convergence is completely analogous to that given in section 4.

6 Concluding remarks

Because note issue was fairly widespread and because it was often suggested as a remedy for currency shortages even when it did not occur, it ought to be rationalized easily. We have shown that three historically plausible assumptions make it uniquely optimal. First, outside money is indivisible and scarce. Second, a tangible medium of exchange is essential because of a lack of double coincidences, an absence of commitment, and some privacy of individual trading histories. Third, the privacy of individual trading histories is not uniform over people; more is known about the trading histories of some people than others. In those circumstances, our model implies that those whose trading histories are known should issue notes, a kind of inside money, that can be used in trade among those whose trading histories are not known. As we noted, that result holds for our welfare criterion and for any criterion that is increasing in the welfare of non bankers and that of bankers.

Our model does contain a number of extreme assumptions that deserve comment. In sections 4 and 5, we assumed that there is no outside money. However, the results we obtained there can also be regarded as limiting results as the amount of indivisible outside money approaches zero. If outside money is indivisible, then the limit, as the amount of outside money approaches zero, of solutions to the optimum problems we posed are the solutions with a zero amount of outside money.

There is a sense in which the assumed unit upper bound on individual holdings of notes or outside money forces us to assume that the amount of outside money is small. If there were no bound on individual holdings,

then there is a sense in which any amount of indivisible outside money per type implies scarcity; the indivisibility implies that small enough transactions do not occur.[15] Therefore, absent a bound on individual holdings, there would seem to be a role for note issue without any special assumptions about the amount of indivisible outside money. However, the detailed descriptions of optima in propositions 2 and 3 do depend on the unit upper bound on holdings of assets.

Also, despite the kind of local robustness to initial conditions that we noted in sections 4 and 5, our focus on steady states is a limitation of our analysis. Although having the society choose an initial stock of bank notes in the hands of non bankers does not violate ordinary technological constraints in the way, for example, that a choice of an initial capital stock would, such a choice is uncomfortably close to having the society choose an initial stock of outside money in the hands of non bankers. We have assumed that the society cannot choose the amount of outside money, because if it could, then notes would seem to be unnecessary. Having ruled out the choice of an amount of outside money, a preferable analysis would study optima starting from an initial condition with no notes in the hands of non bankers.[16]

Finally, there are, of course, some glaring discrepancies between the banks in the model and historically observed banks. Because the history of a banker is known, there are analogs of balance sheets for the banks in our model, but the balance sheets are very simple and they play almost no role. One kind of balance sheet is implied by treating banker production undertaken in connection with redemption of notes as an injection of capital and treating banker consumption that occurs in connection with the issue of notes as a depletion of bank capital, a payout of dividends. In section 4, because of the stationarity we imposed, the status of balance sheets constructed in that way play no role. Even in section 5, the only feature of such balance sheets that

[15] Wallace and Zhou (forthcoming) study a model in which people are diverse in terms of their productivities and in which, therefore, there is an explicit role for transactions of different magnitudes.

[16] The focus on steady states also accounts for a seeming discontinuity at $b = 0$. If $b = 0$, then, as in section 3, there are no notes, and, with no outside money, there is autarky or $v = 0$. However, in both sections 4 and 5, we showed that the limit of the optimum is positive as $b \to 0$. That discontinuity is an artifact of our focus on steady states. An analysis that begins with an initial condition of no notes in the hands of non bankers will imply that $v \to 0$ as $b \to 0$. The reason is that starting from an initial condition with no notes, a small enough b implies that the quantity of notes in the hands of non bankers is arbitrarily close to zero for an arbitrarily long time.

matters for our two-state allocations is the result of the most recent potential injection of capital.

A substantial generalization of the model would be required to get bankers to have more usual balance sheets — for example, balance sheets with loans as assets. While one can regard banker production for other bankers as the granting of loans, such loans do not match up in magnitude with notes issued, because notes are issued to non bankers. More generally, loans can be made only to people whose histories are, in some sense, known. In our simple model, every one with a known history is a banker. That feature is not out of line with what we see in actual economies; those with known histories often try to be bankers (see Eichenbaum and Wallace 1985).

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Appendix

Proof of Lemma 1.

