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The long-run properties of a dynamic Mirrlees' model with aggregate shocks

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A DYNAMICS MIRRLEES' MODEL WITH AGGREGATE SHOCKS: LONG-RUN PROPERTIES .

Dissertação apresentada ao Curso de Mestrado em Economia da Escola de Pós-Graduação em Economia para obtenção do grau de Mestre em Economia.

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Resumo

Nós abordamos a existência de distribuições estacionárias de promessas de utilidade em um modelo *Mirrlees* dinâmico quando o governo tem *record keeping* imperfeito e a economia é sujeita a choques agregados. Quando esses choques são iid, provamos a existência de um estado estacionário não degenerado e caracterizamos parcialmente as alocações estacionárias. Mostramos que a proporção do consumo agregado é invariante ao estado agregado. Quando os choques agregados apresentam persistência, porém, alocações eficientes apresentam dependência da história de choques e, em geral, uma distribuição invariante não existe.

Palavras chave: Tributação ótima; Incerteza agregada; Abordagem Mirrleesiana.
J.E.L. codes: E6; H3; J2.

Abstract

We assess the existence of a long run stationary distribution of expected utilities in a dynamic Mirrlees's (1971) incentive structure when the government has only imperfect record keeping and the economy is subject to aggregate shocks. When aggregate shocks are i.i.d., we prove the existence of such a distribution and partially characterize the steady-state allocations. We show that the consumption share of each cohort is invariant to the aggregate state. In contrast, when aggregate shocks are persistent, efficient allocations display history dependence, and an invariant distribution need not exist.

Keywords: Optimal taxation; Aggregate uncertainty; Mirrleesian Approach.
J.E.L. codes: E6; H3; J2.

The long-run properties of a dynamic Mirrlees' model with aggregate shocks

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Abstract

We assess the existence of a long run stationary distribution of expected utilities in a dynamic Mirrlees's [1971] incentive structure when the government has only imperfect record keeping and the economy is subject to aggregate shocks. When aggregate shocks are i.i.d., we prove the existence of such a distribution and partially characterize the steady-state allocations. We show that the consumption share of each cohort is invariant to the aggregate state. In contrast, when aggregate shocks are persistent, efficient allocations display history dependence, and an invariant distribution need not exist. **Keywords:** Optimal Taxation; Aggregate Uncertainty; Mirrleesian approach. **J.E.L. codes:** E6; H3; J2.

1 Introduction

The analytical structure of the New Dynamic Public Finance—NDPF— literature has allowed a better understanding of efficient provision of social insurance and redistribution, two of the most important issues of Macroeconomics and Public Finance. Another important issue pertaining to the two areas which has received less attention by this literature is tax smoothing: the optimal spreading of distortions across time and states of nature.

[*]This is a joint work with Carlos da Costa.

This is hardly surprising. Characterizing efficient allocations in a dynamic Mirrlees' setting, the paradigm of the NDPF literature, is a notoriously difficult problem. This is true even if one considers a partial equilibrium setting or if one assumes aggregate risk away.[1] Few works have, as a consequence, embedded a Mirrlees' economy in an economy hit by aggregate shocks.[2]

Indeed, few attempts to combine incentive provision and aggregate risk sharing were made in a dynamic setting, to the best of our knowledge. Phelan [1994], probably the most important reference so far, defines a setting for which many of the technical difficulties can be avoided, but never focuses on the policy questions. da Costa and Luz [2011], on the other hand, focuses exactly on the policy implications that is induced by the combination of aggregate risk and constrained efficient insurance. A limitation of their work is that their model produces no steady-state, in which policies may be evaluated. First, one of da Costa and Luz's [2011] main findings is that efficient allocations display memory of aggregate uncertainty, despite their assumption of i.i.d. aggregate shocks. Long run allocations therefore depend on the whole history of aggregate uncertainty. Beyond that, there is the typical long run behavior of constrained efficient insurance problems, in which, degeneracy of long-run distributions of expected utility is to be expected. [3]

In this paper we slightly change da Costa and Luz's [2011] setting in order to avoid degeneracy by assuming that the planner's record keeping technology is imperfect. The way we introduce limited record keeping in our model is isomorphic to Phelan's [1994] perpetual youth structure. Every period a fixed fraction of the planner's files are lost and the individuals' private histories erased from the planner's perspective. As a consequence, at each moment, groups of agents differentiated by the length of their records must share the aggregate risk of the economy.

[1] Even in partial equilibrium and in the absence of aggregate uncertainty the characterization of optimal allocations is not trivial. It is greatly facilitated when one assumes that the stochastic processes are i.i.d.. For the case of persistent idiosyncratic shocks, such characterization has only recently been made possible through the works of Farhi and Werning [2013], that account for the evolution of wedges through time and Golosov et al. [2011], who explore its cross-sectional properties. Both works rely on the first order approach developed by Kapička [2011] and Pavan et al. [2012].

[2] Notable exceptions are Phelan [1994], Kocherlakota [2005] and Werning [2007].

[3] E.g., Green [1987], Spear and Srivastava [1987], Atkeson and Lucas [1992]. For a thorough discussion of the issue, see Phelan [1998].

Because incentive provision varies with the length of one's records, one must consider how incentive provision affects risk sharing across groups facing different incentive problems, as in Demange [2008]. This motivates our second departure from da Costa and Luz [2011]. We change the timing of choices, from that in da Costa and Luz [2011] where agents learn their private shocks after knowing the aggregate state to the one in Phelan [1994] in which private shocks are learned prior to the resolution of aggregate uncertainty.[4] According to Demange [2008] this timing leads to a better behaved risk sharing problem. As we shall explain, this change in the timing of choices turns out to have important consequences for memory of aggregate history.

Because we aim at understanding the long run properties os efficient allocations, we take advantage of the recursive nature of the problem by using the distribution of expected utilities, or utility promises, as one of the states. The ultimate consequence of the difference in timing combined with the perpetual youth structure, is that we are able to prove the existence of a non-degenerate steady state, and provide a partial characterization of long run allocations which are proven not to depend on the aggregate history.

The difference in timing has important consequences for implementation, as well. In particular, we show that agents need not be restricted in their trades of aggregate state contingent markets, although they ought not to be able to assess self-insurance through, for example, risk free bonds. This result resonates the finding of Scheuer [2013]. Second, because agents choose effort before the aggregate state is realized, it is the risk adjusted expected marginal tax rate that matters for incentives. In particular, state-invariant marginal tax rates may be used despite the fact that 'measured wedges' vary across states.[5]

All this simplification comes at a cost: labor supply responses are muted at the business cycle frequency. This shortcoming may, however, be circumvented to a large extent by allowing aggregate shocks to exhibit persistence. That is, although

[4]Phelan's [1994] is a moral hazard model in which case 'actions' are chosen prior to the resolution of uncertainty. Here, as in Phelan [1994], agents must choose how much effort to make before knowing the aggregate state, whereas in da Costa and Luz [2011] this choice is only made after the aggregate state is realized.

[5]These are the wedges calculated by the econometrician after the aggregate state is realized. They are pro-cyclical for risk aversion less than one and counter-cyclical for risk aversion greater than one, exactly as in da Costa and Luz [2011].

labor supply does not respond to the current aggregate state, it does respond to past shocks. By making the length of a period sufficiently small we approximate arbitrarily well the timing in da Costa and Luz [2011].[6]

When aggregates shocks exhibit persistence, allocations display memory, and there will be no invariant distribution of promises, in general.

We characterize long run allocations in few special cases, and suggest a definition of steady state which we find to be more suitable for the environment with persistent shocks. Such definition allows the distribution to vary with the current aggregate state. When preferences for consumption are of the log type, we prove the existence of a steady state in this latter sense despite the fact that no invariant distribution exists. Once again, because preferences are of the log type, labor supply does not respond to the aggregate state.

The question we ask now is whether one may ever obtain both labor supply variations and a steady state. We provide an example in which labor supply varies during the transition, and there is convergence to an invariant distribution. History still plays a role in determining the fate of a society: the *level* of expected utility at which each society settles is a function of the entire history of aggregate shocks. The effect of aggregate shocks on labor supply vanishes in the steady state.

The rest of the paper is organized as follows. After the literature review that still belongs to this introduction, Section 2 presents the primitives of our economy. Section 3 describes the Planner's program. In Section 4 we present the component planners problems with recursive structure and begin the characterization of equilibrium allocations. Section 5 defines invariant distributions and characterizes steady state allocations for the i.i.d. case. The case of persistent aggregate shocks is studied in Section 6. Section 7 concludes the paper.

Literature Review

Most work in the normative question of optimal fiscal policies under aggregate uncertainty has been done under the restrictive assumption of complete markets — Barro [1979], Lucas and Stokey [1983], Chari et al. [1994], Aiyagari et al. [2002],

[6]Note, however, that we do not converge to their setting, since the persistence of aggregate shock will play a role in future promises.

Werning [2007]. When markets are incomplete, an important role played by fiscal policy is to substitute for these missing markets. Hence, the type of orthogonality between aggregate policies and insurance that would justify the use of a representative agent needs to be assessed.[7]

Despite the recognition by the profession of this important limitation of most studies, progress in our understanding of optimal policies slow. And it has been slow for a good reason: these models become intractable so rapidly that as of this moment it has not been possible to fully solve them. Partial characterizations that allow for isolating the forces at work has then arisen as a promising compromise. Representative of this strand is a recent work by da Costa and Luz [2011] which addresses tax smoothing using a dynamic Mirrlees economy that combines privately observed idiosyncratic shocks and publicly observed aggregate shocks.

By restricting their attention to iso-elastic preferences and very simple endowment shocks da Costa and Luz [2011] are able to partially characterize allocations. They isolate two potential properties of allocations, namely the absence of memory of aggregate shocks and a form of separability which are reminiscent of complete markets allocations. Next they identify the forces that determine the direction and the strength of the departure from these properties when not all idiosyncratic risk may be insured against. This enables them to show how wedges vary along the business cycle and how this relates to persistence of allocations.

An important limitation of da Costa and Luz's [2011] work is that their analysis is restricted to the transition to a degenerate steady state. da Costa and Luz [2011] consider a dynastic model which adds aggregate uncertainty to a standard dynamic Mirrlees' economy — e.g., Golosov et al. [2003, 2011], Farhi and Werning [2013]. These models are known to lead to degenerate steady-state distributions, which counter-factual nature has led to the emergence of an interesting body of work aiming at reverting this result — e.g., Phelan [2006], Farhi and Werning [2007], **?**. From a practical perspective, the non-existence of a steady state renders quantitative analysis hard to interpret.

Phelan [1994] is the pioneering and, as far as we know, one of the few works to combine constrained efficient insurance and aggregate uncertainty in a dynamic

[7] Although we speak of a representative agent, a similar criticism may be made to works that take into account heterogeneity but rule out idiosyncratic risk — e.g. Werning [2007].

setting. To avoid degeneracy, Phelan [1994] works with a perpetual youth framework. The idea is that, although the long run distributions for each cohort inherits the degeneracy property of dynastic models, at each moment, the cross-sectional distributions place very small weight on these ever divergent distributions. The assumption of limited record keeping leads to an economy which is isomorphic to the one with a perpetual youth demography, but which is endowed with a standard welfare criterion.[8]

The timing of agent's choices in our model is also similar to Phelan's [1994] and in contrast with da Costa and Luz's [2011]. Namely, effort must be chosen prior to the realization of aggregate uncertainty. This means, in practice, that effort may not be affected by the contemporaneous aggregate state. This is no minor issue.

Imperfect record keeping leads to the separation of individuals according to the length of their recorded history. This leads to a risk sharing problem between groups differentiated by the severity of the incentive problems they face. This is very similar to the problem studied by Demange [2008] in a static setting.

The consequence that the timing of choices has for the problem is best summarized in her own words. "Our results suggest that it is not moral hazard per se that hamper macro-economic risk to be shared, but rather the timing at which effort is exerted. With an irreversible effort, the impact of moral hazard exists but is not dramatic."

Here, we show how this difference in timing induces a very different outcome with respect to both risk sharing and memory when compared to the timing used by da Costa and Luz [2011]. We show that allocations are independent of past aggregate shocks when aggregate shocks are i.i.d., whereas in da Costa and Luz [2011] allocations depend on the whole history of aggregate even when these are i.i.d.. Here, as in da Costa and Luz [2011] or Phelan [1994], allocations display memory of past *idiosyncratic* shocks. In bothda Costa and Luz [2011] and Phelan [1994] they also depend on past *aggregate* shocks whereas here they only depend on the current *aggregate* state. This also suggests that it is not only the CARA assumption on preferences that simplifies the derivation of efficient allocations by Phelan [1994]. A lot of mileage is gained from the specific choice of timing.

[8] Things are less clear cut, if we take the concept of dynasty seriously – e.g. Phelan [2006], Farhi and Werning [2007].

In a sense, the contrasting results regarding memory in da Costa and Luz [2011] and those found herein are the dynamic counterparts of Demange's [2008] findings in a static setting. The difference in timing is also important to understand the contrasting results in Phelan [1994] and da Costa and Luz [2011], when it comes to memory of aggregate uncertainty. Memory of aggregate shocks only arises in Phelan [1994] because aggregate shocks are assumed to be informative about effort, in the sense of Holmström [1979]. In da Costa and Luz [2011], in contrast, memory arises despite the independence assumption they adopt. State-dependent promises are inefficient but useful for alleviating incentive problems when these must be provided after the realization of aggregate uncertainty. The results herein make the contrast in these two papers stark.

