FUNDAÇÃO GETULIO VARGAS
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João Paulo Valente

A Study of the Impacts of Quantitative Easing on the Macroeconomics Variables

Rio de Janeiro
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Dissertação para obtenção do grau de mestre apresentada à Escola de Pós-Grauação em Economia

Área de concentração: Economia Monetária

Orientador: Tiago Couto Berriel

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JOÃO PAULO VALENTE

A STUDY OF THE IMPACTS OF QUANTITATIVE EASING ON THE MACROECONOMICS VARIABLES .

Dissertação apresentada ao Curso de Mestrado em Economia da Escola de Pós-Graduação em Economia para obtenção do grau de Mestre em Economia.

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Tiago Couto Berriel
Orientador (a)

Eduardo Zilberman

Gabriel Chequer Hartung

À minha família...

Agradecimentos

Agradeço ao meu orientador, aos meus amigos, à minha família e ao meu grande amor.

Resumo

Neste trabalho, propusemos um modelo DSGE que busca responder algumas questões sobre políticas de afrouxamento monetário (Quantitative Easing - QE) recentemente implementadas em resposta à crise de 2008. Desenvolvemos um modelo DSGE com agentes heterogêneos e preferred-habitat nas compras de títulos do governo. Nosso modelo permite o estudo da otimalidade da compra de portfolio (em termos de duration dos títulos) para os bancos centrais quando estão implementando a política. Além disso, a estrutura heterogênea nos permite olhar para distribuição de renda provocada pelas compras de títulos. Nossos resultados preliminares evidenciam o efeito distributivo do QE. No entanto, nosso modelo expandido apresentou alguns problemas de estabilidade.

Palavras-chave: Zero-lower bound, Quantitative Easing, Preferred-habitat

Abstract

In this paper, we proposed a DSGE model that seeks to answer some questions about the recent implemented Quantitative Easing (QE) programs. Our framework is a DSGE model with heterogeneous agents and preferred-habitat in purchases of government bonds. It allows the study of optimality purchasing portfolio (in terms of duration of the bonds) for central banks when they are implementing the policy. Furthermore, the heterogeneous structure allows us to look at income distribution caused by purchases of these securities. Our preliminary results show some distributive effect of QE. However, our expanded model showed some stability problems.

Keywords: Zero-lower bound, Quantitative Easing, Preferred-habitat

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1 Introduction

The main policy tool used by Central Banks in the last decades has been the determination of short-term rates. They used open market operations as its primary instrument to influence the supply of bank reserves in order to achieve their target for the rate that clears the reserve market. However, for some countries this scenario has changed. Their short-term rates have reached (or are near to) the Zero Lower Bound (ZLB) and this important policy lost its utility in the short-term.

The first notable case was presented by Japan in the end of 90's. Since April 1998, its call rate (the overnight cash rate analogous to the US federal funds rate) has been within 50 basis points of zero (Figure 1). Together with these low rates, Japan's growth was still anemic, and prices have continued to fall in most of this period.

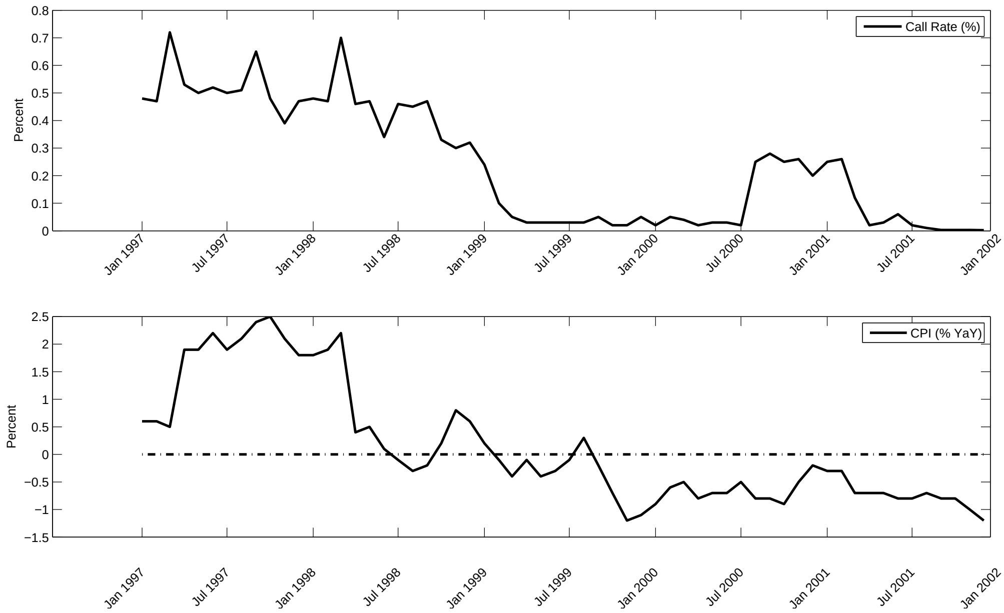

Figure 1: Japan's short-term and inflation rates - 1997-2001

Recently, other countries found themselves in the same problem. In response to the financial crisis in 2008, many central banks reduced their short-term rates. However, some of them reached the ZLB even with very slow economies. The Figure 2 illustrates this scenario. Countries such as United States, Japan, England and Euro Area are very close to this bound.

Bernanke et al. (2004) presented three monetary policies alternatives to a country that have already reached the ZLB: 1) shape policy expectations; 2) increase the size of the Central Bank's balance sheet (Quantitative Easing- QE); and 3) change the composition of the Central Bank's balance sheet. In this work we focus in the second option. This policy constitutes in the

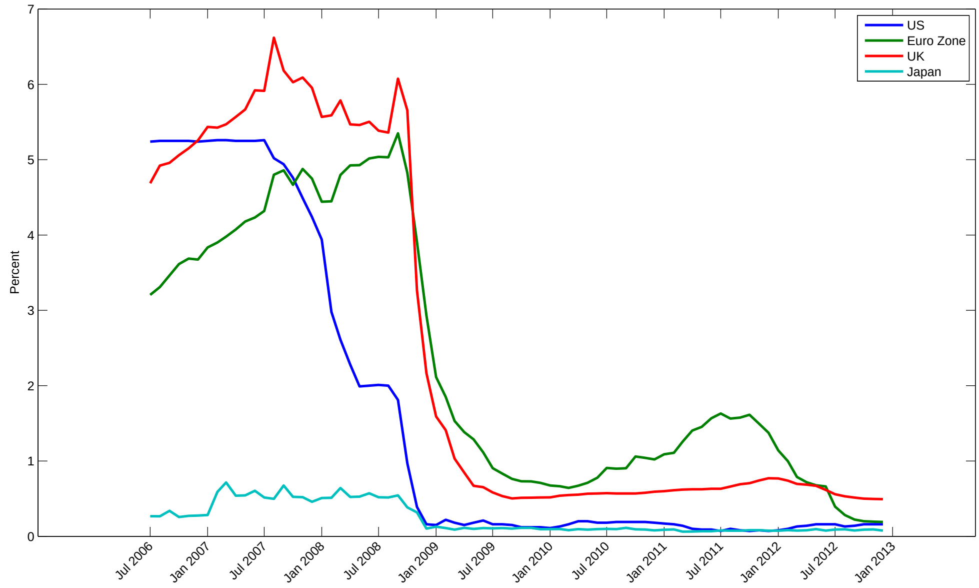

Figure 2: Short-term rates of US, Euro Zone, UK and Japan - 2006-2012

expansion of the liquidity available in the market through the acquisition of financial assets by the Central Bank. Bernanke et al. (2004) argues that it can potentially affect the economy through three channels. The first is built up in the premise that money and other financial assets are not perfect substitutes, which implies that QE may reduce other yields in the economy. The second is the fiscal channel. Since the monetary expansion can cause a relief in the government budget, QE can create expectations of fiscal expansionary policies. Finally, there is the signaling channel. The QE can be seen as a serious effort demonstration of the government to escape from the deflationary scenario. Then, the agents revise their expectation in favor to higher inflation path. Based on these possible effects, some governments used this policy in order to improve their economic environment.