(There are seven possible steady state magnitudes of m_1 consistent with (8)-(10): 0, m_0, $2m_0$, $(N-2)m_0$, $(N-1)m_0$, Nm_0, and $\frac{1}{N}-b$. Moreover, if $(\alpha, \gamma, \delta, \theta)$ is constant at its steady state magnitude, then each interior steady state (one satisfying $m_1 \in (0, \frac{1}{N}-b)$) is stable.)

The set of steady states is obvious from (8)- (10). As regards stability, the law of motion for m_1 is

$$m_1(t+1) = m_1(t) + m_0(t)[b\alpha + b(N-2)\gamma + b\theta] - m_1(t)b\delta, \tag{31}$$

where the second term on the right side is the inflow into note holdings of non bankers and the third is the outflow. Using (8) to express $m_0(t)$ in terms of $m_1(t)$, (50) becomes

$$m_1(t+1) = \phi m_1(t) + (\frac{1}{N} - b)b[\alpha + (N-2)\gamma + \theta], \tag{32}$$

where $\phi = 1 - b[\alpha + (N-2)\gamma + \theta + \delta]$. If a steady state is interior, then $\delta = 1$ and $\max\{\alpha, \gamma, \theta\} = 1$. It follows that $\phi \in [1-(N+1)b, 1-2b]$, which implies $|\phi| < 1$. □

Proof of lemma 2.

(A solution to problem 2 exists. If $v^*(b)$ denotes the maximized value of v in problem 2 and if $G(b)$ denotes the set of maximizers, then $v^*(b)$ is a continuous function and G is *uhc* in b.)

As noted, the boundedness of u implies that the v_i's and w are bounded and those bounds and the incentive constraints involving the production components of s imply bounds on production. Let $v(b,s)$ and $\Gamma(b)$ denote, respectively, the values for the objective function and constraint set of problem 2, associated with the parameter $b \in (0, \frac{1}{N}]$ and coordinated action s. The bounds on production and the weak inequalities describing the feasibility conditions thus imply that $\Gamma(b)$ is compact. Since $v(b,\cdot)$ is clearly continuous, problem 2 describes the maximization of a continuous function over a compact set which thus has a solution. The proof is completed by showing that Γ is continuous and by appealing to the Theorem of the Maximum. Notice that the constraint set depends on b only because the v_i's and w are functions of b. Since the v_i's and w enter linearly in Γ and are continuous in b, then Γ is clearly continuous. □

Proof of proposition 1.

(If $b \in (0, \frac{1}{N})$, then any solution to problem 2 has bankers both creating and destroying notes.)

If $\max\{\alpha, \gamma, \theta\} = 0$ or $\delta = 0$, then, by (8)-(10), either $m_1 = 0$ and $\max\{\alpha, \gamma, \theta\} = 0$ (no non banker has a note and none are being created),

or $m_0 = 0$ and $\delta = 0$ (every non banker has a note and none are being destroyed). Suppose that s' is incentive feasible and satisfies one of these conditions. It follows that s' is such that non bankers do not produce and that v and w are determined by two levels of output: production of bankers for bankers, y', and production of bankers for non bankers, denoted z'. It follows that

$$(1-\beta)v(s') = (1-\beta)v_0(s') = (1-\beta)v_1(s') = bu(z')$$

and

$$(1-\beta)w(s') = b[u(y') - y'] - (\frac{1}{N} - b)z'.$$

We let s'' have $m_0'' = m_1'' = \frac{1}{2}(\frac{1}{N} - b) \equiv m$, $\gamma'' = \theta'' = 0$, $\alpha'' = \delta'' = 1$, $x_n'' = x_b'' = x$, $y'' = y'$, $y_0'' = z'$, and $y_1'' = z' + x$, for an x to be determined. We show that there exist values of $x > 0$ that imply incentive feasibility of s'', $v(s'') > v(s')$, and $w(s'') > w(s')$.