We also show that the implementation results found by Scheuer [2013] are also valid here. Once again, the timing of incentive provision is key; because incentives need not be provided state by state, free access to side markets of state contingent consumption leads to no loss. More precisely, we find that individuals may be allowed to freely insure themselves against aggregate shocks but must be precluded from self-insuring through savings.

It is also important to mention yet another timing explored by **?**. In this paper, aggregate uncertainty is generated by the assumption of a finite number of agents. The planner chooses allocations only after collecting all announcements, and an agent does not learn about others announcements until the allocations are defined. It is hard, however, to relate the timing of this model with actual choices in a real world economy. We conjecture that optimal allocations in this setting displays no memory of aggregate uncertainty and has (in the case of iso-elastic preferences) the separable properties which da Costa and Luz [2011] find not to be valid in their setting.

With regards to the long run behavior of the economy, the dynamic insurance literature has typically associated the idea of a steady state as the existence of an invariant distribution of utility promises. The main issue has typically been how to overturn the degeneracy results that characterizes most models. The very idea of looking for an invariant distribution may, however, be too restrictive when the economy is also hit by aggregate shocks. Phelan [1994] shows, for example, that no invariant distribution exists in his setting, despite all effort made to guarantee

that each aggregate history leads to a non-degenerate steady-state. We explore an alternative definition of steady state that we find more suitable for the environment we study.

2 The Environment

The economy is inhabited by a continuum measure one of agents that live for an infinite number of periods.[9]

Each agent in this economy has preferences defined over sequences of consumption and effort, $\{c_t, n_t\}_t$, with flow utility represented by the time and labor/consumption separable utility function,

$$U(c, n; \theta) = v(c) - \theta\eta(n),$$

where v and η are C^2 and $v' > 0$, $v'' < 0$, $\eta' > 0$ and $\eta'' > 0$. The parameter θ affects the amount of effort (measured in 'utils') that an individual must incur to generate l efficient units of labor. It the fact that this parameter is subject to shocks that drives the idiosyncratic risk faced by the agents.

As it turns, it is more convenient to work directly with the flow utilities $u = v(c)$ and $h = \eta(n)$. We recover c and n using $c = C(v(c))$ and $n = N(\eta(n))$. Naturally, $N(.) = h^{-1}(.)$ and $C(.) = u^{-1}(.)$.

Agents have von Neumann-Morgenstern preferences and discount utility flows at a rate $\delta \in (0, 1)$.

Technology Technology is very simple. One unit of labor is transformed in z units of consumption goods if the aggregate state of the economy is z. That is, aggregate uncertainty in this economy is captured by a variable z_t, which history we represent by $z^t = (z_1, z_2, ..., z_t)$. We let $z^(s) = (z_{i_1}, ..., \theta_{i_s})$ denote an arbitrary partial history of aggregate shocks, of length s starting in an arbitrary period i_1 and ending in i_s. We say that z^{t+s} is a continuation of z^t if there is $z^{(s)}$ such that $z^{t+s} = (z^t, z^{(s)})$. $\pi(z^t)$ is the probability of aggregate history z^t and $\pi(z^{t+s}|z^t)$ the

[9] One may also think of these as dynasties.

probability of a history z^{t+s} given that a history z^t has been observed up to period t.

Idiosyncratic Risk At the beginning of each period each agent draws his or her temporary skill, θ, from the set $\Theta \equiv \{\theta(1) < \cdots < \theta(N)\}$. These shocks are an agent's private information.

We also define $\theta^{(s)}$ analogously to $\theta^{(s)}$ and say that θ^{a+s} is a continuation of θ^a if there is $\theta^{(s)}$ such that $\theta^{a+s} = (\theta^a, \theta^{(s)})$. $\mu(\theta^a)$ is the probability of private history θ^a and $\mu(\theta^{a+s}|\theta^a)$ the probability of a history θ^{a+s} given that a history θ^a has been observed.

It is important to distinguish chronological time, t, from length of recorded history a do to limited record keeping, as we explain next.

The Insurance Problem Because there is risk in this economy there is scope for an institution to provide insurance. We assume the existence of a benevolent planner whose objective is to maximize agents' expected utilities but who is limited by two sources of frictions. First is the aforementioned fact that the skill shock is private information. Second, we assume that the planner's technology for keeping records of agents' histories is imperfect. That is we assume that every period a fraction $1 - \Delta$ of agents have their files destroyed.

Imperfect record keeping To understand the consequences of imperfect record keeping some additional notation is needed. Let θ^t be an agent's history of skill realization from period 1 until period t. Now, let $t - a$ be the last time at which an agent's files were lost. We use $\theta^a = (\theta_1, \theta_2, ..., \theta_a)$ to denote the sequence of private history of skills shocks from period $t - a$ up to period t. That is $\theta^a = \theta^{(a)}$ when i_1 is the last period at which an individual's files was lost. Hence a, is the 'length' or 'age' of an individual's file. Moreover, because θ^a is all the individual history in the records, we shall refer to it as his private history.

We say that θ^{a+s} is a continuation of θ^a, denoted $\theta^{a+s} \succ \theta^a$ if there is $\theta^{(s)}$ such that $\theta^{a+s} = (\theta^a, \theta^{(s)})$. We write $\mu(\theta^a)$ to denote the probability associated with history θ^a we let $\mu(\theta^{a+s}|\theta^a)$ denote the probability of observing a history θ^{a+s} after

the occurrence of a history θ^a. Naturally, $\mu(\theta^{a+s}|\theta^a) > 0$ only if $\theta^{a+s} \succ \theta^a$.[10]

For $a \leq t$, let $\theta_a h(\theta^a, z^{t-1})$ be the flow disutility associated with effort chosen by an individual with private history θ^a living in an economy t periods old that has experienced a history of aggregate shocks z^t, and let $u(\theta^a, z^t)$ be the equivalent definition for flow utility from consumption.

Finally, note that preferences over sequences of utility flows from consumption and labor, $\{u_t, h_t\}_t$, must take into account the fact that the further in the future a flow is the less likely it is for their records not to be lost by the planner.

We may, then, represent preferences over sequences $\{u_t, h_t\}_t$ by

$$\mathcal{U}(\{u_t, h_t\}_t) = \mathbb{E} \sum \beta^t \{u_t - \theta_t h_t\}$$

where $\beta = \delta\Delta \in (0,1)$ and $\mathbb{E}$ is the expectation operator.

Resource Constraints At each point in time, we may divide the society into different groups characterized by the length of their history, i.e., according to the date at which their records were lost for the last time. This defines a cohort structure for the economy in each point in time. We assume that preference shocks are independent across individuals and are independent of the risk of loosing one's records. The cross-sectional distribution of histories of each group coincides, therefore, with the ex-ante expected distribution of histories.[11]

We write the aggregate labor supply as

$$\bar{N}(z^{t-1}) = \sum_{a=1}^{t} \Delta^a \sum_{\theta^a} \mu(\theta^a) N(h(\theta^a, z^{t-1})).$$

Aggregate labor supply is converted into aggregate consumption,

$$\bar{C}(z^t) = \sum_{a=1}^{t} \Delta^a \sum_{\theta^a} \mu(\theta^a) C(u(\theta^a, z^t))$$

[10]There is some abuse in notation here since, we should have used $\mu_a(\theta^a)$, instead. However, since we believe that the understanding is not compromised, we drop the subscript for economy.

[11]I.e., under our independence assumption, individuals history length a that have experienced a history $\theta^a = (\theta_1, \theta_2, ..., \theta_a)$ represent a proportion $\mu_a(\theta^a) = \prod_{j=1}^{a} \mu(\theta^j)$ of their cohort.

at a rate $z_t = \bar{C}(z^t)/\bar{L}(z^{t-1})$.

This defines the period by period resource constraint, $\bar{N}(z^{t-1})z_t \geq \bar{C}(z^t)$.

3 The Planner's Problem

Individuals in our economy are exposed to three types of risk: skill shocks, aggregate shocks and the risk of having their files lost.

The planner tries to maximize a utilitarian social welfare function by providing as much insurance as possible to the agents.

The Mechanism in the Sequence Space In pursuing its objective the planner is restricted by the informational structure of the environment. We shall, therefore, set up the planner's problem as a mechanism design program.

Let $\{u(\theta^a, z^{t+a}), h(\theta^a, z^{t+a-1})\}_{\theta^a, z^{t+a}, a}$ be an allocation for the period t cohort. To apply the revelation principle in our problem we first define a reporting strategy as a function $\sigma : \Theta^\infty \times Z^\infty \to \Theta^\infty$, adapted to the filtrations generated by the two stochastic processes $\{\theta^a\}_a$ and $\{z^{t+a}\}_{t+1}$. The set of all such strategies is defined by Σ.

A strategy σ^* is truth-telling if an agent plans to reveal his true type in all dates and under all possible contingencies, that is, $\sigma^*(\theta^a, z^{t+a}) = \theta^a$, for all θ^a, a, z^{t+a}. An allocation is *incentive-compatible* if the truth-telling strategy yields no less utility than any other strategy σ.

Fix a sequence of state-contingent transfers for the cohort whose records were lost in t for the last time, $\varsigma = \{\varsigma_a(z^{t+a})\}$. We may, then, define

$$(1) \qquad \mathcal{U}_t(\varsigma) \equiv \max \sum_a \sum_{\theta^a} \sum_{z^{t+a}} \beta^a \mu(\theta^a) \pi(z^{t+a}|z^t) \left\{ u(\theta^a, z^{t+a}) - \theta_a h(\theta^a, z^{t+a-1}) \right\}$$

subject to

$$(2) \qquad \sum_a \sum_{\theta^a} \sum_{z^{t+a}} \beta^a \mu(\theta^a) \pi_t(z^{t+a}|z^t) \left\{ u(\theta^a, z^{t+a}) - \theta_a h(\theta^a, z^{t+a-1}) \right\} \geq$$

$$\sum_a \sum_{\theta^a} \sum_{z^t} \beta^a \mu(\theta^a) \pi(z^{t+a}|z^t) \left\{ u(\sigma(\theta^a, z^{t+a-1}), z^{t+a}) - \theta_a h(\sigma(\theta^a, z^{t+a-1}), z^{t+a-1}) \right\}, \quad \forall \sigma$$

and

$$(3)\quad \sum_{\theta^a} \mu(\theta^a) C(u(\theta^a, z^{t+a})) \leq z_{t+a} \sum_{\theta^a} \mu(\theta^a) N(h(\theta^a, z^{t+a-1})) + \varsigma_a(z^{t+a}) \quad \forall z^{t+a}, a.$$

The solution of this problem defines the optimal allocation for a cohort initiated at t for the state contingent transfer plan $\varsigma = \{\varsigma_a(z^{t+a})\}$. We note that here the strategy σ is measurable with respect to the filtration generated by θ^a and z^{t+a-1}.[12] Labor supply is chosen before z^{t+a} is known.

Connecting the cohorts is the economy's resource constraint,

$$(4)\qquad \sum \Delta^a \varsigma_a(z^t) \leq 0 \quad \forall z^t, t.$$

We let $\varsigma = \{\varsigma_a(.)\}_a$ be an inter-cohort contingent transfer scheme, and define $\mathbb{T}$ as the set of all ς such that (4) holds.

For further reference, before we move on to the planner's problem note that an allocation defines at each node of an individual's history her expected continuation utility. That is, for an individual with a personal history θ^a, when the aggregate history is z^t, his or her continuation utility, $w(\theta^a, z^t)$, is

$$(5)\quad \sum_j \sum_{\theta^{a+j}} \sum_{z^{t+j}} \beta^j \mu(\theta^{a+j}|\theta^a) \pi(z^{t+j}|z^t) \left\{ u(\theta^{a+j}, z^{t+j}) - \theta_{a+j} h(\theta^{a+j}, z^{t+j-1}) \right\},$$

which we may also write $w(\theta^a, z^t) =$

$$\sum_\theta \sum_z \mu(\theta^a, \theta|\theta^a) \pi(z^t, z|z^t) \left\{ u(\theta^a, \theta, z^t, z) - \theta h(\theta^a, \theta, z^t) + \beta w(\theta^a, \theta, z^t, z) \right\}.$$

3.1 Preliminary Findings

With as little structure as we have imposed so far it is still possible to derive some results that are valid both at the steady state (if there is one) or in the transition.

[12] In contrast, it is measurable with respect to the filtration generated by θ^a and z^{t+a} in da Costa and Luz [2011]. To be precise, since in da Costa and Luz [2011] record keeping is perfect there is no distinction between t and a. Announcement strategies are, therefore, simply measurable with respect to the filtration generated by (θ^t, z^t).

The first thing we should note is that it is always possible to transfer consumption between any two individuals in such a way as to preserve expected utility for both individuals in an incentive compatible way. This results from the separability — between consumption and effort — only. Naturally, this type of reform cannot save resources at the optimum. This type of reasoning restricts optimal allocations to satisfy the restrictions defined in Lemma 1.

