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MARCIA M. G. DE ANDRADE
Secretária-Geral/EPGE

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ON THE CONVERGENCE OF BAYESIAN PRIORS TO RATIONAL EXPECTATIONS IN COMPLETE MARKETS

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Comments Welcome.

§1 Introduction

The purpose of this paper is to study the convergence to rational expectations of arbitrary Bayesian priors in infinite horizon economies with complete markets. The main idea explored in the paper is that if agents can trade over future events then they must have the same measure zero sets. And therefore, by a result of Blackwell and Dubins (62) their posteriors must also converge.

Some of the ideas in this papers can be traced back to Hayek and also appears in Grossman (81). They say that equilibrium prices reveal information to market participants. In our case this means revealing measure zero sets which is enough, by the Blackwell-Dubins result, to garantee convergence to rational expectations. More formally we start in Section 2 with the presentation of the three models studied in the paper: The purely extrinsic uncertainity, the sunspot and the temporary equilibrium one, the most general of the three. By Bewley's existence theorem (72), equilibrium exists in very general situations and in particular with non convergence probability priors. However, if one allows Brankruptcy that is not the case anymore and that is the content of Theorem 1 in Section 3. It says that the posteriors converges to equilibrium prices with probability one with respect to the measure generated by equilibrium prices. In particular it says that in the long run expectations become homogeneous. Corollary 2 gives an assumption to garantee convergence to rational expectations. In particular, in the long run temporary equilibrium disapear.

Corollary 3 stablishes the convergence of assets prices. Theorem 2 however says that as long as expectations are not "disjoint" equilibrium exists. Hence the convergence region might have as small probability as one wishes.

In Section IV we briefly expose some work in progress: the incomplete market case, the convergence with probability one when the priors are gaussians, the convergence of the prices and the convergence of goods allocations.

§2 The model and basic facts

Consider on infinite time horizont economy with a finite set I of economic agents. At every period $t \in \mathbf{N}_+$, the economic agents trade ℓ goods and receive a publicy observable signal that belongs to a set S_t. These signals can be classified in three cases.

In the first case, the one of purely extrinsic uncertainty, the signals are exogenous and determine future endowments. The price realization is assumed to be common Knowlewdge conditional on the signals.

In the second case, the signals, called sunspots, are still exogenous variables but do not affect endowments. Threfore, the uncertainty is intrinsical although the prices are still common knowledge conditional on the suspots realization.

In the third case, the signals are future prices and so endogenous variables that do not affect endowments. Similarities exists between cases 1, 2 and 3. In fact, cases 1 and 2 will be indentified with the traditional general Equilibrium model and case 3 with the Temporary equilibrium model.[1] In order to reduce the number of simbols involved we will keep some notation in common. Whenever we feel that this notational abuse might misguide the reader an index $k = 1,2,3$ will be used to indicate the proper case in consideration.

The signal set is $S = \prod_{n=1}^{+\infty} S_n$. For every $B \subset \prod_{n=1}^{t} S_n$, let $\overline{B} = \{s \in S|(s, \ldots, s_t) \in B\}$ be a cylinder with base B. Consider the σ-algebra $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$. $\mathcal{F}_t$ is a σ-algebra whose elements are cylinders with bases on $\prod_{n=1}^{t} S_n$. $\mathcal{F}$ is the σ-algebra generated by the algebra $\mathcal{F}^0 = \bigcup_{t \in \mathbf{N}_+} \mathcal{F}_t$.

The functions interpretated as economic variables are assumed to be non-antecipative. Let $x_t^i: S \to R_+^\ell$ and $w_t^i: S \to R_+^\ell$ $\mathcal{F}_t$-mensurable functions be respectively the i-th agent demand and initial endowment at period t. In cases $k = 2,3$ w_t^i is a constant function

[1] In temporary equilibrium models signals may also be exogenous but since this is studied in cases 1 and 2 there is no loss of generality in assuming that in case 3 signals are only endogenous.

$\forall t \in \mathbf{N}_+$ and $\forall i \in I$. Let L_+ be the space of bounded functions

$$f: S \times \mathbf{N}_+ \longrightarrow \mathbf{R}_+^\ell$$
$$(s,t) \longrightarrow f_t(s)$$

such that $\forall t \in \mathbf{N}_+$, $f_t: S \to R_+^\ell$ is $\mathcal{F}_t$-mensurable.

Let $x^i: S \times \mathbf{N}_+ \to R_+^\ell$ and $w^i: S \times \mathbf{N}_+ \to R_+^\ell$ be respectively the i-th agent demand and initial endowments. We assume

II.1 $\forall i \in I$, $(x^i, w^i) \in L_+ \times L_+$.

II.2 $\forall i \in I$, $w^i >>> 0$.

II.3 In cases $k = 1,2$, S_t is a finite set.

Each agent i has a utility function $v^i: (R_+^\ell)^{\mathbf{N}_+} \to R$ and a subjective probability P^i defined on $(S, \mathcal{F})$. We assume

II.4 In case $k = 3$, $\forall t \in \mathbf{N}_+$ $\forall i \in I$ there are only a finite number of elements $A_t^i \in \mathcal{F}_t$ such that $P^i(A_t^i) > 0$.

That is, we assume that in case $k = 3$ all agents believe that at every period only a finite number of prices might occur. For instance, take all prices in cents inside a bounded set.

By III.4, we may assume that S_t is, in case $k = 3$, also finite.[2] In cases $k = 1,2$ $p_t: S \to R_+^\ell$ a $\mathcal{F}_t$-mensurable function represents the price of goods at period t.

Let

$$p: S \times \mathbf{N}_+ \longrightarrow R_+^\ell$$
$$(s,t) \longrightarrow p_t(s)$$

be the price of goods.

In the spirit of overlapping generation models we make no assumption about the space where p is.

In case $k = 3$, an element $s \in S$, $s = (s_1, s_2, \dots)$ represent the price of goods. Clearly, $s_t \in R_+^\ell$ is the price of goods at period t.

[2] We plan to drop assumptions III.3 and III.4 to also consider the continuous case.

Let

$$\overline{v}(x^i): S \longrightarrow \mathbf{R}$$
$$s \longrightarrow v^i(x_1^i(s), x_2^i(s), \ldots)$$

be the utility agent i gets if x^i is his consumption function and s occurs. We assume

II.5 $\forall i \in I$, $\overline{v}^i(x^i)$ is P^i integrable.

Each agent i has a Von Newmann-Morgenstern expected utility function

$$v^i: L \longrightarrow \mathbf{R}$$
$$x^i \longrightarrow \int_S \overline{v}^i(x^i) P^i$$

We assume that v^i satisfies

II.6 a) v^i is concave.

II.6 b) v^i is continuous in the mackey topology.

II.6 c) v^i is bounded.

II.6 d) (monotonicity assumption). $\forall i \in I$, if $\overline{x}^i \geq x^i$ and $P^i\left(\bigcup_{t\in\mathbf{N}_+} \{s \in S \mid \overline{x}_t^i - x_t^i > 0\}\right) > 0$ then $v^i(\overline{x}^i) > v^i(x^i)$.

II.7 At period 0, there exists a complete set of Arrow securities.

Let $d_t^i: S \rightarrow \mathbf{R}$ and $\mu_t: S \rightarrow \mathbf{R}_+$ $\mathcal{F}_t$-mensurable functions be respectively the i-th agent asset demand and the asset price at period t.

Let

$$d^i: S \times \mathbf{N}_+ \longrightarrow \mathbf{R}$$
$$(s,t) \longrightarrow d_t^i(s)$$

be the i-th agent asset demand and let

$$\mu: S \times \mathbf{N}_+ \longrightarrow \mathbf{R}_+$$
$$(s,t) \longrightarrow \mu_t(s)$$

be the asset price.

We write $\langle \mu, d\rangle = \sum_{t=1}^{+\infty} \int_S \mu_t d_t^i \#.$ [3].

These Arrow securities have different interpretation according to the case considerated. In case $k = 1$ they are traditional Arrow securities. In case $k = 2$, we are assuming complete markets for sunspots. In case $k = 3$, they are Arrow securities for prices that at least one agent conjectures as possible. Althrough there is no reason for the non existence of such markets (a kind of market of bets on future prices) it would be better if the assets in case $k = 3$ could be interpretaded as options. Chi-fu Huang and R. H. Litzenberger (88), Foundation for Financial Economics, cap. 5, contains models that relates Arrow securities with options.

We define a feasible asset allocations as a vector $d = (d^1, \ldots, d^{\# I})$ such that $\sum_{i\in I} d^i = 0$.

In cases $k = 1, 2$, we define a feasible allocation as a vector $(x^1, \ldots, x^{\# I}) \in L_+^{\# I}$ such that $\sum_{i\in I}(x^i - w^i) = 0$.