From (11) and (12), we have

$$\begin{aligned} & (1-\beta)[m_1''v_1(s'') + m_0''v_0(s'')] \\ = \ & m^2[u(x) - x] + mbu(z' + x) + mbu(z') - mbx \\ \geq \ & m^2[u(x) - x] - mbx + 2mbu(z'). \end{aligned}$$

Therefore, to set the stage for the comparison between $v(s'')$ and $v(s')$, we have

$$\begin{aligned} (1-\beta)v(s'') &\geq \frac{1}{2}[mu(x) - (m+b)x] + bu(z') \\ &= \frac{1}{2}[mu(x) - (m+b)x] + (1-\beta)v(s'). \end{aligned} \tag{33}$$

From (11) and (12) we also have

$$(\rho + \frac{1}{N} + b)\beta(v_1'' - v_0'') = m[u(x) + x] + b[u(z' + x) - u(z') + x], \tag{34}$$

where $\rho = (1-\beta)/\beta$. Because $u(z' + x) - u(z') > 0$ for any $x > 0$, it follows from (34) that $x \leq \beta(v_1'' - v_0'')$ if

$$(\rho + \frac{1}{N} + b)x \leq m[u(x) + x] + bx,$$

or, equivalently, if

$$\frac{\rho + b + m}{m}x \leq u(x). \tag{35}$$

By the concavity of u, $u(z' + x) - u(z') \leq u(x)$. Therefore, it follows from (34) that $u(x) \geq \beta(v_1'' - v_0'')$ if $(\rho + \frac{1}{N} + b)u(x) \geq (m + b)[u(x) + x]$. This is implied by (35).

Now we consider w. From (13), we have

$$\begin{aligned}(1-\beta)w(s'') &= b[u(y') - y'] - mu(x) - mz' - m(z' + x) \\ &= b[u(y') - y'] - 2mz' + m[u(x) - x)] \\ &= (1-\beta)w(s') + m[u(x) - x].\end{aligned} \tag{36}$$

Because feasibility of s' implies $w(s') \geq 0$, (36) implies that the incentive constraint, $x \leq \beta w(s'')$, is satisfied provided

$$\rho x \leq m[u(x) - x],$$

which is equivalent to

$$(1 + \frac{\rho}{m})x \leq u(x). \tag{37}$$

Because $u'(0) = \infty$, it is evident that there exists $x > 0$ that satisfies the two incentive constraints, (35) and (37), and that also make the terms in x on the far right sides of (33) and (36) positive. In particular, if we define k by

$$k = \max\{1 + \frac{b}{m}, \frac{\rho + \frac{1}{N} - m}{m}, 1 + \frac{\rho}{m}\},$$

then $k > 1$ and there exists a unique $x_k > 0$ that satisfies $u(x_k) = kx_k$. Any $x \in (0, x_k)$ implies that s'' is incentive feasible and that $v(s'') > v(s')$ and $w(s'') > w(s')$. □

Proof of Lemma 3.

(Let x' denote the solution to $\frac{du(x)}{dx} = 1$, let $x''(b)$ denote the unique positive solution for x to $(1 + \frac{2\rho}{\frac{1}{N}+b})x = u(x)$, and let $\hat{x}(b) = \min\{x', x''(b)\}$. Then $\hat{x}(b)$ is the x that solves problem 2 from among no-gift allocations. Moreover, $\hat{x}(b)$ is weakly increasing in b and if $\hat{v}(b)$ and $\hat{w}(b)$ denote the expected discounted utilities of non bankers and bankers attained by $\hat{x}(b)$, then $\hat{v}(b) = \frac{\frac{1}{N}+b}{4(1-\beta)}[u(\hat{x}(b)) - \hat{x}(b)]$ and $\hat{w}(b) = 2\hat{v}(b)$.)

For any no-gift allocation, x, (11) and (12) imply that v is the right-side of (20) evaluated at x, while (13) implies that $w = 2v$. It follows that the constraint $w \geq v$ in problem 2 is not binding for any no-gift allocation satisfying $u(x) \geq x$. Also, $w = 2v$ and $2v = v_1 + v_0$ imply that satisfaction of the non banker production incentive constraint (14) implies satisfaction of the banker incentive constraint (19). Consequently, solving problem 2 by choice of a no-gift allocation amounts to choosing x to maximize v subject only to (16) (and $x_n = x$), because such a solution necessarily satisfies (14) and $u(x) \geq x$. For no-gift allocations, (16) becomes $(1 + \frac{2\rho}{\frac{1}{N}+b})x \leq u(x)$. Therefore, $x''(b)$ is the unique positive solution for x to (16) at equality. It follows that $\hat{x}(b)$ solves problem 2 among no-gift allocations. That $x''(b)$ is increasing in b is immediate from its definition. That, in turn, implies that $\hat{x}(b)$ is weakly increasing.□

Proof of Proposition 2.

