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Escola de Pós-Graduação em Economia

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da Fundação

Getulio Vargas

N° 748 ISSN 0104-8910

On the Welfare Costs of Business-Cycle Fluctuations and Economic-Growth Variation in the 20th Century and Beyond

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Outubro de 2013

URL: http://hdl.handle.net/10438/11273

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     Teixeira de Carvalho Guillén, Osmani
        On the Welfare Costs of Business-Cycle Fluctuations
and Economic-Growth Variation in the 20th Century and Beyond/
Osmani Teixeira de Carvalho Guillén, João Victor Issler,
Afonso Arinos de Mello Franco-Neto - Rio de Janeiro :  FGV,EPGE,
2013
        36p. - (Ensaios Econômicos; 748)

        Inclui bibliografia.

                                         CDD-330
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On the Welfare Costs of Business-Cycle Fluctuations and Economic-Growth Variation in the 20th Century and Beyond*

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JEL Codes: E32; C32; C53.

Keywords: Business cycles fluctuations, economic-growth variation, welfare costs, Beveridge-Nelson Decomposition, Unobserved-component model.

October, 2013.

Abstract

The main objective of this paper is to propose a novel setup that allows estimating separately the welfare costs of the uncertainty stemming from business-cycle fluctuations and from economic-growth variation, when the two types of shocks associated with them (respectively,

*We gratefully acknowledge the comments and sugggestions given by an anonymous referee, Caio Almeida, Isabel Correia, Larry Christiano, Carlos E. da Costa, Wouter den Haan, Robert F. Engle, Pedro C. Ferreira, Antonio Fiorencio, Clive Granger, Rodolfo Manuelli, Marcelo Moreira, Chris Otrok (Editor-in-Charge), Samuel Pessoa, Pedro Telles, and Octavio Tourinho on this or on earlier versions of this paper. All errors are ours. We thank CNPq-Brazil, FAPERJ and INCT for financial support. We gratefully acknowledge the excellent research assistence given by Rafael Burjack, Marcia Waleria Machado, and Marcia Marcos.

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transitory and permanent shocks) hit consumption simultaneously. Separating these welfare costs requires dealing with degenerate bivariate distributions. Levi's Continuity Theorem and the Disintegration Theorem allow us to adequately define the one-dimensional limiting marginal distributions. Under Normality, we show that the parameters of the original marginal distributions are not affected, providing the means for calculating separately the welfare costs of business-cycle fluctuations and of economic-growth variation.

Our empirical results show that, if we consider only transitory shocks, the welfare cost of business cycles is much smaller than previously thought. Indeed, we found it to be negative – -0.03% of per-capita consumption! On the other hand, we found that the welfare cost of economic-growth variation is relatively large. Our estimate for reasonable preference-parameter values shows that it is 0.71% of consumption – US$ 208.98 per person, per year.

1. Introduction

The main objective of this paper is to propose a novel setup that allows estimating separately the welfare costs of the uncertainty stemming from business-cycle fluctuations and from economic-growth variation, when the two types of shocks associated with them (respectively, transitory and permanent shocks) hit consumption simultaneously. Permanent shocks arise naturally in the consumption literature – e.g., Hall (1978) and Flavin (1983) – where it is shown that consumption should follow a martingale. It also arises in the stochastic discount factor literature – e.g., Alvarez and Jermann (2005), and Hansen and Scheinkman (2009). Indeed, as stressed by Alvarez and Jermann, "for many cases where the pricing kernel is a function of consumption, innovations to consumption need to have permanent effects." Thus, we model the trend in (log) consumption as a martingale process to accommodate this need. Fluctuations about the trend (the cycle) are modelled as a stationary and ergodic zero-mean process, and can have a variety of sources, e.g., monetary policy; see the discussion on the nature of both shocks in Issler and Vahid (2001).

Separating the welfare effects of permanent and transitory shocks requires dealing with degenerate bivariate distributions. Levi's Continuity Theorem and the Disintegration Theorem allow us to adequately define the one-dimensional limiting marginal distributions. Under Normality, we show that the parameters of the original marginal distributions are not affected, providing the means for calculating separately the welfare costs of business-cycle fluctuations and of economic-growth variation. We measure welfare costs using the *equivalent variation* in consumption, a common practice in this literature. Despite that, in computing equivalent variation, we employ a novel counter-factual exercise, where consumption sequences are devoid of the effects of either permanent or transitory shocks, one at a time. Our solution is consistent with two usual assumptions on the bivariate distribution for transitory and permanent shocks – when their innovations are independent and when they are dependent with a general correlation coefficient.

From the perspective of a representative consumer, who dislikes systematic risk, it makes sense

for macroeconomic policy to try to reduce the variability of pervasive shocks affecting consumption. The best known approach to this issue was put forth by Lucas (1987, 3), who calculates the amount of extra consumption a rational consumer would require in order to be indifferent between an infinite sequence of consumption under cyclical uncertainty alone (aggregate consumption under deterministic growth) and a consumption sequence with the same deterministic growth and no cyclical variation. In his setup, business-cycle shocks are the only source of randomness for aggregate consumption. Thus, Lucas' measure is known as the *welfare cost of business cycles*. For 1983 figures, using a reasonable parametric utility function (CRRA), and post-WWII data, the extra annual consumption is about US$ 8.50 per person in the U.S., a surprisingly low amount.

Several papers have been written just after Lucas first presented his results. For example, Imrohoroglu (1989) and Atkeson and Phelan (1995) recalculated welfare costs using models with specific types of market incompleteness. Van Wincoop (1994), Pemberton (1996), Dolmas (1998), and Tallarini (2000) have either changed preferences or relaxed expected utility maximization. In some of them, welfare costs of business cycles reached up to 25% of per-capita consumption. Regarding preferences, Otrok(2001) notes that "it is trivial to make the welfare cost of business cycle as large as one wants by simply choosing an appropriate form for preferences." To avoid this critique, we keep preferences as in Lucas, but allow for an additional source of uncertainty – randomness on the growth rate of the economy.

Regarding the original setup, as in Zellner's (1992) version of the KISS principle, Lucas *Keeps It Sophisticatedly Simple*: if only transitory shocks hit consumption, the best a *macroeconomist* can hope to achieve in terms of welfare improvement is to eliminate completely its cyclical variation, which is equivalent to eliminating all systematic risk. Of course, the implicit counter-factual exercise being performed is rather limited in scope. First, no one expects that this trained *macroeconomist* can indeed eliminate all cyclical variation in consumption. Second, it dismisses any sources of uncertainty affecting long-term growth. Regarding the latter, Lucas recognizes that the setup could also include permanent shocks, which has lead Obstfeld (1994) to compute welfare costs in this context; see also Dolmas (1998), Tallarini (2000), Issler, Franco, and Guillén (2008) and Reis (2009).

Making the counter-factual exercise more realistic to the representative consumer is the object of study in Alvarez and Jermann (2004) – a very interesting paper in this literature. The consumer is offered a convex combination of observed consumption and its conditional mean, but not a deterministic sequence *a priori*. Because they consider the conditional mean, they can deal with non-stationary consumption. Their setup encompasses the *total* and the *marginal* welfare costs. Total costs are obtained using the method in Lucas, while marginal welfare costs are a first-order approximation of total costs when we consider small changes in welfare costs in the neighborhood of observed consumption.

More recently, the welfare-cost literature in macroeconomics has focused on rare disasters –

Barro (2009); on the effects of model uncertainty on the welfare cost of business cycles – Barillas, Hansen, and Sargent (2009); on how the stochastic properties of aggregate consumption affects welfare cost estimates – Reis; on the distinction between individual and aggregate consumption risk in computing welfare costs – De Santis (2009); and on the difference between welfare costs based on preference-parameter values that fit or not asset-pricing data – Melino (2010).

In our view, despite the existence of a seemingly mature body of work, separating the effects of permanent and transitory shocks in welfare-cost analysis is an important issue still to be addressed in this literature. Simply compare it with the macroeconometrics literature, where separating the effects of these shocks is standard, allowing the implementation of impulse-response-function and variance-decomposition exercises. In welfare analysis, this is important because the nature and sources of these shocks are completely different. Permanent shocks are usually associated with the productivity process in aggregate supply, while transitory shocks are often associated with demand factors such as monetary policy. So far, what some authors have done (e.g., Issler, Franco and Guillén) was to lump all uncertainty together, computing the welfare costs of what we have labelled *macroeconomic uncertainty*. Others have computed inconsistent estimates of the welfare cost of business cycles due to the lack of an appropriate method for separating the effects of permanent and transitory shocks. Indeed, the dichotomy between shocks with short- and long-run effects on economic variables has been key in macroeconomics since the seminal work of Phelps (1967, 1968, 1970).

This paper has also two other minor contributions. First, we show how to implement *marginal* welfare cost analysis using the consumption process itself – recall that Alvarez and Jermann implement it based on asset prices. Second, almost all of the previous literature has computed welfare costs for the post-WWII period[1]. Here, for historical interest, we also compute welfare costs for the pre-WWII period, which has lead us to compute here "Welfare Costs in the 20th Century and Beyond."

The paper is divided as follows. Section 2 provides a theoretical and statistical framework to evaluate the welfare costs of business cycles and of economic-growth variation. We compute welfare costs under two distinct assumptions on the bivariate distribution for transitory and permanent shocks – when their innovations are independent and when they are dependent with general correlation. Section 3 discusses identification issues under these two different assumptions. Section 4 provides the calculations results, and Section 5 concludes.

[1] The only exception is Alvarez and Jermann, who also estimated welfare costs including the pre-WWII period (1889-2001 and 1927-2001), although they do not present separate pre- and post-WWII results. In any case, their emphasis is on the post-WWII period (1954-2001).

2. The Problem

In this section we discuss the original problem as proposed by Lucas (1987), where transitory shocks and the welfare cost of business cycles are identified by assuming that the trend to consumption is deterministic. At least since Obstfeld (1994), the literature has considered explicitly the existence of permanent shocks. We discuss that too, and note that, so far, no satisfactory attempt has been made to separate the welfare costs associated with permanent and transitory shocks when consumption is hit by both of them. Finally, we propose a novel way to separate the effects of permanent and transitory shocks to consumption.

2.1. Previous Literature on Welfare Costs

In 1987, Lucas wrote the pioneering paper on evaluating the welfare gains of cycle smoothing – or the welfare costs of business cycles. Suppose that consumption (c_t) is *log-Normally* distributed about a deterministic trend:

$$c_t = \alpha_0 (1+\alpha_1)^t \exp\left(-\frac{1}{2}\sigma_z^2\right) z_t, \tag{2.1}$$

where $\ln(z_t) \sim N(0, \sigma_z^2)$ is the stationary and ergodic cyclical component of consumption. Cycle-free consumption is the sequence $\{c_t^*\}_{t=0}^{\infty}$, where $c_t^* = \mathbb{E}(c_t) = \alpha_0 (1+\alpha_1)^t \exp\left(-\frac{1}{2}\sigma_z^2\right) \mathbb{E}(z_t) = \alpha_0(1+\alpha_1)^t$. Notice that $\{c_t^*\}_{t=0}^{\infty}$ is the resulting sequence when we replace the random variable c_t with its unconditional mean. Hence, for any particular time period, c_t represents a mean-preserving spread of c_t^*:

$$c_t^* = \lim_{\sigma_z^2 \to 0} c_t = \lim_{\sigma_z^2 \to 0} \alpha_0 (1+\alpha_1)^t \exp\left(-\frac{1}{2}\sigma_z^2\right) z_t = \alpha_0 (1+\alpha_1)^t.$$

Hence, c_t^* is a degenerate random variable with all the mass of its distribution at $\alpha_0(1+\alpha_1)^t$.