In case $k = 3$, let $E \subset S$ be the set of all prices s such that $\forall t \in \mathbf{N}_+$, $\sum_{i\in I}(s^i - w^i)(s,t) = 0$. If $E \neq \phi$ we say that the allocation x is feasible with respect to E.

Given two functions $f^j : S \times \mathbf{N}_+ \to \mathbf{R}_+^\ell$, $j = 1, 2$. We define

$$f^1.f^2 : S \times \mathbf{N}_+ \longrightarrow \mathbf{R}_+^\ell$$

$$(s,t) \longrightarrow f_t^1(s).f_t^2(s)$$

We also define the function

$$p^0 : S \times \mathbf{N}_+ \longrightarrow \mathbf{R}_+^\ell$$

$$(s,t) \longrightarrow s_t$$

In cases $k = 1, 2$, we define an equilibrium (no bankruptcy) as a pair of feasible goods and assets allocations (x, d) and a price system (μ, p), such that $\forall i \in I$, (x^i, d^i) maximizes v^i in the i-th budget set

$$\{(x^i, d^i) \mid \langle \mu, d^i\rangle = 0, \quad p(x^i - w^i) = d^i\}.$$

[3]The integral means the price of the assets d_t^i in period t and is well defined since $\mathcal{F}_t$ is finite

In case $k = 3$, we define a temporary equilibrium (no bankruptcy) as a feasible asset allocation d, an asset price μ and a feasible foods allocation x with respect to E such that $\forall i \in I$, (x^i, d^i) maximizes v^i in the i-th budget set

$$\{(x^i, d^i) \mid \langle \mu, d^i \rangle = 0; \quad p^0(x^i - w^i) = d^i\}.$$

In cases $k = 1,2$ the existence of equilibrium was proved by Bewley (72). Below, we restate his theorem in our notation.

Theorem II.1. *In cases $k = 1,2$, under II.1 - II.7 there exists an equilibrium (no bankruptcy). Also, there exists a function $\pi: S \times \mathbf{N}_+ \to \mathbf{R}_+$, such that if $\mu.p = \pi$ than (μ, p) is an equilibrium price system (no bankruptcy).*

Clearly, in this model, every sunspot equilibrium such that $\forall t \in \mathbf{N}_+$ p_t is an injection can also be interpretated as a temporary equilibrium such that all agents knows the future price that can or can not occur but not necessarily the true distribution of the future prices. So, by Theorem II.1 one can also prove the existence of temporary equilibrium (no bankruptcy).

It is interesting to notice that there exists an equilibrium (no bankruptcy) for arbitrary priors P^i, $i \in I$. In particular, there exists (in cases $k = 1,2,3$) an equilibrium (no bankruptcy) such that the posteriors of the agents are non convergent in all $s \in S$. So, it is not possible to get a general convergence of beliefs theorem in the no bankruptcy case. However, consider the following example: Suppose that for some $A_t \in \mathcal{F}_t, P^1(A_t) = 0$ and $P^2(A_t) > 0$. That is, agent 1 is sure that A_t cannot occur but agent 2 is not.[4] Then one

[4] A very similar intuition one could also get from the Blackwell-Dubins theorem and the temporary equilibrium literature since it is prove there (see, for instance, Grandmont (77) and Green (77)) that a suficient conditions for the existence of temporary equilibrium is that the supports of the priors are not disjoint. If one wants a direct application of these ideas the easiest way to do so would be to consider a finite time horizont model with an infinite number of trade periods and a future market for every possible signal. In this case every element in the uncertainty space would be "defined in time" and so it would not be necessary to restrict trade to $\mathcal{F}^0$ and a direct proof would state that the introduction of bankruptcy implies the convergence of the posteriors in cases $k = 1,2,3$. Unfortunately, this do not seem (to us) natural because then we would have to assume trade and beliefs revision in infinitesimal time. A model like that would dramatically increase the number of future markets from an enumerable one to a non enumerable one. This would not be without gain because then one could prove expectations convergence with probability one and if future markets are restricted to $\mathcal{F}^0$ then convergence, in general, will be only with positive probability. This will reinforce one of the central points of this paper: the importance of the existence of future markets to spread out the private information to the economic agents.

could argue that no equilibrium exists because if the asset price μ is strictly positive at (A_t, t) then the economic agent 1 would sell arbitrary large amount of Arrow securities that pays conditional to A_t because agent 1 believes that A_t can not occur and so he believes that he will never have to pay anything for the resourses he is getting at time zero. On the other hand, if $\mu(A_t, t) = 0$ then since agent 2 believes that A_t might accur, he has some indirect utility with the Arrow securites that pays conditional on A_t and so, at no cost, agent 1 will certainly buy arbitrary large amounts of Arrow securities that pays conditional to A_t. In particular, we get that if an equilibrium exists then P^i, $i \in I$ has the same null gets in $\mathcal{F}^0$ (but not necessarily in $\mathcal{F}$) and this is a contradiction with Theorem II.1. This apparent paradox can be easily explained if it is observed that since $x_t^i(A_t) \geq 0$ and $p_t(A_t)(x_t^i(A_t) - w^i(A_t)) = d_t^i(A_t)$, $\forall A_t \in \mathcal{F}^0$, $\forall i \in I$ then $d_t^i(A_t) \geq -p_t(A_t)w^i(A_t)$ $\forall A_t \in \mathcal{F}^0$, $\forall i \in I$. That is, the agents are forbidden to offer arbitrary amounts of assets as in the above argument because they are restricted by all budget restrictions even the zero subjective probability ones. In brief, if short soles (or bankruptcy) are introduced in a general Equilibrium model then some equilibriums are to be ruled out because the economic agents becomes no more restricted by zero subjective probability restrictions. In the following a theorem due to Blackwell and Dubins garantees that if two priors have the same null sets in an infinite certain product space (as S) then with probability one (according to both priors) the posteriors converge in the sup norm. This suggest that short sales rules out exactly the non convergent equilibriums. [5]

However, to prove this, for technical reason, it is better to deal explicitely with bankruptcies instead of the fiction of negative consumption. To achieve this goal it is introduced in the model a riskless asset, called money, that can be borrowed or lended throughout time paying a fix interest rate. There is a penalty function that will be computed according to the present value of the agents debt.[5]

$\forall i \in I$, let $m_t^i \colon S \to \mathbf{R}$ a $\mathcal{F}_t$-mensurable function, be the i-th agent money demand at $t \in \mathbf{N}_+$. Let $m_0^i \colon S \to \mathbf{R}$, a constant function, be the i-th agent money demand at period

[5] the interest rate plays almost no role in this model and although it is determined endogenouly one may easily see that it could be exogenously fixed at zero or any other positive number

0 and let

$$m^i: S \times \mathbf{N} \longrightarrow \mathbf{R}$$
$$(s,t) \longrightarrow m_t^i(s)$$

be the i-th agent money demand. We define

$$m^{*i}: S \times \mathbf{N} \longrightarrow \mathbf{R}$$
$$(s,t) \longrightarrow m_t^i(s)(1+r)^{-t}$$

We assume

II.9 $\quad \forall i \in I,\ m^{*i} \in L.$

We also define

$$h^i: S \longrightarrow \mathbf{R}$$
$$s \longrightarrow \liminf_{t \to +\infty} m^{*i}(s,t)$$

$h^i(s)$ is the lim inf of the present value of the i-th agent debt if s occurs. The "lim inf" is used in some models to compute the debt in order to avoid "ponzi games". For a reference see Wilson (80).

Let $\mathcal{I}: \mathbf{R}_- \to \mathbf{R}_+$ be the penalty function. We assume

II.10 $\quad \mathcal{I}$ is a continuous and non increasing function.

Lemma II.1. *$\forall i \in I$, $\mathcal{I}(\min\{h^i, 0\}): S \to \mathbf{R}_+$ is P^i-integrable.*

Proof: See Appendix. ∎

$\mathcal{I}(\min\{h^i(s), 0\})$ is the penalty agent i will get if he chooses h^i and s occurs. The min function is used to give no profit in holding money in the limit.

We define $b(h^i) = \int_S \mathcal{I}(\min\{h^i, 0\}) P^i$ the i-th agent expected penalty or the expected disutility for going bankrupt.

In cases $k = 1, 2$, we define a feasible money allocation as a vector $m = (m^1, m^2, \ldots, m^{\#I})$ such that $\sum_{i \in I} m^i = 0$ and $(m^{*1}, m^{*2}, \ldots, m^{*\#I}) \in L^{\#I}$.

In case $k = 3$, let $E' \subset S$ be the set of all prices s such that $\sum_{i \in I} m_t^i(s) = 0\ \forall t \in \mathbf{N}_+$.