(Let $s^*(b)$ be a solution to problem 2 and let $v^*(b)$ be the maximized objective in that problem. Then, (i) if b is sufficiently close to zero, then $s^*(b)$ satisfies $\alpha = \delta = 1$ and $\gamma = \theta = 0$; (ii) $\lim_{b \to 0} v^*(b) = \hat{v}(0)$; and (iii) $\lim_{b \to \frac{1}{N}} v^*(b) = \hat{w}(\frac{1}{N})$.)

Proof of (i). We begin by maximizing v without regard to the constraint $w \geq v$. From (11) and (12) and the definition of v,

$$(1-\beta)(m_0 + m_1)v = m_0 m_1[u(x_n) - x_n] + b[m_1 u(y_1) + m_0 u(y_0) - m_0 \alpha x_b]. \quad (38)$$

We first show that the first term on the right side of (38), $m_0 m_1[u(x_n) - x_n]$, is maximized at $m_1 = m_0$ and that it is larger by a finite amount at

$m_0 m_1[u(x_n) - x_n]$ than at any other feasible choice of m_1. From (11) and (12),

$$\begin{aligned}
& [\rho + \frac{1}{N} - b + b\delta(1 + \frac{m_1}{m_0})]\beta(v_1 + v_0) \\
= \; & m_0 u(x_n) + m_1 x_n + b[u(y_1) - u(y_0) + \alpha x_n], \qquad (39)
\end{aligned}$$

where $\rho = \frac{1-\beta}{\beta}$. Therefore, the incentive constraint , $x_n \leq \beta(v_1 - v_0)$, is equivalent to

$$\begin{aligned}
& [\rho + \frac{1}{N} - b + b\delta(1 + \frac{m_1}{m_0})]x_n \\
\leq \; & m_0 u(x_n) + m_1 x_n + b[u(y_1) - u(y_0) + \alpha x_n]. \qquad (40)
\end{aligned}$$

Let $X(b; y_0, y_1, \delta, \alpha)$ denote the set of non-negative x_n's that satisfy (40). This set is continuous in b and because the production levels are bounded independently of b, $X(0; y_0, y_1, \delta, \alpha)$ is given by non negative x_n's that satisfy

$$(\rho + \frac{1}{N})x_n \leq m_0 u(x_n) + m_1 x_n, \qquad (41)$$

or, equivalently,

$$(1 + \frac{\rho}{\frac{1}{N} - m_1})x_n \leq u(x_n). \qquad (42)$$

Because $1 + \frac{\rho}{\frac{1}{N} - m_1}$ is positive and increasing in m_1 for $m_1 \in (0, \frac{1}{N})$, and because $m_1 = m_0$ is the smallest feasible and positive magnitude of m_1, it follows that for b sufficiently close to zero, the set of x_n's satisfying (40) is larger at $m_1 = m_0$ than at any other feasible and positive magnitude of m_1. It also follows, exactly as in section 3, that if x_n satisfies (40), then for sufficiently small b it satisfies $u(x_n) \geq \beta(v_1 - v_0)$. Thus, for b sufficiently close to zero, the maximum of $u(x_n) - x_n$ is attained when $m_1 = m_0$. Moreover, with $m_1 = m_0$, (42) becomes

$$(1 + \frac{2\rho}{\frac{1}{N} + b})x_n \leq u(x_n). \qquad (43)$$

It follows that the x_n that maximizes $u(x_n) - x_n$ subject to $x_n \in X(b; y_0, y_1, \delta, \alpha)$ approaches $\hat{x}(0)$ of lemma 3 as $b \to 0$. Therefore, the maximized value of $u(x_n) - x_n$ is positive for sufficiently small b.

The maximum of $m_0 m_1$ is uniquely attained at $m_1 = m_0$ and the maximum is larger by a finite amount than at any other feasible choice of m_1. Therefore, for b sufficiently close to zero, $m_0 m_1[u(x_n) - x_n]$ is maximized at $m_1 = m_0$ and the maximum is larger by a finite amount than at any other feasible choice of m_1. It follows that $m_1 = m_0$ is necessary for maximization of v when b is sufficiently close to zero.

We now consider w. By (13), if $\alpha = 0$, then $w \to 0$ as $b \to 0$. Therefore, $\alpha = 1$ is necessary for attainment of a positive value of w that is bounded away from zero as $b \to 0$.