On March 19, 2001, the Bank of Japan (BoJ) initiated an unprecedented monetary policy in an attempt to stimulate the nation's stagnant economy. Bernanke et al. (2004) argue that the BOJ's quantitative easing policy might initially have been interpreted as a recommitment to the policy of keeping the short-term rate at zero. However, the BOJ raised its target for current account balances at commercial banks (essentially, bank reserves) a number of times, to the point that reserves substantially exceeded the level needed to pin the call rate at zero.

The initial current account balances target was set at ¥5 trillion (approximately $62 billion) level, ¥1 trillion above the previous month level. However, it was increased during the program to reach ¥30-35 trillion in January 2004, and remained unchanged until the end of the program in March 2006. The policy was withdrawn after the Japanese CPI was consolidated in the

positive side. This was the first great move of a country toward a quantitative easing policy.

In the United States, the Federal Reserve (FED) carried out three rounds of large-scale asset purchases during the Great Recession. Two rounds of "QE1" took place in November 2008 and March 2009, during the financial crisis. The third round followed the "QE2" announcement in November 2010. The Federal Reserve purchased a total of US$1.75 trillion in agency debt, mortgage-backed securities (MBS) and Treasuries in the "QE1", followed by a second Treasury-only program of US$600 billion in the fall of 2010.

Recently, on 13 September 2012, the Federal Reserve launched a new US$40 billion a month, open-ended, bond purchasing program of agency mortgage-backed securities. The Committee also announced that it would continue through that year its program to extend the average maturity of its holdings of Treasury securities and its policy of reinvesting principal payments from its holdings of agency debt and agency MBS in agency MBS. These actions would increase the Committee's holdings of longer-term securities by about US$85 billion each month. This program is know as QE3. Extending the easing policy, on 12 December 2012, the FOMC announced that it would purchase longer-term Treasury securities at a pace of US$45 billion per month. Then, the monthly total purchase would stand at US$85 billion of assets, since the extension and reinvestment policies would be finished by the end of 2012.

In March 2009, the Bank of England (BoE) announced it would purchase £75 billion of U.K. gilts, which, after subsequently increases, expanded to £375 billion in July 2012. The European Central Bank (ECB) initiated purchases of up to €60 billion in Euro area covered bonds in mid-2009. In October 2010, the Bank of Japan unveiled a ¥5 trillion asset purchase program, which had been progressively increased to ¥70 trillion by July 2012. In order to achieve a 2 percent annual inflation rate, on 4 April 2013, the BoJ announced a plan to purchase ¥7.5 trillion of bonds a month and double the monetary base. It is expected the BoJ's balance sheet size be equivalent to 60% os Japan's GDP by the end od 2014. This program is part of a new set of policies known as Abenomics. Figure 3 illustrates these expansions through the comportment of the Central Bank's balance sheets.

Although Quantitative Easing programs have been used by many policy makers in the recent years, many questions about its functioning and effects are still unanswered. Here we will deal with some operational issues of QE and its possible consequences on the individual choices.

Programs such as quantitative easing aims to affect the yield curve through the purchase of longer-term assets using reserves. They focus in the reduction of long-term yields relative to the short-term ones. This is made by the change in the relative prices of the bonds. Using the preferred-habitat framework, where some agents are "restricted" to some kind of assets.

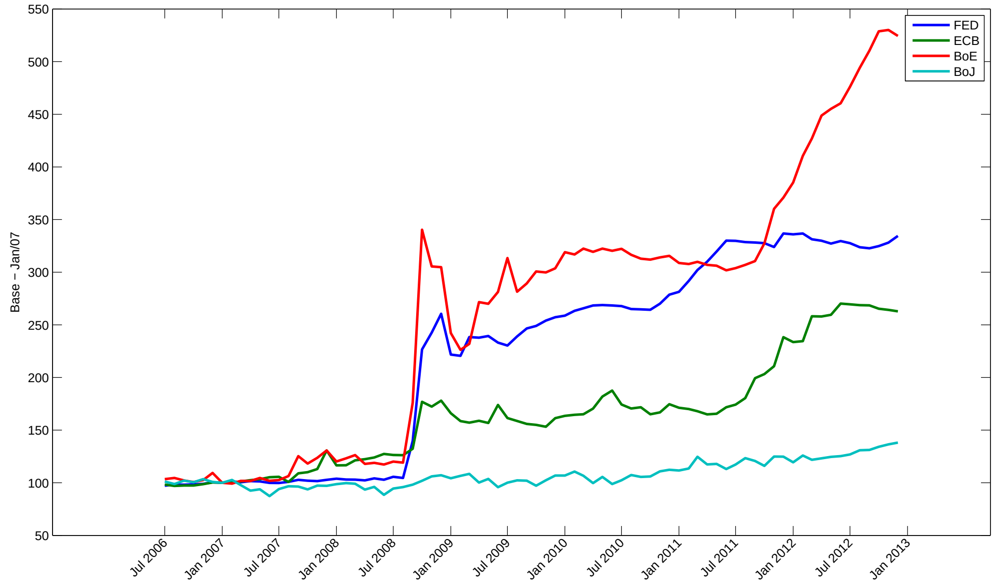

Figure 3: Central Banks' Balance Sheets - Total Assets

We can conjecture that LSAP programs also play a role on the wealth distribution of the economy. Changing the relative prices of the bonds, QE programs may cause direct effects such as redistribute wealth trough the agents (investors) and indirect through the intertemporal budget allocation of the government. We propose a framework that allows a deep study of this effect through a heterogeneous agent model.

The other question we will try to answer is if the duration of the bonds bought by Central Bank is important in the final output of the program. When implementing the QE the Central Bank decide the quantity of each bonds it will acquire in the market. We developed a model with different bonds durations in its framework which allows to answer if a concentrated duration purchase can achieve the same effect os a disperse once. Answering this may bring up some operational possibilities in the terms of quantity bought and liquidity management at QE implementation.

In order to developed these ideas, in the next section we present the relate literature of our work. Section 3 contains a detailed description of the used model. On Section 4 we show our main results. Finally, the final remarks are in the las section of the work.

2 The Literature

The first important theoretical model to study the effects of quantitative easing in the economy was introduced by Wallace (1981). Using a two-period lived overlapping generations individuals in a complete and competitive market economy, the author showed that QE is ineffective as a policy tool. His central point was an irrelevance theorem that shows that different paths of government portfolio do not change the economy's equilibrium consumption allocation and the path of the price level.

Wallace's Irrelevance Theorem is based on the fact that alternatives paths of the government's portfolio don't change the economy's equilibrium consumption allocation and the path of the price level. In other words, the change in the hedging demand of the representative household precisely offsets the change in the central bank's portfolio, so that asset markets continue to clear at the same prices as before. Then, the open-markets operations are neutral when the fiscal policy is held constant.