If $E' \neq \phi$ and $(m^{\bullet 1}, m^{\bullet 2}, \ldots, m^{\bullet \# I}) \in L^{\# I}$ the money allocation m is said to be feasible with respect to E'.

In case $k = 1, 2$, given the price system (μ, p) and the interest rate r, let the i-th agent budget set be defined by:

$$\{(x^i, d^i, m^i) \mid \langle \mu, d^i \rangle + m_0^i = 0; \quad m_t^i + p_t(x_t^i - w_t^i) = d_t^i + m_t^i(1+r) \quad \forall t \in \mathbf{N}_+\}.$$

In case $k = 3$, given the asset price μ and the interest rate r, let the i-th agent budget set be defined by:

$$\{(x^i, d^i, m^i) \mid \langle \mu, d^i \rangle + m_0^i = 0; m_t^i + p_t^0(x_t^i - w_t^i) = d_t^i + m_{t-1}^i(1+r) \quad \forall t \in \mathbf{N}_+\}.$$

Let Q, a probability defined on $(S, \mathcal{F})$, be the true uncertainty distribution. Q is an exogeneous variable in cases $k = 1, 2$ but an endogeneous one in case $k = 3$ since the prices are endogenously determined.

In cases $k = 1, 2$, we define an equilibrium as a triple (x, d, m) of feasible goods, assets and money allocations, a price system (μ, p) and an interest rate r such that $\forall i \in I \{x^i, d^i, m^i\}$ maximizes $v^i - b$ in the i-th agent budget set.

In case $k = 3$, we define a temporary equilibrium as an asset price μ, an interest rate r, a feasible asset allocation d, a feasible goods allocation x with respect to $E \subset \mathcal{F}$, a feasible money allocation m with respect to $E' \subset \mathcal{F}$, such that $Q(E \cap E') = 1$ and $\forall i \in I$, $\{x^i, d^i, m^i\}$ maximizes $v^i - b$ in the i-th agent budget set.

For any two probabilities ν_1 and ν_2 on the same σ-field $\mathcal{T}$ the distance $d(\nu_1, \nu_2)$ between ν_1 and ν_2 is defined by least upper bound of $|\nu_1(D) - \nu_2(D)|$ over $D \in \mathcal{T}$.

For any prior ν defined on $(X, \mathcal{T})$, $X = \prod_{n=1}^{+\infty} X_n$, $\nu^{(n)}_{(s_1, \ldots, s_n)}$ denote the posteriors distributions of ν given the past observation $(s_1, \ldots, s_t) \in \prod_{n=1}^{t} X_n$.

The priors P^i, $i \in I$, are said asymptotically homogenous in a set $\Omega \in \mathcal{F}$ if for every $s \in \Omega$, $s = (s_1, s_2, \ldots, s_n, \ldots)$ and for every pair $(i, j) \in I \times I$ the distance between $P^{i(n)}_{(s_1, \ldots, s_n)}$ and $P^{j(n)}_{(s_1, \ldots, s_n)}$ converges to 0 as n converges to ∞.

In cases $k = 1, 2$, the priors P^i, $i \in I$, are said to be convergent to rational expectations in a set $\Omega \in \mathcal{F}$ if for every $s \in \Omega$, $s = (s_1, s_2, \ldots, s_n, \ldots)$ and every $i \in I$, the distance between $P^{i(n)}_{(s_1,\ldots,s_n)}$ and $Q^{(n)}_{(s_1,\ldots,s_n)}$ converges to zero to 0 as n converges to ∞.

In case $k = 3$, a temporary equilibrium is said to be convergent to rational expectations in a set $\Omega \in \mathcal{F}$ if for every $s \in \Omega$, $s = (s_1, s_2, \ldots, s_n, \ldots)$ and for every agent $i \in I$, the distance between $P^{i(n)}_{(s_1,\ldots,s_n)}$ and $Q^{(n)}_{(s_1,\ldots,s_n)}$ converges to zero as n converges to ∞.

Notice that in cases $k = 1, 2$ since the signals are exogenous, convergence to rational expectations means that the agents learn to predict correctly the exogenous probability distribution of the future signals.

In case $k = 3$, agents learn to predict whether a price might occur or not in the future and its probability of occurrence although the "true" probability Q is endogenously given and so it might depend on the forecasting priors P^i.

III – The results:

Theorem III.1. *In cases $k = 1, 2, 3$, under II.1 – II.9. If there exists en equilibrium then there exists a measure μ^* defined on $\mathcal{F}$ such that:*

1) $\mu^* << P^i \; \forall i \in I$.

2) $\forall A \in \mathcal{F}^0$, $\mu^*(A) = 0 \Leftrightarrow P^i(A) = 0 \; \forall i \in I$.

Theorem III.2. *In cases $k = 1, 2$. Under II.1 – II.9 if there exists a measure ν defined on $\mathcal{F}$ such that:*

1) $\nu << P^i \; \forall i \in I$

2) $\forall A \in \mathcal{F}^0$, $\nu(A) = 0 \Leftrightarrow P^i(A) \; \forall i \in I$.

Then there exists an $M > 0$ such that if the penalty satisfies: $\mathcal{I}(0) = 0$ and $\mathcal{I}(z) \geq -Mz$, then there exists an equilibrium.

The proofs of Theorems III.1 and III.2 depends on Lemmas III.1 – III.7 and so we will present them latter on in this section.

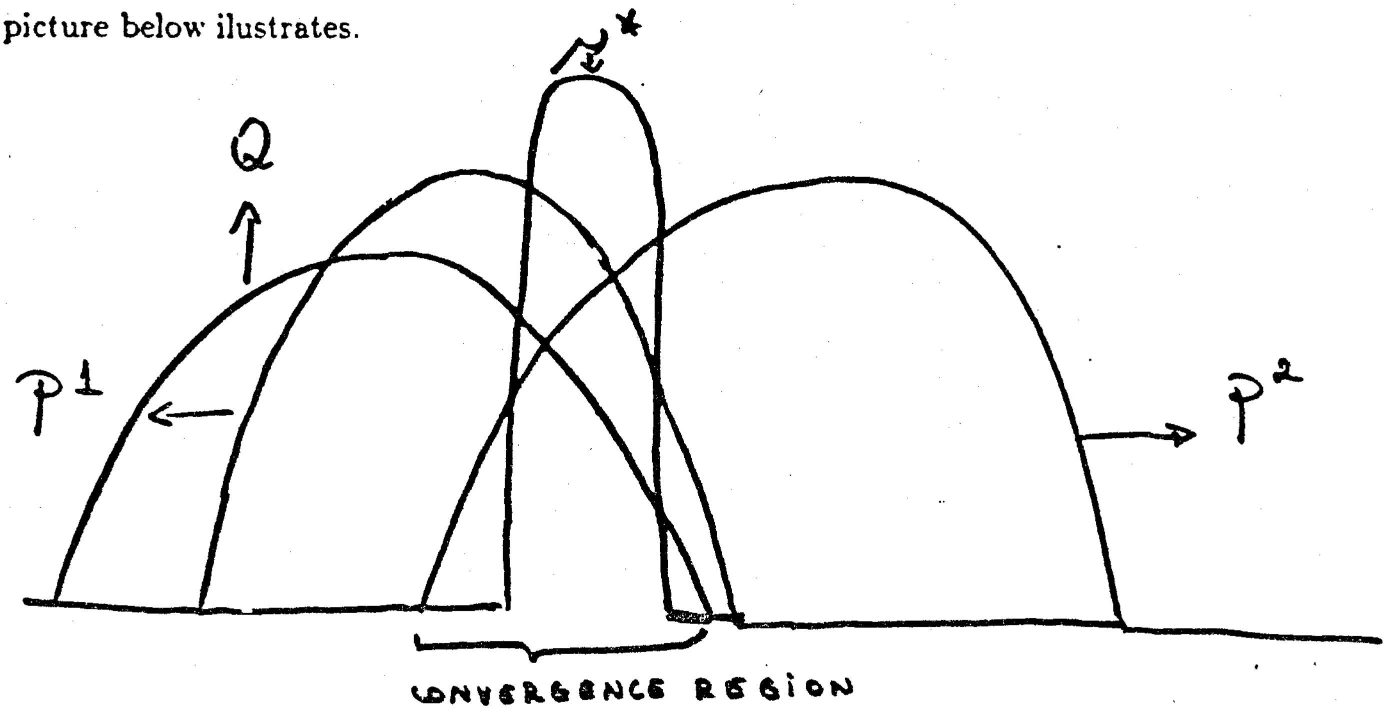
μ^* depends only on the equilibrium asset price hence we call it the price measure.

Theorem III.1 in fact holds with no condition on (v^i, w^i), $i \in I$ except for the monotonicity assumption.

Lemma III.4 caracterize conditions 1) and 2).