Risk averse consumers prefer $\{c_t^*\}_{t=0}^{\infty}$ to $\{c_t\}_{t=0}^{\infty}$. Evaluating the welfare costs of business cycles amounts to calculating λ, which solves[2]:

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u((1+\lambda) c_t) = \sum_{t=0}^{\infty} \beta^t u(c_t^*), \tag{2.2}$$

where $u(\cdot)$ is the utility function of the representative agent who discounts future utility at the rate β, and $\mathbb{E}(\cdot)$ is the unconditional expectation operator. Then, the welfare cost is expressed as the *equivalent variation* λ, that consumers would require at all dates and states of nature, which makes them indifferent between the uncertain stream $\{c_t\}_{t=0}^{\infty}$ and the risk-free stream $\{c_t^*\}_{t=0}^{\infty}$.

[2] Notice that Lucas (1987) uses the unconditional mean operator instead of the conditional mean operator in (2.2). The same problem can be proposed using the conditional expectation instead. This is exactly how we proceed in this paper, after Obstfeld (1994).

In Lucas, uncertainty comes in the form of a stochastic business cycle alone, since the trend in consumption is purely deterministic. One important limitation of this setup is that it does not consider the existence of permanent shocks to consumption. Of course, at least since Nelson and Plosser (1982), macroeconomists have benefitted from econometric models with the dichotomy between permanent and transitory shocks, the first being associated with permanent factors influencing economic growth - such as productivity, population, etc., and the second being associated with transient factors - such as monetary policy.

Since Lucas modelled consumption trend as deterministic, eliminating *all the cyclical variability* in $\ln(c_t)$ is equivalent to eliminating *all* its variability. Under difference stationarity for (log) consumption, this equivalence is lost. Moreover, $\mathbb{E}(c_t)$ is not defined, since the stochastic component of $\ln(c_t)$ is neither stationary nor ergodic. This led Obstfeld (1994) to use $c_t^* = \mathbb{E}_0(c_t)$ in defining welfare costs, where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \mid \mathcal{I}_t)$ is the conditional expectation operator of a random variable using $\mathcal{I}_t$ as the information set:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left((1+\lambda) c_t\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t^*\right). \tag{2.3}$$

Here, λ is the welfare cost associated with *all* the uncertainty in consumption, not just the uncertainty associated with its business-cycle component. Indeed, on an earlier paper (Issler, Franco, and Guillén (2008)), we have labelled it the *welfare cost of macroeconomic uncertainty* as opposed to the *welfare cost of business cycles.*

An interesting generalization of the setup in Lucas is due to Alvarez and Jermann (2004), who proposed offering the consumer a convex combination of $\{c_t^*\}_{t=0}^{\infty}$ and $\{c_t\}_{t=0}^{\infty}$: $(1-\alpha)c_t + \alpha c_t^*$, where $c_t^* = \mathbb{E}_0(c_t)$. They make the welfare cost to be a function of the weight α, $\lambda(\alpha)$, which solves:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left((1+\lambda(\alpha)) c_t\right) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left((1-\alpha) c_t + \alpha c_t^*\right). \tag{2.4}$$

In their setup $\lambda(0) = 0$, and λ, as defined by Lucas, is obtained as $\lambda = \lambda(1)$, when using $\mathbb{E}(\cdot)$ instead of $\mathbb{E}_0(\cdot)$ in (2.4). They label $\lambda(1)$ as the *total cost of business cycles* and define the *marginal cost of business cycles*, obtained after differentiating (2.4) with respect to α as[3]:

$$\lambda'(0) = \frac{\mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t u'(c_t) \times \mathbb{E}_0(c_t)\right]}{\mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t u'(c_t) \times c_t\right]} - 1. \tag{2.5}$$

As stressed by Alvarez and Jermann, there is a straightforward interpretation for $\lambda'(0)$. Consider a Taylor-expansion argument for $\lambda(\alpha)$ around zero. We have: $\lambda(\alpha) \approxeq \lambda(0) + \lambda'(0)\alpha$. Recall that $\lambda(0) = 0$. Thus, $\lambda(\alpha) \approxeq \lambda'(0)\alpha$, which makes $\lambda'(0)$ the first-order approximation of $\lambda(1)$

[3]We have to assume that the usual regularity conditions hold in exchanging the integral and derivative signs; see the conditions in Amemiya (1985, Theorem 1.3.2).

around zero, i.e., around observed consumption. Their setup relies solely on asset-pricing data to compute $\lambda'(0)$, which avoids completely the specification of preferences. However, as seen in (2.5), and implemented in Issler, Franco, and Guillén, there is a simpler preference counterpart to their formulas.

It is clear from (2.2), (2.3), and (2.5), that the total and the marginal cost of business cycles can be computed when consumption is stationary and ergodic and also when it is not. A permanent-transitory decomposition of consumption shocks allows to explicitly isolate transient and permanent sources of welfare fluctuations, which could, in principle, be associated with the welfare costs of business cycles and the welfare costs of economic-growth variation. This is certainly a useful distinction for guidance of policy efforts directed at different sources of aggregate uncertainty.

Regarding difference-stationary consumption, recall that Beveridge and Nelson (1981) show that every linear difference-stationary process can be decomposed as the sum of a deterministic term, a martingale trend, and a stationary and ergodic cycle ($ARMA$ process). Jumping straight to our empirical results, we find support for a Beveridge-Nelson decomposition of $\ln(c_t)$ in a bivariate setup, where the analogue of (2.1) is:

$$\ln(c_t) = \ln(\alpha_0) + \ln(1+\alpha_1)\cdot t - \frac{\omega_t^2}{2} + \sum_{i=1}^{t}\varepsilon_i + \sum_{j=0}^{t-1}\psi_j\mu_{t-j}, \tag{2.6}$$

where $\ln\left[\alpha_0(1+\alpha_1)^t\cdot\exp\left(-\omega_t^2/2\right)\right]$ is the deterministic term, $\sum_{i=1}^{t}\varepsilon_i$ is the Martingale trend component, $\sum_{j=0}^{t-1}\psi_j\mu_{t-j}$ is the $MA(\infty)$ representation of the stationary part (cycle), which entails $\psi_0 = 1$ and $\sum_{j=0}^{\infty}\psi_j^2 < \infty$. The permanent shock ε_t and the transitory shock μ_t are assumed to have a bivariate Normal distribution as follows[4]:

$$\begin{pmatrix}\varepsilon_t\\ \mu_t\end{pmatrix} \sim i.i.d.\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}\right). \tag{2.7}$$

In general, under $\sigma_{12}\neq 0$, $\omega_t^2 = \sigma_{11}\cdot t + \sigma_{22}\sum_{j=0}^{t-1}\psi_j^2 + 2\cdot\sigma_{12}\sum_{j=0}^{t-1}\psi_j$ is the conditional variance of $\ln(c_t)$, where it becomes clear that ε_t and μ_t have two very different roles in terms of uncertainty: the uncertainty of ε_t grows without bound with t, whereas that of μ_t also increases with t but is bounded from above by the unconditional variance $\sigma_{22}\sum_{j=0}^{\infty}\psi_j^2$. If $\sigma_{12}=0$, $\omega_t^2 = \sigma_{11}\cdot t + \sigma_{22}\sum_{j=0}^{t-1}\psi_j^2$, and the two shocks are independent due to the distributional assumption of Normality. As noted by Morley, Nelson, and Zivot (2003), the restriction that $\sigma_{12}=0$ is not innocuous. In fact, assuming that $\ln(c_t)$ is generated within the $ARIMA$ class, there will be processes for which it is infeasible

[4] Since the Beveridge-Nelson decomposition is implemented in a multivariate setting, the correlation coefficient between ε_t and μ_t needs not be unity in absolute value, which prevents a degenerate distribution for $\begin{pmatrix}\varepsilon_t\\ \mu_t\end{pmatrix}$ prior to implementing the counter-factual exercise.

to decompose $\ln(c_t)$ as in (2.6). We present a broader discussion of this issue at the beginning of Section 3 and test the assumption that $\sigma_{12} = 0$ in Section 4.

2.2. Separating the Welfare Effects of Permanent and Transitory Shocks

A main objective of this paper is to isolate the welfare costs of business cycles and the welfare costs of economic growth. If one does not separate the effects of these shocks, she/he is forced to examine the welfare cost of *all* macroeconomic uncertainty, or to work with a tainted measure of welfare cost of business cycles which encompasses some or all of the cost associated with economic-growth factors. A previous attempt to deal with this issue includes only examining consumption fluctuations at *business-cycle horizons*; see, e.g., Alvarez and Jermann (2004). In our view, this strategy is best viewed as an approximation, since some business-cycle variation in consumption will be due to permanent shocks. Recall that one of the main features of the real-business-cycle literature was that permanent shocks alone could generate business-cycle fluctuations; see, *inter alia*, King, Plosser and Rebelo (1987), King, Plosser, Stock and Watson (1991), and Issler and Vahid (2001).

Consider the bivariate distribution:

$$\begin{pmatrix} \varepsilon_t \\ \mu_t \end{pmatrix} \sim i.i.d.\mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \right). \tag{2.8}$$

In analogy to the common practice of the previous literature, we propose to measure welfare costs using the *equivalent variation* in consumption, albeit our consumption counter-factual sequences are devoid of the effects of either ε_t (permanent shocks) or μ_t (transitory shocks), one at a time. In technical terms, we want to be able to take the bivariate distribution (2.8) and make it degenerate in one dimension alone. In doing so, the key issue is what will the resulting "residual" distribution be like.

Under suitable continuity and differentiability properties of the characteristic function (or the moment-generating function), we are able to compute exactly what is the resulting degenerate distribution in one dimension, which answers this question. The Normality assumption is sufficient to guarantee that these continuity and differentiability properties are met, as the next proposition makes clear.

Proposition 2.1. *Consider a bivariate random variable distributed as* $\mathcal{N}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right]$. *Its moment-generating function is:*

$$M(t_1, t_2) = \exp\left[t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\left(t_1^2\sigma_{11} + 2t_1t_2\rho\sqrt{\sigma_{11}\sigma_{22}} + t_2^2\sigma_{22}\right) \right],$$

where ρ, $|\rho| < 1$, is the correlation coefficient between the two random variates. As the variance in

one dimension vanishes, e.g., as $\sigma_{11} \to 0$, we are able to compute its limit as:

$$\lim_{\sigma_{11}\to 0} M(t_1,t_2) \equiv M_2(t_1,t_2) = \exp\left[t_1\mu_1 + t_2\mu_2 + \frac{1}{2}t_2^2\sigma_{22}\right].$$

From Levi's Continuity Theorem, the first two moments of the limiting distribution can be determined respectively by:

$$\begin{aligned}
\frac{d}{dt}M_2(0,0) &= M_2(t_1,t_2)\left.\begin{bmatrix} \mu_1 \\ (\mu_2 + t_2\sigma_{22}) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \\
\frac{d^2}{dt^2}M_2(0,0) &= M_2(t_1,t_2)\left.\begin{bmatrix} \mu_1^2 & \mu_1(\mu_2 + t_2\sigma_{22}) \\ \mu_1(\mu_2 + t_2\sigma_{22}) & \sigma_{22} + (\mu_2 + t_2\sigma_{22})^2 \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \mu_1^2 & \mu_1\mu_2 \\ \mu_1\mu_2 & \sigma_{22} + \mu_2^2 \end{bmatrix},
\end{aligned}$$

implying that the distribution converges in law to the degenerate Normal $\mathcal{N}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{pmatrix}\right]$.