The environment used in the theorem is compounded by a complete and competitive market economy. His model has a two-period lived, overlapping generations with a single consumption item storable via constant returns to scale and produced with stochastic technology. One necessary condition to the theorem is that there be equilibria in which the private sector voluntary holds real capital and unbacked government liabilities. This is made possible by the overlapping-generations structure and risky real capital (storable consumption good).

More than twenty years after Wallace's work, when Japan already presented the ZLB issue, Eggertsson and Woodford (2003) argued that: "Wallace (1981)'s analysis is often supposed to be of little practical relevance for actual monetary policy because his model is one in which money serves only as a store of value, so that an equilibrium in which short-term Treasury securities dominate money in terms of rate of return is not possible, although this is routinely observed. "

Based on this critique, Eggertsson and Woodford (2003) modeled the demand for money and the role of financial assets (including monetary base) in an intertemporal equilibrium model. Nevertheless, even in this new framework, open-market operations conducted at the zero bound (when the liquidity services provided by money balances at the margin have fallen to zero) are neutral, which corroborates the idea that an analysis of the kind proposed by Wallace is correct.

Curdia and Woodford (2011) generalized the above result using an extended basic New Keynesian model of the monetary transmission mechanism and introduced the central bank's balance sheet as part of the model. In their model, the agents are heterogeneous and financial intermediation has an important role, even though it has some imperfections. This framework gives rise to the possibility of central bank to improve this intermediation. Once again, the

irrelevance result was corroborated, but now in an extended model with heterogeneity and credit friction.

Using a similar DSGE model with nominal rigidities, Gertler and Karadi (2011) added financial intermediaries (between firms and households) to their framework and found a potential effect of QE when private assets are bought by Central Banks. In their model, there is an agency problem that constraint the ability of financial intermediaries to obtain funds from households. So, in the event of a financial crisis, leveraged banks assets are depreciated and its ability to finance the productive sector is reduced. Then, the central bank can act buying private assets which in turns increase their prices and improve the intermediaries' financial health. This policy helps to bring the economy back to the steady state path.

Using the basic framework of the above work, Gertler and Karadi (2012) introduced the long-term government bonds in their model. These bonds have a fixed supply and are part of intermediaries' and Central Bank's balance sheets. This new approach allowed them to study the effects of QE1 and QE2 in some numerical exercises. They found that the QE1 would be less effective if the private bonds purchases would be replace by governments bonds acquisitions. They also concluded that a similar program to QE2 has an effect of forty base points reduction in the short-term rates. As in their previous work, the QE policy seems to improve the economic conditions.

Negro et al. (2011) analyzed the economic and financial crisis of 2008 based on shocks to liquidity private paper. They incorporated a set of financial frictions into a standard DSGE where nominal prices and wages are sticky. The first friction is a limit on the amount a firm can borrow when it faces an investment opportunity. The firm can borrow only up to a fraction of the value of its current investment. Second, a firm that faces an investment opportunity can sell only up to a certain fraction of the "illiquid" assets on its balance sheet in each period. It can be understood as a haircut in the repo market. The government paper is not subject to this resaleability constraint. This framework gives a chance to composition of liquid and illiquid assets in the hands of the private sector affects the equilibrium, differently from Wallace (1981).

As said before, one of the channels through which QE can affect the yields is the non perfect substitutability. In this way, Vayanos and Vila (2009) focused on the behavior of the bond yields term structure and developed a preferred-habitat model of term structure of interest rates that gives rise to the possibility of demand shocks affect the yield curve. In their model, the term structure results from the interaction between investors' clienteles with preferences for specific maturities and risk averse arbitrageurs. In this context, when there is a shock in a bond demand of a specific maturity, the arbitrageurs' portfolio rebalance does not offset all this shock effect

and the curve inclination changes. As a consequence, there is a possibility of QE be effective, through the reduction specific yields in the economic.

In their model, the term structure results from the interaction between investors' clienteles with preferences for specific maturities and risk averse arbitrageurs. Chen et al. (2012) used this basic framework in an New Keynesian DSGE to study the macroeconomics effects of government bonds focused LSAP programs, such as QE2. They found that these programs increase the GDP growth and inflation path. However, these effects are small.

Based on this preferred-habitat framework and in a standard medium-scale new Keynesian DSGE model, Chen et al. (2012) tested the macroeconomic effects of QE2. The model has a set of restricted households that can only invest in long-term bonds and a set of unrestricted households who can invest in both short and long-term bonds. These unrestricted households face transaction costs on their purchases of long-term bonds. Asset purchases can have effects on the macroeconomic variables through changing the long-term interest rate. When the Central Bank buys long-term bonds the transaction costs (a premium over the quantity of long-term issued by the government)reduces and, consequently, the long-term yields fall. This affects the intertemporal choice of the restricted household. Estimating the model, they found that the effects of QE2 are likely to be modest, although with a lasting impact on the level of GDP.

The empirical works are also very important to the study of QE policy. Since it was an unconventional policy, the lacking of available data made its study very difficult. But in the advent of Japan's crises in the beginning of the XXI century and the "Great Recession" years later, some studies started to be available. Bernanke et al. (2004) was one of the studies that made use of the Japanese case to study QE. Based on an event study of policies of the Bank of Japan, and a Term Structure model, they tried to verify the effects of the QE. Their study provided some support for the view that asset prices respond to quantitative easing policy. However, this result was based in a small sample size. Their estimated term structure model does suggest that yields in Japan were noticeably lower during the policy's period than the model would have predicted. As the following studies, their work did not provide any concrete conclusion of the QE effect on other variables than yields.

Many works used the recent Large-scale Asset Programs Purchases programs to develop an empirical literature for these programs. Doh (2010) used a simplified version of the preferred-habitat model developed by Vayanos and Vila (2009) and concluded that the QE1 announcement reduced the 10 year Treasury yield by 39 base points. Also using this theoretical framework and CUSIP level data, D'Amico and King (2010) found the yield drop caused by the Treasury purchases in the first FED's LSAP program was between 30 and 50 base points. Hamilton and

Wu (2011) used this preferred-habitat framework in a discrete time version and concluded that a US$400 billion dollar purchase of long-term treasuries would reduce the 10 year yields by 13 base points.

In Gagnon et al. (2011), the authors made use of event study and time series data to analyze the effects of QE1 on the yields of the economy. In the event study they found that the term premium in the 10 year Treasuries yields was reduce by 50-100 base points, and the biggest drop was in the Agency debt yields. Complementing these results, their time series analyses showed a 52 base points drops in the term premium. Krishnamurthy and Vissing-Jorgensen (2011) also found reductions in the nominal rates of Treasuries, Agencies debt, corporate bonds and Mortgage Backed Securities (MBS). These reductions varied across bonds types, maturities and programs (QE1 and 2).

In a different approach, Carvalho et al. (2012) used cross-country data for the period between February and December 2009 to study these effects. They found that a ten percent increase in the Central Bank's balance sheet above the trend is associated with a 18.2 basis point in the inflation expectation. Nevertheless, this expansion does not seem to affect the growth expectations. Finally, using their estimates results, D'Amico et al. (2012) concluded that the scarcity and duration channels contributed to the reduction of the ten year Treasury yields of 35 basis point in the QE1 and 45 in QE2, as expected by the policy makers.

This literature helps us to understand our problem a give tools to answer the question. Our model uses a basic New Keynesian added with preferred-habitat investors structure. We expand these structures and tries to explore these framework to answer questions about the optimal central bank's portfolio purchase and wealth distribution effects. We also intended to elucidate the QE effects in the activity, since the empirical works are not very developed in this issue¿ Based on this, in the next section we describe the model and how we expect it will work.