Clearly, if penalty is zero or "weak" no equilibrium exists because the agents will choose to go bankrupt.

Theorems III.1 and III.2 gives necessary and suficient conditions for the existence of equilibrium when bankruptcy is introduced. By the Blackwell-Dubins theorem, we can determine (depending on the parameters) the "convergence region" of the priors as the picture below ilustrates.



Figure

Lemma III.1. *A) In cases $k = 1,2,3$ if μ is an equilibrium asset price then $\forall A_t \in \mathcal{F}_t$:*

$$\mu_t(A_t) = 0 \Leftrightarrow P^i(A_t) = 0 \quad \forall i \in I.$$

b) In cases $k = 1,2$, let π be as in Theorem II.1. If the priors have the same null sets in $\mathcal{F}^0$ then $\forall A_t \in \mathcal{F}_t$:

$$\pi_t(A_t) = 0 \Leftrightarrow P^i(A_t) = 0 \quad \forall i \in I.$$

Proof: For the proof of this lemma just use the arguments given in Section II.

Lemma III.2. *Let $f: \mathbf{R}_+ \to \mathbf{R}$ be a concave, nondecreasing function and $\bar{\varepsilon} > 0$ then there exists an $\overline{M} \geq 0$ such that $f(\varepsilon + \bar{\varepsilon}) - f(\bar{\varepsilon}) \leq M\bar{\varepsilon} \quad \forall \bar{\varepsilon} \geq 0$.*

Proof: See Aubin and Ekeland (84) Applied nonlinear analysis, chapter 4, sections 3, proposition 3.

Lemma III.3. *Let ν^1 and ν^2 be two measures defined on a σ-algebra $\mathcal{T}$ generated by the algebra $\mathcal{T}'$. If $\nu^1(D) > \varepsilon$ and $\nu^2(D) = 0$ for some $D \in \mathcal{T}$ then there exists $A^1, A^2, \ldots,$ elements of $\mathcal{T}'$ such that $\nu^1(A^n) > \varepsilon$ $\forall n \in \mathbb{N}_+$ and $\nu^2(A^n) \underset{n \to +\infty}{\longrightarrow} 0$*

Proof: See Appendix.

For any two measures ν^1 and ν^2 on a σ-algebra $\mathcal{T}$ we write, as usual, that $\nu^1 << \nu^2$ if $\forall D \in \mathcal{T}$, $\nu^2(D) = 0 \Rightarrow \nu^1(D) = 0$.

We write $\nu^1 \leq \nu^2$ if $\forall D \in \mathcal{T}$ $\nu^1(D) \leq \nu^2(D)$.

Lemma III.4. *Let $\nu^1, \nu^2, \ldots, \nu^L$ be arbitrary positive measures defined on a space X and a σ-algebra $\mathcal{T}$. Let $\mathcal{T}^0$ be any subset of $\mathcal{T}$ such that $X \in \mathcal{T}^0$. Then A), B) and C) bellow are equivalent. If $\mathcal{T}^0 = \{X\}$ then A), B), C) and D) are equivalent.*

A) Let $C^j \in \mathcal{T}$ be full measures sets of ν^j, $1 = 1, \ldots, L$ and $C = \bigcap_{j=1}^{L} C^j$. $\forall D \in \mathcal{T}^0$, if there exists an $\epsilon, 1 \leq \epsilon \leq L$ such that if $\nu^\epsilon(D) > 0$ then $\nu^j(D \cap C) > 0$ $j = 1, \ldots, L$.

B) There exists a positive measure $\nu^{L+1} \neq 0$ such that:

1) $\nu^{L+1} << \nu^j$ $j = 1, \ldots, L$.

2) $\forall D \in \mathcal{T}^0$, $\nu^{L+1}(D) = 0 \Leftrightarrow \nu^j(D) = 0 \quad j = 1, \ldots, L$.

C) There exists a positive measure $\nu^{L+2} \neq 0$ such that: $\nu^{L+2} \leq \nu^j$ $j = 1, \ldots, L$ and

1) $\nu^{L+2} << \nu^j$ $j = 1, \ldots, L$.

2) $\forall D \in \mathcal{T}^0$, $\nu^{L+2}(D) = 0 \Leftrightarrow \nu^j(D) = 0 \quad j = 1, \ldots, L$

D) Let $C^j \in \mathcal{T}$ be full measures sets of ν^j $j = 1, \ldots, L$ and $C = \bigcap_{j=1}^{L} C^j$.

Then there exists an $\epsilon, 1 \leq \epsilon \leq L$, $\nu^\epsilon(C) \not> 0$.

Proof: See Appendix.

Lemma III.5. *In cases $k = 1, 2, 3$, if μ and r are equilibrium asset prices and interest rate respectively. Then, $\mu_1(S) = (1 + r)^{-1}$ and $\forall A_t \in \mathcal{F}_t$, $\mu_{t+1}(A_t) = \mu_t(A_t)(1 + r)^{-1}$.*

Proof: See Appendix.

We define

$$\mu^0 : \mathcal{F}^0 \longrightarrow \mathbf{R}_+$$
$$A_t \longrightarrow \mu_t(A_t)(1+r)^t.$$

By Lemma III.5 μ^0 is a well defined probability measure on $\mathcal{F}^0$.

Lemma III.6. *In cases $k = 1, 2, 3$, if μ is an equilibrium asset price and for some sequence of elements of $\mathcal{F}^0$, $A^1, A^2, \ldots$ $P^{i^0}(A^j) \underset{j \to +\infty}{\longrightarrow} 0$, $i^0 \in I$ then $\mu^0(A^j) \underset{j \to +\infty}{\longrightarrow} 0$.*

Proof: (We will make the proof in cases $k = 1, 2$, the proof in case $k = 3$ is totally analogous). Suppose not, then there exists an $\varepsilon > 0$ and a subsequence $A^{k(j)}$ such that $\mu^0(A^{k(j)}) \geq \varepsilon$ $\forall j \in \mathcal{N}_+$ and $P^{i^0}(A^{k(j)}) \underset{j \to +\infty}{\longrightarrow} 0$. Let $(\overline{x}, \overline{d}, \overline{m})$ be the equilibrium allocations, $\overline{p}$ the equilibrium price and $\overline{r}$ the equilibrium interest rate and let $\delta \in \mathbf{R}^\ell_+$ be such that $\delta_n = \frac{\varepsilon(1+\overline{r})}{\ell . \sup_{s \in S} \|p_1(s)\|}$ for every n-coordinate, $1 \leq n \leq \ell$. $\forall i \in I$, let $x^i \in L_+$ be such that

$$x^i_1 : S \longrightarrow \mathbf{R}^\ell_+$$
$$s \longrightarrow x^i_1(s) + \delta$$
$$x^i_t = \overline{x}^i_t \text{ if } t \neq 1.$$

By the monotonicity assumption, there exists an $\varepsilon' > 0$ such that $v^i(x^i) - v^i(\overline{x}^i) > \varepsilon'$.

Let $R_1 \in \mathbf{R}_-$ be such that $\overline{h}^{i^0} \geq R_1$. Let $R_2 \in \mathbf{R}_+$ be such that $I(z-1) - I(z) \leq R_2$ if $0 \geq z \geq R_1$.

Let $A^{k(j^0)} \in \mathcal{F}_{t^0}$ be such that $P^{i^0}(A^{k(j^0)}) < \frac{\varepsilon'}{R_2}$. Notice that it is no loss of generality to assume that $t^0 > 1$.

Define (d^i, m^i) such that $\forall t \in \mathcal{N}_+$

$$d^i_t : S \longrightarrow \mathbf{R}$$
$$s \longrightarrow \overline{d}^i_t(s) - (1+\overline{r})^t \quad \text{if } s \in A^{k(j^0)}, \quad t = t^0$$
$$s \longrightarrow \overline{d}^i_t(s) \quad \text{if } s \notin A^{k(j^0)} \text{ or } t \neq t^0$$

$$m_t^i : S \longrightarrow \mathbf{R}$$

$$s \longrightarrow \overline{m}_t^i(s) \quad \text{if } s \notin A^{k(j^0)} \text{ or } t \neq t^0$$

$$s \longrightarrow \overline{m}_t^i(s) - (1+\overline{r})^t \quad \text{if } s \in A^{k(j^0)}, \quad t = t_0$$

$$m_0^i = -\left\langle \mu, d^i \right\rangle.$$

To get a contradiction we prove two facts:

I) d^i can be bought at period zero by the i-th agent. With (m^i, d^i), x^i can be bought by the i-th agent. This give to agent i^0 an extra expected utility strictly bigger then ε'.

II) m^{i^0} implies that agent i^0 will have a new h^{i^0} such that the extra increase in his penalty is strictly smaller then ε'.