By the Disintegration Theorem, a density can be defined for such distribution on a restriction of the original 2-dimensional Lebesgue measure to the affine subspace $\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2\right\}$. *It is given by:*

$$\begin{aligned}
f(x_1,x_2) &= \frac{1}{\sqrt{2\pi\sigma_{22}}}\exp\left\{-\frac{1}{2}[x_1-\mu_1, x_2-\mu_2]\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix}^{+}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma_{22}}}\exp\left\{-\frac{1}{2}[x_1-\mu_1, x_2-\mu_2]\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22}^{-1} \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma_{22}}}\exp\left\{-\frac{1}{2}\frac{[x_2-\mu_2]^2}{\sigma_{22}}\right\},
\end{aligned}$$

where $\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix}^{+}$ *is the Moore-Penrose (pseudo or generalized) inverse matrix of* $\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{22} \end{bmatrix}$, *which pseudo-inverse determinant*[5] *is given by σ_{22}. As the last line makes clear, this is the density of univariate random variable $\mathcal{N}[\mu_2,\sigma_{22}]$. A symmetric result applies when $\sigma_{22} \to 0$.*

Therefore, degenerating the bivariate Normal distribution in one dimension does not affect either the mean or the variance of the other dimension, allowing to determine exactly what the remaining univariate distribution will be. As expected, the covariance will vanish as well, since the covariance of a constant with a random variable should be zero.

The result in Proposition 1 allows performing counter-factual exercises when we shut out either permanent or transitory shocks. As it becomes clear in the next section, we evaluate the welfare

[5] See Magnus and Neudecker (1988) for the definition and properties of pseudo-inverse matrices and determinants.

cost of business cycles using μ_t and the welfare cost of economic growth using ε_t, after applying the results in Proposition 1. We consider two cases of interest, covered by Proposition 1. The first imposes that the these two shocks are independent, i.e., that $\sigma_{12} = 0$, while the other relaxes this assumption.

2.2.1. Independent shocks μ_t and ε_t: $\sigma_{12} = 0$

We start with the case where the two shocks are independent, i.e., that $\sigma_{12} = 0$, and the conditional variance of logged observed consumption is $\omega_t^2 = \sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2$. Consider the two processes below, where we start with (2.6) and shut out permanent and transitory shocks, respectively, as follows:

$$\ln\left(c_t^T\right) = \ln(\alpha_0) + \ln(1+\alpha_1) \cdot t - \frac{\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2}{2} + \sum_{j=0}^{t-1} \psi_j \mu_{t-j}, \text{ and,} \tag{2.9}$$

$$\ln\left(c_t^P\right) = \ln(\alpha_0) + \ln(1+\alpha_1) \cdot t - \frac{\sigma_{11}}{2} t + \sum_{i=1}^{t} \varepsilon_i. \tag{2.10}$$

From (2.6), we can think of $\ln\left(c_t^T\right)$, and $\ln\left(c_t^P\right)$ as limit cases, respectively:

$$\lim_{\sigma_{11}\to 0} \ln(c_t) = \ln\left(c_t^T\right), \text{ and } \lim_{\sigma_{22}\to 0} \ln(c_t) = \ln\left(c_t^P\right).$$

Note that $\ln\left(c_t^T\right)$ only exposes the consumer to transitory shocks, whereas $\ln\left(c_t^P\right)$ only exposes the consumer to permanent shocks. However, $\ln(c_t)$ exposes the consumer to both shocks. Comparing the exposure to risk of c_t and c_t^P allows measuring solely the welfare effects of business cycles and the comparison between c_t and c_t^T allows measuring solely the welfare effects of economic-growth variation.

We propose measuring the welfare cost for the representative consumer of bearing the uncertainty associated with $\{\mu_t\}$ alone (business cycles) through the use of c_t^P. Notice that the conditional means of c_t^P and c_t are identical: $\mathbb{E}_0\left(c_t^P\right) = \mathbb{E}_0(c_t) = \alpha_0 (1+\alpha_1)^t$. When $\sigma_{12} = 0$, the uncertainty of the consumption stream $\{c_t\}_{t=1}^{\infty}$ is larger than that of $\left\{c_t^P\right\}_{t=1}^{\infty}$, since the conditional variance of $\ln(c_t)$ is $\sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2$ (a sum of variances) and that of $\ln\left(c_t^T\right)$ is $\sigma_{11} \cdot t$. Thus, c_t is a mean-preserving spread of c_t^P[6].

Risk averse consumers prefer the stream $\left\{c_t^P\right\}_{t=1}^{\infty}$ over $\{c_t\}_{t=1}^{\infty}$. Thus, we measure the welfare cost associated with $\{\mu_t\}$ alone using λ_P, which solves:

$$\mathbb{E}_0\left[\sum\nolimits_{t=0}^{\infty} \beta^t u\left((1+\lambda_P)\, c_t\right)\right] = \mathbb{E}_0\left[\sum\nolimits_{t=0}^{\infty} \beta^t u\left(c_t^P\right)\right], \tag{2.11}$$

[6] Notice that $\sigma_{12} = 0$ guarantees that c_t is a mean-preserving spread of c_t^P, since the variance of $\ln(c_t)$ is a sum of variances, which must be greater than each of its terms.

i.e., we can think of λ_P as the welfare cost of bearing the risks associated with transitory shocks alone. Thus, we label it the welfare cost of business cycles.

It is important to stress that, in computing the right-hand-side of (2.11), we use the fact that $\ln\left(c_t^P\right)$ has a conditional Normal distribution with mean $\ln\left[\alpha_0\left(1+\alpha_1\right)^t \cdot \exp\left(-\frac{\sigma_{11}}{2}t\right)\right]$ and conditional variance $\sigma_{11}\cdot t$, a result that comes out directly from Proposition 2.1. Without that knowledge, we would not be able to compute λ_P.

In order to solve (2.11) for λ_P, we assume that the utility function is in CES class, with risk aversion coefficient ϕ:

$$u(c_t) = \frac{c_t^{1-\phi}-1}{1-\phi}, \tag{2.12}$$

where $u\left(c_t\right)$ approaches $\ln\left(c_t\right)$ as $\phi \to 1$. After straightforward but tedious algebra we get,

$$\lambda_P = \begin{cases} \exp\left(\phi\widetilde{\sigma}_{22}/2\right) - 1 & \text{for } \phi \neq 1 \\ \\ \exp\left(\widetilde{\sigma}_{22}/2\right) - 1 & \text{for } \phi = 1 \end{cases} \tag{2.13}$$

where, for the sake of simplicity in computation, we replace $\sigma_{22}\sum_{j=0}^{t-1}\psi_j^2$ by its respective unconditional counterpart $\widetilde{\sigma}_{22} = \sigma_{22}\sum_{j=0}^{\infty}\psi_j^2$. We also assume that the convergence condition $\beta\cdot\left(1+\alpha_1\right)^{(1-\phi)}\cdot \exp^{(-\phi(1-\phi)\sigma_{11}/2)} < 1$ holds. Notice that the welfare cost of business cycles does not depend on the uncertainty associated with permanent shocks. However, it depends on σ_{22} – the uncertainty behind transitory shocks – as well as on the degree of persistence of these shocks, captured by $\sum_{j=0}^{\infty}\psi_j^2$, and on the relative risk-aversion coefficient ϕ.

Analogously, we propose measuring the welfare cost of bearing the uncertainty associated with $\{\varepsilon_t\}$ alone (economic-growth variation) through the use of c_t^T. Recall that $\mathbb{E}_0\left(c_t^T\right) = \mathbb{E}_0\left(c_t\right) = \alpha_0\left(1+\alpha_1\right)^t$, and c_t is a mean-preserving spread of c_t^T when $\sigma_{12} = 0$. Hence, we measure the welfare cost associated with $\{\varepsilon_t\}$ alone by using λ_T, which solves:

$$\mathbb{E}_0\left[\sum\nolimits_{t=0}^{\infty}\beta^t u\left(\left(1+\lambda_T\right)c_t\right)\right] = \mathbb{E}_0\left[\sum\nolimits_{t=0}^{\infty}\beta^t u\left(c_t^T\right)\right] \tag{2.14}$$

Using (2.12), and applying Proposition 1, one can show that:

$$\lambda_T = \begin{cases} \left[\frac{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\right)}\right]^{\frac{1}{(1-\phi)}} - 1 & \text{for } \phi \neq 1 \\ \\ \exp\left(\frac{\beta\sigma_{11}}{2(1-\beta)}\right) - 1 & \text{for } \phi = 1 \end{cases}, \tag{2.15}$$

where we assume that the convergence condition $\beta\cdot\left(1+\alpha_1\right)^{(1-\phi)} < 1$, holds. Notice that λ_T does not depend on σ_{22} – i.e., on how uncertain transitory shocks are. However, it depends on β, ϕ, σ_{11} and α_1.

Finally, we can compute welfare costs for the representative consumer of bearing the uncertainty associated with both $\{\varepsilon_t\}$ and $\{\mu_t\}$ by introducing c_t^D:

$$\lim_{\sigma_{11}\to 0, \sigma_{22}\to 0} c_t = c_t^D = \alpha_0 (1+\alpha_1)^t . \tag{2.16}$$

Here, $c_t^D = \mathbb{E}_0 (c_t) = \alpha_0 (1+\alpha_1)^t$, making c_t a mean-preserving spread of c_t^D. We measure the welfare cost associated with both $\{\varepsilon_t\}$ and $\{\mu_t\}$ using λ_D, which solves:

$$\mathbb{E}_0 \left[\sum\nolimits_{t=0}^{\infty} \beta^t u\left((1+\lambda_D) c_t\right)\right] = \mathbb{E}_0 \left[\sum\nolimits_{t=0}^{\infty} \beta^t u\left(c_t^D\right)\right] = \sum\nolimits_{t=0}^{\infty} \beta^t u\left(c_t^D\right). \tag{2.17}$$

Using (2.12), we obtain:

$$\lambda_D = \begin{cases} \left[\frac{\exp(\phi(1-\phi)\widetilde{\sigma}_{22}/2)\cdot\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\right)}\right]^{\frac{1}{(1-\phi)}} - 1 & \text{for } \phi \neq 1 \\ \\ \exp\left(\frac{\beta\sigma_{11}+(1-\beta)\widetilde{\sigma}_{22}}{2(1-\beta)}\right) - 1 & \text{for } \phi = 1 \end{cases}, \tag{2.18}$$

where we assume that the convergence condition $\beta \cdot (1+\alpha_1)^{(1-\phi)} < 1$, holds.