3 The Model

To study the effects of different maturity composition purchases and effects of QE in the optimization of different individuals it's clear that our model must contain different duration bonds and heterogenous households. Then, we developed a framework that is a basic New Keynesian model as Christiano et al. (2001) and Smets and Wouters (2007) using the preferred-habitat concept as done by Chen et al. (2012).

Our model is compounded by 3 types of households, one unrestricted (denoted by u) and two restricted (denoted by r_m and r_l), that populate the economy and supply differentiated labor inputs. As usual in the literature, competitive labor agencies combine these inputs into a homogeneous composite. The production side of economy is similar to standard New Keynesian models and has 3 types of firms: competitive capital producers, monopolistic competitive intermediate goods producers and competitive final goods firms. The government sets monetary and fiscal policy. These policies also include the supply of short-term and all types of long-term bonds.

3.1 Households

In the model, there is a continuum of measure one of households in the economy. Household $j = \{u, r_m, r_l\}$ enjoys consumption C_t^j (relative to productivity Z_t) and dislikes hours worked L_t^j. Households supply differentiated labor inputs indexed by i but perfectly share consumption risk within each group. The life-time utility function for a generic households j is

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta_j^s b_{t+s}^j \left[\frac{1}{1-\sigma_j} \left(\frac{C_{t+s}^j}{Z_{t+s}} - h \frac{C_{t+s-1}^j}{Z_{t+s-1}} \right)^{1-\sigma_j} - \frac{\varphi_{t+s}^j (L_{t+s}^j(i))^{1+\upsilon}}{1+\upsilon} \right] \tag{3.1}$$

where $\beta_j \in (0,1)$ is the individual discount factor, b_t^j is a preference shock, $\sigma_j > 0$ is the coefficient of relative risk aversion, $h \in (0,1)$ is the habit parameter, $\nu \geq 0$ is the inverse elasticity of labor supply and φ_t is a labor supply shock. The preference and labor supply shocks both follow stationary AR(1) processes in logs.

As said before, there are 3 types of bonds in this economy. Short-term bonds, B_t, are one-period securities purchased at time t that pay a nominal return R_t at time $t+1$. Medium-term bonds (B_t^{Lm}) and long-term bonds (B_t^{Ll}) are perpetuities that cost $P_{Lm,t}$ and $P_{Ll,t}$, respectively, at time t. They pay an exponentially decaying coupon κ_{Lm}^s and κ_{Ll}^s at time $t+s+1$, for $0 < \kappa_{Lm} \leq \kappa_{Ll} \leq 1$. The fraction ω_u of unrestricted households trade in both short-term and long-term government bonds markets. However, they pay a transaction cost $\zeta_{Lm,t}$ and $\zeta_{Ll,t}$ per-unit of medium-term and long-term bond purchased. The fraction of the population of type r_m,

ω_{r_m}, can only trade in medium-term bonds and pays no transaction costs when buying then. In this way, they can only transfer wealth along time using this type of bonds. The fraction ω_{r_l} of households are restricted to the trade of long-term bonds. Note that the unrestricted households are the arbitrageurs of the economy, since only them can buy all bonds types. In this way, they will determine the yield curve of the bond market. Their budget constraint is as follows:

$$P_t C_t^u + B_t^u + (1+\zeta_{Lm,t}) P_{Lm,t} B_t^{Lm,u} + (1+\zeta_{Ll,t}) P_{Ll,t} B_t^{Ll,u} \leq$$
$$R_{t-1} B_{t-1}^u + \textstyle\sum_{s=1}^{\infty} \kappa_{Lm}^{s-1} B_{t-s}^{Lm,u} + \sum_{s=1}^{\infty} \kappa_{Ll}^{s-1} B_{t-s}^{Ll,u} + W_t^u(i) L_t^u(i) + \overline{\mathcal{P}_t^u} - T_t^u$$

and for the households restricted to the medium-term and long-term bonds are, respectively:

$$P_t C_t^{r_m} + P_{Lm,t} B_t^{Lm,r_m} \leq \sum_{s=1}^{\infty} \kappa_{Lm}^{s-1} B_{t-s}^{Lm,r_m} + W_t^{r_m}(i) L_t^{r_m}(i) + \overline{\mathcal{P}_t^{r_m}} - T_t^{r_m}. \tag{3.2}$$

$$P_t C_t^{r_l} + P_{Ll,t} B_t^{Ll,r_l} \leq \sum_{s=1}^{\infty} \kappa_{Ll}^{s-1} B_{t-s}^{Ll,r_l} + W_t^{r_l}(i) L_t^{r_l}(i) + \overline{\mathcal{P}_t^{r_l}} - T_t^{r_l}. \tag{3.3}$$

where P_t is the price of the final consumption good, $W_t^j(i)$ is the wage set by a household of type j who supplies labour of type i, $\overline{\mathcal{P}_t^j}$ is the sum of $\mathcal{P}_t^j$, $\mathcal{P}_t^{cp,j}$, and $\mathcal{P}_t^{fi,j}$ that are, respectively, profits of household j from ownership of intermediate goods producers, capital producers and financial firms and T_t^j are lump-sum taxes. The budget constraints make clear the differences among the individuals. The unrestricted household is allowed to buy all types of bonds in the economy, as can be seen on his equation. However, he must pay an extra fee of $\zeta_{i,t}$ on each dollar spent in purchases of bonds of maturity i. Notice that since the bonds are perpetuities and pay decaying coupons, we are allowed to aggregate the different types as if every period the government replace the old bonds by new ones. This make clear that the difference in the bonds is tracked by its maturity. This approach makes the numerical solution easier, since it will have much less state variables in our problem. In this way, we made use of a useful technique of the literature to simplify our problem and that changes the budget constraints:

$$P_t C_t^u + B_t^u + (1+\zeta_{Lm,t}) P_{Lm,t} B_t^{Lm,u} + (1+\zeta_{Ll,t}) P_{Ll,t} B_t^{Ll,u} \leq$$
$$R_{t-1} B_{t-1}^u + P_{Lm,t} R_{Lm,t} B_{t-1}^{Lm,u} + P_{Ll,t} R_{Ll,t} B_{t-1}^{Ll,u} + W_t^u(i) L_t^u(i) + \overline{\mathcal{P}_t^u} - T_t^u$$

$$P_t C_t^{r_m} + P_{Lm,t} B_t^{Lm,r_m} \leq P_{Lm,t} R_{Lm,t} B_{t-1}^{Lm,r_m} + W_t^{r_m}(i) L_t^{r_m}(i) + \overline{\mathcal{P}_t^{r_m}} - T_t^{r_m}. \tag{3.4}$$

$$P_t C_t^{r_l} + P_{Ll,t} B_t^{Ll,r_l} \leq P_{Ll,t} R_{Ll,t} B_{t-1}^{Ll,r_l} + W_t^{r_l}(i) L_t^{r_l}(i) + \overline{\mathcal{P}_t^{r_l}} - T_t^{r_l}. \tag{3.5}$$

where $R_{Lm,t} = \frac{1}{P_{Lm,t}} + \kappa_{Lm}$ and $R_{Ll,t} = \frac{1}{P_{Ll,t}} + \kappa_{Ll}$ are the gross yield to maturity at time t

on the medium-term and long-term bonds.