By the m_0^i definition, d^i can be bought at period zero.

$$\overline{p}_1(x_1^i - w_1^i) = \overline{p}_1(\overline{x}_1^i - w_1^i) + \overline{p}_1.\delta = \overline{d}_1^i + \overline{m}_0^i(1+r) - \overline{m}_1^i + p_1.\delta$$

$$= d_1^i - m_1^i - \left\langle \mu, \overline{d}^i \right\rangle (1+r) + p_1\delta \leq d_1^i - m_1^i - \left\langle \mu, \overline{d}^i \right\rangle (1+r) + (1+r)\varepsilon$$

$$= d_1^i - m_1^i + m_0^i(1+r) + (\varepsilon - \left\langle \mu, \overline{d}^i - d^i \right\rangle)(1+r) < d_1^i - m_1^i + m_0^i(1+r)$$

So, in period 1 x_1^i can be bought mith (d^i, m^i). By analogous proof one may see that in period $t \neq 1$ x_t^i can also be bought mith (d^i, m^i). To prove II, it is enough to observe that $(h^i - \overline{h}^i)(s) = 0$ if $s \in A^{k(j^0)}$ and $(h^i - \overline{h}^i)(s) = -1$ if $s \in A^{k(j^0)}$. So, the extra increase in the i^0-th agent penalty is less then $R_2.P^{i^0}(A^{k(j^0)})$. ∎

Lemma III.7. *In cases $k = 1, 2, 3$ if μ is an equilibrium asset price μ^0 is a σ-aditive probability measure on $\mathcal{F}^0$.*

Proof: If $A^j \downarrow \phi$, $A^j \in \mathcal{F}^0$, $j \in \mathbf{N}_+$, then $\forall i \in I$ $P^i(A^j) \downarrow 0$. By Lemma III.6, $\mu^0(A^j) \downarrow 0$.

By Lemma III.7 and the Carathéodory extension theorem, if μ is an equilibrium asset price then there exists a measure μ^* on $(S, \mathcal{F})$ such that μ^* estends μ^0.

Theorem III.3. *(Blackwell-Dubins) - If ν^1 and ν^2 are two probabilities measures defined on $(S, \mathcal{F})$ and ν^1 is absolutely continuous with respect to ν^2 then there exists a set $\Omega \subseteq S$*

such that $\nu^1(\Omega) = 1$ *and* $\forall s \in \Omega$, $s = (s_1, s_2, \ldots, s_n, \ldots)$ *the distance between* $\nu^{1(n)}_{(s_1,\ldots,s_n)}$ *and* $\nu^{2(n)}_{(s_1,\ldots,s_n)}$ *converges to 0 as n converges to* ∞.

Proof of Theorem III.1: Condition 2) hold by Lemma III.1.

Suppose by contradiction that for some $A \in \mathcal{F}$, $\mu^*(A) > \varepsilon$ and $P^{i^0}(A) = 0$, $i^0 \in I$. By Lemma III.3 there exists a sequence $A^1, A^2, \ldots$, such that $A^j \in \mathcal{F}^0$ $\forall j \in \mathrm{N}_+$, $\mu^*(A^j) > \varepsilon$ $\forall j \in \mathrm{N}_+$ and $P^{i^0}(A^j) \underset{j\to+\infty}{\longrightarrow} 0$. By Lemma III.6 $\mu^*(A^j) = \mu^0(A^j) \underset{j\to+\infty}{\longrightarrow} 0$. ∎

Corollary 1: In cases $k = 1, 2, 3$, if there exist an equilibrium then the priors are asumptotically homogeneous in a full μ^* measure set $\Omega \in \mathcal{F}$. Moreover, Ω has a strictly positive probability according to all agents.

Proof: Immediate from Theorems III.1 and III.3. ∎

One can see by Corollary 1 the difference between models with and without bankruptcies. In the last case the non convergent equilibriums are ruled out and it is possible to prove convergence to homogeneous expectations. However, this convergence is with full measure with respect to the price measure and positive measure with respect to the agents priors but not necessarily with respect to the true measure Q. Clearly if no relationship is imposed between the agents prior and true distribution then the true distribution can be completely "disconected" from all agents priors. Intuitively, in a case such that all agents are mistaken even if the markets spread out all the private information the agents will not become perfectly informed, especially if they use myopic bayesian learning as their revision process.

III.1 Let C^i be a full measure set of P^i, $i \in I$. If $A \subset \bigcap_{i\in I} C^i$ and $\forall i \in I$ $P^i(A) > 0$ then $Q(A) > 0$.

III.2 Let C^j be a full measure set of P^j, $j \in I$. If $\forall i \in I$ $P^i(\bigcap_{j\in I} C^j) > 0$ then $Q(\bigcap_{j\in I} C^j) > 0$.

Essentially III.2 assumes that if the supports of the priors are not disjoint then its intersection is also not disjoint from the support of Q.

For instance III.2 is satisfied if there exists an $s \in S$ such that $Q(s) > 0$ and $P^i(s) > 0$ $\forall i \in I$.

Essentially III.1 assumes that if an event A can not happen then there exists an agent (not necessarily the same agent) that knows it. Clearly III.1 is a stronger condition then III.2 especially in case $k = 3$ because in this case, III.1 implies that if a future price is not an equilibrium price then there exists an agent that knows it.

Corollary 2: In cases $k = 1, 2$, Under II.1 - II.9 and III.2, if there exists an equilibrium then the agents prior P^i, $i \in I$ are convergent to rational expectations in a strictly positive Q-measure set.

In case $k = 3$, Under II.1 - II.9 if there exists a temporary equilibrium that satisfies III.2 then it is convergent to rational expectations in a strictly positive Q measure set.

Proof: The proof is the same in cases $k = 1, 2, 3$. By Theorem III.1, $\forall i \in I$ $\mu^* << P^i$. Let C^i be a full measure set of P^i. By Lemma III.4 (Take $\mathcal{T}^0 = \{S\}$), $P^i(\bigcap_{j \in I} C^j) > 0$ $\forall i \in I$. By III.2, $Q(\bigcap_{j \in I} C^j) > 0$. So if $\overline{C}$ is a full Q measure set, then $Q((\bigcap_{j \in I} C^j) \cap \overline{C}) > 0$.

Appling again Lemma III.4 ($\mathcal{T}^0 = \{S\}$) one can see that there exists a measure ν on $(S, \mathcal{F})$ such that $\forall i \in I$, $\nu << P^i$ and $\nu << Q$. By Theorem III.3 distance between the posteriors of P^i and Q converges to zero in a set $D \in \mathcal{F}$, $\nu(D) = 1$. But $\nu << Q$ implies that $Q(D) > 0$. ∎

Grossman and Stiglitz (80) and Grossman (81) have pointed out in very known papers that prices could be used to inform the economic agents about the realization of the uncertainty. Corollary 2 (in cases $k = 1, 2$) also relates prices with information because it states that observing the equilibrium prices the agents can eventually learn the true distribution of the uncertainty. However there are some differences between this model and Grossman's. In those models the bijection between prices and uncertainty reveals information only if the economic agents knows the bijection. In this model, this assumption is not necessary so, if there exists an informed agent the non informed agents will "figure out" this information only in the limit. As there are trade thought the learning process during some time an agent may profit from private information. That is why there is no

non existence of equilibrium paradox in this model.

Clearly Theorem III.1 can be used to prove that if $s \in \Omega$ occurs then in the long run the "psycology" of the markets do not matter and one must focus only on the "fundamentals". As a example we will (in Corollary 3) state that in the long run the relative asset price depends only on Q and converges to the "true uncertainty ratio". In particular since μ is, in the long run, independent from (u^i, P^i, w^i), $i \in I$ it is also independent from the risk aversions of the agents.

Consider $(s_1, \ldots, s_n) \in \prod_{t=1}^{n} S_t$. Fix $\delta > 0$. We say that the pairs of events $(A, A') \in \mathcal{F}^0 \times \mathcal{F}^0$ subsequents to $(s_1, \ldots, s_n)$ satisfies $(^*)$ if one of them, say $A' \in \mathcal{F}^0$, is such that $Q^{(n)}_{(s_1,\ldots,s_n)}(A') \geq \delta$. Fortunately, the pairs of events that do not satisfies $(^*)$ are very unlikely to occur.

The relative asset price is said to be convergent to the true uncertainty ratio with respect to $s \in S$ if $\forall \varepsilon > 0$, there exists on $N(\varepsilon) \in \mathbf{N}_+$ such that for all pair of events $(A_t, A'_t) \in \mathcal{F}_t \times \mathcal{F}_t$ subsequents to $(s_1, \ldots, s_{N(\varepsilon)})$ that satisfies $(^*)$ are such that

$$\left| \frac{\mu_t(A_t)}{\mu_t(A'_t)} - \frac{Q(A_t)}{Q(A'_t)} \right| \leq \varepsilon.$$

Corollary 3: In cases $k = 1, 2, 3$, Under II.1 – II.9 and III.1. If there exist on equilibrium then. There exists a set $\Omega' \in \mathcal{F}$, $Q(\Omega') > 0$, $\mu^*(\Omega') = 1$ such that if the relative asset price is convergent to the true uncertainty ration with respect to every $s \in \Omega'$.