Measures λ_P, λ_T, and λ_D are what Alvarez and Jermann have labelled *total welfare costs*. Here, we are also interested in measures of what they have labelled *marginal welfare costs*, i.e., $\lambda'_P(0)$, $\lambda'_T(0)$, and $\lambda'_D(0)$. Starting from (2.4), using (2.12), and the results in Proposition 1, when $\sigma_{12} = 0$, we measure marginal welfare costs of business cycles, economic-growth variation, and macroeconomic uncertainty, by using c_t^P, c_t^T, and c_t^D, respectively, as:

$$\lambda'_P(0) = \begin{cases} \exp\left(\phi\widetilde{\sigma}_{22}\right) - 1 & \text{for } \phi \neq 1, \\ \\ \exp\left(\widetilde{\sigma}_{22}\right) - 1 & \text{for } \phi = 1. \end{cases} \tag{2.19}$$

$$\lambda'_T(0) = \begin{cases} \frac{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(\phi(1+\phi)\sigma_{11}/2)\right)} - 1 & \text{for } \phi \neq 1 \\ \\ \frac{1-\beta}{1-\beta\cdot\exp(\sigma_{11})} - 1 & \text{for } \phi = 1 \end{cases}, \text{ and,} \tag{2.20}$$

$$\lambda'_D(0) = \begin{cases} \frac{\exp(\phi\widetilde{\sigma}_{22})\cdot\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(\phi(1+\phi)\sigma_{11}/2)\right)} - 1 & \text{for } \phi \neq 1 \\ \\ \frac{\exp(\widetilde{\sigma}_{22})(1-\beta)}{1-\beta\cdot\exp(\sigma_{11})} - 1 & \text{for } \phi = 1 \end{cases}, \tag{2.21}$$

where we assume that the usual specific convergence conditions apply in computing $\lambda'_P(0)$, $\lambda'_T(0)$ and $\lambda'_D(0)$, respectively, and replace $\sigma_{22}\sum_{j=0}^{t-1}\psi_j^2$ by its respective unconditional counterpart $\widetilde{\sigma}_{22} = \sigma_{22}\sum_{j=0}^{\infty}\psi_j^2$ in computing $\lambda'_P(0)$.

Finally, we give some intuition behind the measures of welfare costs proposed above. One way to think about (2.10) is:

$$\lim_{\sigma_{22}\to 0} c_t = c_t^P = \mathbb{E}\left[c_t \mid \mathcal{I}_0, \{\varepsilon_t\}_{t=0}^{\infty}\right],$$

which shows that c_t^P is the conditional expectation of c_t when we have *perfect foresight* of the sequence $\{\varepsilon_t\}_{t=0}^{\infty}$ of permanent shocks. Thus, in computing the welfare costs of business cycles, we *control* for the existence of permanent shocks to consumption in the whole planning horizon. This shows that the welfare-cost measures λ_P and $\lambda'_P(0)$ only take into account the uncertainty that goes beyond permanent shocks, i.e., transitory shocks alone. A similar reasoning applies to c_t^T and c_t^D.

2.2.2. Relaxing Independence between shocks μ_t and ε_t: $\sigma_{12} \neq 0$

In the present section we generalize the results in the previous section, in which we allow for $\sigma_{12} \neq 0$. When $\sigma_{12} \neq 0$, there is an additional complication regarding the setup of the previous section. As we discuss next, we cannot guarantee that c_t will always be a mean-preserving spread of c_t^P (or that c_t will always be a mean-preserving spread of c_t^T). This happens because the variance of c_t is $\sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j$ while that of c_t^P is $\sigma_{11} \cdot t$. To assure that the variance of c_t is positive, we must have $\sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > 0$. This implies that $\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > -\sigma_{11} \cdot t$, but not necessarily that $\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > 0$, which would guarantee that c_t is a mean-preserving spread of c_t^P[7].

In cases where c_t is not a mean-preserving spread of c_t^P, in offering the consumer the sequence $\left\{c_t^P\right\}_{t=1}^{\infty}$ in exchange to $\{c_t\}_{t=1}^{\infty}$, we know from theory that welfare costs will be negative. Intuitively, when the variance of business cycles is small, the direct gains of eliminating it, going from c_t to c_t^P, are small. Because there is usually a negative covariance between trend and cycles in the Beveridge-Nelson decomposition, eliminating small business cycles has also the unintended indirect effect of increasing the variance of counter-factual consumption c_t^P, since the covariance will vanish as well when the cyclical component is degenerate. If the covariance is large vis-a-vis the variance of business cycles, the direct gains are outweighed by unintended indirect costs, and the consumer will indeed prefer c_t to c_t^P. So, welfare costs will be negative.

Whenever $\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > 0$, we obtain positive welfare costs when we offer the consumer the sequence $\left\{c_t^P\right\}_{t=1}^{\infty}$ in exchange to $\{c_t\}_{t=1}^{\infty}$. And, whenever $\sigma_{11} \cdot t + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > 0$, we obtain positive welfare costs when we offer the consumer $\left\{c_t^T\right\}_{t=1}^{\infty}$ in exchange to $\{c_t\}_{t=1}^{\infty}$.

[7] There is a similar result involving c_t and c_t^T.

Using (2.11), with CES utility and recalling that the conditional variance of logged observed consumption is $\omega_t^2 = \sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j$, we can compute the welfare cost of business cycle when $\sigma_{12} \neq 0$. The analogous formula to (2.13) is:

$$\lambda_P = \begin{cases} \exp\left(\phi\left(2\widetilde{\sigma}_{12} + \widetilde{\sigma}_{22}\right)/2\right) - 1 & \text{for } \phi \neq 1, \\ \\ \exp\left(\left(2\widetilde{\sigma}_{12} + \widetilde{\sigma}_{22}\right)/2\right) - 1 & \text{for } \phi = 1. \end{cases} \qquad (2.22)$$

Similarly, using (2.14), we can compute the welfare cost of economic-growth variation. The analogous formula to (2.15) is:

$$\lambda_T = \begin{cases} \left[\frac{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)\cdot\exp(-\phi(1-\phi)\widetilde{\sigma}_{12}/2)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\right)}\right]^{\frac{1}{(1-\phi)}} - 1 & \text{for } \phi \neq 1, \\ \\ \exp\left(\frac{\beta\sigma_{11}+2(1-\beta)\widetilde{\sigma}_{12}}{2(1-\beta)}\right) - 1 & \text{for } \phi = 1. \end{cases} \qquad (2.23)$$

Finally, using (2.17), we can compute the welfare cost of all macroeconomic uncertainty. The analogous formula to (2.18) is:

$$\lambda_D = \begin{cases} \left[\frac{\exp(\phi(1-\phi)(2\widetilde{\sigma}_{12}+\widetilde{\sigma}_{22})/2)\cdot\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp(-\phi(1-\phi)\sigma_{11}/2)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\right)}\right]^{\frac{1}{(1-\phi)}} - 1 & \text{for } \phi \neq 1, \\ \\ \exp\left(\frac{\beta\sigma_{11}+(1-\beta)(2\widetilde{\sigma}_{12}+\widetilde{\sigma}_{22})}{2(1-\beta)}\right) - 1 & \text{for } \phi = 1. \end{cases} \qquad (2.24)$$

The key difference between (2.13), (2.15), and (2.18), and their respective counterparts in the previous section is their dependence on $\widetilde{\sigma}_{12} = \sigma_{12} \sum_{j=0}^{\infty} \psi_j$, the unconditional counterpart of σ_{12}.

Finally, it is straightforward to compute the marginal welfare costs of business cycles, economic-growth variation, and all macroeconomic uncertainty. They are, respectively:

$$\lambda_P'(0) = \begin{cases} \exp\left(\phi\left(2\widetilde{\sigma}_{12} + \widetilde{\sigma}_{22}\right)\right) - 1 & \text{for } \phi \neq 1 \\ \\ \exp\left(2\widetilde{\sigma}_{12} + \widetilde{\sigma}_{22}\right) - 1 & \text{for } \phi = 1 \end{cases} \qquad (2.25)$$

$$\lambda_T'(0) = \begin{cases} \frac{\exp(\phi 2\widetilde{\sigma}_{12})\cdot\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp\left(\frac{-\phi(1-\phi)\sigma_{11}}{2}\right)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp\left(\frac{\phi(1+\phi)\sigma_{11}}{2}\right)\right)} - 1 & \text{for } \phi \neq 1 \\ \\ \frac{\exp(2\widetilde{\sigma}_{12})\cdot(1-\beta)}{1-\beta\cdot\exp(\sigma_{11})} - 1 & \text{for } \phi = 1 \end{cases}, \text{ and,} \qquad (2.26)$$

$$\lambda_D'(0) = \begin{cases} \frac{\exp(\phi(2\widetilde{\sigma}_{12}+\widetilde{\sigma}_{22}))\cdot\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp\left(\frac{-\phi(1-\phi)\sigma_{11}}{2}\right)\right)}{\left(1-\beta\cdot(1+\alpha_1)^{(1-\phi)}\cdot\exp\left(\frac{\phi(1+\phi)\sigma_{11}}{2}\right)\right)} - 1 & \text{for } \phi \neq 1 \\ \\ \frac{\exp(2\widetilde{\sigma}_{12}+\widetilde{\sigma}_{22})\cdot(1-\beta)}{1-\beta\cdot\exp(\sigma_{11})} - 1 & \text{for } \phi = 1 \end{cases}. \qquad (2.27)$$

3. Identification and Estimation of Structural Parameters used in Computing λ_T, λ_P, λ_D, $\lambda'_T(0)$, $\lambda'_P(0)$, and $\lambda'_D(0)$

Next, we discuss the estimation of λ_T, λ_P, λ_D, $\lambda'_T(0)$, $\lambda'_P(0)$, and $\lambda'_D(0)$ under the two cases considered in the previous section: $\sigma_{12} = 0$, and $\sigma_{12} \neq 0$. When $\sigma_{12} \neq 0$, we employ the multivariate Beveridge and Nelson (1981) decomposition in the same fashion as it was implemented by Issler, Franco, and Guillén (2008), where the starting point is a vector autoregression (VAR), where possible cointegrating restrictions are imposed in estimation. As is well known, VAR-model components have an $ARMA$ representation with identical $AR(\cdot)$ polynomials for each series. When there are unit roots in them, VAR components fall into the $ARIMA$ class, thus allowing the representation in (2.6), where the identification of trend and cycle is always guaranteed. An added bonus is the fact that the Beveridge-Nelson decomposition in this context does not impose the restriction that innovations to trends and cycles are either perfectly correlated or orthogonal, allowing testing the latter.

When $\sigma_{12} = 0$, we estimate the key parameters using the structural time-series model proposed by Harvey (1985b) and Koopman et al. (2009), where the unobserved components are assumed to be Normal and uncorrelated, i.e., independent. As noted by Morley, Nelson, and Zivot (2003), when dealing with the trend-cycle (log) GDP decomposition, constraining the innovations to trend and cycle to be uncorrelated, within the $ARIMA$ class, may lead to lack of identification of trends and cycles. As they note, "[T]his reflects a basic theme of this paper: the trend process is always identified from the univariate properties of the series, though the cycle process may not be[8]."