In our model, perfectly competitive labor agencies combine differentiated labor inputs into a homogenous labor composite L_t according to the technology

$$L_t = \left[\int_0^1 L_t(i)^{\frac{1}{1+\lambda_w}} di \right]^{1+\lambda_w} \tag{3.6}$$

where $\lambda_w \geq 0$ is the steady state wage markup.

Households are monopolistic suppliers of differentiated labor inputs $L_t(i)$ and set wages on a staggered basis as in (Calvo (1983)) taking the demand as given. In this way, there is a probability of resetting the wage, $(1 - \zeta_w)$, at each period t. In the other side, at a probability of ζ_w wage is readjusted by the the steady state rate of inflation (Π) and of productivity growth (e^γ). Then, a household of type j who can reset the wage at time t chooses $\widetilde{W}_t^j(i)$ to maximize:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta_j \zeta_w)^s \left[\Xi_{t+s}^{j,p} \widetilde{W}_t^j(i) L_{t+s}^j(i) - \frac{\varphi_{t+s}^j (L_{t+s}^j(i))^{1+\upsilon}}{1+\upsilon} \right], \tag{3.7}$$

s.t.

$$L_t(i) = \left[\frac{W_t(i)}{W_t} \right]^{-\frac{1+\lambda_w}{\lambda_w}} L_t, \tag{3.8}$$

$$W_{t+s}^j(i) = (\Pi e^\gamma)^s \widetilde{W}_t^j(i). \tag{3.9}$$

where $\Xi_t^{j,p}$ is the marginal utility of consumption in nominal terms, $s > 0$ and 3.8 comes from labor agencies profit maximization. This determines the supply of labor in our framework.

Since the transaction costs are exogenous and paid to the financial firm, its profit is:

$$\mathcal{P}_t^{fi} = \zeta_{Lm,t} P_{Lm,t} B_t^{Lm,u} + \zeta_{Ll,t} P_{Ll,t} B_t^{Ll,u}. \tag{3.10}$$

3.2 Production Side

Now we turn to the production side of the economy. As said before, our production framework is very usual in the literature. Final goods producers combine differentiated intermediate goods $Y_t(f)$, supplied by a continuum of firms f of measure 1, into a homogenous good Y_t according to technology

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{1}{1+\lambda_f}} df \right]^{1+\lambda_f}, \tag{3.11}$$

where $\lambda_f \geq 0$ is the steady-state price markup.

These firms maximize profits in a perfectly competitive environment. Their problem is

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \tag{3.12}$$
$$\text{s.t. } Y_t = [\textstyle\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_f}} di]^{1+\lambda_f}$$

where P_t is the final good price sold to the households and $P_t(i)$ is the intermediate good i price.

The intermediate goods producers sector is a continuum of measure one of monopolistic competitive firms that combine rented capital and hired labor to produce intermediate goods according to a standard Cobb-Douglas technology. The firm i chooses labor, $L_t(i)$, and capital quantities, $K_t(i)$, according to the following problem

$$\max_{L_t(i), K_t(i)} P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t(i) \tag{3.13}$$
$$\text{s.t. } Y_t(i) = Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha}$$

where Z_t is a labor-augmenting technology process which evolves according to

$$log\left(\frac{Z_t}{Z_{t-1}}\right) = (1-\rho_z)\gamma + \rho_z log\left(\frac{Z_{t-1}}{Z_{t-2}}\right) + \epsilon_{z,t} \tag{3.14}$$

Their product prices are sticky as in (Calvo (1983)) with probability $(1-\zeta_p)$ to adjust the price in each period. Then, the problem for a firm that can adjust at time t is to choose the price $\widetilde{P}_t(f)$ that maximizes

$$\max_{\widetilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \Xi_{t+s}^p [\widetilde{P}_t(i)\Pi^s - \lambda_{f,t+s} MC_{t+s}] Y_{t+s}(i) \tag{3.15}$$
$$\text{s.t. } Y_{t+s}(i) = \left[\frac{\widetilde{P}_t(i)\Pi^s}{P_{t+s}}\right]^{-\frac{1+\lambda_f}{\lambda_f}} Y_{t+s},$$

where Ξ_{t+s}^p is the marginal utility of consumption in nominal terms of average household, Π is the inflation rate at the steady-state, MC_{t+s} is the marginal cost derived from the problem (3.13), and $\lambda_{f,t}$ is a goods markup shock that follows a stationary $AR(1)$ process in logs.

In the capital production sector there is a representative firm, owned by all households, that operates under perfect competition, invests in capital, chooses utilization and rents it to intermediate firms. By choosing the utilization rate u_t, capital producers end up renting in each period t an amount of effective capital equal to

$$K_t = u_t \bar{K}_{t-1} \tag{3.16}$$

Utilization, however, subtracts real resources measured in terms of the consumption good

$$a(u_t)\bar{K}_{t-1}$$

Capital producers maximize expected discounted stream of dividends to their shareholders:

$$\max_{\overline{K}_t,u_t,I_t} \mathbb{E}_t \sum_{s=0}^{\infty} \Xi^p_{t+s}[R^k_{t+s}u_{t+s}\overline{K}_{t+s-1} - P_{t+s}a(u_{t+s})\overline{K}_{t+s-1} - P_{t+s}I_{t+s}] \tag{3.17}$$

$$\text{s.t. } \overline{K}_t = (1-\delta)\overline{K}_{t-1} + \mu_t \left[1 - S\left(\tfrac{I_t}{I_{t-1}}\right)\right] I_t$$

where R^k_t is the return per unit of effective capital, I_t is the firm investment at t, the problem restriction is the law of motion of capital, $\delta \in (0,1)$ is the depreciation rate and $S(\cdot)$ is the cost of adjusting investments $(S'(\cdot) > 0, S''(\cdot) > 0)$.

3.3 Government

The government has an important role in the economy, since it determines the path of many variables. The first one is the short-term interest rate, which is set by the central bank as the following Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_m} \left[\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}}\right)^{\phi_y}\right]^{1-\rho_m} e^{\epsilon_{m,t}}. \tag{3.18}$$

where $\Pi_t = P_{t+1}/P_t$ is the inflation rate, $\rho_m \in (0,1)$, $\phi_\pi > 1$, $\phi_y \geq 0$ and $\epsilon_{m,t}$ is an i.i.d. innovation.

The government budget constraint is:

$$B_t + P_{Lm,t}B^{Lm}_t + P_{Ll,t}B^{Ll}_t = R_{t-1,t}B_{t-1} + (1+\kappa_{Lm}P_{Lm,t})B^{Lm}_{t-1} + (1+\kappa_{Ll}P_{Ll,t})B^{Ll}_{t-1} + P_tG_t - T_t. \tag{3.19}$$

The supply of of medium-term and long-term bonds follows a simple autoregressive rule for their detrended market value in real terms:

$$\frac{P_{Lm,t}B^{Lm}_t}{P_tZ_t} = \left(\frac{P_{Lm,t-1}B^{Lm}_{t-1}}{P_{t-1}Z_{t-1}}\right)^{\rho_{B_{Lm}}} e^{\epsilon_{B_{Lm,t}}} \tag{3.20}$$

where $\rho_{B_{Lm}} \in (0,1)$ and $\epsilon_{B_{Lm,t}}$ is an i.i.d. exogenous shock.

$$\frac{P_{Ll,t}B^{Ll}_t}{P_tZ_t} = \left(\frac{P_{Ll,t-1}B^{Ll}_{t-1}}{P_{t-1}Z_{t-1}}\right)^{\rho_{B_{Ll}}} e^{\epsilon_{B_{Ll,t}}} \tag{3.21}$$

where $\rho_{B_{Ll}} \in (0,1)$ and $\epsilon_{B_{Ll,t}}$ is an i.i.d. exogenous shock.