Proof: Suppose that $\mu^*(B) > 0$, $B \in \mathcal{F}$. Consider a full measure set C^i to P^i. Let C be $\bigcap_{i \in I} C^i$. By Theorem III.2 $\mu^* << P^i$ $\forall i \in I$. So $\mu^*(\bigcup_{i \in I}(C^i)^c) = 0$ and then $\mu^*(C) = 1$, $\mu^*(B \cap C) > 0$. Since $\mu^* << P^i$, $P^i(B \cap C) > 0$ $\forall i \in I$. By III.1, $Q(B \cap C) > 0$ and so, $Q(B) > 0$.

Therefore, $\mu^* << Q$.

We define

$$f\colon [0,1] \times [\frac{\delta}{2}, 1] \longrightarrow R_+$$

$$(x, y) \longrightarrow \frac{x}{y} \tag{1}$$

Since f is uniformily continuous there exists an $\Delta > 0$ such that

$$\text{if } |x - x'| \leq \Delta \text{ and } |y - y'| \leq \Delta \text{ then } \left|\frac{x'}{y'} - \frac{x}{y}\right| \leq \varepsilon \tag{2}$$

On the other hand, by the μ^0 and μ^* definitions, if $(A_t, A'_t) \in \mathcal{F}_t \times \mathcal{F}_t$,

$$\frac{\mu_t(A_t)}{\mu_t(A'_t)} = \frac{\mu^0(A_t)}{\mu^0(A'_t)} = \frac{\mu^*(A_t)}{\mu^*(A'_t)} \tag{3}$$

By (1) and Theorem III.1, for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbf{N}$ such that if $D \in \mathcal{F}$ is subsequent to $(s_1, \ldots, s_{N(\varepsilon)})$ then

$$\left|\mu^{N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(D) - Q^{N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(D)\right| \leq \Delta \tag{4}$$

Since D is contained in the cylinder with base $(s_1, \ldots, s_{N(\varepsilon)})$, by the Baye's rule:

$$\frac{\mu^*(A_t)}{\mu^*(A'_t)} = \frac{\mu^{*N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(A_t)}{\mu^{*N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(A'_t)} \text{ and } \frac{Q(A_t)}{Q(A'_t)} = \frac{Q^{N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(A_t)}{Q^{N(\varepsilon)}_{(s_1,\ldots,s_{N(\varepsilon)})}(A'_t)} \tag{5}$$

But (4), (2) and (5) implies that

$$\left|\frac{\mu^*(A_t)}{\mu^*(A'_t)} - \frac{Q(A_t)}{Q(A'_t)}\right| \leq \varepsilon.$$

Notice that it is no less of generality to assume that $|\Delta| < \frac{\delta}{2}$. By (3),

$$\left|\frac{\mu_t(A_t)}{\mu_t(A'_t)} - \frac{Q(A_t)}{Q(A'_t)}\right| \leq \varepsilon \quad \blacksquare$$

The main criticism we can make to corollaries 1,2 and 3 is that these statements hold only with positive probability (according to Q). Unfortunatelly, Theorem III.2 says that under II.1 – II.9 it is impossible to get a better convergence result without imposing further restrictions on P^i, $i \in I$ and Q. So, lack of convergence is also a possibility even under complete markets.

Proof of Theorem III.2: Let π be an equilibrium (no bankruptcy) price as in Theorem II.1. By II.6a), II.6b), II.6c), H_π^i defined by:

$$\begin{aligned} H_\pi^i : \mathbf{R}_+ &\longrightarrow \mathbf{R} \\ \varepsilon &\longrightarrow \sup_{\langle\pi, x^i\rangle = \varepsilon} \{v^i(x^i)\} \end{aligned}$$

is a concave, nondecreasing function. Consider $\bar{\varepsilon}_i = \langle \pi, w^i \rangle > 0$, by Lemma III.2 there exist an $M^i \in \mathbf{R}_+$ such that:

$$H_\pi^i(\varepsilon + \bar{\varepsilon}_i) - H_\pi^i(\bar{\varepsilon}_i) \leq M^i \varepsilon \quad \forall \varepsilon > 0 \tag{A}$$

By Theorem III.1 and Lemma III.4 there exists a measure ν_1 on $(S, \mathcal{F})$ such that $\nu_1 \leq P^i$ $\forall i \in I$.

We define M by $(\max_{i \in I} M^i)/\nu_1(S)$.

Let $r \geq 0$ be the equilibrium interest rate.

We define the equilibrium price system (μ, p) by: $\forall t \in \mathbf{N}_+, \forall A_t \in \mathcal{F}_t$

$$\mu_t(A_t) = \frac{\nu_1(A_t)}{\nu_1(S)}(1+r)^{-t}$$

$$p_t(A_t) = \begin{cases} \frac{\pi_t(A_t)}{\mu_t(A_t)} & \text{if } \mu_t(A_t) > 0 \\ 0 & \text{if } \mu_t(A_t) = 0 \end{cases}$$

We define the equilibrium goods allocation x as being the same as in the no bankruptcy case. We define the equilibrium assets and money allocation (d, x) by: $\forall i \in I, \forall t \in \mathbf{N}_+$,

$$\begin{aligned} d_t^i &= p_t(x_t^i - w_t^i) \\ m_t^i &= 0 \\ m_0^i &= 0 \end{aligned}$$

By the definitions condition 2) and Lemma III.1, $\mu_t p_t = \pi_t$ $\forall t \in \mathbf{N}_+$, $\langle \mu, d^i \rangle = \langle \pi, x^i - w^i \rangle$, (x, d, m) are feasible allocations of goods, assets and money and (x^i, d^i, m^i) gives to agent i maximal utility and minimal penalty between all feasible allocations such that

$\left\langle \pi, \overline{x}^i - w^i \right\rangle \leq 0$. So, to conclude the proof we must prove that if $(\overline{x}^i, \overline{d}^i, \overline{m}^i)$ is in the i-th agent budget set and $\left\langle \pi, \overline{x}^i - w^i \right\rangle = \varepsilon > 0$ then $v^i(x^i) \geq v^i(\overline{x}^i) - b(\overline{h}^i)$.

But $\forall t \in \mathbb{N}_+$,

$$\overline{m}_t^i + p_t(\overline{x}_t^i - w^i) = \overline{d}_t^i + \overline{m}_{t-1}^i(1+r)$$

Multiplying both sides by μ_t,

$$\mu_t(\overline{m}_t^i - \overline{m}_{t-1}^i(1+r)) + \pi_t(\overline{x}_t^i - w^i) = \mu_t \overline{d}_t^i.$$

Since ν_1 is a measure, $\mu_{t+1} = \mu_t(1+r)$. So, adding the above equation,

$$\mu_{t_0}\overline{m}_{t_0}^i - \mu_1.\overline{m}_0^i(1+r) + \sum_{t=1}^{t_0} \pi_t(\overline{x}_t^i - w_t^i) = \sum_{t=1}^{t_0} \mu_t \overline{d}_t^i$$

Since $\mu_1((S) = (1+r)^{-1}$ and integrating

$$\int \frac{\mu_{t_0}}{(1+r)^{t_0}}.\overline{m}_{t_0}^{*i}\# \quad - \quad \overline{m}_0^i + \sum_{t=1}^{t_0} \int_S \pi_t(\overline{x}_t^i - w_t^i)\# \quad = \quad \sum_{t=1}^{t^0} \int_S \mu_t \overline{d}_t^i \#.$$

taking limits one gets:

$$\lim_{t_0 \to +\infty} \int_S \overline{m}_{t_0}^{*i} \frac{\nu_1}{\nu_1(S)} - \overline{m}_0^i + \left\langle \pi, \overline{x}^i - w^i \right\rangle = \left\langle \mu, \overline{d}^i \right\rangle$$

So,

$$\lim_{t_0 \to +\infty} \int_S \overline{m}_{t_0}^{*i} \frac{\nu_1}{\nu_1(S)} = -\varepsilon.$$

Since $\overline{h}^i$ is bounded. by Fatou's lemma:

$$\int_S \overline{h}^i \nu_1 \leq \liminf_{t_0 \to +\infty} \int_S \overline{m}_{t_0}^{*i} \nu_1 = -\varepsilon \nu_1(S).$$

But, $\min\{\overline{h}^i, 0\} \leq \overline{h}^i$. So, $\int_S \min\{\overline{h}^i, 0\}\nu_1 \leq -\varepsilon\nu_1(S)$. Since $\nu_1 \leq P^i$, $\forall i \in I$ $\int_S \min\{\overline{h}^i, 0\} P^i \leq -\varepsilon \nu_1(S)$.