Lemma 1. *An efficient allocation must satisfy*

$$\sum_{\theta^{a+s}\succ\theta^{a}} \mu(\theta^{a+s}|\theta^{a})\frac{C'(u(\theta^{a+s},z^{t+s}))}{C'(u(\theta^{a},z^{s}))} = \sum_{\theta^{\hat{a}+s}\succ\theta^{\hat{a}}} \mu(\theta^{\hat{a}+s}|\theta^{\hat{a}})\frac{C'(u(\theta^{\hat{a}+s},z^{t+s}))}{C'(u(\theta^{\hat{a}},z^{t}))}, \tag{6}$$

for all $z^{t+s}, a, \hat{a}, \theta^{a}, \theta^{\hat{a}}$.

This result will be important in all that follows. Equation (6) is a form of inverse Euler equation with aggregate uncertainty, akin to that found in Kocherlakota [2005], but derived without the assumption that aggregate resources may be transferred across periods. It defines the relative shadow price of resources and is associated with the prices in the relaxed component planning program.

Another relevant result concerns the division of aggregate risk in the economy. Lemma 2 shows how the marginal rate of substitution vary across agents in different realizations of the aggregate shock. To derive this result we assume the usual Pareto problem stated in (7)

$$\max_{\varsigma\in\mathbb{T}} \mathbb{E}\sum \delta^{t}\mathcal{U}_{t}(\varsigma).$$

Lemma 2. *Let* $a, \hat{a} \in N$*. For every history* $z^{t}, \hat{z}^{t}$*, an efficient allocation satisfies*

$$\frac{C'(u(\theta^{a},z^{t}))}{C'(u(\theta^{a},\hat{z}^{t}))} = \frac{C'(u(\theta^{\hat{a}},z^{t}))}{C'(u(\theta^{\hat{a}},\hat{z}^{t}))}, \quad \forall a,\hat{a} \le t\ \theta^{a},\theta^{\hat{a}} \tag{7}$$

This result is directly related with efficient nature of the division of aggregate risk in our economy. This result will have several implications on the asset markets and on taxation.

3.2 Implementation

Taxes and Wedges

A question of great interest for policy makers is to determine how labor wedges ought to vary across aggregate states. As it turns, for the environment we study here two types of wedges are defined.

First, define the ex-post (or measured) wedges through

$$1 - \tau^{EP}(\theta^a, z^t) = \frac{\theta_a C'(u(\theta^a, z^t))}{N'(h(\theta^a, z^{t-1}))z}.$$

This is the wedge that is calculated by someone, an econometrician, who has data on an individual's consumption, labor supply and the aggregate labor productivity. The important thing to note is that this is merely an accounting exercise, with no necessary connection with incentives.

As mentioned, ex-post wedges are not directly associated with the provision of incentives. For this, we define ex-ante (or incentive-relevant) wedges,

$$N'(h(\theta^a, z^{t-1})) = \left\{ \sum \pi(z_t) \frac{1 - \tau^{EA}(\theta^a, z^t)}{C'(u(\theta^a, z^t))} z_t \right\}^{-1}.$$

Note that, in practice, provided that one is not restricted to using linear taxes, one may choose $\tau^{EA}(\theta^a, z^t)$ in a very arbitrary manner. In particular, one can choose $\tau^{EA}(\theta^a, z^t) = \bar{\tau}^{EA}(\theta^a, z^{t-1})$, such that

$$1 - \bar{\tau}^{EA}(\theta^a, z^{t-1}) = \sum \pi(z_t) \frac{N'(h(\theta^a, z^{t-1}))}{C'(u(\theta^a, z^t))} z_t.$$

Anonymous asset markets

Scheuer [2013] has explored the effect on the provision of incentives of allowing agents to access financial markets. Although he focuses on a static moral hazard environment, there is a clear connection between his question and some of the issues we explore here. In this Section we explore the effects on incentives of allowing agents to trade on financial markets. We assume that in this economy there are two types of assets: *risk free bonds*, and *state-contingent* securities.

Let z^t be the history of aggregate shocks for the economy. We assume that at the end of each period, i.e., after state z_t is realized, a market for one period risk free bonds opens. These bonds which price in state z^t we denote by $b_t(z^t)$, pay one unit of consumption in period $t+1$ in all states of the world.

This is not all that one can do. In the first sub-period of each period one may hedge against aggregate shocks by exchanging these risk free bonds for state contingent securities that pay one unit of consumption in one and only one state (z^t, z). We denote the price following history z^t of a one period Arrow security paying in state (z^t, z) by $q_t(z|z^t)$.

Figure , below, illustrates the timing of the model.

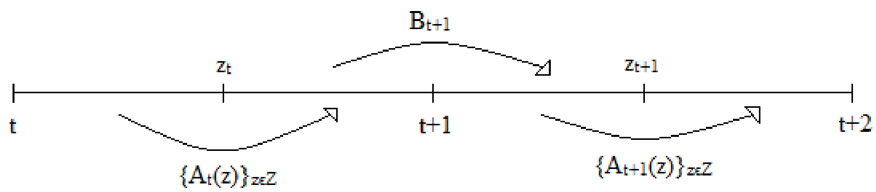

Figure 1: This figure displays the subdivision of a period and illustrates the timing of the model.

This asset structure, which follows Phelan [1994], allows for a separation between trade for inter-temporal smoothing reasons and aggregate risk sharing reasons. To anticipate our results, they are the same as Scheuer's [2013]: individuals can freely hedge against aggregate uncertainty but should not be allowed to smooth inter-temporally.

To understand what is at stake here, note that, if an agent has direct access to financial markets, the planner can no longer be assured that an announcement $\tilde{\theta}$ following history (θ^a, z^t) will lead to flow consumption utilities $\{u(\theta^a, \theta, z^t, z)_z\}$. By trading in the state-contingent asset markets, the agent can attain any other flow utility $\{\tilde{u}(\theta^a, \theta, z^t, z)_z\}$ provided that

$$\sum_z q(z|z^t)\left[C(u(\theta^a, \tilde{\theta}, z^t, z)) - C(\tilde{u}(\theta^a, \tilde{\theta}, z^t, z))\right] \geq 0.$$

A similar consideration applies to the inter-temporal distribution of flow utilities.

The question is whether this will preclude the optimal allocation to be attained.

Let bonds holdings and contingent assets holdings be denoted J_t, and A_t, respectively. With free access to asset markets, if an agent is offered the contract $(u(\theta^j, z^t), h(\theta^j, z^{t-1}))_{j,t}$ in t, he or she will choose an announcement strategy along with a portfolio strategy $\{A_t^\sigma, J_{t+1}^\sigma\}_t$ which defines her equilibrium utility given the assignment (u, h) as $\mathcal{V}(\{u, h\}) \equiv$

$$(8) \quad \max_{\sigma, \{A_t, J_{t+1}\}_t} \sum_t \beta^t \sum_{\theta^a} \mu(\theta^a) \sum_{z^t} \pi(z^t) \left\{ \tilde{u}(\sigma(\theta^a, z^{t-1}), z^t)) - \theta_a h(\sigma(\theta^a, z^{t-1}), z^{t-1}) \right\}$$

subject to

$$C(\tilde{u}(\sigma(\theta^j, z^{t-1}), z^t)) = C(u(\sigma(\theta^j, z^{t-1}), z^t)) + A_{t-1}(z) - \sum_z q(z) A_t(z) + J_{t-1} - b_t J_t, \quad \forall z \in Z.$$

Definition 1. An equilibrium in the re-trading market, given a contract (u, h), is comprised of prices $q_t = \{q_t(z|z^t)\}$ and $b_t(z_t)$, for all t, z^t, reporting strategies $\sigma : \Theta^\infty \times Z^\infty \to \Theta^\infty$ and an allocation $\{\tilde{u}_t, A_t^\sigma, J_{t+1}^\sigma\}_t$, such that:

1. Given $\{u, h\}$ and $\{q_t, b_t\}_t$, $\{\tilde{u}_t, A_t^\sigma, J_{t+1}^\sigma\}_t$ solve the agent's problem above.

2. The asset markets clear for every t:

$$\sum_{j=1}^t \Delta^j \sum_{\theta^j} \mu(\theta^j) A_t^\sigma(\theta^j, z^t) = 0, \quad \forall z^t \in Z^t$$

$$\sum_{j=1}^t \Delta^j \sum_{\theta^t} \mu(\theta^t) J_t^\sigma(\theta^j, z^{t-1}) = 0, \quad \forall z^{t-1} \in Z^{t-1}$$

Let $\tilde{\mathcal{V}}(\{u, h\}, \sigma)$ be the equilibrium utility attained by an agent following strategy σ, when all other agents are following the truth telling strategy. The planner now chooses contracts $\{u, h\}$ to solve the usual problem with the endogenous incentive restriction

$$(9) \quad \sum_{j=0}^\infty \beta^j \sum_{\theta^j} \mu(\theta^j) \sum_{z^{t+j}} \pi(z^{t+j}|z^t) \left\{ u(\theta^a, z^t) - \theta_a h(\theta^a, z^{t-1}) \right\} \geq \tilde{\mathcal{V}}(\{u, h\}, \sigma),$$

instead of (2).

State-contingent assets market Let us start by assuming that only the market for state-contingent assets exist. We shall then formally prove our statement regarding the irrelevance of side markets in our setting. To do so, assume that after realizing θ_a an agent announces her type and enters the market of aggregate state contingent assets where firms are trading.

Proposition 1. *The no-trade equilibrium is the unique equilibrium in the state-contingent assets markets.*

The proposition is a direct consequence of lemma 2. Since the division of aggregate risk equalizes the marginal rates of substitutions, the state-contingent asset market is irrelevant to the agents. Uniqueness is due to strong monotonicity and strict convexity of preferences.

Bonds market Let us now consider bonds markets. More precisely, assume that after the realization of z^t agents may trade risk free bonds at a price $b(z^t)$.

Lemma 3. *Define the shadow value for an agent with history θ^a in state z^t, at his efficient allocation, of one unit of consumption in the beginning of the next period by $\hat{b}(z^t, \theta^a)$. Then,*

$$\hat{b}(z^t, \theta^a) = \frac{\hat{b}(z^t)}{\delta} = \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) \frac{C'(u(\theta^a, z^t))}{\sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) C'(u(\theta^{a+1}, z^{t+1}))}. \tag{10}$$

for all θ^a, a.

Lemma 3 is an immediate consequence of (1). It first establishes that the shadow value of a bond is equalized across individuals at the optimal allocation. The first part of the lemma establishes $\hat{b}(z^t)$ as a natural candidate for a bond price, by showing that $\hat{b}(\theta^a, z^t)$ is identical across agents.

The second part of the lemma expresses this shadow value as an expected, in z^{t+1}, ratio between the inverse of the marginal utility of consumption at z^t and the

harmonic mean of the marginal utility of consumption next period in state z^{t+1}. The first order condition for the same individual's savings problem is

$$\frac{b(z^t)}{\delta} = \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) \frac{C'(u(\theta^a, z^t))}{C'(u(\theta^{a+1}, z^{t+1}))}. \tag{11}$$

Hence, unless there is no idiosyncratic risk, (3) and (11) cannot be satisfied at the same time. The standard inefficiency result in (Rogerson's [1985] and Golosov et al.'s [2003]) holds, which implies that the access to such markets should be precluded. This is formally stated below.

Proposition 2. *Free access to the bonds markets cannot be allowed at the optimum.*

3.3 A Relaxed Program

Let us now consider a relaxed program for which the period by period, state by state resource constraint (4) is replaced by a single inter-temporal constraint,

$$\sum \mathcal{Q}_t(z^t) \sum_a \Delta^a \varsigma_a(z^t) = 0,$$

for some $(\mathcal{Q}_t(.))_t$, $\mathcal{Q}_t : Z^t \to R_+$ with $\lim_{t\to\infty} \sum_t \mathcal{Q}_t(z^t) < \infty$ for all $z^\infty = \lim_{t\to\infty} z^t$.

Assume that, at the solution of this relaxed program, (4) is satisfied in all states and all periods. Then the allocation that solves the relaxed program solves the original planner's program as well.

We assume that there exist such prices and prove the existence of steady state allocations which are supported by prices of the form $\mathcal{Q}^* = \delta$, thus justifying our assuming $\lim_{t\to\infty} \sum_t \mathcal{Q}_t(z^t) < \infty$. In a sense, this shows that our guess of prices (in the l^1 space) doesn't arrive to any inconsistency.

Component Problems Note that the relaxed program is separable in component programs of the form

$$\max \sum_a \sum_{\theta^a} \sum_{z^{t+a}} \beta^a \mu(\theta^a) \pi(z^{t+a}|z^t) \left\{ u(\theta^a, z^{t+a}) - \theta_a h(\theta^a, z^{t+a-1}) \right\},$$

subject to (2) and

$$\sum \mathcal{Q}_t(z^t) \sum_{\theta^a} \mu(\theta^a) \left\{C(u(\theta^a, z^t)) - N(h(\theta^a, z^{t-1}))z_t\right\} = 0.$$

4 The Recursive Problem with i.i.d. Aggregate Shocks

As of this moment, no assumptions were made on the stochastic processes of θ or z beyond the fact that they were independent of one another.[13] In what follows we shall address memory and assess the existence (or not) of invariant steady state distributions. To accomplish this, we restrict the stochastic processes in such a way as to give recursivity a chance.