Finally, the government adjusts the real primary fiscal surplus in response to the lagged real value of its debt

$$\frac{T_t}{P_t Z_t} - \frac{G_t}{Z_t} = \Phi\left(B_{t-1} + \frac{P_{Lm,t-1}B_{t-1}^{Lm}}{P_{t-1}Z_{t-1}} + \frac{P_{Ll,t-1}B_{t-1}^{Ll}}{P_{t-1}Z_{t-1}}\right)^{\phi_T} e^{\epsilon_{T,t}} \tag{3.22}$$

where $\phi_T > 0$ and $\epsilon_{T,t}$ follows a stationary AR(1) process.

3.4 Equilibrium and Comments

We impose that transaction costs of bond i are function of the detrended real value of long-term bonds plus an error

$$\zeta_{i,t} \equiv \zeta\left(P_{i,t}\frac{B_t^i}{P_t Z_t}, \epsilon_{\zeta_i,t}\right) \tag{3.23}$$

It is also assumed that $\zeta(P_i B_z^i, 0) > 0$ and $\zeta'(P_i B_z^i, 0) > 0$, for each bond type for $i = Lm, Ll$. This is part of channel through which the bond purchases affect the yields in the economy.

In equilibrium, households and firms maximize their objectives subject to their constraints and all markets clear. In particular, the resource constraint of this economy is

$$Y_t = \omega_u C_t^u + \omega_{r_m} C_t^{r_m} + \omega_{r_l} C_t^{r_l} + I_t + G_t + a(u_t)\bar{K}_{t-1}$$

We solve the model by taking a first-order log-linear approximation around a steady state in which quantities are normalized by the level of productivity Z_t and relative prices are expressed as function of P_t.

In our framework, the short-term bonds include the less than one period maturity bonds and central banks's liabilities such as vault cash, deposits and reserves. In this way, the QE is set as a shock in the medium and long-term bond policies. Note that a reduction of the availability of these bonds by exogenous shocks will be compensated by an increase in the amount of short-term bonds (3.19). Then we can replicate the QE easily with this structure.

When the government buys long-term debt, it's transaction cost reduces and this affects the agents'consumption and asset allocation. The following equations help to understand how this work in our model. There you have the unrestricted households' Euler equations.

$$1 = \beta_u \mathbb{E}_t\left[e^{-\gamma - z_{t+1}}\frac{\Xi_{t+1}^u}{\Xi_t^u}\frac{R_t}{\Pi_{t+1}}\right] \tag{3.24}$$

$$(1+\zeta_{i,t}) = \beta_u \mathbb{E}_t \left[e^{-\gamma - z_{t+1}} \frac{\Xi^u_{t+1}}{\Xi^u_t} \frac{P_{i,t+1}}{P_{i,t}} \frac{R_{i,t}}{\Pi_{t+1}} \right] \tag{3.25}$$

for $i = Lm, Ll$. Here, Ξ^u_t is unrestricted household's marginal utility of real detrended consumption at time t and $e^{-\gamma - z_{t+1}}$ is the correction factor due to productivity growth. The next equation is the similar equation for the restricted agents.

$$1 = \beta_i \mathbb{E}_t \left[e^{-\gamma - z_{t+1}} \frac{\Xi^i_{t+1}}{\Xi^i_t} \frac{P_{i,t+1}}{P_{i,t}} \frac{R_{i,t}}{\Pi_{t+1}} \right] \tag{3.26}$$

for $i = Lm, Ll$.

First, note that since only unrestricted households can buy short-term bonds, equation (3.24) determine their price. Now, the two last equations make possible to understand how the LSAP programs can affect the real economy. When the government buys longer-duration bonds, it reduces the outstanding market value of the bond and through (**??**), the transaction cost ($\zeta_{i,t}$) is reduced. In this way, the yield demanded by the unrestricted households (the arbitrageurs of the economy) to hold these assets will decrease. Then, the interest rate faced by restricted households fell, which encourages more consumption. It's important to remember that the short-term rate set by the central bank is been hold constant in our example. Finally, more household demand will boost the production side of the economy, which tends to generate economic growth and inflation. Based on this mechanism we have made our exercises.

As we mentioned before, our model is based on Chen et al. (2012). However, we used a more flexible environment, which allow us to answer our proposed questions. Nevertheless, the model proposed by the above work can be simulated by ours. Note that, assuming that the duration of medium-term and long-term bonds are the same and that both restricted households share the same characteristics (parameters), our model collapse to the one proposed by Chen et al. (2012).

Our flexible model allows a deeper study of the distributive consequences of QE. Having more heterogeneity and a better defined bond yield curve permit us to track the response of a wider range of individuals. This specification also allows to study the effects of different composition of bonds purchases. This is not possible under the model presented by Chen et al. (2012).

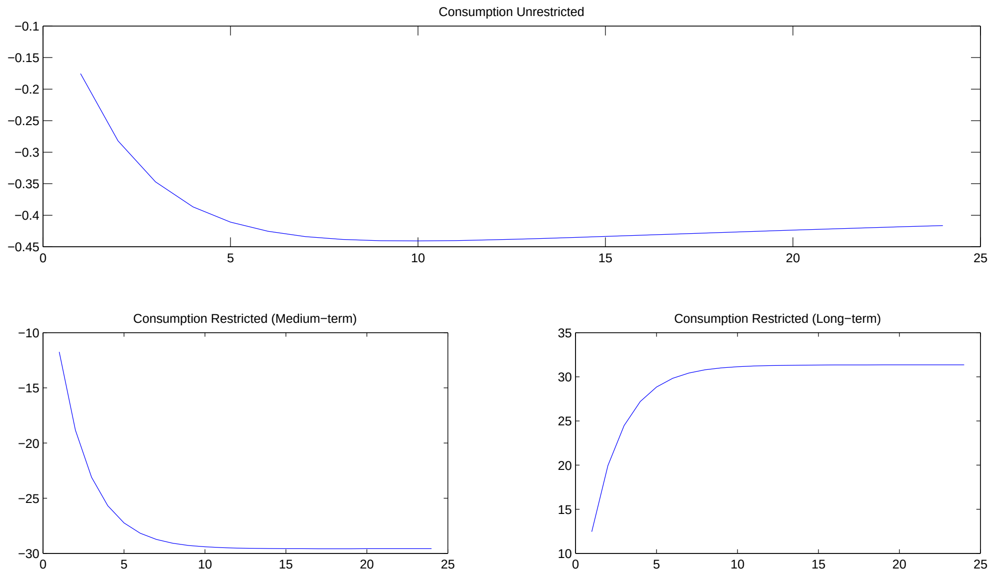

Figure 4: Consumption response to a simulation of Quantitative Easing

4 Results

We calibrated the model following the posterior estimation of Chen et al. (2012), the parameters values are in the Appendix. The biggest differences are in the calibration of bonds' characteristics. The implied medium-term bond duration was set at 20 quarters and for long-term 37. The amount of outstanding debt and other government variables were set based on the data available at FRED database of the Federal Reserve Bank of St. Louis.