Because of this potential lack of identification, we emphasize the results using the multivariate Beveridge and Nelson decomposition. Despite that, in both cases, we show how to identify the key elements in λ_T, λ_P, λ_D, $\lambda'_T(0)$, $\lambda'_P(0)$, and $\lambda'_D(0)$, given each model's estimates of the trend and cycle in consumption.

3.1. VAR Estimation with Possible Long-Run Constraints

A full discussion of the econometric models employed here can be found in Beveridge and Nelson (1981), Stock and Watson (1988), Engle and Granger (1987), Campbell (1987), and Proietti (1997). Denote by $y_t = (\ln(c_t), \ln(I_t))'$ a 2×1 vector containing respectively the logarithms of consumption and income per-capita. We assume that both series contain a unit-root and are possibly cointegrated as in $\alpha' y_t$ because of the Permanent-Income Hypothesis (Campbell(1987)). A vector error-correction

[8] See p. 237. As a simple example of lack of identification, consider the case where $\ln(c_t)$ is generated by an $ARIMA(0,1,1)$ with positive first-order autocorrelation in $\Delta \ln(c_t)$. This implies a negative variance for the transitory component.

model ($VECM(p-1)$) is:

$$\Delta\, y_t = \Gamma_1 \,\Delta\, y_{t-1} + \ \ldots\ + \Gamma_{p-1}\,\Delta\, y_{t-p+1} + \gamma\alpha' y_{t-p} + \eta_t, \tag{3.1}$$

where the variance-covariance matrix of the error terms $\mathbb{E}\left(\eta_t \eta_t'\right)$ is not necessarily diagonal, and independence among shocks to consumption is not imposed.

We turn now to the discussion of how to extract trends and cycles using (3.1). To simplify notation, we jump straight to our empirical results, where we found that the system (3.1) is well described by a $VECM(1)$. It can be put in state-space form, as discussed in Proietti (1997):

$$\begin{aligned} \Delta y_{t+1} &= Z f_{t+1} \\ f_{t+1} &= T f_t + Z' \eta_{t+1}, \end{aligned} \tag{3.2}$$

where,

$$f_{t+1} = \begin{bmatrix} \Delta y_{t+1} \\ \Delta y_t \\ \alpha' y_{t-1} \end{bmatrix},\ T = \begin{bmatrix} \Gamma_1 & -\gamma\alpha' & -\gamma \\ I_2 & 0 & 0 \\ 0 & \alpha' & 1 \end{bmatrix}$$

with the associated VECM being,

$$\begin{aligned} \Delta y_t &= \Gamma_1 \Delta y_{t-1} + \gamma\alpha' y_{t-2} + \eta_t, \text{ and,} \\ Z &= \begin{bmatrix} I_2 & 0 & 0 \end{bmatrix}. \end{aligned}$$

From the work of Beveridge and Nelson (1981), and Stock and Watson (1988), and ignoring initial conditions and deterministic components, the series in y_t can be decomposed into a trend (τ_t) and a cyclical component (φ_t), as follows:

$$y_t = \tau_t + \varphi_t,$$

where,

$$\begin{aligned} \tau_t &= y_t + \lim_{l\to\infty} \sum_{i=1}^{l} \mathbb{E}_t\left[\Delta y_{t+i}\right], \text{ and,} \\ \varphi_t &= -\lim_{l\to\infty} \sum_{i=1}^{l} \mathbb{E}_t\left[\Delta y_{t+i}\right]. \end{aligned}$$

It is straightforward to show that τ_t is a martingale. Using the state-space representation (3.2), we can compute the limits above. The cyclical and trend components will be, respectively:

$$\begin{aligned} \varphi_t &= -Z\left[I_m - T\right]^{-1} T f_t, \\ \tau_t &= y_t - \varphi_t. \end{aligned} \tag{3.3}$$

Apart from an irrelevant constant, the trend innovation in logged consumption ε_t is simply $\underset{1\times 2}{[1,0]} \times \underset{2\times 1}{\Delta\tau_t}$. Its variance σ_{11} equals VAR($[1,0] \times \Delta\tau_t$). Notice that:

$$\ln(c_t) - \mathbb{E}_{t-1}(\ln(c_t)) = [1,0] \times \eta_t = \varepsilon_t + \mu_t,$$

identifies μ_t up to an irrelevant constant using $[1,0] \times (\varepsilon_t - \Delta\tau_t) = \mu_t$, which allows computing σ_{12} and σ_{22}. A similar approach allows computing $\widetilde{\sigma}_{12}$ and $\widetilde{\sigma}_{22}$ using the cycle in consumption. It is easy to identify $\ln(1+\alpha_1)$ by employing $\mathbb{E}(\Delta\tau_t)$.

3.2. Structural Time-Series Models with Long-Run Constraints

To impose the constraint that shocks to consumption are independent, we rely on the *structural* time-series model of Harvey (1985b) and Koopman et al. (2009). We start the discussion using a univariate framework to gain some intuition. The main objective is to decompose a single integrated series ($I(1)$) in a trend and a cycle, treating both as latent variables to be estimated by maximum likelihood, which guarantees consistent and asymptotically Normal parameter estimates, a key property in our case.

We decompose a single economic series x_t as:

$$x_t = \tau_t + \varphi_t$$

where τ_t is the $I(1)$ trend and φ_t is the stationary cycle. Shocks to each of these two components are independent of each other and also across time. The trend evolves as:

$$\tau_t = \tau_{t-1} + \delta + \upsilon_t, \tag{3.4}$$

where υ_t has variance σ_υ^2, and the cyclical component evolves as a bivariate $VAR(1)$:

$$\begin{bmatrix} \varphi_t \\ \varphi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix} \begin{bmatrix} \varphi_{t-1} \\ \varphi_{t-1}^* \end{bmatrix} + \begin{bmatrix} w_t \\ w_t^* \end{bmatrix} \tag{3.5}$$

where the component φ_t^* is an auxiliary variable needed to characterize the cycle. Both w_t and w_t^* are orthogonal white noise errors with variances given by σ_w^2 and σ_w^{*2}, respectively. Some restrictions on key parameter are:

$$0 \le \lambda \le \pi \text{ and } 0 < \rho \le 1,$$

where λ is the frequency of the cycle and ρ is the discount factor for its amplitude.

The cyclical component obeys:

$$\varphi_t = \frac{(1 - \rho\cos\lambda L)\, w_t + (\rho\sin\lambda L)\, w_t^*}{1 - 2\rho\cos\lambda L + \rho^2 L^2}$$

where L is the lag operator, $L^k x_t = x_{t-k}$. Under $\sigma_w^2 = \sigma_w^{*2}$, we can put the last equation in an $ARMA$ format as:

$$\left(\rho^2 L^2 - 2\rho \cos \lambda L + 1\right) \varphi_t = (1 + \Phi L)\, w_t,$$

i.e., φ_t is an $ARMA\,(2,1)$, with $\Phi = \rho\,(\sin\lambda - \cos\lambda)$. Note that this is a restriction into the $ARMA$ class of models, leading to one of the criticisms of the *structural* time-series model approach. The other stringent assumption is that v_t and w_t are independent, which may not fit the data when one considers an unrestricted estimate of of their covariance and tests the hypothesis of zero covariance implied by independence.

In a multivariate setting, we can represent $y_t = (\ln(c_t), \ln(I_t))'$ as having a common trend as:

$$\begin{bmatrix} \ln(c_t) \\ \ln(I_t) \end{bmatrix} = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \tau_t + \varphi_t, \tag{3.6}$$

where the scalar $I\,(1)$ trend component τ_t follows (3.4). Here, the bivariate system in y_t is modelled with just a single stochastic trend, where the cointegrating vector $[\theta, -1]'$ removes it, but cycles need not be common. The 2×1 vector φ_t of cyclical components evolves as:

$$\begin{bmatrix} \varphi_t \\ \varphi_t^* \end{bmatrix} = \left[\rho \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \otimes \mathbf{I}_2\right] \begin{bmatrix} \varphi_{t-1} \\ \varphi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}_t^* \end{bmatrix}, \tag{3.7}$$

where φ_t^*, $\mathbf{w}_t$ and $\mathbf{w}_t^*$are 2×1 vectors, λ is a 1×2 vector, and we impose the restriction that $\mathbb{E}\left(\mathbf{w}_t \mathbf{w}_t'\right) = \mathbb{E}\left(\mathbf{w}_t^* \mathbf{w}_t^{*\prime}\right) = \Sigma_\omega$, making $\mathrm{VAR}\begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}_t^* \end{bmatrix} = \mathbf{I}_2 \otimes \Sigma_\omega$.

These models can be easily put in state-space form with Normal disturbances, where the Kalman Filter can be used to compute the likelihood function through the one-step prediction error decomposition.

The identification of the key parameters in the welfare-cost formulas of Section 2 can be done as follows: σ_{11} can be identified using VAR($\Delta\tau_t$), recalling that the variance is invariant to addition of a constant. The parameter $\widetilde{\sigma}_{22} = \sigma_{22} \sum_{j=0}^{\infty} \psi_j^2$ can be identified by using VAR(φ_t). Identification of $\widetilde{\sigma}_{22}$ is straightforward by using VAR($[1,0] \times \varphi_t$). It is easy to identify $\ln(1 + \alpha_1)$ by employing $\mathbb{E}\left(\Delta\tau_t\right)$.

4. Empirical Results

4.1. The Data

Data for annual consumption of nondurables, services, and for annual gross national product (GNP) were extracted from FED's FRED database from 1929 through 2012, all supplied by the Bureau of Economic Analysis (BEA) of the U.S. Department of Commerce. Prior to 1929, we had to rely on

data for consumption of perishables, semi-durables and services, from 1901 to 1929, obtained from Kuznets (1961), as well as GNP data from the same source.

We are interested in constructing real consumption for nondurables and services, following almost all of the literature on consumption. Unfortunately, there is no deflator for nondurables plus services. Thus, from 1929 onwards, we aggregated nondurables and services using Irving Fisher's ideal price index – an equally weighted geometric average of the Laspeyres and Paasche price indices. Prior to 1929, we used the same technique for perishables, semi-durables and services. Intuitively, by employing Fisher's method, we allow rebalancing the weights of the parts on the sum of the components. Simply summing up the deflated parts implies keeping these weights fixed throughout the whole sample – from 1901-2012 – which is obviously inappropriate.

As stated above, our motivation for using consumption and income comes from the permanent-income literature. Thus, preferably, our measure of income would be personal disposable income, which is total income received from all persons, net of current taxes. Indeed, this series exists from 1929 to 2012, although our source of pre-war data – Kuznets (1961) – does not have it. Moreover, we are unaware of any other data sources that would have it for the pre-war period. An alternative to personal disposable income is Gross National Income, but it is also unavailable for the pre-1929 period. Thus, we opted to use GNP. It is similar to Gross National Income, except that in measuring GNP one does not deduct indirect business taxes[9].