By III.4, $b(\overline{h}^i) = \int_S \mathcal{I}(\min\{\overline{h}^i, 0\})P^i \geq +M\varepsilon\nu_1(S) \geq M^i\varepsilon$.

By the definition of H^i_π, $v^i(\overline{x}^i) \leq H^i_\pi(\varepsilon + \overline{\varepsilon}_i)$.

By Lemma III.5, $v^i(\overline{x}^i) \leq H^i_\pi(\overline{\varepsilon}_i) + M^i\varepsilon \leq v^i(x^i) + b(\overline{h}^i)$. So $\forall i \in I$, $v^i(\overline{x}^i) - b(\overline{h}^i) \leq v^i(x^i)$. ∎

IV - Further results and conjectures[6]

Once the case of complete markets are studied a natural conjecture is if all those markets are necessary to garantee convergence to rational expectations. The authors have the proof that there exists a certaint N (endogenously given) such that the asset $\mu_t, t \geq N$ are not necessary. These reduce the number of markets from a countable to a finite one.

This generalization is not important in cases $k = 1, 2$ since one could consider the case of dynamically complete markets.[7] But, in case $k = 3$, it makes no sense to assume that the future asset prices are common knowledge since the prices themselves are not. So, it seems that in this case the generalization for the dynamically complete market are not so easy and it is possible that some redudancy in the assets markets are important to "generate an information market". (The role of redundant assets were pointed out by Mas-Colell (89) to avoid nontrivial sunspot equilibrium).

To consider the case of in complete markets, another possibility is to assume that μ_t, d^i_t are $\mathcal{F}'_t$-mensurable and that $\mathcal{F}'_t$ is a less refined σ-algebra than $\mathcal{F}_t$. We conjecture that there is convergence only with respect to the less refined stochastic process. That is, there is convergence only where "there exists a market".

We also conjecture that assumptions II.3 and II.4 are not necessary and one may consider the continuous case. Dropping assumption III.2 and assuming that $\forall A \in \mathcal{F}$:

$$P^i(A) = 0 \quad \forall i \in I \Rightarrow Q(A) = 0,$$

then one might prove that if priors are gaussians the convergence set has full measure according to Q. The convergence in full measure could be obtained in view of the zero one law valid for Gaussian probability measures: Any measurable subspace has measure zero or one. Finally, we conjecture that under III.2 the goods allocations converges to an

[6] The authors have already proved many of the conjectures presented.

[7] In fact the original proofs were done considering the dynamically complete market case.

"Arrow-Debreu equilibrium allocation". In particular, in the long run, sunspots disapear. Under III.1, we conjecture that the prices also converge and becomes deterministically given conditional to the signal observation.

APPENDIX

Proof of Lemma II.1: m_t^{*i} is $\mathcal{F}$-mensurable because it is $\mathcal{F}_t$-mensurable. $h^i = \liminf m^{*i}$ is $\mathcal{F}$ mensurable by well known properties of mensurable functions and since $\mathcal{I}$ is continuous, $\mathcal{I}(\min\{h^i,0\})$ is also $\mathcal{F}$ mensurable. By II.8 h^i is a bounded function. So, $\mathcal{I}(\min\{h^i,0\})$ is also bounded. Since P^i is a probability measure, $\mathcal{I}(\min\{h^i,0\})$ is P^i integrable.

Proof of Lemma III.3: If $\mathcal{T}$ is the σ-algebra generated by an algebra $\mathcal{T}^1$ then by the definition of the Lesbegue extension if ν is a measure on $\mathcal{T}$ then $\forall B \in \mathcal{T}$,

$$\nu(B) = \inf\left\{\sum_{r=1}^{+\infty}\nu(B^r)/B \subset \sum_{r=1}^{+\infty} B^r; B^r \in \mathcal{T}^1 \quad \forall r \in \mathbb{N}_+\right\}$$

(Rembember that B^r, $r \in \mathbb{N}_+$ are assumed to be disjoint sets).

If $\nu^2(D) = 0$ then $\forall j \in \mathbb{N}$ there exists a set $\{B^r \in \mathcal{T}^1, r \in \mathbb{N}_+\}$ such that $\sum_{r=1}^{+\infty}\nu^2(B^r) \leq \frac{1}{j}$ and $D \subset \sum_{r=1}^{+\infty} B^r$.

If $\nu^1(D) > \varepsilon$ then $\sum_{r=1}^{+\infty}\nu^1(B^r) = \nu^1\left(\sum_{r=1}^{+\infty}B^r\right) > \varepsilon$. So there exists an j such that $\sum_{r=1}^{j}\nu^1(B^r) > \varepsilon$ and $\sum_{r=1}^{j}\nu^2(B^r) \leq \frac{1}{j}$. Take $A^j = \sum_{r=1}^{j} B^r$.

$A^j \in \mathcal{T}^1$ and by construction, $\nu^1(A^j) > \varepsilon$ $\forall j \in \mathbb{N}$ and $\nu^2(A^j) \underset{j \to +\infty}{\longrightarrow} 0$.

Proof of Lemma III.4: A) $\Rightarrow$ C). Let ν be $\frac{1}{L}\sum_{j=1}^{L}\nu^j$.

Clearly $\nu^j << \nu$ $j = 1, \ldots, L$. Let $f^j: X \to \mathbb{R}$ be the Radon-Nikodyn derivative with respect to ν that is, $\nu^j = f^j\nu$ $j = 1, \ldots, L$.

Define $\nu^{L+2} = \min_{1\leq j\leq L} f^j.\nu$. Clearly $\nu^{L+2} \leq \nu^j$ $j = 1, \ldots, L$. Let $\overline{C}^j$ be $\{w \in X; f^j(w) > 0\}$ and $\overline{C} = \bigcap_{j=1}^{L}\overline{C}^j$. Since $\nu^j(\overline{C}^j) = 1$ $j = 1, \ldots, L$ and $\nu^j(X) > 0$

$j = 1, \ldots, L$ then by A) $\nu^j(\overline{C}) > 0$ $j = 1, \ldots, L$. So, $\nu(\overline{C}) > 0$. But, if $w \in \overline{C}$ then $\min_{1\le j\le L} f^j(w) > 0$. Therefore,

$$\nu^{L+2}(\overline{C}) = \int_C \min_{1\le j\le L} f^j \nu > 0. \quad \nu^{L+2}(\overline{C}) \not\ll 0.$$

Notice that by the same proof if $\mathcal{T}^0 = \{X\}$ then D) $\Rightarrow$ C) because to prove $\nu(\overline{C}) > 0$ it is only necessary that there exists an $\ell, 1 \le \ell \le L$ such that $\nu^\ell(\overline{C}) > 0$.

Repeting the argument, if $D \in \mathcal{T}^0$ and $\nu^\ell(D) > 0$ for some $\ell, 1 \le \ell \le L$ then by A) $\nu^j(D \cap C) > 0$ $j = 1, \ldots, L$. So, $\nu(\overline{C} \cap D) > 0$. Therefore,

$$\nu^{L+2}(D) \ge \nu^{L+2}(D \cap \overline{C}) = \int_{D\cap\overline{C}} \min_{1\le j\le L} f^j \nu > 0.$$

On the other hand if $\nu^j(D) = 0$ $j = 1, \ldots, L$, $D \in \mathcal{T}$ then as $\nu^{L+2} \le \nu^j$ $j = 1, \ldots, L$. $\nu^{L+2}(D) = 0$. So, $\forall D \in \mathcal{T}^0$, $\nu^{L+2}(D) = 0 \Leftrightarrow \nu^j(D) = 0$ $j = 1, \ldots, L$.

C) $\Rightarrow$ B). Obvious.

B) $\Rightarrow$ A).

Let C^j $j = 1, \ldots, L$ be such that $\nu^j(C^j) = 1$ $j = 1, \ldots, L$ and $C = \bigcap_{j=1}^{L} C^j$.

Consider $D \in \mathcal{T}^0$ and suppose that there exists an $\ell, 1 \le \ell \le L$ such that $\nu^\ell(D) > 0$. By B), $\nu^{L+1}(D) > 0$ and $\nu^{L+1} << \nu^j$ $j = 1, \ldots, L$. So, $\nu^{L+1}(\bigcup_{j=1}^{L}(C^j)^c) = 0$. That is, C has full measure according to ν^{L+1} and so, since $\nu^{L+1}(D) > 0$ then $\nu^{L+1}(D \cap C) > 0$. But, $\nu^{L+1} << \nu^j$ $j = 1, \ldots, L$ and $\nu^{L+1}(D \cap C) > 0$, and so $\nu^j(D \cap C) > 0$ $j = 1, \ldots, L$.