Our first exercise was to verify the consumption response of households after purchases of medium and long-term bonds by the Central Bank. As said before, we do this by simulating a shock in the medium and long-term bond policy. The calibrated shocks affect both maturity in the same magnitude, corresponding to an equal fraction of the outstanding market value of bonds been purchased.

Figure 4 shows the impulse response functions of the households after the QE simulation in the calibrated model. Note that they have different reactions to the program. While the household restricted to the long-term bond increases its consumption, the other types reduce. This result indicates that our previous hypothesis might be confirmed by our framework. The relative consumption, which we treat as a proxy for wealth, is affect when the central bank buys longer-term bonds. While the consumption of unrestricted and medium-term restricted households decrease relative to the one in the steady state, the variation of the long-term restricted individuals is positive. Then, at a first moment, we might conclude that the program benefited the long-term restricted investor in detriment to the other types.

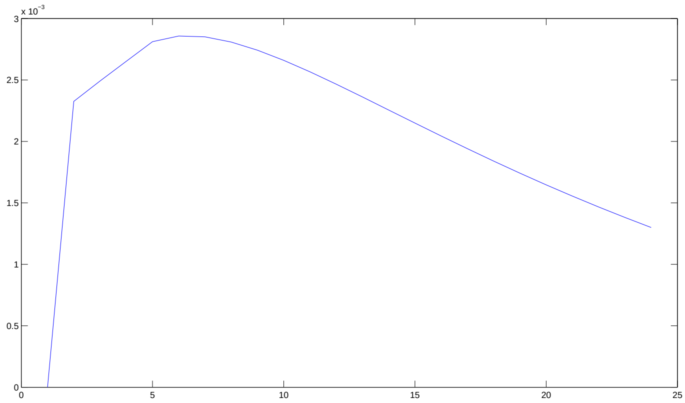

Figure 5: Tax rate response to a simulation of Quantitative Easing

Analyzing these results deeply we could think that we have a counterintuitive result, since it was expected an increase for both restricted households. We expected that QE would affect the consumption positively, as the prices of bonds should go up, reducing yields and increasing individuals wealth. The fall in the unrestricted type consumption was also a surprise, since the short-term rate was hold constant in the first four periods. A possible explanation for this behavior might come from the expected tax rate path reaction (Figure 5).

As the tax rate policy respond to the path of the total debt, this change affect its future behavior. Then, some households might be more affected by these variation in the future and the present value of their wealth may be reduced, causing the consumption fall. In order to investigate this fact, we will track the responses after the QE varying the relative tax paid by each individual.

In our basic calibration we do not set a value for the steady state tax paid by each household type. We only define the whole economy tax. However this relation comes under the calibration of consumption rate among the types. Using the household's budget constraints in the steady state we have that:

$$\frac{C^{r_m}}{C^u} = \frac{\left(\frac{R_{Lm}-1}{R_{Lm}-\kappa_{Lm}}\right) B^{Lm,r_m} + \int w^{r_m} L^{r_m} + \overline{\mathcal{P}^{r_m}} - T^{r_m}}{(R-1)\, B^u + \left(\frac{R_{Lm}-1-\zeta_{Lm}}{R_{Lm}-\kappa_{Lm}}\right) B^{Lm,u} + \left(\frac{R_{Ll}-1-\zeta_{Ll}}{R_{Ll}-\kappa_{Ll}}\right) B^{Ll,u} + \int w^u L^u + \overline{\mathcal{P}^u} - T^u} \tag{4.1}$$

$$\frac{C^{r_l}}{C^u} = \frac{\left(\frac{R_{Ll}-1}{R_{Ll}-\kappa_{Ll}}\right) B^{Ll,r_l} + \int w^{r_l} L^{r_l} + \overline{\mathcal{P}^{r_l}} - T^{r_l}}{(R-1)\, B^u + \left(\frac{R_{Lm}-1-\zeta_{Lm}}{R_{Lm}-\kappa_{Lm}}\right) B^{Lm,u} + \left(\frac{R_{Ll}-1-\zeta_{Ll}}{R_{Ll}-\kappa_{Ll}}\right) B^{Ll,u} + \int w^u L^u + \overline{\mathcal{P}^u} - T^u} \tag{4.2}$$

The above equations allow us to conclude that $\frac{C^{r_m}}{C^u} = f_m\left(\frac{wealth_{r_m}}{wealth_u}, \frac{T^{r_m}}{T^u}\right)$ and $\frac{C^{r_l}}{C^u} =$

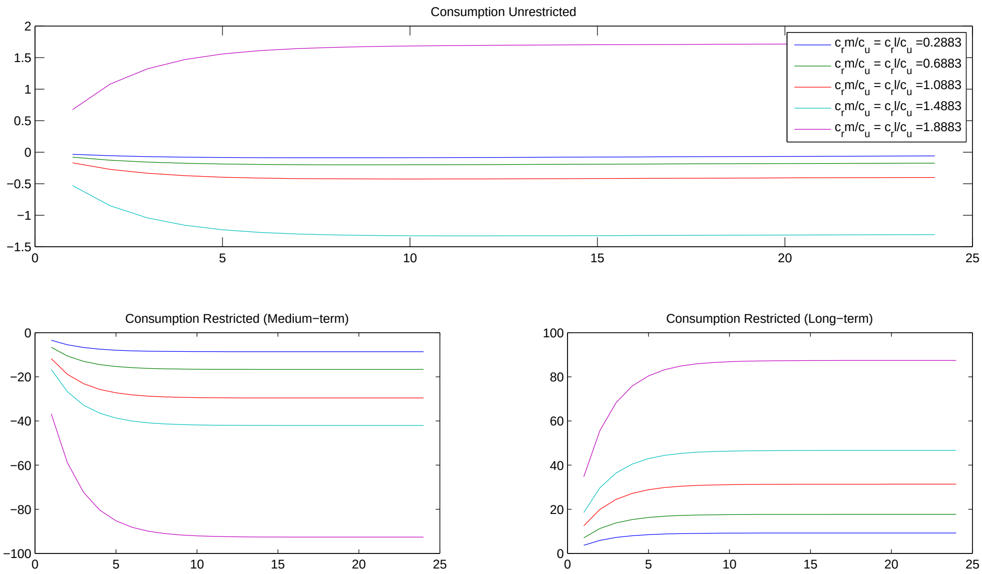

Figure 6: Consumption response to a simulation of Quantitative Easing - varying the relatives taxes for unrestricted households

$f_l\left(\frac{wealth_{r_l}}{wealth_u}, \frac{T^{r_l}}{T^u}\right)$, with $f'_m\left(\cdot, \frac{T^{rm}}{T^u}\right) < 0$ and $f'_l\left(\cdot, \frac{T^{r_l}}{T^u}\right) < 0$. Then, we are able to calibrate the ratio of tributes indirectly by the steady state consumption ratio.

In our base scenario, these consumption rates are $\frac{C^{rm}}{C^u} = \frac{C^{r_l}}{C^u} = 1.0883$. Our next exercise is a simulation of the responses of the consumption under different calibrations to these ratios. Figure 6 shows the results for different tax levels for the unrestricted households. Note that the higher is the relative tax, more negative is the consumption response after the QE for the unrestricted. However, in the the simulation which the relative tax is the lowest (highest consumption rate) the response to the shock became positive. This contradicts our hypothesis.

Figure 7 shows a similar result. Reducing the tax paid by the medium-term restricted household seems to increase her consumption response to QE. However, in some cases, it fell. The long-term restricted household presented the same behavior (Figure 8).