Throughout this section we assume that the aggregate shocks are i.i.d..[14] In all that follows we also assume that the idiosyncratic shocks, θ, are i.i.d. Adding persistence to idiosyncratic shocks does not change the main findings in this section. [15]

A further restriction on preferences will make all the discussion that follows simpler. We shall assume that preferences are of the iso-elastic type, i.e.,$\upsilon(c) = c^{1-\rho}(1-\rho)^{-1}$, and $\eta(n) = n^{\gamma}\gamma^{-1}$, for $\gamma > 1$, $\rho > 0$, $\rho \neq 1$, or $u(c) = \ln c$. Equivalently,

$$C(u) = (1-\rho)^{\frac{1}{1-\rho}} u^{\frac{1}{1-\rho}}, \text{ and } N(h) = \gamma^{\frac{1}{\gamma}} h^{\frac{1}{\gamma}}. \tag{12}$$

4.1 The Component Problem

We have seen in the previous section that the planner's program may be separated in a sequence of component programs, one for each generation. The planner's program does not, however, have a natural recursive structure. The dual problem of minimizing the cost of delivering a certain expected utility does. There-

[13]Relaxing independence would lead, among other things, to the need for intervention in the market for state contingent assets, e.g., Scheuer [2013].

[14]In Section 6, we relax this assumption by allowing for Markovian shocks.

[15]However, it increases the notation burden and adds complexity to some of the proofs, e.g. Fernandes and Phelan [2000], da Costa and Luz [2011]. The same need not be true in Section 6, where persistence of the two types of shocks are bound to interact in subtle ways. We leave this interesting issue for another paper.

fore, in what follows we discuss how to go from the utility maximization to the cost minimization problem in our setting.[16]

Let $\mathcal{Q}_t(z^t|z^{t-1})$ be the price in z^{t-1} of one unit of consumption in period t, state z^t. Note that $\mathcal{Q}_t(z^t|z^{t-1})$ may depend on the whole history of aggregate shocks up to period z^{t-1}. When aggregate shocks are i.i.d., it is still possible to write the problem recursively using the distribution of promises, ψ_t, an object to be defined momentarily, as a state.

Our focus, however, is on steady state allocations. Therefore, we shall simply assume that there are functions $Q : Z \to R_+$ and $B : Z \to R_+$ such that, for all z^t, as $t \longrightarrow \infty$, $\mathcal{Q}_t(z^t, z|z^t) = \sum_{\tilde{z}} B(\tilde{z})Q(z)$, for all z. After solving the problem under the assumption that steady-state prices are of the guessed form, we show that prices that support steady state allocations must be of the form we have guessed.

Let us now consider the dual component planner's problem. Under the assumption that $\lim_{t\to\infty} \mathcal{Q}_t(z^t, z|z^t) = \sum_{\tilde{z}} B(\tilde{z})Q(z)$, we define

(13)

$$V(w(\theta^a, z^t)) \equiv \min \sum_j \beta^j \mathcal{Q}_{t+j}(z^{t+j}, z|z^{t+j}) \left\{ c(\theta^{a+j}, \theta, z^{t+j}, z) - n(\theta^{a+j}, \theta, z^{t+j}) z \right\}$$

subject to delivering the promised utility w defined by (5), and to respecting the incentive constraint (2). The omission of z^t as a state is only possible because of our assumption $\lim_{t\to\infty} \mathcal{Q}_t(z^t, z|z^t) = Q(z)$.

The value function, $V(.)$ satisfies the Bellman equation

$$V(w) \equiv \min \sum_\theta \sum_z \mu(\theta) Q(z) \left\{ B(z) V(W(\theta, w, z)) + C(\bar{u}(\theta, w, z)) - z N(\bar{h}(\theta, w)) \right\}$$

subject to

$$\sum_\theta \mu(\theta) \left\{ \sum_z \pi(z) \left[\bar{u}(\theta, w, z) + \beta W(\theta, w, z) \right] - \theta \bar{h}(\theta, w) \right\} = w. \tag{14}$$

[16] An alternative approach for order writing the planner's problem recursively would be to consider the original planner's program and to try to adapt Atkeson and Lucas's [1992] approach to our setting. As we have seen, however, there is a relaxed separable program which solution coincides with the original planner's program solution. It is easiest in this case to proceed as Farhi and Werning [2007] and rely on the results in Atkeson and Lucas's [1992] Section 7, to work with the dual component planning problems for appropriately chosen supporting prices.

and

$$\sum_z \pi(z)\left[\bar{u}(\theta,w,z)+\beta W(\theta,w,z)\right]-\theta\bar{h}(\theta,w)\geq$$

$$\sum_z \pi(z)\left[\bar{u}(\hat{\theta},w,z)+\beta W(\hat{\theta},w,z)\right]-\theta\bar{h}(\hat{\theta},w)\forall\theta,\hat{\theta}. \tag{15}$$

The recursive planner may be understood as follows. At the beginning of each period, each agent is identified by, w, the continuation expected utility she is entitled to. Each component planner chooses policy functions — allocation rules, in the language of Atkeson and Lucas [1992] — which, with some abuse in notation we denote, $\bar{u}(\theta,w,z)$, $\bar{h}(\theta,w)$, and $W(\theta,w,z)$. $\bar{h}(\theta,w)$ maps an announcement of skill θ from an agent who entered the period with a utility promise w into effort $\hat{\theta}\bar{h}(\theta,w)$ that the agent must endure.[17] $\bar{u}(\theta,w,z)$ maps the same announcement utility promise pair to a flow utility $\bar{u}(\theta,w,z)$ from consumption if state z occurs. Finally, $W(\theta,w,z)$ determines the utility promise that the agent will enter next period under the same circumstances.

As in Farhi and Werning [2007], these policy functions — allocation rules in the terminology of Atkeson and Lucas [1992] — maximize $V(w)$, the value function that arises from the sequence problem.

One may go back from policy functions to allocations as follows. Starting from a utility promise at $a=1$, w_0, it is possible to recover the whole allocation through

$$c(\theta^a,z^t)=C(\bar{u}(\theta_a,w(\theta^{a-1},z^{t-1}),z_t)), \tag{16}$$

and

$$n(\theta^a,z^{t-1})=N(\bar{h}(\theta_a,w(\theta^{a-1},z^{t-1}))). \tag{17}$$

The allocations generated by the policy functions $\bar{u}$, $\bar{h}$ and W according to (16) and (17) solve the relaxed component problem. The proof, as well as the proof that the value function (13) solves the Bellman equation follow Farhi and Werning

[17]The planner does not observe the effort the agent makes, but can recover $\bar{h}(\theta,w)$ from effective labor, y. Hence, if an agent of type $\hat{\theta}$ announces θ, the planner believes she is making effort $\theta\bar{h}(\theta,w)$. This, of course, never occurs in the equilibrium of the truthful mechanism.

[2007] closely. We omit both for brevity.

The Distribution of Promises

The key insight that allowed the use of recursive methods for these types of dynamic contracts was that the promised expected utility, $w(\theta^a, z^t)$, encoded all the relevant information about the individual when idiosyncratic shocks are i.i.d. — Green [1987], Spear and Srivastava [1987]. In a partial equilibrium setting this was all that was needed to study optimal allocations. The same is true, here, given the supporting prices $\mathcal{Q}_t(z^t, z|z^t)$. The difference, here is that prices depend on the whole history of aggregate shocks.

With i.i.d. aggregate shocks, however, past shocks do not provide any information about z's future behavior. It is only through the endogenous persistence of allocations that history plays a role. But, as we have seen, all relevant information for allocations is encoded in $w(\theta^a, z^t)$. Therefore, all relevant information of prices is encoded in the distribution of utility promises, ψ_t.

To understand what this distribution is, recall that in period t, following an aggregate history z^t, an individual of age a who has experienced a personal history θ^a, is entitled to an expected utility $w(\theta^a, z^t)$. Naturally another individual of this cohort, would be entitled to a $w(\hat{\theta}^a, z^t)$ if his or her personal history was $\hat{\theta}^a$. Considering all personal histories for the individuals of this cohort we would obtain the distribution of utility promises, $\psi_t^{(a)}$, for the cohort of age a when the economy is t periods old and has lived through history z^t. Adding across cohorts we obtain the economy's period t distribution of promises, ψ_t.

It is important to emphasize that ψ_t is generated from ψ_{t-1} and z_{t-1}, only. Indeed, let us start by noting that the allocation rule $W(\theta, w, z)$ induces a transition, $\Lambda(\psi, z)$, from one period's distribution of promises, ψ, to the next period's distribution, ψ', through

$$\Lambda(\psi, z)(A) = \sum \mu(\theta) \int_{\omega(A)} d\psi,$$

where $\omega(A) = \{(\theta, w) \in W \times \Theta; W(\theta, w, z) \in A\}$.

$\psi_t^{(a)}$, the distribution of utility promises for cohort a, may thus be built using successive applications of $\Lambda(\psi, z)$, $\psi_t^{(a)} = \Lambda(\psi_{t-1}^{(a-1)}, z)$, starting from $\psi_{t-a}^{(0)}$. The equal

treatment assumption implies that $\psi_t^{(0)}$ is degenerate for all t.[18] The economy's complete distribution of utility promises in t is simply $\psi_t(w) = \sum \Delta^a \psi_t^{(a)}(w)$, $\forall w$.[19]

Steady State

Most of the literature on dynamic optimal taxation or social insurance more generally has abstracted from aggregate uncertainty. A steady-state is then defined as an invariant distribution of promises ψ^*.[20] In this section we shall follow this literature ask whether a steady state exists in this sense of the word. We shall later argue that this is too restrictive a concept to be useful in most situations and offer an alternative definition which is more appropriate to the presence case of Markovian aggregate shocks.

For now, let us just clarify the definition we shall use. Let $\Psi_t(w|z^t) = \sum_{\hat{w} \leq w} \psi_t(w)$, where ψ_t is obtained recursively $\psi_s = \Lambda(\psi_{s-1}, z_s)$ along the history z^t. Next, let

$$\Gamma_t(z^t) = \sup_s \left\{ \sup_w \left| \Psi_{t+s}(w|z^{t+s}) - \Psi_t(w|z^t) \right| \right\}. \tag{18}$$

We say that a limiting distribution exists if $\lim_{t\to\infty} |\Gamma_t(z^t) - \Gamma_t(\hat{z}^t)| = 0$ for all z^t, $\hat{z}^t$.

Although our focus is on the steady-state, a brief comment on the transition is due. There are two different sources of transition. First, the passage of time increases the 'number' of alternative histories for each cohort. For any economy with finite life this means that $\psi' \neq \psi$. For any given history, z^∞, this effect vanishes as t increases, since the addition of new 'histories' occurs for a cohort of size $\Delta^t(\sum_{s=1}^t \Delta^s)$. This is true for a given, z^∞, but the planner may choose to vary promises with z. This latter effect need not disappear even in the steady-state.

[18] If at $t = 0$ agents were heterogeneous at birth with regards to utility promises, heterogeneity would also characterize the distribution of promises of each cohort at $a = 1$. We shall work with the special case of a $t = 0$ degenerate distribution. The concavity of the problem leads to a degenerate distribution for each cohort at $a = 1$.

[19] We shall focus on long run allocations, i.e., we consider a measure $\lim_{t\to\infty} \sum_t \Delta^t = (1 - \Delta)^{-1}$ for the population size, and write $\psi(w) = \lim_{t\to\infty} \sum_{a=1}^t \Delta^a \psi_t^{(a)}(w), \forall w$.

[20] See Phelan [1994].

4.2 Component Planners' Allocations

Let us now characterize the efficient allocations associated with the component planner's program. The concavity of $V(.)$, proven in the appendix, implies that the recursive program may be handled with Lagrangian methods.

Associate the multipliers $\lambda(w)$ to constraint (14) and $\gamma(\hat{\theta}, w|\theta)$ to (15). The first order conditions for the component planner's recursive problem are

$$\frac{Q(z)}{\pi(z)} \sum_{\theta} \mu(\theta) C'(\bar{u}(\theta, w, z)) = \lambda(w), \tag{19}$$

with respect to $\bar{u}(\theta, w, z)$,

$$\frac{Q(z)}{\pi(z)} B(z) \mu(\theta) V'(W(\theta, w, z)) = \beta \lambda(w), \tag{20}$$

with respect to $W(\theta, w, z)$, and

$$\sum_{\theta} \mu(\theta) N'(\bar{h}(\theta, w)) \sum_{z} Q(z) z = \lambda(w) \sum_{\theta} \mu(\theta) \theta, \tag{21}$$

with respect to $\bar{h}(\theta, w)$.

Our strategy in all that follows consists in: i) guessing a functional form for the policy functions compatible with the satisfaction of all constraints; ii) using Lemma 3 to establish price formats compatible that support these allocations, and; iii) showing that the first order conditions of the Lagrangian are all satisfied.

Using this procedure, we show in the appendix that there are functions $\tilde{u} : \Theta \times W \longrightarrow \mathbb{R}$ and $\eta : Z \to \mathbb{R}_{+}$ such that $\bar{u}(\theta, w, z) = \tilde{u}(\theta, w)\eta(z)$. We may then, write (19) as

$$\frac{Q(z)}{\pi(z)} \sum_{\theta} \mu(\theta) C'(\tilde{u}(\theta, w)) = \lambda(w) z^{-\rho}.$$

This allows us to prove Proposition 3, below.

Proposition 3. *Arrow security prices are of the form* $Q(z) = q(\psi)\pi(z) z^{-\rho}$.