It remains to show that it is incentive feasible to have $m_1 = m_0$, $\alpha = 1$, and $w \geq v$. However, that is immediate from lemma 3. In particular, a no-gift allocation with $x = \hat{x}(b)$ is such an allocation.

Proof of (ii). If we insert into (38) the necessary conditions for a maximum of v when b is sufficiently small and take the limit as $b \to 0$, the result is as asserted; namely $\lim_{b \to 0} v^*(b) = \hat{v}(0)$. Because $\hat{v}(b)$ is consistent with a no-gift allocation which implies $w \geq v$, conclusion (ii) follows.

Proof of (iii). From (13) and (19), if $b = \frac{1}{N}$, then $x(b)''$, as defined in lemma 3, is the largest amount of production by a banker consistent with (19). It follows that the incentive feasible y that maximizes w when $b = \frac{1}{N}$ is given by $y = \hat{x}(\frac{1}{N})$ and that the implied w is $\hat{w}(\frac{1}{N})$ as given in lemma 3. By the constraint $w \geq v$, $v^*(\frac{1}{N})$ cannot exceed $\hat{w}(\frac{1}{N})$. Therefore, the conclusion follows if there is an incentive feasible allocation with $v = \hat{w}(\frac{1}{N})$. It is incentive feasible for non bankers to never produce and to consume any amount that does not exceed $\hat{x}(\frac{1}{N})$ in a meeting with a banker producer. It follows that it is incentive feasible to have v be anything in the interval $\frac{1}{N(1-\beta)}[0, u(\hat{x}(\frac{1}{N}))]$. Because the upper endpoint of this interval exceeds $\hat{w}(\frac{1}{N})$, it follows that there is an incentive feasible allocation with $v = \hat{w}(\frac{1}{N})$.$\square$

Proof of Lemma 4.

(Let b_i denote the fraction of each type who are bankers in state i divided by N, so that $b_0 + b_1 = b$. If people behave according to definition 4 whenever their production realizations permit such behavior, then there exists a unique constant distribution of notes in the hands of non bankers and of bankers over states given by $m_1 = m_0 = \frac{1}{2}(\frac{1}{N} - b)$ and $b_0 = pb$, $b_1 = (1-p)b$. Moreover, this constant distribution is locally stable.)

The law of motion implied by definition 4 is as follows:

$$m_1(t+1) = m_1(t) + m_0(t)(1-p)b - m_1(t)(1-p)b \tag{44}$$

and

$$b_1(t+1) = b_1(t) + b_0(t)(1-p)[b + m_1(t)] - b_1(t)p[b + m_1(t)], \tag{45}$$

where the second term in each equation is the inflow and the third is the outflow. In (44), the inflow results from meetings between non bankers without a note who are able to produce and bankers who consume what they produce, while the outflow results from meetings between non bankers with a note who meet bankers who are able to produce what the non banker consumes. In (45), the inflow is the result of meetings between bankers in state 0 who are able to produce and potential consumers, while the outflow is the result of meetings between bankers in state 1 who are unable to produce and the same potential consumers. Substituting for $m_0(t)$ from (8) and for $b_0(t)$ from $b_0(t) + b_1(t) = b$, these can be written as

$$m_1(t+1) = m_1(t)[1 - 2(1-p)b] + (\frac{1}{N} - b)(1-p)b \tag{46}$$

and

$$b_1(t+1) = b_1(t)\,[1 - [b + m_1(t)]] - (1-p)[b + m_1(t)]. \tag{47}$$

The steady state claims follow directly. As regards local stability, (46) and (47) imply that a linear approximation (evaluated at the steady state) of deviations from the steady state, denoted with "tildes", satisfy

$$\begin{bmatrix} \tilde{m}_1(t+1) \\ \tilde{b}_1(t+1) \end{bmatrix} = A \cdot \begin{bmatrix} \tilde{m}_1(t) \\ \tilde{b}_1(t) \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

and where $a_{11} = 1 - 2(1-p)b$ and $a_{22} = 1 - \frac{1}{2}(\frac{1}{N} + b)$. Because a_{11} and a_{22} are the characteristic roots of A and because both are strictly between 0 and 1, the steady state is locally stable.□

Proof of Proposition 3.