After obtaining the growth rates of real consumption of nondurables and services, and from GNP in both sub-samples, we chained both of them to represent real consumption and GNP at 2009 prices. Data on population were extracted from Kuznets and FRED, and then chained. Figure 1 presents the data on consumption of nondurables and services and GNP per-capita for the whole period 1901-2012. The peculiar features are first the magnitude of the great depression in both consumption and income behavior, and second the fact that pre-WWII data present much more volatility than post-WWII data.

4.2. VAR Estimation

For the post-war sample 1947 through 2012, we fitted a bivariate vector autoregression for the logs of consumption and income. Lag length selection criteria – Hannan-Quinn and Schwarz – indicated that a VAR(2) with an unrestricted constant term was an appropriate description of the dynamic system, although the Akaike criterium chose a higher lag length. Since the latter is known to be an inconsistent lag-lenght selection criterium, we kept a VAR with two lags, after a battery of diagnostic tests confirmed our initial choice. Testing for cointegration between consumption of

[9] Including a measure of income in modelling consumption is important for computing welfare, since the latter is an expected present-discounted value of the form $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u\left(\cdot\right)\right]$ and the income measure enlarges the information set in computing it.

nondurables and services and GNP finds cointegration at the 1% level, although we do not find long-run proportionality – a cointegrating vector $(1, -1)'$.

Using the whole sample 1901-2012, we still find that a VAR(2) is the best representation for the dynamic system. Again, in testing for cointegration we found similar results, although we reject the null of no cointegration now at the 7.1% significance level instead of 1%. For the pre-war period, 1901-1941, we still chose a VAR(2) using these same procedures, but we did not find cointegration at the usual significance levels using Johansen's (1991) test. Despite that, because we found cointegration for the whole sample 1901-2012, and the sample 1901-1941 is rather small, we kept the cointegrated VAR(2) for the pre-war period as well[10].

Table 4.1, panel A, presents VAR-based parameter estimates associated with the representation of log of consumption in equation (2.6) for different sub-periods. These are based on cointegrated VAR(2) estimates, as explained above. We considered three distinct periods: pre-war data – 1901-1941, post-war data – 1947-2012, and whole period (20th Century and beyond) data – 1901-2012. Figure 2 plots the results of the VAR-based trend-cycle decomposition.

4.3. Structural Time-Series Model Estimation

Following our findings in VAR analysis, we model the logs of consumption and income as having a common trend, but do not impose that it affects the two series identically in the long run, i.e., $\theta \neq 1$ in equation (3.6). We do not impose common cycles either for consumption and income, although we impose homogeneity for estimated parameters in (3.7) – the frequency of the cycle and the discount factor for its amplitude.

Table 4.1, panel B, presents (2.6) parameter estimates, when we employ the structural time-series model approach. For the post-war period the trend in consumption is relatively smooth and close to observed consumption. This is also true for the whole period 1901-2012, but not for the pre-war period 1901-1941. Figure 3 plots the results of the trend-cycle decomposition based on the structural time-series model.

4.4. Welfare Costs λ_T, λ_P, λ_D, $\lambda_T'(0)$, $\lambda_P'(0)$, and $\lambda_D'(0)$, for the Post-WWII Era

In the general case where $\sigma_{12} \neq 0$, the estimates of the *total and marginal* welfare costs of business cycles for the post-war period, 1947-2012, are presented in Table 4.2, panel A. These are all based on VAR estimates (Beveridge and Nelson (1981) decomposition), discussed in Section 4.2. As we noted in Section 2.2.2, when $\sigma_{12} \neq 0$, we cannot guarantee that c_t will always be a mean-preserving spread of either c_t^P or c_t^T. For example, although the mean of c_t and of c_t^P is the same, the variance

[10] Imposing cointegration makes little difference in terms of welfare-cost estimates for the pre-war period. Results without cointegration are available upon request.

Table 4.1: Estimated Parameters in Beveridge-Nelson (BN) and Unobserved-Component (UC) Decomposition

Panel A: BN Decomp.				**Panel B: UC Decomp.**			
Sample	1901-1941	1901-2012	1947-2012	Sample	1901-1941	1901-2012	1947-2012
$\ln(1+\alpha_1)$	0.01324	0.01930	0.02071	$\ln(1+\alpha_1)$	0.01376	0.01928	0.02073
σ_{11}	0.00072	0.00073	0.00028	σ_{11}	0.00017	0.00033	0.00013
$\tilde{\sigma}_{12}$	0.00008	-0.00009	-0.00009	$\tilde{\sigma}_{12}$	0.00000	0.00000	0.00000
$\tilde{\sigma}_{22}$	0.00081	0.00019	0.00005	$\tilde{\sigma}_{22}$	0.00212	0.00106	0.00004

of c_t is $\sigma_{11} \cdot t + \sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j$ while that of c_t^P is $\sigma_{11} \cdot t$. A positive variance for c_t implies $\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > -\sigma_{11} \cdot t$, but not necessarily that $\sigma_{22} \sum_{j=0}^{t-1} \psi_j^2 + 2 \cdot \sigma_{12} \sum_{j=0}^{t-1} \psi_j > 0$, which guarantees that c_t is a mean-preserving spread of c_t^P. If the variance of c_t is smaller, the consumer actually prefers c_t to c_t^P, and we know from theory that welfare costs will be negative.

For the post-war period, 1947-2012, this was the case. Our estimates are: $\widehat{\sigma_{11}} \cdot t = 0.0002417 \cdot t$, $\widehat{\sigma_{22} \sum_{j=0}^{\infty} \psi_j^2} = 0.0000216$, and $\widehat{2 \cdot \sigma_{12} \sum_{j=0}^{\infty} \psi_j} = -0.00005139$, implying that $\widehat{VAR(c_t)} = 0.0002417 \cdot t - 0.00002979$, while $\widehat{VAR(c_t^P)} = 0.0002417 \cdot t$. So, the welfare cost of business cycles is indeed negative.

With that in mind, all our estimates of the welfare cost of business cycles (Table 4.2, panel A) are negative. Considering reasonable preference parameters ($\beta = 0.985, \phi = 5$), our estimate of the welfare cost of business cycles is -0.03% of consumption. Hence, the consumer needs not be compensated to bear the risk of business-cycle fluctuations. On the contrary, she/he pays to bear it. Intuitively, when the variance of business cycles is small, the direct gains of eliminating it are small. Because there is usually a negative covariance between trends and cycles in the Beveridge-Nelson decomposition, eliminating small business-cycle fluctuations has the unintended indirect effect of increasing the variance of counter-factual consumption c_t^P, since the covariance will vanish in the limit as well. Because the latter is relatively large, welfare costs are negative. Notice that Lucas' estimate is 0.04%, but he treats the growth rate of consumption as deterministic. Hence, part of his *measured* cyclical uncertainty incorporates growth-rate uncertainty, i.e., Lucas overestimates the uncertainty associated with business-cycle fluctuations alone.

The *total and marginal* welfare costs of economic-growth variation for the post-war period (1947-2012) are presented in Table 4.3, panel A, when we consider VAR-based estimates, consistent with $\sigma_{12} \neq 0$. Welfare costs of economic-growth variation, λ_T, are sizable: 0.71% of per-capita con-

Table 4.2: Welfare Cost of Business-Cycle Fluctuations (1947-2012) – % Consumption

PANEL A ($\sigma_{12} \neq 0$)								
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	-0.01	-0.03	-0.07	-0.14	-0.01	-0.07	-0.14	-0.27
0.97	-0.01	-0.03	-0.07	-0.14	-0.01	-0.07	-0.14	-0.27
0.99	-0.01	-0.03	-0.07	-0.14	-0.01	-0.07	-0.14	-0.27
PANEL B ($\sigma_{12} = 0$)								
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	0.002	0.01	0.02	0.04	0.004	0.02	0.04	0.07
0.97	0.002	0.01	0.02	0.04	0.004	0.02	0.04	0.07
0.99	0.002	0.01	0.02	0.04	0.004	0.02	0.04	0.07

Note: σ_{12} is the covariance between ε_t (permanent shocks) and μ_t (transitory shocks) defined in equation (2.9)

sumption for reasonable discount values and preference parameters ($\beta = 0.985, \phi = 5$), US\$ 208.98 at 2009 prices. Marginal welfare costs – $\lambda_T'(0)$ – are about twice the size of total welfare costs for reasonable discount values and preference parameters ($\beta = 0.985, \phi = 5$): 1.32% of per-capita consumption.

In Table 4.4, panel A, we present *total and marginal* welfare cost of all macroeconomic uncertainty. For λ_D, our results are very close to those of the welfare costs of economic growth for ($\beta = 0.985, \phi = 5$) – 0.65% of consumption. We find the same regarding marginal welfare costs $\lambda_D'(0)$: they are 1.34% of consumption for identical parameter values.

In panel B of Tables 4.2, 4.3, and 4.4, we present results of *total and marginal* welfare costs when we impose the constraint that $\sigma_{12} = 0$ in estimating the parameters in (2.6). They cover the post-war period, 1947-2012, and are all based on structural-time-series model estimates, which was discussed in Section 4.3. From the work of Morley, Nelson, and Zivot (2003) on GDP, one should expect that imposing $\sigma_{12} = 0$ yields a much smoother trend than the one found when we let the data estimate freely σ_{12}, i.e., the VAR-based approach of Section 4.2. Roughly speaking, this should reduce the welfare cost of economic-growth variation and increase the welfare costs of business cycles.

Indeed, these are exactly our findings. Comparing panels A and B of Table 4.2 for ($\beta = 0.985, \phi = 5$), shows that although the welfare costs of business cycles are negative for VAR-based estimates ($\sigma_{12} \neq 0$), reaching -0.03% of consumption, our estimate when $\sigma_{12} = 0$ is imposed is 0.01% of

Table 4.3: Welfare Cost of Economic-Growth Variation (1947-2012) – % Consumption

	PANEL A ($\sigma_{12} \neq 0$)							
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	0.25	0.51	0.58	0.63	0.51	0.91	0.93	0.81
0.97	0.45	0.62	0.64	0.67	0.91	1.12	1.07	0.89
0.99	0.90	0.71	0.69	0.69	1.82	1.32	1.17	0.95
	PANEL B ($\sigma_{12} = 0$)							
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	0.13	0.23	0.25	0.25	0.25	0.47	0.52	0.52
0.97	0.22	0.28	0.28	0.26	0.45	0.57	0.58	0.55
0.99	0.44	0.33	0.30	0.28	0.88	0.66	0.62	0.58

Note: σ_{12} is the covariance between ε_t (permanent shocks) and μ_t (transitory shocks) defined in equation (2.9)

consumption – more than twice of the original estimate in Lucas. On the other hand, the welfare costs of economic-growth variation is roughly slashed in half: from 0.71% to 0.33% of consumption.

A key issue in deciding which estimate is more appropriate relates to whether or not $\sigma_{12} = 0$. For the post-war period, we test this hypothesis by using Granger and Newbold's (1985) approach, applied for testing zero covariance between errors, based on the product of trend and cycle innovations. The *t-ratio* test for $H_0 : \sigma_{12} = 0$ is -4.43, rejecting the null at usual levels, whereas the estimated correlation coefficient between the two innovations is -0.8048. Thus, we conclude that the appropriate trend-cycle decomposition estimates are VAR-based, i.e., estimates reported in panel A of Tables 4.2, 4.3, and 4.4.