Using expression (40), in the appendix, the first order condition with respect to

$W(\theta, w, z)$ may be written,

$$B(z)z^{-\rho}\mu(\theta)V'(W(\theta, w, z)) = \beta \sum_{\theta} \mu(\theta)C'(\tilde{u}(\theta, w)). \tag{22}$$

Define $\overline{W}(\theta, w) = \sum \pi(z)W(\theta, w, z)$. A simple examination of the component planners' problem suffices to note that $\overline{W}(\theta, w)$ determines whether (14) and (15) are valid independently of the specific way in which one spread promises across aggregate states, $W(\theta, w, z)$.

A natural guess for the optimal $W(\theta, w, z)$ is $W(\theta, w, z) = \overline{W}(\theta, w)$. For this choice to solve (22) we need $B(z)$ to be of the form bz^{ρ}, for some $b > 0$, exactly what we have assumed to prevail in the steady state: $B(z) = b(\psi) \sum \pi(z')(z/z')^{\rho}$. In proposition 5 we prove that this is exactly the case.

5 Steady-state analysis

In this section we show that with i.i.d. aggregate shocks, $\lim_{t\to\infty} |\Gamma_t(z^t) - \Gamma_t(\hat{z}^t)| = 0$ for all z^t, $\hat{z}^t$, with $\Gamma(.)$ as defined in (18) More precisely, there is a unique invariant distribution of promises, i.e., a distribution ψ such that, for all z, $\psi' = \Lambda(\psi_*, z)$ with $\psi(w) = \psi'(w) \quad \forall w \in D$ — or, equivalently, $\psi_* = \Lambda(\psi_*, z) \, \forall z$. Of course, this implies (18). Moreover, we show that ψ_* is not degenerate.

A well known result for this kind of models in a dynastic setting is the non-existence of a non-degenerated steady state, e.g., Green [1987], Spear and Srivastava [1987], Atkeson and Lucas [1992].[21] In contrast, the perpetual youth structure we are using here guarantees the existence of a non-degenerated steady-state as in Phelan [1994].

We have seen that, for the social preferences we have adopted, and given that shocks are i.i.d, all generations have the same treatment. That is, every cohort is granted the same initial distribution of utility promises, ψ_0, which we have assumed to be concentrated in a single point.

Once again, let $W(\theta, w, z)$ be the utility promise for the agent with former promise

[21] This type of long-run degeneracy may also be averted by the use of less extreme social welfare functions, Farhi and Werning [2007], Phelan [2006], ad hoc restrictions on utility promises, Atkeson and Lucas [1995], Albanesi and Sleet [2006], lack of commitment, **?**, or endogenous demography **?**.

w, skill θ if aggregate shock z happened. Using the no memory result, proved before, we know that the utility promise is only a function of w and θ. That is, we start by recalling that $\psi' = \Lambda(\psi, z) = \Lambda(\psi, \hat{z})\, \forall z, \hat{z}$.

Let ψ_t be the distribution of promises of the economy when the oldest generation is t years old. Since the promises won't depend on z, we have that:

$$\begin{aligned}
\psi_t(\overline{w}) \quad &= \quad \psi_0(\overline{w}) \\
&+ \quad \Delta \int_D \sum_\Theta \mu(\theta_1) \mathbb{I}(W(w, \theta_1) \leq \overline{w}) d\psi_0(w) + \\
&+ \quad \Delta^2 \int_D \sum_\Theta \mu(\theta_1) \sum_\Theta \mu(\theta_2) \mathbb{I}(W(W(w, \theta_1), \theta_2) \leq \overline{w}) d\psi_0(w) \\
+ \cdots &+ \quad \Delta^t \int_D \sum_\Theta \mu(\theta_1) \cdots \sum_\Theta \mu(\theta_t) \mathbb{I}(W(w, \theta^t) \leq \overline{w}) d\psi_0(w)
\end{aligned}$$

Where $W(w, \theta^t) = W(W(W \cdots W(W(w, \theta_1), \theta_2) \cdots, \theta_{t-1})\theta_t)$ is the utility promise of an individual with history length t with initial utility promise w and skill history θ^t.

We now define our steady state distribution of promises by ψ^{ss}. This distribution is simple the limit when The following proposition guarantees the existence of a non-degenerated promises distribution.

Proposition 4. *Let $\psi^{ss}(w) = \lim_{t\to\infty} \psi_t(w) \quad \forall w \in D$. This limit exists and is an invariant distribution of promises for this economy. Moreover, this distribution is non-degenerate.*

5.1 Steady-State Prices and Allocations.

We have seen that for i.i.d. aggregate shocks a unique invariant distribution of promises, $\psi^{ss}(.)$ obtains. As a consequence, steady-state consumption and labor supply only depend on an individual's private history.

Proposition 5. *Steady-state prices are of the form $Q(z) = \pi(z)z^{-\rho}$ and $B(z) = \delta \sum \pi(z')(z'/z)^\rho$.*

Beyond this very simple structure of prices, steady-state allocations are also special in that they are characterized by a cross-sectional restriction on the distribution of consumption. Indeed, we know that in the steady-state $c(\theta^a, z^t) = \tilde{c}(\theta^a) z_t$ and $l(\theta^a, z_{t-1}) = \tilde{l}(\theta^a)$. Moreover, for all a, θ^a, we have

$$\sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1} | \theta^a) c(\theta^{a+1})^\rho = \tilde{c}(\theta^a)^\rho,$$

from (42).

Therefore,

$$\left\{ \sum_{\theta^{a+1}} \mu(\theta^{a+1}) c(\theta^{a+1})^\rho \right\}^{1/\rho} = \sum_{\theta^a} \mu(\theta^a) \tilde{c}(\theta^a),$$

for all a.

An interesting immediate consequence of this expression is Proposition 6 below.

Proposition 6. *In the steady-state, average consumption share is increasing (decreasing) in the lenght of one's history, depending only on whether $\rho > 1$ ($\rho < 1$).*

The specification of the model with i.i.d. shocks allows for very sharp predictions regarding both the short and long run behavior of allocations. In particular, by erasing every period any memory of aggregate uncertainty, the model allows for a long run invariant distribution of expected utility. The drawback is that the model does not generate any action in labor supply.

Allowing for persistence of aggregate shocks potentially recovers cyclical behavior of aggregate labor supply and greatly improves the scope of the model. That is, even though there cannot be contemporary responses of labor supply to the aggregate state by construction, allowing for persistence may lead to lagged responses which we have proved not to be optimal in the case of i.i.d.. Indeed, by making the length of a period sufficiently small, we may approximate arbitrarily well the contemporaneous reactions.

6 Persistent Aggregate Shocks

To allow for persistence while keeping the recursive structure of the problem we shall assume that aggregate shocks, z, follow a first order Markov process which we represent with the transition $\pi(z|z_{-})$.

Recall that in the case of i.i.d. shocks the entire history of aggregate shocks may be summarized by a single state, ψ, the distribution of promises. I.e., there are functions $Q : Z \times \Psi \to R_{+}$ and $B : Z \times \Psi \to R_{+}$ such that, for all t, z^{t-1} and all z, $\sum_{\tilde{z}} B(\tilde{z}, \psi) Q(z, \psi) = \mathcal{Q}_t(z^{t-1}, z|z^{t-1})$.

The fact that prices depend on ψ, means that ψ must be included as one of the states. The reason is immediate for those familiar with the work of Krusell and Smith [1998]. The component planners must forecast future prices, and this requires knowledge of the whole history of aggregate shocks up to that moment. Since the history of aggregate shocks up to period t is summarized in the distribution of utility promises at that period, using ψ_t suffices.

In the case of i.i.d. aggregate shocks, we have shown that steady-state prices are independent of ψ, whereas in the transition prices may depend on the age of the economy, t, but not on its history, z^t. When the aggregate shocks process exhibits persistence, however, there is a direct impact of history on the program since past shocks help forecast future shocks.

One must now include a new state variable, z_{-}, in the component planner's. By doing so we recognize that w is no longer a sufficient statistic for the effects of z_{-} on the choices of efficient allocations.

Taking this explicitly into account requires re-defining the value function through

$$\begin{aligned} V(w, \psi, z_{-}) \equiv \quad & \min \textstyle\sum_{\theta} \sum_{z} \mu(\theta) Q(z) \{ B(z) V(W(\theta, w, z), \psi', z, z_{-}) + \\ & C(\bar{u}(\theta, w, z, z_{-})) - z N(\bar{h}(\theta, w, z_{-})) \} \end{aligned} \tag{23}$$

subject to

$$\sum_{\theta} \mu(\theta) \left\{ \sum_{z} \pi(z|z_{-}) \left[\bar{u}(\theta, w, z, z_{-}) + \beta W(\theta, w, z, z_{-}) \right] - \theta \bar{h}(\theta, w, z_{-}) \right\} = w, \tag{24}$$

$$\sum_z \pi(z|z_-)\left[\bar{u}(\theta, w, z, z_-) + \beta W(\theta, w, z, z_-)\right] - \theta \bar{h}(\theta, w, z_-) \geq$$

$$\sum_z \pi(z|z_-)\left[\bar{u}(\hat{\theta}, w, z, z_-) + \beta W(\hat{\theta}, w, z, z_-)\right] - \theta \bar{h}(\hat{\theta}, w, z_-), \forall \theta, \hat{\theta} \tag{25}$$

and the transition $\psi' = \Lambda(\psi, z)$.[22,23]

Once again, we have omitted the potential dependence of Q and B on both ψ and z_- for brevity. Without persistence, all history of aggregate uncertainty is encoded in the distribution of promises, ψ. Here, however, the cost of delivering expected utility w is directly affected by z_- through $\pi(z|z_-)$.

The role of z_- Let us first discuss the sense in which z_- must be explicitly taken into account in the allocation rules.

Let us consider two aggregate states z and $\hat{z}$ such that $\pi(z'|z) \neq \pi(z'|\hat{z})$ for some z', $z \neq z'$, and such that $\pi(.|\hat{z})$ first order stochastically dominates $\pi(.|z)$. For further reference, note that this implies $\sum \pi(z'|\hat{z}) z'^{1-\rho} > \sum \pi(z|z_-) z'^{1-\rho}$.

It must, then, be the case that there are w and θ for which $W(w, \theta, z, z_-) \neq W(w, \theta, \hat{z}, z_-)$. Suppose not. And consider that the incentive-feasible allocation which is our candidate optimum has the property that $W(w, \theta, z, z_-) = W(w, \theta, \hat{z}, z_-)$ for all (w, θ).

From Lemma 2, we know that the flow utility that an agent who enters the period with promise w and realizes θ obtains is

$$-\theta h(w, \theta, z_-) + \tilde{u}(w, \theta, z_-) \sum \pi(z|z_-) z^{1-\rho}, \tag{26}$$

if the previous period aggregate shock was z_-. $\hat{z}_-$ substitutes for z_- in (26) when the aggregate shock is $\hat{z}_-$, instead. Now, replace the utility flow in the node follow-

[22] The fact that the component planner is recursive in w is well known for this class of problems with i.i.d. shocks at least since Green [1987] and Spear and Srivastava [1987].

[23] In Krusell and Smith [1998], it is the distribution of wealth that matters. The underlying idea is the same, here: to form expectations about future prices one needs to know the whole distribution of promises. Current prices are also important, but, again knowledge of the current distribution of promises suffice.

ing $\hat{z}_-$ by the following one. For all (w, θ), let $h(\theta, w, z_-) = h(\theta, w, \hat{z}_-)$, $\tilde{u}(\theta, w, z_-) = \tilde{u}(\theta, w, \hat{z}_-)$. For all agents, expected utility is greater at this new allocation than at the candidate optimum. The problem is that this allocation is not incentive compatible. Since, following Lemma 2 the difference in expected utility from consumption in this period becomes

$$\tilde{u}(\theta, w, z_-) \sum \{\pi(z|\hat{z}_-) - \pi(z|z_-)\} z^{1-\rho},$$

which is a (non-trivial) linear function of $\tilde{u}(\theta, w, z_-)$ whenever $\sum \{\pi(z|\hat{z}_-) - \pi(z|z_-)\} z^{1-\rho} \neq 0$.

The next step is to redistribute consumption across agents in such a way that for the new utility flows from consumption, $\dot{u}(\theta, w, z, \hat{z}_-)$, we have $\sum \pi(z|\hat{z}_-)\dot{u}(\theta, w, z, \hat{z}_-) - \tilde{u}(\theta, w, z_-) \sum \pi(z|z_-)z^{1-\rho} = \kappa > 0$.

It is, therefore, possible to find an alternative incentive-feasible allocation that generates a higher expected utility for all agents following the aggregate state $\hat{z}_-$, then following state z_-. Of course, the efficient allocation must also generate different expected utilities following the two aggregate shocks. This shows how w, henceψ, must depend on aggregate shocks. This is the indirect role of z_-that arises through w.

Next, by inspection of (26) one readily sees that from a temporary utility perspective a higher z_- is isomorphic to a higher aggregate productivity in a world where announcements are made after one knowing the aggregate state.[24] Hence, with regards to effort, the traditional ambiguity induced by income and substitution effects moving in opposite directions is present here. If shocks are persistent, then following a good aggregate state one expects to obtain a better outcome per unit of efficiency labor. Leisure becomes, therefore, more expensive. On the other hand, the whole society is richer since it expects to generate more consumption even if labor supply is not adjusted. The parameter ρ determines whether it is the income or the substitution effect that prevails.