If $\mathcal{T}^0 = \{X\}$ then as was proved D) $\Rightarrow$ C) and clearly A) $\Rightarrow$ D). But A), B) and C) are equivalent and so, in this case, A), B), C) and D) are equivalent. ∎

Proof of Theorem III.5: This proof is a generalization of Cass and Shell (83) Proposition 1. The lemma is a consequence of the "no arbitrage" conditions between assets and money. If $\mu_t(A_t) = 0$ then by Lemma III.1 $P^i(A_t) = 0$ $\forall i \in I$. Since $A_t \in \mathcal{F}_{t+1}$ again by Lemma III.1 $\mu_{t+1}(A_t) = 0$.

Suppose by contradition that $\mu_{t+1}(A_t) > \mu_t(A_t)(1+r)^{-1}$ and $\mu_t(A_t) > 0$ for some $A_t \in \mathcal{F}_t$. Consider the i-th agent optimal allocations $(\overline{x}^i, \overline{d}^i, \overline{m}^i)$ and for arbitrary $M > 0$ let (d^i, m^i) be such that:

$$d_t^i : S \longrightarrow \mathbf{R}$$

$$s \longrightarrow \overline{d}_t^i(s) + M\frac{\mu_{t+1}(s)}{\mu_t(s)} \quad s \in A_t$$

$$s \longrightarrow \overline{d}_t^i(s) \quad s \notin A_t$$

$$d_{t+1}^i : S \longrightarrow \mathbf{R}$$

$$s \longrightarrow \overline{d}_{t+1}^i(s) - M \quad s \in A_t$$

$$s \longrightarrow \overline{d}_{t+1}^i(s) \quad s \notin A_t$$

$d_j^i = \overline{d}_j^i$ if $j \neq t$ or $j \neq t+1$.

$$m_t^i : S \longrightarrow \mathbf{R}$$

$$s \longrightarrow \overline{m}_t^i(s) + M(1+r)^{-1} \quad s \in A_t$$

$$s \longrightarrow \overline{m}_t^i(s) \quad s \notin A_t$$

$m_j^i = \overline{m}_j^i$ if $j \neq t$.

One can see that the new demands correspond to take more assets, less money at (A_t, t) and less asset at the subsequents events. Clearly,

$$p_{t+1}(\overline{x}_{t+1}^i - w_{t+1}^i) = \overline{d}_{t+1}^i + \overline{m}_t^i(1+r) - \overline{m}_{t+1}^i = d_{t+1}^i + m_t^i(1+r) - m_{t+1}^i.$$

The last equality hold by (m^i, d^i) definition.

$\forall i \in I$, h^i and the penalty won't change since $m_j^i = \overline{m}_j^i$ $j > t$. But at (A_t, t), (d^i, m^i) allows the i-th agent to buy more goods since $\forall s \in A_t$,

$$\begin{aligned} p_t(s)(x_t^i(s) - w_t^i(s)) &= d_t^i(s) + m_t^i(s)(1+r) - m_{t+1}^i(s) \\ &= \overline{d}_t^i(s) + \frac{M}{(1+r)}\left(\frac{\mu_{t+1}(s)}{\mu_t(s)}(1+r) - 1\right) \\ &+ \overline{m}_t^i(s)(1+r) - \overline{m}_{t+1}^i(s) > p_t(s)(\overline{x}_t(s) - w_t^i(s)). \end{aligned}$$

So, By the monotonicity assumption, (d^i, m^i) makes the i-th economic agent better off and d^i can be bought at period zero by the i-th agent since the definition of d^i implies that $\langle \mu, d^i \rangle = \left\langle \mu, \overline{d}^i \right\rangle$. We get a similar contradiction if we suppose that $\mu_{t+1}(A_t) < \mu_t(A_t)(1+r)^{-1}$. The proof of $\mu_1(S) = (1+r))^{-1}$ is also similar. ∎

REFERENCES

AZARIADIS, C. (1981) "Self-fulfilling prophecies". Journal of Economic theory 25, 380-396.

ARROW, K.J. (1964) "The role of sewrities in the optimal allocation of Risk-bearing", Review of Economic Studies, 31, 91-96.

BEWLEY, T. (1972) "Existence of Equilibria in Economies with Infinitely many Comodities". Journal of Economic theory 4, 514-540.

BLACKWELL, D. and DUBINS, L. (1962) "Merging of opinions with increasing information". Berkley Simposium, 883-886.

BLUME, L.E., BRAY, M.M. and EASLEY, D. (1982) "Introduction to the Stability of Rational Expectations Equilibrium". Journal of Economic theory 26, 313-317.

BRAY, M. (1982) "Learning. Estimation and the Stability of Rational Expectations". Journal of Economic theory, 26, 318-339.

BRAY, M. and N. SAVIN (1986) "Rational Expectations Equilibria Learning and Model Specification". Econometrica, 54, 1129-1160.

CANIO, de, S. (1979) "Rational Expectations and Learning from Experience", Quarterly Journal of Economics, 93, 47-57.

CASS, D. and K. SHELL "Do Sunspots Matter?" (1983). Journal of Political Economy. 91-2, 193-227.

CHI-FU HUANG and R.H. LITZINBERGER Foundations for Financial Economics. North-Holland, ed.

EVANS, G. (1985) "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models", Quarterly Journal of Economics, 100, 1217-1234.

FELDMAN, M. (1987) "An Example of Convergence to Rational Expectations with Heterogeneous beliefs", International Economic Review, 28: 3, 635-650.

FELDMAN, M. (1987) "Bayesian Learning and convergence to Rational Expectations". Journal of Mathematical Economics, 16, 297-313.

FOURGEAUD, C., GOURIÉROUX C. and J. PRADEL (1986) "Learning procedures and convergence to rationality", Econometrica, 54, 854-868.

FRYDMAN, R. (1982) "Toward an Understanting of Market Processes", American Economic Review, 72, 652–668.

FUCHS, G. (1979b) "Is error Learning Behavior Stabilizing?", Journal of Economic theory, 20, 300-317.

FUCHS, G. and G. LAROQUE (1976) "Dynamics of Temporary Equilibria and Expectations", Econometrica, 44, 1157-1178.

GRANDMONT, J.M. (1977) "Temporary General Equilibrium Theory", Econometrica, 45, 535–572.

GRANDMONT, J.M. and G. LAROQUE (1986) "Stability of Cycles and Expectations", Journal of Economic theory, 40, 138–151.

GRANDMONT, J.M. and G. LAROQUE (1990) "Economic Dynamics with Learning: Some instability Examples". Mimeo. Cepre map.

GREEN, J.R. (1973) "Temporary General Equilibrium in a Sequential Trading Model with Spot and futures Transactions". Econometrica 41, 1103–1123.

GROSSMAN, S.J. (1981) "An introduction to the theory of rational expectations under asymetric information". Review of Economics Studies. XLVIII, 541–559.

GROSSMAN, S.J. and STIGLITZ, J.E. (1980) "On the impossibility of Informationally Efficient Markets". American Economic Review, 70 (3), 393–408.

HELLWIG, M. (1982) "Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example". Journal of Economic theory 26, 279–312.

JORDAN, J.S. (1980) "On the predictability of Economic Events". Econometrica, 48, 955-972.

JORDAN, J.S. (1990) "Convergence to Rational Expectations in a Stationary Linear Games". Mimeo. University of Minnesota.

LUCAS, R.E. (1972) "Expectations and the neutrality of money", Journal of Economic theory, 4 (2), 103–124.

LUCAS, R.E. (1986), "Adaptive Behavior and Economic theory", Journal of Business, 59, 401–426.

MARCET, A. and T. SARGENT (1987) "The fate of systems with "Adaptive" Expectations". American Economic Review, 78 (2).

MARCET, A. and T. SARGENT (1989) "Convergence of Least Squares Learning Mechanisms in Self-Referential linear Stochastic Models", Journal of Economic Theory, 48, 337–368.

MAS-COLLEL, A. (1985) The theory of general economic equilibrium: a differentiable approach. Econometric Society Publication nº 9.

MAS-COLELL, A. (1989) "Three observations on sunspots". Mimeo. Harvard University.

TOWNSEND, R. (1978) "Market anticipations, Rational Expectations and Bayesian Analysis". International Economic Review, 19 (2), 481–494 .

WILSON, C.A. (1981) "Equilibrium in Dynamic Models with an Infinity of Agents". Journal of Economic theory 24, 95–111.

WOODFORD, M. (1990), "Learning to Believe in Sunspots". Econometrica 58-2, 277–308.