We now compare our empirical results so far with those in Reis (2009). He does not separate the effects of transitory and permanent shocks, i.e., he computes the welfare cost of *all* macroeconomic uncertainty. Thus, we compare Table 4 in Reis (sample 1947-2003), where a unit root is imposed for consumption, with our results for λ_D for post-war data (sample 1947-2012) in Table 4.4, panel A, where $\sigma_{12} = 0$ is not imposed due to our testing results above. Using an ARMA model for the instantaneous growth rate of consumption, Reis finds welfare costs to be roughly between 0.5% and 5% of consumption. Here, we find much lower estimates – roughly between 0.25% and 0.90%. For $(\beta = 0.985, \phi = 5)$, we found λ_D to be 0.65%, whereas Reis finds 5.33% – almost one order of magnitude higher.

A plausible explanation for the difference in results is the fact that Reis fits an ARMA model

Table 4.4: Welfare Cost of All Macroeconomic Uncertainty (1947-2012) – % Consumption

PANEL A ($\sigma_{12} \neq 0$)								
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	0.26	0.45	0.47	0.40	0.51	0.93	0.98	0.90
0.97	0.46	0.56	0.53	0.44	0.92	1.15	1.11	0.99
0.99	0.90	0.65	0.58	0.46	1.83	1.34	1.22	1.05
PANEL B ($\sigma_{12} = 0$)								
	Welfare Cost				**Marginal Welfare Cost**			
ϕ	1	5	10	20	1	5	10	20
β								
0.95	0.13	0.24	0.27	0.29	0.26	0.49	0.55	0.59
0.97	0.22	0.29	0.30	0.30	0.45	0.59	0.61	0.63
0.99	0.44	0.34	0.32	0.31	0.88	0.68	0.66	0.65

Note: σ_{12} is the covariance between ε_t (permanent shocks) and μ_t (transitory shocks) defined in equation (2.9)

which uses only lagged consumption growth as information set, whereas we fitted a bivariate model, and found evidence of Granger (1969) causality from income to consumption (a t-statistic of 2.50 for income growth in consumption's equation). This implies different results for single equation and multivariate analyses, and reinforces the importance of the latter, since the consumer ought to look beyond consumption to form expectations in computing welfare.

Finally, it is worth comparing our marginal welfare costs estimates with those found by Alvarez and Jermann (2004). For the post-war era, our marginal welfare costs of business cycles are all negative – our best estimate being -0.07% of consumption. Compare this to the estimates of marginal welfare costs in Alvarez and Jermann for the period 1954-2001: between 0.08% and 0.49% of consumption – computed at business-cycle frequencies alone. As we argued above, if one does not properly disentangle the effects of permanent and transitory shocks to consumption, there is the risk of biasing the estimate of the welfare costs of business cycles. Although Alvarez and Jermann try to remedy this problem by using various filters to isolate gains from eliminating fluctuations at business cycle frequencies, permanent shocks can induce short-term variation in consumption, which will not be filtered out by their procedure. So, we should expect an upward bias. Notwithstanding the slight difference in sample periods in estimation, the marginal welfare costs in their paper are very different than ours for the post-war period[11].

[11] Since our sample includes the great recession of 2008, but theirs do not, the fact that they find a much larger estimate is even more problematic.

All in all, the contrast between our estimates and those in the literature show the importance of the new techniques advanced here, where we control for one source of uncertainty in computing the welfare cost of the other source.

4.5. Welfare Costs λ_T, λ_P, λ_D, $\lambda'_T(0)$, $\lambda'_P(0)$, and $\lambda'_D(0)$, Including the Pre-WWII Era

This section focuses on welfare-cost estimates when we use pre-war data – either the pre-war sample 1901-1941, or the whole sample 1901-2012. We put much less weight on these results because pre-war data for consumption and income are not as reliable as post-war data. Indeed, Kuznets (1961) has to resort to moving-average schemes to estimate some annual pre-war macroeconomic aggregates, which induces measurement error. Despite that, for historical reasons, we believe that computing welfare costs for this period has some value added.

Table 4.5, panel A, reports the *total and marginal* welfare costs for the pre-war and whole sample periods, respectively 1901-1941 and 1901-2012, when we let the data estimate the covariance between transitory and permanent innovations, i.e., when we do not impose that $\sigma_{12} = 0$. In general, the welfare costs of business cycles and of economic-growth variation are much higher in the pre-war period (1901-1941) than in the post-war period (1947-2012). This is a direct consequence of the fact that the estimates of σ_{11} and σ_{22} are higher for the former period. For example, the welfare costs of business cycles increases from -0.03% to 0.24% of consumption, and the welfare costs of economic growth from 0.71% of per-capita consumption to 2.73%, evaluated at $(\beta = 0.985, \phi = 5)$. We also observe an increase if we compare the results in Table 4.5, panel B, with those in Table 4.2, panel B, where we impose $\sigma_{12} = 0$ in estimation. Finally, results for the whole sample 1901-2012 are a combination of pre- and post-war results.

5. Conclusion

The main contribution of this paper is to propose a novel setup that allows estimating separately the welfare costs of the uncertainty stemming from business-cycle fluctuations and from economic-growth variation, when permanent and transitory shocks hit consumption simultaneously. Separating their welfare effects requires dealing with degenerate bivariate distributions using measure-theoretic results. We resort to Levi's Continuity Theorem, when the limit of the bivariate distribution is degenerate in one dimension. By using the Disintegration Theorem, we compute the resulting marginal distribution for the remaining dimension. Under Normality, we show that the parameters of the original marginal distribution are not affected in the limit, which provides the theoretical basis for separating welfare costs. This result holds under two key cases: independent and dependent permanent and transitory shocks.

Since permanent and transitory shocks have different sources and they affect welfare in a different way, separating welfare costs allows consistent estimation of the welfare costs of business cycles and

Table 4.5: Welfare Costs Including Pre-WWII Data – % Consumption

PANEL A ($\sigma_{12} \neq 0$)												
	Business-Cycle Fluctuations				**Economic-Growth Variation**				**All Macro Uncertainty**			
					Welfare Cost (1901-1941)							
$\beta\backslash\ \phi$	1	5	10	20	1	5	10	20	1	5	10	20
0.95	0.05	0.24	0.48	0.97	0.69	1.70	2.16	2.79	0.73	1.96	2.69	3.87
0.97	0.05	0.24	0.48	0.97	1.22	2.21	2.56	3.15	1.26	2.47	3.09	4.23
0.99	0.05	0.24	0.48	0.97	2.39	2.73	2.91	3.44	2.43	3.00	3.44	4.52
					Marginal Welfare Cost (1901-1941)							
0.95	0.10	0.48	0.97	1.95	1.40	3.75	5.31	9.12	1.48	4.17	6.16	10.90
0.97	0.10	0.48	0.97	1.95	2.48	4.93	6.42	10.74	2.56	5.36	7.28	12.54
0.99	0.10	0.48	0.97	1.95	4.97	6.20	7.43	12.13	5.05	6.63	8.30	13.96
					Welfare Cost (1901-2012)							
0.95	0.00	0.00	0.01	0.02	0.69	1.42	1.66	1.92	0.70	1.40	1.62	1.85
0.97	0.00	0.00	0.01	0.02	1.23	1.74	1.87	2.07	1.24	1.73	1.84	2.00
0.99	0.00	0.00	0.01	0.02	2.43	2.04	2.04	2.18	2.44	2.02	2.01	2.11
					Marginal Welfare Cost (1901-2012)							
0.95	0.00	0.01	0.02	0.04	1.40	2.87	3.46	4.54	1.42	2.97	3.66	4.94
0.97	0.00	0.01	0.02	0.04	2.50	3.59	4.00	5.05	2.52	3.69	4.20	5.45
0.99	0.00	0.01	0.02	0.04	5.05	4.28	4.45	5.43	5.07	4.38	4.64	5.84
PANEL B ($\sigma_{12} = 0$)												
	Business-Cycle Fluctuations				**Economic-Growth Variation**				**All Macro Uncertainty**			
					Welfare Cost (1901-1941)							
$\beta\backslash\ \phi$	1	5	10	20	1	5	10	20	1	5	10	20
0.95	0.11	0.53	1.07	2.15	0.16	0.39	0.46	0.50	0.27	0.92	1.54	2.66
0.97	0.11	0.53	1.07	2.15	0.29	0.50	0.54	0.55	0.40	1.03	1.61	2.71
0.99	0.11	0.53	1.07	2.15	0.57	0.61	0.60	0.59	0.67	1.14	1.68	2.75
					Marginal Welfare Cost (1901-1941)							
0.95	0.21	1.07	2.15	4.34	0.33	0.79	0.96	1.09	0.54	1.87	3.13	5.47
0.97	0.21	1.07	2.15	4.34	0.58	1.01	1.12	1.20	0.79	2.09	3.29	5.58
0.99	0.21	1.07	2.15	4.34	1.14	1.24	1.25	1.28	1.36	2.32	3.43	5.67
					Welfare Cost (1901-2012)							
0.95	0.05	0.27	0.53	1.07	0.31	0.61	0.68	0.71	0.37	0.88	1.22	1.78
0.97	0.05	0.27	0.53	1.07	0.55	0.75	0.77	0.76	0.60	1.02	1.31	1.83
0.99	0.05	0.27	0.53	1.07	1.08	0.88	0.84	0.80	1.14	1.14	1.37	1.87
					Marginal Welfare Cost (1901-2012)							
0.95	0.11	0.53	1.07	2.14	0.63	1.26	1.44	1.59	0.73	1.79	2.52	3.76
0.97	0.11	0.53	1.07	2.14	1.11	1.54	1.63	1.71	1.22	2.08	2.71	3.89
0.99	0.11	0.53	1.07	2.14	2.20	1.81	1.78	1.81	2.31	2.35	2.86	3.99

Note: σ_{12} is the covariance between ε_t (permanent shocks) and μ_t (transitory shocks) defined in equation (2.9)

of economic-growth variation. This is a step forward vis-a-vis previous studies, which have either lumped together the welfare effects of these distinct shocks or have inconsistently estimated the welfare costs associated with each of them separately.

Our empirical results show that, if we consider only transitory shocks, the welfare cost of business cycles is much smaller than previously thought. Indeed, Lucas (1987, 3) posits that it is 0.04% of consumption in the post-war period, whereas we found it to be negative – -0.03% of per-capita consumption! This means that the representative consumer actually pays to be indifferent between actual consumption and cycle-free consumption. Since Lucas assumed that consumption growth is purely deterministic, his estimate can be thought to be an upper bound for the true welfare cost of business cycles, because it must account for the uncertainty associated with economic-growth variation – or at least some of it[12]. The idea of an upper bound is indeed present in Lucas (2003), who discusses these issues with a historical perspective.

Going one step forward, if we link the source of business cycles to monetary policy, as is done in the short-run macroeconomic literature, e.g., Galí (1999) or Clarida, Galí and Gertler (1999), our results clearly put a negative value on how much a society benefits from trying to manage business cycles through the use of monetary policy. This is true despite the potential importance of monetary policy in generating business cycles.