Finally, Lemma 2 shows that $u(\theta, w, z_-, z) = \tilde{u}(\theta, w, z_-)z^{1-\rho}$. Hence, for $\rho \neq 1$, holding $h(w, \theta, z_-)$ and $\tilde{u}(w, \theta, z_-)$ invariant across z_- does not preserve incentive

[24]As discussed in the introduction, the analogy with da Costa and Luz [2011] is never perfect. Although we are able to reduce the flow utility to a similar structure, the optimal level of promised utility will naturally depend on current shock, z, through the additional channel, $\pi(.|z)$.

compatibility across z_- unless $\sum \pi(z|z_-)W(w,\theta,z,z_-)$ is a function of z_-. Note that this is beyond the indirect impact of z_- that goes on through w. Moreover, in this latter case, promises depend not only (potentially) on the current aggregate state, but on past aggregate states. This opens the possibility that allocations may depend on more lags of aggregate history.

To further characterize efficient allocations let us examine the first order condition with respect to $\bar{u}(\theta,w,z,z_-)$,

$$\mu(\theta)Q(z)C'(\bar{u}(\theta,w,z,z_-))-\left\{\lambda(w,\psi,z_-)\mu(\theta)-\sum_{\hat{\theta}}\left[\gamma(\theta|\hat{\theta})-\gamma(\hat{\theta}|\theta)\right]\right\}\pi(z|z_-)=0.$$

Adding over θ, we have

$$\frac{Q(z|z_-)}{\pi(z|z_-)}z^{\rho}=\lambda(w,\psi,z_-)\left\{\sum_{\theta}\mu(\theta)C'(\tilde{u}(\theta,w,z_-))\right\}^{-1}. \tag{27}$$

Note that the right hand side of (27) does not depend on z, so neither does the left hand side.

Beyond that, not much can be said about (27) without the imposition of further restrictions on preferences or the nature of the stochastic process of z. Indeed, a consequence of Lemma **??** is that w and, consequently, ψ vary with z_-, which makes equation (27) very non-informative.

Moreover, once persistence of aggregate shocks is introduced it becomes harder for a stationary distribution of promises, as defined in section 5, to be obtained. Since, with the exception of Phelan [1994] the literature is yet to deal with this case, we shall try to find a sensible definition of steady state for our economy. To motivate our definition, we consider the special case of $\ln$ preferences for consumption, in which the dependence of utility promises on the whole history of aggregate shocks can be made explicit.

6.1 Log Preferences

Let us now focus on the specific case of $\rho=1$, i.e., we assume that preferences for consumption are of type $u(c)=\ln c$. We also let Π be an $n\times n$ transition matrix

for the process, i.e., the matrix created by placing $\pi(.|z)$ one in the top of the other.

What we are able to formally show in the appendix is that allocation rules are of the form $h(\theta, w, z_-) = \tilde{h}(\theta, w)$, $u(\theta, w, z_-, z) = \tilde{u}(\theta, w)z$, and $W(\theta, w, z_-, z) = \tilde{W}(\theta, w, z)$. The dependence of $\tilde{W}(\theta, w, z)$ on z implies that $\Lambda(\psi, z) \neq \Lambda(\psi, z')$ for$z \neq z'$.

A consequence of this finding is that there is no invariant distribution,

$$\sup_w \left|\Psi_t(w|z^t) - \Psi_{t-1}(w|z^{t-1})\right| > \epsilon$$

for some $\epsilon > 0$, whenever $z_t \neq z_{t-1}$.

However, provided that there is a stationary distribution, $\pi_*(.)$ associated with the transition, $\pi(z'|z)$, there will be a distribution of distributions in the sense used by Phelan [1994]. This is the idea we now formalize.

Definition 2. A steady state is a vector, Ψ^* of distributions of promises $\Psi = \{\psi(z)\}_z$, such that, for all $\tilde{z}, z \in Z$

$$\psi^*(\tilde{z}) = \Lambda(\psi^*(z), \tilde{z})$$

Then, it is immediate to see that if, for each $z \in Z$, we define $\lim_{t\to\infty} \psi_t(z^{t-1}, z) = \psi^*(z)$, we have that $\Psi^* = \{\psi^*(z)\}_{z\in Z}$ is a steady state in the $\ln$ case.

What we have seen so far is that promises may depend on the whole history of aggregate shocks, since $w(\theta^a, z^t) = W(\theta^{a-1}, z^{t-1}, \theta_a, z_t)$. For a steady state to obtain even as we allow for a broader definition as the one used in Section 6.1, we had to restrict ourselves to the case of $\ln$ preferences for consumption. With $\ln$ utility, however, labor supply does not vary with changes in productivity. This means that in all cases for which we were able to define a steady state, labor supply was invariant to the aggregate state. One may, then, guess that for a steady state, not alone an invariant distribution, to exists, labor supply responses must be muted.

The next example shows that this is not the case. We are able to find a stochastic process for which labor supply responds to (lagged) aggregate shocks and the long run of the economy is characterized by an invariant distribution of promises.

6.2 A very particular stochastic process

Define the variable

$$\varphi(z_-) = \left[\sum_z \pi(z|z_-) z^{1-\rho}\right]^{\frac{\gamma}{\gamma+\rho-1}}$$

and assume that φ is a martingale with $\sup_t E[|\varphi(z^t)|] < \infty$.

Under these assumptions about the stochastic process governing z, consider the following guess for the functional form of efficient allocation rules: $w = \tilde{w}\varphi(z_-)$, $\psi = \tilde{\psi}\varphi(z_-)$, $u(\theta, w, z) = \tilde{u}(\theta, \tilde{w})\varphi(z_-)^{\frac{1-\rho}{\gamma}} z^{1-\rho}$, $h(\theta, w) = \tilde{h}(\theta, \tilde{w})\varphi(z_-)$, and $W(\theta, w, z) = \tilde{W}(\theta, \tilde{w})\varphi(z)$.

Rewriting the recursive problem under these assumptions, one obtains the new promise keeping constraint,

$$\sum_\theta \mu(\theta) \left[\tilde{u}(\theta, \tilde{w}) + \beta \tilde{W}(\theta, \tilde{w}) - \theta \tilde{h}(\theta, \tilde{w})\right] = \tilde{w}, \tag{28}$$

and the new incentive compatibility constraint

$$\tilde{u}(\theta, \tilde{w}) + \beta \tilde{W}(\theta, \tilde{w}) - \theta \tilde{h}(\theta, \tilde{w}) \geq \tilde{u}(\hat{\theta}, \tilde{w}) + \beta \tilde{W}(\hat{\theta}, \tilde{w}) - \theta \tilde{h}(\hat{\theta}, \tilde{w}) \quad \forall \theta, \hat{\theta} \in \Theta \tag{29}$$

Neither the promise keeping nor the incentive compatibility constraint depends on z or z_-, whereas the objective function, becomes

(30)

$$\begin{array}{rcl} V(\varphi(z_-)\tilde{w}, \psi, z_-) & = & \sum_\theta \mu(\theta) \sum_z Q(z) B(z) V(\varphi(z)\tilde{W}(\theta, \tilde{w}), \psi', z) \\ & + & \sum_\theta \mu(\theta) \left[C(\tilde{u}(\theta, \tilde{w})) - N(\tilde{h}(\theta, \tilde{w}))\right] \varphi(z_-)^{\frac{1}{\gamma}} \sum_z Q(z) z. \end{array}$$

From lemmas 1 and 2 it is apparent that prices must be of the form $Q(z) = \pi(z|z_-) z^{-\rho} \varphi(z_-)^{\frac{-\rho}{\gamma}}$ and $B(z) = \delta z^\rho \varphi(z_-)^{\frac{\rho}{\gamma}}$ to support the efficient allocations.

The objective function, thus, becomes

$$\begin{array}{rcl} V(\varphi(z_-)\tilde{w}, \psi, z_-) & = & \delta \sum_\theta \mu(\theta) \sum_z \pi(z|z_-) V(\varphi(z)\tilde{W}(\theta, \tilde{w}), \psi', z) \\ & + & \sum_\theta \mu(\theta) \left[C(\tilde{u}(\theta, \tilde{w})) - N(\tilde{h}(\theta, \tilde{w}))\right] \varphi(z_-). \end{array} \tag{31}$$

Using all the findings above it is immediate to verify that the value function is of the form $V(w, \psi, z_-) = \varphi(z_-)\tilde{V}(\tilde{w}, \tilde{\psi})$, which allows us to finally write the planner's objective as

$$V(\tilde{w}, \tilde{\psi}) = \delta \sum_\theta \mu(\theta) V(\tilde{W}(\theta, \tilde{w}), \tilde{\psi}) + \sum_\theta \mu(\theta) \left[C(\tilde{u}(\theta, \tilde{w})) - N(\tilde{h}(\theta, \tilde{w})) \right]. \tag{32}$$

It is apparent that this equation does not depend on z or z_- in any way. Indeed, the first order conditions,

$$\sum_\theta \mu(\theta) C'(\tilde{u}(\theta, \tilde{w})) = \lambda(\tilde{w})$$

with respect to $u(\theta, w, z)$, and

$$\sum_\theta \mu(\theta) N'(\tilde{h}(\theta, \tilde{w})) = \lambda(\tilde{w})\mathbb{E}(\theta)$$

with respect to $h(\theta, w)$, are consistent with our choices.

We have assumed that $\varphi(z_-)$ is a martingale, with $\sup_t \mathbb{E}\left[|\varphi(z_t)|\right] < \infty$. By Doob's martingale convergence theorem,[25] $\varphi(z_-)$ converges to a random variable, φ^* almost surely. There will be, then, a long run invariant distribution of promises $\psi^*(w) = \varphi^* \tilde{\psi}(w)$.

In other words along almost all histories, z^∞,

$$\lim_{t \to \infty} \left\{ \sup_s \left\{ \sup_w \left| \Psi_{t+s}(w | z^{t+s}) - \Psi_t(w | z^t) \right| \right\} \right\} = 0.$$

That is, even though the allocation displays memory, an invariant distribution of promises exists. Unlikely the i.i.d. case, however, this existence of steady state does *not* mean that the steady state allocation is independent of history. If we identify a society with its history, what this result shows is that societies may end up with very different stationary distributions of income despite the fact that they start with the same expected long run distribution. That is, the fate of a society is decided by the history of aggregate shocks z^∞.

[25] See, e.g., Davidson [1994].

7 Conclusion

In this paper we have examined the long run properties of efficient allocations for a dynamic Mirrlees's [1971] economy exposed to aggregate risks. Our focus was on assessing whether an invariant non-degenerate distribution of utility entitlements or promieses, the relevant individual state variable for this type of environment, exists in general.

The key assumption that we used to avoid degeneracy was to endow the planner with only an imperfect record keeping technology. Alhough we have used a somewhat artificial notion of imperfection, this highlights the fact that if we deviate from the extreme assumptions about the planner's capacity to retain information about everyone's individual history, the generacy results disappear.[26] This paper combines elements of two other works that have addressed optimal policy in a setting where idiosyncratic private information and aggregate public information shocks are both present: Phelan [1994] and da Costa and Luz [2011].

An invariant distribution obtains when aggregate shocks are i.i.d., or for very special stochastic processess. As we add persistence to the aggregate shocks our model behaves ever more similar to da Costa and Luz [2011], where memory of aggregate shocks is a generic property of the model. This will in general, but not always, lead to the non-existence of an invariant distribution of utility promises.

References

S. Rao Aiyagari, Albert Marcet, Thomas J. Sargent, and Juha Seppala. Optimal taxation without state-contingent debt. *Journal of Political Economy*, 110(6):1220–1254, 2002. 1

Stefania Albanesi and Christopher Sleet. Dynamic optimal taxation with private information. *Review of Economic Studies*, 73(1):1–30, 2006. 21

Andrew Atkeson and Robert E. Lucas. On efficient distribution with private information. *Review of Economic Studies*, 59:427–453, 1992. 3, 16, 4.1, 5

[26]We claim no originality with this regard. As we have already explained, our economy is isomorphic to Phelan's [1994]. Beyond perfect record keeping, the extreme long run results found in the literature also require commitment by the planner — see Sleet and Yeltekin [2006], **?**.