(Assume that a solution to problem 3 exists and that the maximized value of the objective, denoted $v^*(b)$, is a continuous function. Let p_0 denote the unique solution for p of $(1-2p)u(x') = x'$ and let ρ_0 denote the unique solution for ρ of $(1-p_0)u(x') = (1+2N\rho)x'$, where x' solves $\frac{du(x)}{dx} = 1$. If $p < p_0$ and $\rho < \rho_0$, then there exists a two-state allocation with $x_n = x_{b1} = y = y_1 = x'$ that is feasible for problem 3 and that implies a magnitude of v, denoted $\tilde{v}(b)$, satisfying $\lim_{b\to 0}[v^*(b) - \tilde{v}(b)] = 0$.)

As noted in section 5, the best that can happen to non bankers in the limit as $b \to 0$ is that x' is produced whenever the potential producer is able to produce. Hence $\lim_{b\to 0}\tilde{v}(b) \geq \lim_{b\to 0} v^*(b)$, and it remains to be shown that the proposition 3 allocation is incentive feasible and satisfies $Ew \geq \tilde{v}(b)$ for b close to zero. For a fixed $x_{b0} > 0$, the expected utilities attained by this two-state allocation are uniquely defined by (22-25) with $x_n = x_{b1} = y = y_1 = x'$ and are continuous in b. We can thus appeal to this continuity and complete the proof by showing feasibility and $Ew \geq \tilde{v}(b)$ for some positive x_{b0} when $b = 0$.

Regarding no-defection conditions, the key inequalities under the proposed two-state allocation are $-x' + \beta(v_1 - v_0) \geq 0$, from (27), and $-x' + \beta(w_1 - w_0) \geq 0$, from (29). The remaining no-defection conditions follow trivially. If $b = 0$ then, according to lemma 4, $m_1 = m_0 = \frac{1}{2N}$, so that (22-25) correspond to

$$v_1 = \beta v_1 + \frac{1-p}{2N}[u(x') + \beta(v_0 - v_1)], \tag{48}$$

$$v_0 = \beta v_0 + \frac{1-p}{2N}[-x' + \beta(v_1 - v_0)], \tag{49}$$

$$w_1 = \beta w_1 + \frac{1-p}{2N}[u(x') - x'] + \frac{p}{2N}\beta(w_0 - w_1) \tag{50}$$

and

$$w_0 = \beta w_0 + \frac{1-p}{2N}[u(x_{b0}) - x' + \beta(w_1 - w_0)]. \tag{51}$$

We can now use (48-51) to write the no-defection conditions in terms of primitives. The inequality $-x' + \beta(v_1 - v_0) \geq 0$ is equivalent to

$$u(x') \geq (1 + \frac{2N\rho}{1-p})x', \tag{52}$$

while $-x' + \beta(w_1 - w_0) \geq 0$ is equivalent to

$$(1-p)u(x') \geq (1 + 2N\rho)x' + (1-p)u(x_{b0}). \tag{53}$$

It is now clear that the restrictions on (p, ρ) assumed in proposition 3 imply that $(1-p)u(x') > (1 + 2N\rho)x'$, so that (53) follows if x_{b0} is chosen close to zero. They are also sufficient for having non bankers produce x' in single coincidence meetings because inequality (52) is implied by inequality (53).

Regarding the $Ew \geq \tilde{v}(b)$ condition, we use lemma 4 to derive Ew under the proposed two-state allocation as $Ew = \frac{b_1 w_1 + b_0 w_0}{b_1 + b_0} = (1-p)w_1 + pw_0$. Its counterpart for non bankers is $\tilde{v}(b) = \frac{1}{2}(v_1 + v_0)$. Using now (48-51), the values $\tilde{v}(b)$ and Ew for $b = 0$ are computed according to

$$\tilde{v}(0) = \frac{1-p}{4N(1-\beta)}[u(x') - x']$$

and

$$Ew = \frac{1-p}{2N(1-\beta)}[(1-p)u(x') + pu(x_{b0}) - x'],$$

so that $Ew > \tilde{v}(0)$ follows from the assumption $(1-2p)u(x') > x'$ of proposition 3.

We have shown that the proposed two-state allocation with a sufficiently small x_{b0} is incentive feasible for b close to zero. As a result, for b close to zero, $\tilde{v}(b)$ is a lower bound for $v^*(b)$. Hence the continuity of $v^*(b)$ and $\tilde{v}(b)$ in b, and the inequality $v^*(0) \leq \tilde{v}(0)$, now imply $\lim_{b \to 0}[v^*(b) - \tilde{v}(b)] = 0$ as asserted. □