On the other hand, we found that the welfare cost of economic-growth variation is relatively large. Our estimate for reasonable preference-parameter values shows that it is 0.71% of consumption – US$ 208.98 per person, per year. Going one step forward, if we link permanent shocks to productivity, policies aimed at reducing its variance should be paid more attention – although at this point it is unclear (at least to us) what these policies might be. This is indeed the message in Lucas (1983).

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[12] In that regard, the reader might conjecture why this upper bound is so small. The answer lies on the fact that, despite assuming a deterministic linear trend for the log of consumption, Lucas extracts its cycle using the Hodrick-Prescott filter, treating it as deterministic. Issler, Franco, and Guillén (2008) have done Lucas' exercise with a linear trend arriving at a welfare cost of 0.27% of consumption.

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A. Appendix – Figures

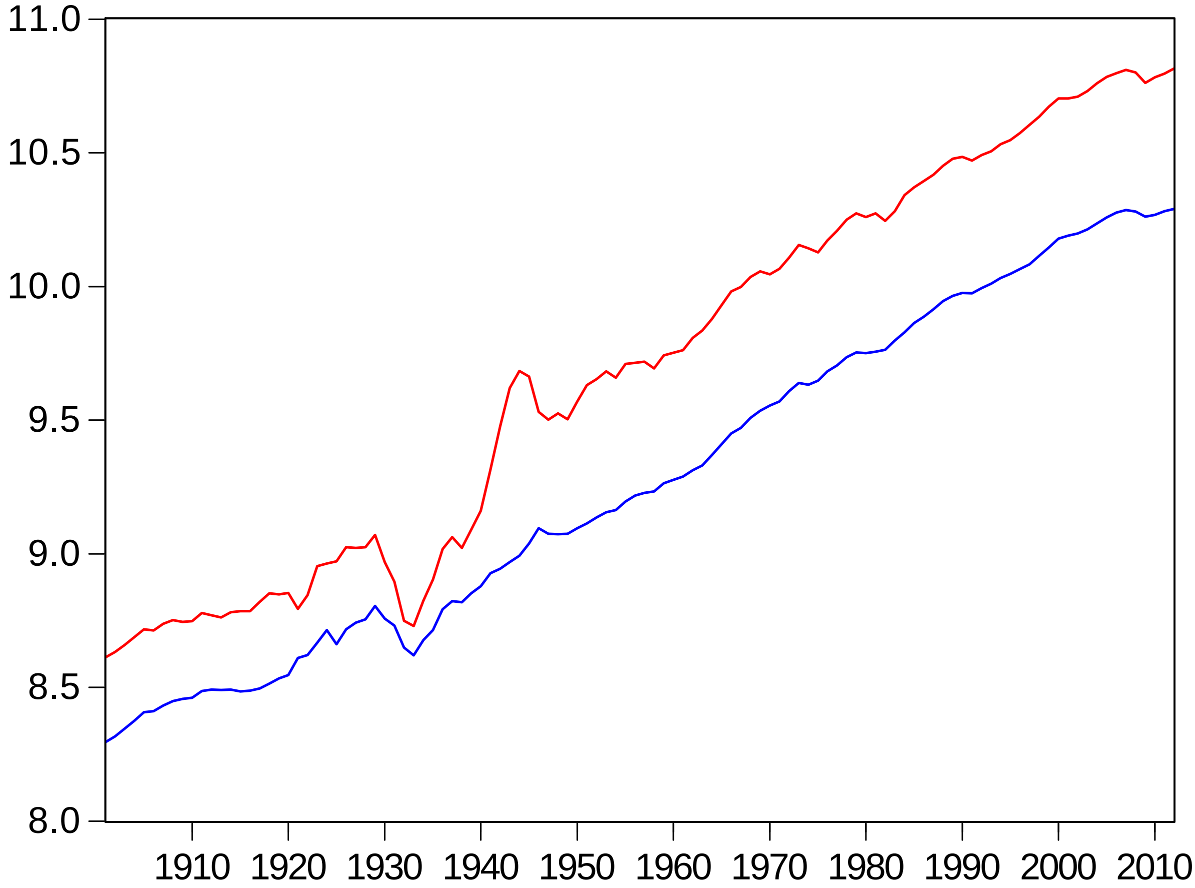

Figure 1: Consumption of nondurables and services and GNP per capita (logs)

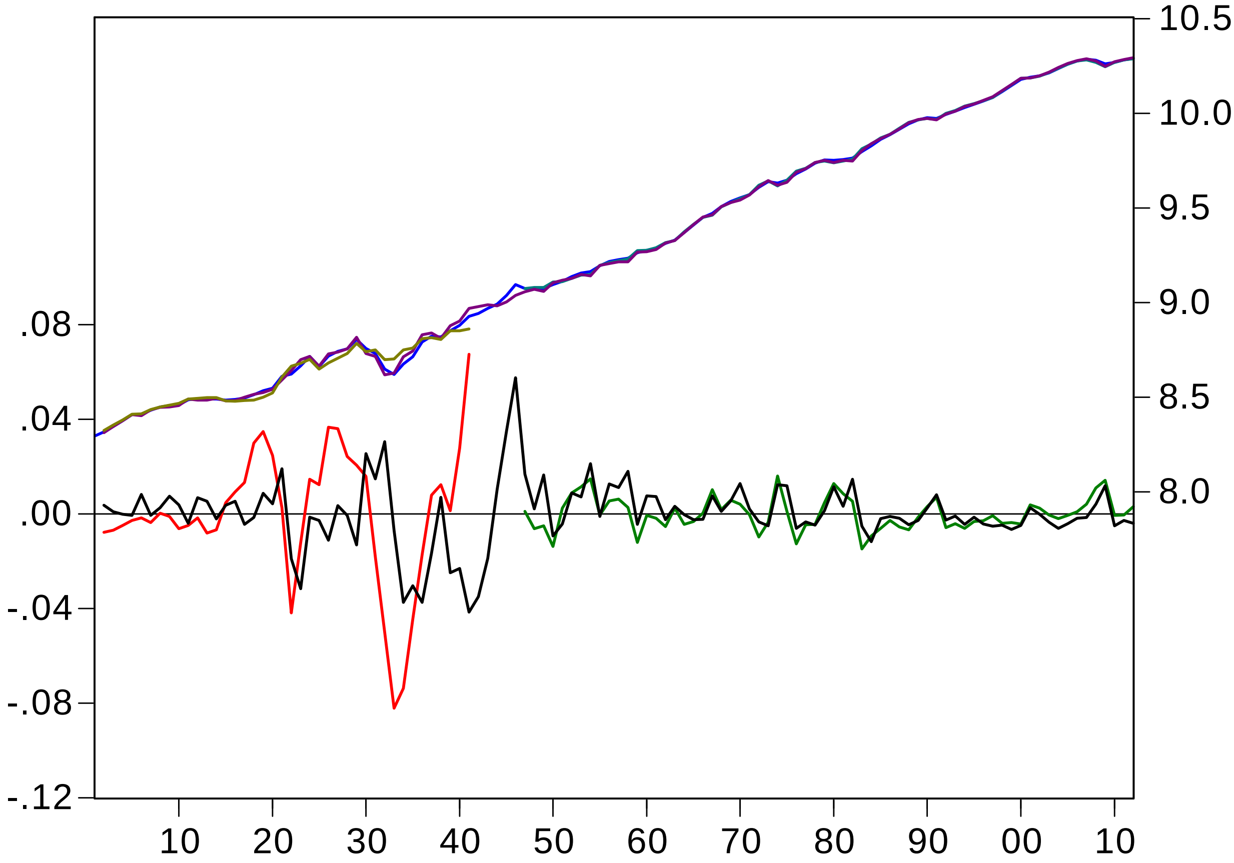

Figure 2: Consumption per-capita (log) and Beveridge-Nelson Trends and Cycles, 1901-2012, 1901-1941, and 1947-2012

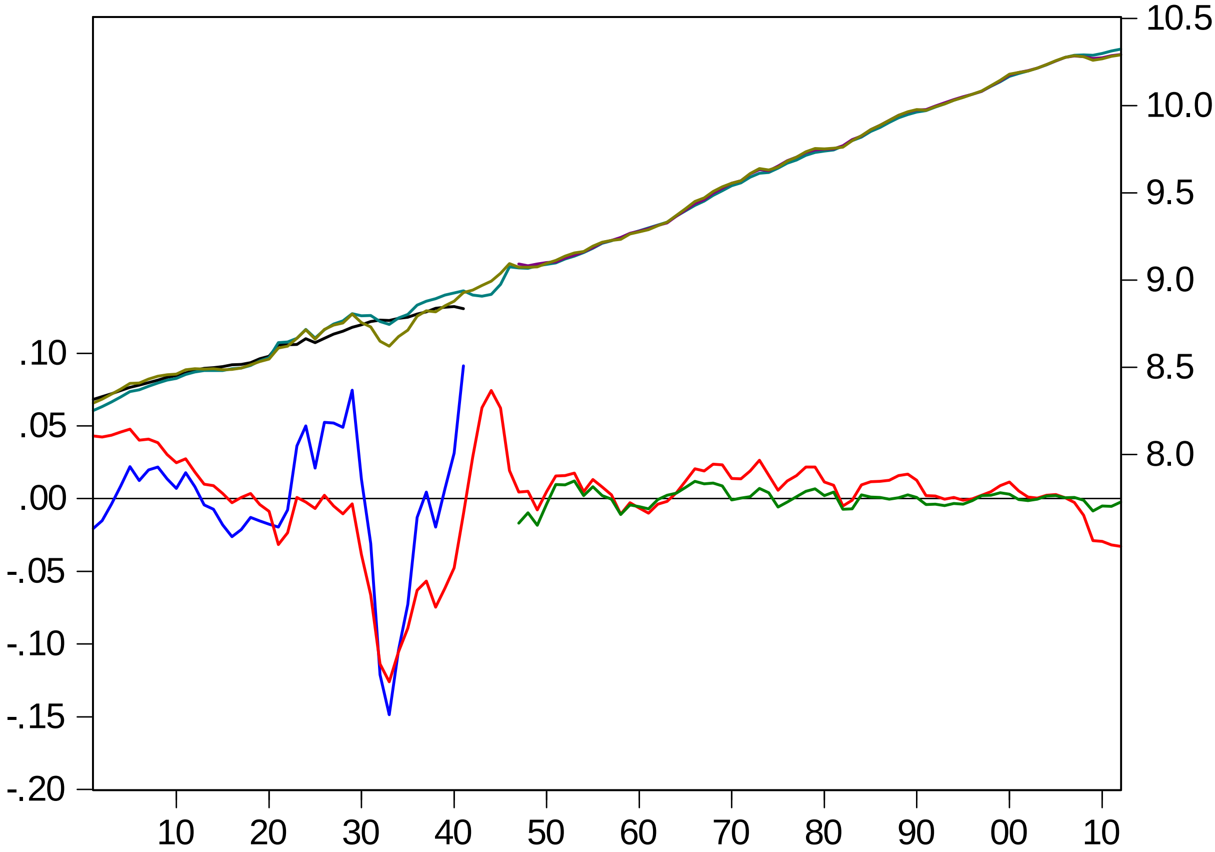

Figure 3: Consumption per-capita (log) and Unobserved-Component Trends and Cycles, 1901-2012, 1901-1941, and 1947-2012