Andrew Atkeson and Robert E. Lucas. Efficient and equality in a simple model of unemployment insurance. *Journal of Economic Theory*, 66:64–88, 1995. 21

Robert Barro. On the determination of the public debt. *Journal of Political Economy*, 87(5):940–971, 1979. 1

Varadarajan V. Chari, Larry J. Christiano, and Patrick J. Kehoe. Optimal fiscal policy in a business cycle model. *Journal of Political Economy*, 102(4):617–652, 1994. 1

Carlos E. da Costa and Vitor F. Luz. The private memory of aggregate uncertainty. mimeo. Fundação Getulio Vargas, August 2011. 1, 4, 5, 1, 12, 15, 24, 7

James Davidson. *Stochastic Limit Theory: An Introduction for Econometricicans*. Oxford University Press, 1994. 25

Gabrielle Demange. Sharing aggregate risk under moral hazard. Technical Report 2008-27, Paris School of Economics, 2008. 1, 1

Emmanuel Farhi and Iván Werning. Social discounting and progressive estate taxation. *Journal of Political Economy*, 115:365–402, 2007. 1, 8, 16, 4.1, 4.1, 21

Emmanuel Farhi and Iván Werning. Insurance and taxation over the life cycle. *Review of Economic Studies*, 2013. forthcoming. 1, 1

Ana Fernandes and Christopher Phelan. A recursive formulation for repeated agency with history dependence. *Journal of Economic Theory*, 91:223–247, 2000. 15

Mikhail Golosov, Narayana Kocherlakota, and Aleh Tsyvinski. Optimal indirect and capital taxation. *Review of Economic Studies*, 70(3):569–587, 2003. 1, 3.2

Mikhail Golosov, Maxim Troshkin, and Aleh Tsyvinski. Optimal dynamic taxes. Working Paper 17642, NBER, 2011. Mimeo. 1, 1

Edward Green. Contractual arrangements for intertemporal trade. In Edward Prescott and Neil Wallace, editors, *Lending and the Smoothing of Uninsurable Income*. Minneapolis: University of Minnesota Press, 1987. 3, 4.1, 5, 22

Bengt Holmström. Moral hazard and observability. *Bell Journal of Economics*, 10(1): 79–91, 1979. 1

Marek Kapička. Efficient allocations in dynamic private information economies with persistent shocks: A first order approach. mimeo. University of Santa Barbara, 2011. 1

Narayana Kocherlakota. Zero expected wealth taxes: A mirrlees approach to dy-

namic optimal taxation. *Econometrica*, 73:1587–1621, 2005. 2, 3.1

Per Krusell and Tom Smith. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998. 6, 23

Robert E. Jr. Lucas and Nancy L. Stokey. Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics*, 12:55–93, 1983. 1

James A. Mirrlees. An exploration in the theory of optimal income taxation. *Review of Economic Studies*, 38:175–208, 1971. (document), 7

Alessandro Pavan, Ilya Segal, and Juuso Toikka. Dynamics mechanism design. mimeo. Northwestern University, 2012. 1

Christopher Phelan. Incentives and aggregate shocks. *The Review of Economic Studies*, 61:681–700, 1994. 1, 2, 4, 1, 3.2, 20, 5, 6, 6.1, 7, 26

Cristopher Phelan. On the long-run implications of repeated moral hazard. *Journal of Economic Theory*, 79:174–191, 1998. 3

Cristopher Phelan. Opportunity and social mobility. *Review of Economic Studies*, 73 (0):487–504, 2006. 1, 8, 21

William P. Rogerson. Repeated moral hazard. *Econometrica*, 53(1):69–76, 1985. 3.2

Florian Scheuer. Pareto-optimal taxation with aggregate uncertainty and financial markets. *Review of Economic Dynamics*, 2013. forthcoming. 1, 1, 3.2, 3.2, 13

Christopher Sleet and Sevin Yeltekin. Credibility and endogenous societal discounting. *Review of Economic Dynamics*, 9:410–437, 2006. 26

Stephen E. Spear and Sanjay Srivastava. On repeated moral hazard with discounting. *Review of Economic Studies*, 54:599–617, 1987. 3, 4.1, 5, 22

Nancy Stokey, Robert Lucas, and Edward Prescott. *Recursive methods in economic dynamics*. Harvard University Press, 1989. A

Iván Werning. Optimal fiscal policy with redistribution. *Quarterly Journal of Economics*, 122:925–967, 2007. 2, 1, 7

A Appendix

Lemma 4. *The value function $V(., \psi)$ is strictly convex in w for all ψ.*

Proof. The result follows from theorem 9.8 of Stokey et al. [1989] $\square$

Lemma 5. *An allocation induced by a policy function* $(\bar{u}(\theta, w, z), \bar{h}(\theta, w), W(\theta, w, z))$ *is incentive compatible if and only if (15) holds for all* (θ, w).

Proof. Necessity of (15) is immediate, for if (15) is violated for some (θ, w), the strategy of telling the truth at all possible nodes with the exception of the one for which a realization θ follows a sequence of shocks that induces expected utility w dominates truth-telling. $\square$

Proof. of Lemma 1 Consider the perturbations of the optimal contract, $\hat{u}_B(\theta^a, z^t) = u(\theta^a, z^t) + B$ and $\hat{u}_B(\theta^{a+j}, z^{t+j}) = u(\theta^a, z^t) - \beta^{-j}B$. For small enough B, this reform increases costs in period t by

$$\mu(\theta^a)\Delta^a C'(u(\theta^a, z^t))B \tag{33}$$

and in $t + j$ by

$$-\sum_{\theta^{a+j} \succ \theta^a} \mu(\theta^{a+j})\Delta^{a+j}\beta^{-j}C'(u(\theta^{a+j}, z^{t+j}))B. \tag{34}$$

Consider a similar reform, for an individual of history length $\hat{a}$ now varying his or her utility by $-A$ in period t and $\beta^{-j}A$ in period $t + j$. Costs are, in this case,

$$-\mu(\theta^{\hat{a}})\Delta^{\hat{a}}C'(u(\theta^{\hat{a}}, z^t))A \tag{35}$$

and reduces costs in $t + j$ by

$$\sum_{\theta^{\hat{a}+j} \succ \theta^{\hat{a}}} \mu(\theta^{\hat{a}+j})\Delta^{\hat{a}+j}\beta^{-j}C'(u(\theta^{\hat{a}+j}, z^{t+j}))A. \tag{36}$$

respectively. Neither reform affects equilibrium utilities or incentives. Choose

$$A = -\frac{\mu(\theta^{\hat{a}})\Delta^{\hat{a}}C'(u(\theta^{\hat{a}}, z^t))}{\mu(\theta^a)\Delta^a C'(u(\theta^a, z^t))}B.$$

Then the reform is feasible in t. The total cost in $t+j$ will be

$$\sum_{\theta^{\hat{a}+j}\succ\theta^{\hat{a}}} \mu(\theta^{\hat{a}+j})\Delta^{\hat{a}+j}\beta^{-j}C'(u(\theta^{\hat{a}+j},z^{t+j}))A+$$
$$\sum_{\theta^{a+j}\succ\theta^{a}} \mu(\theta^{a+j})\Delta^{a+j}\beta^{-j}C'(u(\theta^{a+j},z^{t+j}))B,$$

which, using (), is

$$\sum_{\theta^{\hat{a}+j}\succ\theta^{\hat{a}}} \mu(\theta^{\hat{a}+j})\Delta^{\hat{a}+j}\beta^{-j}C'(u(\theta^{\hat{a}+j},z^{t+j}))A-$$
$$\sum_{\theta^{a+j}\succ\theta^{a}} \mu(\theta^{a+j})\Delta^{a+j}\beta^{-j}C'(u(\theta^{a+j},z^{t+j}))\frac{\mu(\theta^{\hat{a}})\Delta^{\hat{a}}C'(u(\theta^{\hat{a}},z^{t}))}{\mu(\theta^{a})\Delta^{a}C'(u(\theta^{a},z^{t}))}A.$$

Because the sign of A is arbitrary, for there not to be any reform that leads to an allocation that delivers the same expected utility at lower costs we need, therefore,

$$\sum_{\theta^{\hat{a}+j}\succ\theta^{\hat{a}}} \mu(\theta^{\hat{a}+j}|\theta^{\hat{a}})\frac{C'(u(\theta^{\hat{a}+j},z^{t+j}))}{C'(u(\theta^{\hat{a}},z^{t}))} = \sum_{\theta^{a+j}\succ\theta^{a}} \mu(\theta^{a+j}|\theta^{a})\frac{C'(u(\theta^{a+j},z^{t+j}))}{C'(u(\theta^{a},z^{t}))},$$

for all t, j, a, $\hat{a}$, θ^a, $\theta^{\hat{a}}$, z^t, z^{t+j}. □

Proof. of Lemma 2 Associate $\gamma(\hat{\theta}^a|\theta^a)$ and $\lambda_a(z^{t+a})$ with the restrictions ref (???). The first order condition with respect to $c(\theta^a,z^{t+a})$ of the first problem implies:

$$\beta^a\pi(z^{t+a}|z^t)\left\{\sum_{\hat{\theta}^a}\left[\gamma\left(\hat{\theta}^a|\theta^a\right)-\gamma\left(\theta^a|\hat{\theta}^a\right)\right]\right\}=\lambda_a(z^{t+a})C'(u(\theta^a,z^{t+a}))$$

Therefore, for two different histories z^{t+a} and $\hat{z}^{t+a}$:

$$\frac{C'(u(\theta^a,\hat{z}^{t+a}))}{C'(u(\theta^a,z^{t+a}))}=\frac{\pi(\hat{z}^{t+a}|z^t)}{\pi(z^{t+a}|z^t)}\frac{\lambda_a(z^{t+a})}{\lambda_a(\hat{z}^{t+a})}$$

Now, doing the same for agent of cohort $a-j$, we have:

$$\frac{C'(u(\theta^{a-j},\hat{z}^{t+a}))}{C'(u(\theta^{a-j},z^{t+a}))}=\frac{\pi(\hat{z}^{t+a}|z^t)}{\pi(z^{t+a}|z^t)}\frac{\lambda_{a-j}(z^{t+a})}{\lambda_{a-j}(\hat{z}^{t+a})}$$

Let's analyze the second problem. Associate $\alpha(z^t)$ with restriction ref (???). The first order condition with respect to $\varsigma_a(z^{t+a})$ to the second problem implies:

$$\hat{\delta}^t\frac{\partial\mathcal{U}_t(\varsigma^*)}{\partial\varsigma_a(z^{t+a})}=\alpha(z^{t+a})\Delta^a$$

Using the envelope theorem in the first problem, we have that:

$$\frac{\partial\mathcal{U}_t(\varsigma^*)}{\partial\varsigma_a(z^{t+a})}=\lambda_a(z^{t+a})$$

Combining the two above equations and taking two different histories z^{t+a} and $\hat{z}^{t+a}$, both continuations of z^t, we finally have:

$$\frac{\lambda_a(z^{t+a})}{\lambda_a(\hat{z}^{t+a})}=\frac{\lambda_{a-j}(z^{t+a})}{\lambda_{a-j}(\hat{z}^{t+a})},\quad\forall j$$

which yields the result. □

Proof. of Lemma 3 Consider the perturbations of the optimal contract:

$$C(\hat{u}(\theta^a,z^t))=C(u(\theta^a,z^t))-\varepsilon C'(u(\theta^a,z^t)),\quad\forall\theta^a$$

and

$$C(\hat{u}(\theta^{a+1},z^{t+1}))=C(u(\theta^{a+1},z^{t+1}))+\beta^{-1}\varepsilon C'(u(\theta^{a+1},z^{t+1})),\quad\forall\theta^{a+1}\succ\theta^a,\ z^{t+1}\succ z^t$$

Note that the expected utility remains unchanged for ε small. Therefore, the variation in resources must be zero, or else, taking the ε in the appropriate sign would lead to a contradiction. Note that the planner can save by the market price

$b(z^t)$, therefore, the resource variation is:

$$\varepsilon[C'(u(\theta^a, z^t)) - b(z^t)\frac{\Delta}{\beta} \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) C'(u(\theta^{a+1}, z^{t+1}))]$$

Since ε can be either positive or negative, we must have:

$$\frac{b(z^t)}{\delta} = \frac{C'(u(\theta^a, z^t))}{\sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) C'(u(\theta^{a+1}, z^{t+1}))}$$

The expected result. □

Proof. of Proposition 1 We show that, if the planner offers the contract associated with the second best solution, there is no incentive to trade and the endogenous incentive constraint (9) is trivially satisfied.

First note that the Euler equation associated with the state-contingent portfolio problem (8) is

$$q(z^t) = \frac{\pi(z^t|z^{t-1})}{\sum_{\hat{z}^t \succ z^{t-1}} \pi(\hat{z}^t|z^{t-1}) \frac{C'(\tilde{u}(\theta^a, z^t))}{C'(\tilde{u}(\theta^a, \hat{z}^t))}} \tag{37}$$

Lemma 2 implies that in a second best ($\tilde{u} = u$), the right hand side is equal to all agents. This implies that the planner can just offer the second-best contract and there will be no incentives to trades the asset, that is, $A(z^t) = 0$ for all histories z^t.

The endogenous incentive constraint (9) is trivially satisfied, since the no trade implies the equilibrium utility level is the same as the second best. □

Proof. of Proposition 2 The proof is analogous to . Note that, it the agent is allowed to participate in the bond market, her Euler equation associated is

$$b(z^t) C'(\tilde{u}(\theta^a, z^t))^{-1} = \beta \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) C'(\tilde{u}(\theta^{a+1}, z^{t+1}))^{-1}. \tag{38}$$

Using Jensen's inequality in the inverse Euler equation , we have:

$$\sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) C'(\tilde{u}(\theta^{a+1}, z^{t+1}))^{-1} > \left[\sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) C'(\tilde{u}(\theta^{a+1}, z^{t+1})) \right]^{-1}$$

Combining this with the relation found in lemma 3 and adding in z^{t+1}, yields:

$$b(z^t) C'(\tilde{u}(\theta^a, z^t))^{-1} < \beta \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) C'(\tilde{u}(\theta^{a+1}, z^{t+1}))^{-1}$$

A contradiction with the Euler equation derived above.

$\square$

Proof. of Proposition 3 We have already proved in lemma 2 that the marginal rate of substitution between consumption in different states of the world is equalized across agents. In the case of iso-elastic preferences, this implies

$$\frac{u(\theta, z, w)^{-\rho}}{u(\theta, \hat{z}, w)^{-\rho}} = \frac{u(\hat{\theta}, z, \hat{w})^{-\rho}}{u(\hat{\theta}, \hat{z}, \hat{w})^{-\rho}}, \ \forall \theta, \hat{\theta}, w, \hat{w}, z, \hat{z}. \tag{39}$$

Hence $u(\theta, z, w)/u(\theta, \hat{z}, w)$ is equalized across agents. Therefore, $u(\theta, z, w) = \tilde{u}(\theta, w)\varrho(z)$, $\forall \theta, w, z$. Using the feasibility condition (4) we can conclude that $\varrho(z)^{1/(1-\rho)} = z$, that is $\varrho(z) = z^{1-\rho}$ and finally $C(u(\theta, w, z)) = C(\tilde{u}(\theta, w))z$. Under this assumptions, the first order condition (19) becomes

$$\frac{Q(z)}{\pi(z)} \sum_{\theta} \mu(\theta) C'(\tilde{u}(\theta, w)) = \lambda_{\psi}(w) z^{-\rho}. \tag{40}$$

Thus, (40) leads to $Q(z)/\pi(z) = q(\psi) z^{-\rho}$. $\square$

Proof. of Proposition 4 Note that, for $n, m \in \mathbb{N}$ (without loss in generality, suppose

$n > m$):

$$
\begin{aligned}
\sup_{w\in D}|\psi_n(w) - \psi_{m-1}(w)| &= \sup_{w\in D}\sum_{j=m}^{n}\int_D \Delta^j \left[\prod_{k=1}^{j}\sum_{\Theta}\mu(\theta_k)\mathbb{I}(W(w,\theta^j) \le \overline{w})\right] d\psi_0(w) \\
&\le \sum_{j=m}^{n}\int_D \Delta^j \left[\prod_{k=1}^{j}\sum_{\Theta}\mu(\theta_k)1\right] d\psi_0(w) \\
&\le \sum_{j=m}^{n}\Delta^j \int_D d\psi_0(w) \\
&\le \frac{\Delta^m - \Delta^n}{1-\Delta} \\
&\le \frac{\Delta^m(1-\Delta^{n-m})}{1-\Delta}
\end{aligned}
$$

(41)

Take $\epsilon > 0$, since $n > m$, $(1 - \Delta^{n-m}) \in (0,1)$ and $\Delta^m \to 0$, when $m \to \infty$. Take $m \ge N \in \mathbb{N}$ such that $\Delta^N < \epsilon(1-\Delta)$, but this implies that

$$\frac{\Delta^m(1-\Delta^{n-m})}{1-\Delta} < \epsilon, \quad \forall n, m \in \mathbb{N}$$

This sequence is, hence, Cauchy. Therefore, it converges. Let $\psi^{ss}(w) = \lim_{t\to\infty}\psi_t(w) \quad \forall w \in D$ be the limit distribution.

$$\psi^{ss}(\overline{w}) = \psi_0(\overline{w}) + \sum_{j=1}^{\infty}\Delta^j \int_D \left[\prod_{k=1}^{j}\sum_{\Theta}\mu(\theta_k)\mathbb{I}(W(w,\theta^j) \le \overline{w})\right] d\psi_0(w)$$

The fact that this distribution is invariant is straightforward, since there is the same treatment over generations and the law of large numbers hold. The distribution is non-degenerate because even if the generation promises are converging to zero with time, there are always younger generations with positive promises. □

Proof. of Proposition 5. To prove the form of $Q(z)$, note in appendix (**??**) that $q(\psi)$ won't change the solution of the problem directly. The main problem was the interrelation between ψ and ψ', that could affect prices. Since now $\psi^{ss} = \Lambda(\psi^{ss}, z)$,

this is not a problem anymore and we can normalize $q(\psi^*) = 1$. To show that $B(z) = b(\psi') \sum \pi(z')(z/z')^\rho$, note that, using (6) any price to support the efficient allocation must respect

$$\sum \pi(z') \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}|\theta^a) \frac{\tilde{c}(\theta^a, w)^{-\rho}}{\tilde{c}(\theta^{a+1}, w)^{-\rho}} \left[\frac{z'}{z}\right]^\rho = B(z), \tag{42}$$

which is the format we have guessed.

To show this, let $C_a(z^t)$, be the total amount of resources available for the cohort aged a in period t, state z^t, and assume that $b(\psi) \neq \delta$. Now consider, at the steady state, a reform whereby one transfers resources from $\bar{C}_a(z^t)$ to $\bar{C}_{a+1}(z^{t+1})$ for all z^{t+1}.

$$C(\hat{u}(\theta^a, z^t)) = C(u(\theta^a, z^t)) - \varepsilon C'(u(\theta^a, z^t))$$

and

$$C(\hat{u}(\theta^a, z^{t+1})) = C(u(\theta^a, z^{t+1})) + \beta^{-1} \varepsilon C'(u(\theta^a, z^{t+1}))$$

This reform may be made in an incentive compatible way and don't affect the welfare. The resources constraint is changed in for each occurrence of $z \in Z$

$$\varepsilon \left[\sum_{\theta^a} \mu(\theta^a) C'(u(\theta^a, z^t)) + \frac{\Delta}{\beta} \sum_{\theta^{a+1} \succ \theta^a} \mu(\theta^{a+1}) C'(u(\theta^{a+1}, z^{t+1}))\right]$$

Finally, using lemma 1, we know that

$$B(z) = \sum_{\hat{z}} \pi(\hat{z}) \frac{C'(\tilde{u}(\theta^a))}{\sum_\theta \mu(\theta) C'(\tilde{u}(\theta^{a+1}))} \left(\frac{\hat{z}}{z}\right)^\rho$$

Using the two equations above, we have that $b(\psi*) = \delta$ □

A.1 Characterizing Efficient Allocations: i.i.d. case.

We use our guess for prices in Section 4 to re-write the component planners' program. First, however, note that $W(\theta, w, z) = \bar{W}(\theta, w)$ for all z implies $\psi' =$

$\Lambda(\psi, z) = \Lambda(\psi, \hat{z})$ for all z, $\hat{z}$. $V(w, \psi) \equiv$

(43)

$$\min q(\psi) \sum_{\theta} \sum_{z} \mu(\theta)\pi(z)z^{-\rho} \left\{b(\psi')z^{\rho}V(W(\theta, w, z), \psi') + z\left[C(\tilde{u}(\theta, w)) - N(\bar{h}(\theta, w))\right]\right\}$$

subject to

$$\sum_{\theta} \mu(\theta) \left\{ \sum_{z} \pi(z)z^{1-\rho}\tilde{u}(\theta, w) + \beta\bar{W}(\theta, w) - \theta\bar{h}(\theta, w) \right\} = w.$$

$$\sum_{z} \pi(z)z^{1-\rho}\tilde{u}(\theta, w) + \beta\bar{W}(\theta, w) - \theta\bar{h}(\theta, w) \geq$$

$$\sum_{z} \pi(z)z^{1-\rho}\tilde{u}(\theta, w) + \beta\bar{W}(w, \hat{\theta}) - \theta\bar{h}(\hat{\theta}, w), \forall\theta, \hat{\theta} \tag{44}$$

and the transition $\psi' = \Lambda(\psi, z)$.

Let us re-write the objective under the assumption that $W(\theta, w, z) = \bar{W}(\theta, w)$.

$$q(\psi) \sum_{\theta} \mu(\theta) \left\{ \sum_{z} \pi(z)b(\psi')V(\bar{W}(\theta, w), \psi') + \sum_{z} \pi(z)z^{1-\rho}\left[C(\tilde{u}(\theta, w)) - N(\bar{h}(\theta, w))\right] \right\}$$

The first term inside the curly brackets may be written simply $q(\psi)\sum_{\theta}\mu(\theta)b(\psi')V(\bar{W}(\theta, w), \psi')$ if $\psi' = \Lambda(\psi, z) = \Lambda(\psi, \hat{z})$ which is the case if all component planners are choosing $W(\theta, w, z) = \bar{W}(\theta, w)$.

Hence, policy functions of the form $\bar{u}(\theta, w, z) = \tilde{u}(\theta, w)z^{1-\rho}$, $\bar{h}(\theta, w)$ and $W(\theta, w, z) = \bar{W}(\theta, w)$ for all z and the prices of the form we have guessed are mutually consistent.

A.2 Persistent Aggregate Shocks

Here, we show that for persistent shocks, $\Lambda(., z) \neq \Lambda(., \hat{z})$ for all $z \neq \hat{z}$.

Assume that this is not the case, i.e., assume that $W(\theta, w, z) = \tilde{W}(\theta, w)$. Then,

using $\bar{u}(\theta, w, z) = \tilde{u}(\theta, w) z^{1-\rho}$, the incentive compatibility constraint (25) becomes,

$$\tilde{u}(\theta, w) \sum_z \pi(z|z_-) z^{1-\rho} + \beta \tilde{W}(\theta, w) - \theta \bar{h}(\theta, w) \geq$$
$$\tilde{u}(\hat{\theta}, w) \sum_z \pi(z|z_-) z^{1-\rho} + \beta \tilde{W}(w, \hat{\theta}) - \theta \bar{h}(\hat{\theta}, w), \forall \theta, \hat{\theta}$$

If a constraint binds following a state z_- it cannot bind following a state $\hat{z}_-$. This is not compatible with efficiency.

Proof of Proposition ??? As it turns, it is simpler to prove our main result regarding this case in the space of sequences. Consider, in this case, the following transformation of variables. $u(\theta^a, z^{t+a}) = \tilde{u}(\theta^a, z^{t+a}) + \ln z_{t+a}$ and $h(\theta^a, z^{t+a-1}) = \tilde{h}(\theta^a, z^{t+a-1})$. We may, then, define

$$\tilde{\mathcal{U}} \equiv \max \sum_t \sum_{z^t} \sum_a \sum_{\theta^a} \beta^a \mu(\theta^a) \pi(z^t|z^{t-a}) \left\{ \tilde{u}(\theta^a, z^t) - \theta_a \tilde{h}(\theta^a, z^{t-1}) \right\} \tag{45}$$

subject to

$$\sum_a \sum_{\theta^a} \sum_{z^{t+a}} \beta^a \mu(\theta^a) \pi_t(z^{t+a}|z^t) \left\{ \tilde{u}(\theta^a, z^{t+a}) - \theta_a \tilde{h}(\theta^a, z^{t+a-1}) \right\} \geq \tag{46}$$
$$\sum_a \sum_{\theta^a} \sum_{z^{t+a}} \beta^a \mu(\theta^a) \pi(z^{t+a}|z^t) \left\{ \tilde{u}(\sigma(\theta^a, z^{t+a-1}), z^{t+a}) - \theta_a \tilde{h}(\sigma(\theta^a, z^{t+a-1}), z^{t+a}) \right\},$$

$\forall\, t, \sigma$, and

$$\sum_a \sum_{\theta^a} \mu(\theta^a) C(\tilde{u}(\theta^a, z^{t+a})) \leq \sum_a \sum_{\theta^a} \mu(\theta^a) N(\tilde{h}(\theta^a, z^{t+a-1})) \quad \forall z^{t+a}, a. \tag{47}$$

The program is independent of z^{t+a} for all z^{t+a}, t and a, so must the solution be. That is, the allocation that solve the problem above is of the form $\tilde{u}(\theta^a, z^{t+a}) = \tilde{\tilde{u}}(\theta^a)$, $\tilde{h}(\theta^a, z^{t+a-1}) = \tilde{\tilde{h}}(\theta^a)$. The solution of the original problem is, therefore, $u(\theta^a, z^{t+a}) = \tilde{\tilde{u}}(\theta^a) + \ln z_{t+a}$ and $h(\theta^a, z^{t+a-1}) = \tilde{\tilde{h}}(\theta^a)$.

We may then use the definition in (5) to write the utility promise $w(\theta^a, z^t)$ as

$$\sum_j \beta^j \sum_{\theta^{a+j}} \mu(\theta^{a+j}|\theta^a) \left\{ \tilde{\tilde{u}}(\theta^{a+j}) - \theta_{a+j}\tilde{\tilde{h}}(\theta^{a+j}) \right\} + \sum_j \beta^j \sum_{z^{t+j}} \pi(z^{t+j}|z^t) \ln z_{t+j}.$$

Let $z_{t-1} = z_-$, then using the transition matrix Π we may write,

$$\sum_j \beta^j \sum_{z^{t+j}} \pi(z^{t+j}|z^{t-1}) \ln z_{t+j} = (I - \beta\Pi)^{-1}_{z_-} (\ln z)_z,$$

where $(\ln z)_z$ is a vector with $\ln z$ for all possible realizations of z. We may also define

$$\tilde{w}(\theta^a) \equiv \sum_j \beta^j \sum_{\theta^{a+j}} \mu(\theta^{a+j}|\theta^a) \left\{ \tilde{\tilde{u}}(\theta^{a+j}) - \theta_{a+j}\tilde{\tilde{h}}(\theta^{a+j}) \right\},$$

which finally allows us to write (5) as

$$\tilde{w}(\theta^a) \equiv \sum_{\theta^{a+1}} \mu(\theta^{a+1}) \left\{ \tilde{\tilde{u}}(\theta^{a+1}) - \theta_{a+1}\tilde{\tilde{h}}(\theta^{a+1}) + \beta\tilde{w}(\theta^{a+1}) \right\}$$

In this case, $w(\theta^a, z^t) =$

$$\tilde{w}(\theta^a) + \sum_{\pi} (z|z_-) \left\{ \ln z + \beta[I - \beta\Pi]^{-1}_{z} (\ln \tilde{z})_{\tilde{z}} \right\}.$$