One possible explanation for the non-monotonicity of the agents' reactions is the asset allocation. Even if they are restricted to only one type of bonds, restricted households may be altering their allocation between short and long positions as the model calibration is changed. Then, this issue could be better understood by studying the individual asset positions of each household. However, our model does not allow this type of evaluation. Our framework aggregates the individuals positions in securities, as it is done with the taxes. I this way, there could be a gain to implement the method proposed by Devereux and Sutherland (2011) that permits to tracking these investments positions. That said, this work will not advance on this issue, which will be for a later development.

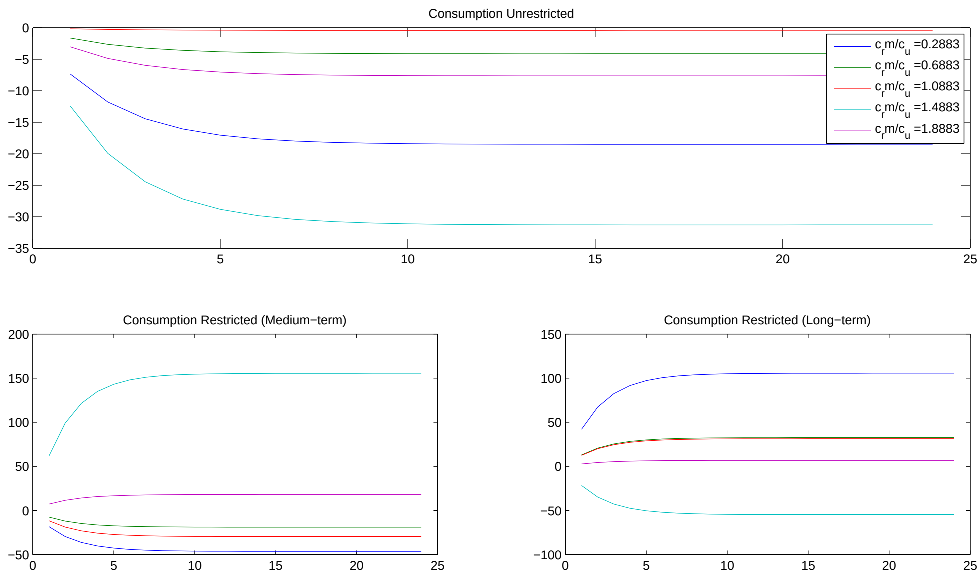

Figure 7: Consumption response to a simulation of Quantitative Easing - varying the relatives taxes for medium-term households

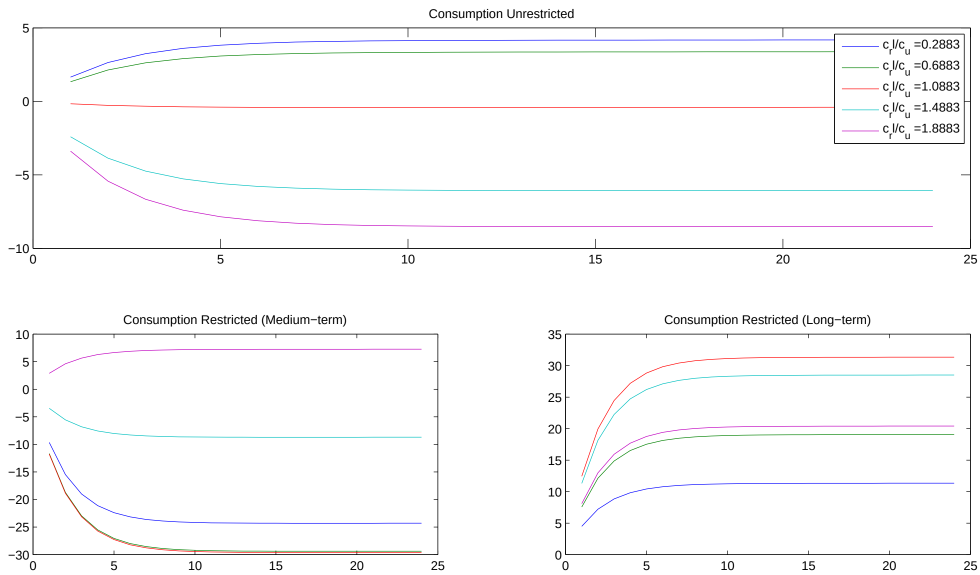

Figure 8: Consumption response to a simulation of Quantitative Easing - varying the relatives taxes for long-term households

We opted to increase the number of bond durations in the economy and for each new bond, a new restricted household type. In this way, we could analyse a more complete economy and the agents' behavior in this environment. After solving and log-linearizing the model, we found some problems on it. The model seems to present some instability. It failed to accomplish the Blanchard-Kahn conditions due to rank problems. Using the Dynare's diagnoses function, the Euler equations of households are accused to have some collinear relations. Note the following log-linearized r_m restricted agent's Euler equation:

$$\hat{\Xi}_t^{r_m} = \frac{R_{r_m}}{R_{r_m} - \kappa_{r_m}} r_{r_m,t} + \mathbb{E}_t \left((1-\tau)\hat{\Xi}_{t+1}^{r_m} - z_{t+1} - \pi_{t+1} - \frac{\kappa_{r_m}}{R_{r_m} - \kappa_{r_m}} r_{r_m,t+1} \right) \tag{4.3}$$

In the model, we have $\tau = 0$, which give us a standard log-linearized Euler equation. When we take $(1 - 10^{-6}) \geq \tau \geq 10^{-6}$, the model works fine and the Dynare's diagnoses function does not accuse any further problem with the model. Based on this findings, we speculate that the problem may be caused by a model's misspecification. However, this is only a guess. Since this issue was not solved yet, we could not present any further results.

These technical problems avoided us to develop further results. We found some preliminary results confirming the hypothesis of wealth distribution. However, these issues did not allow us to expand the experiments in the distributional effects and optimal portfolio purchase.

5 Conclusions

The difficulty of measuring the effects of the QE and its recent implementation gives rise to controversy when this policy is in debate. It is easy to find defenders for both sides of the issue. Some argue that, in this new environment of ZLB, Quantitative Easing policies must be used to avoid the deflationary trap risks, which is more perverse than the possible undesirable effects of QE. On the other side, people say that the consequences of the program may be very harmful, creating an unnecessary risk for the economy. They also defend that QE does not fix the real economy problem.

On the theoretical field, the controversy is not different. On one side, Wallace's proposition and its New Keynesian versions contribute to the infectiveness defenders. However, many works advocating the opposite result were also developed and the theoretical debate is clearly open. The indirect effects of the policy are also not clear. Other points, such as the issues of optimal central bank portfolio purchase and wealth distribution, also deserve the attention and have no conclusive results. In the empirical side, although there is a disagreement in the magnitude of effects, the works seem to converge to the conclusion that the QE policy does reduce the long term yield. However, the effects on other macroeconomics variables are still far from an agreement.

After this work, we can infer that many questions are unanswered and a lot of work still has to be done. There is still demand for answers about: i) the effects of QE not only on the yields, but also in the whole economy; ii) the benefits of the guidance about the purchases in the LSAPs; iii) when should these policies be used; iv) central bank optimal portfolio purchase; and v) the indirect effects of QE on the economy. These and other questions gives the QE research a vast field and ways to be explored.

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Appendix

Table 1: Calibration values used in the exercises

Variable	Value	Variable	Value
100γ	0.6167	ρ_z	0.3857
100π	0.49	ρ_μ	0.7595
β_u	0.9976	ρ_b	0.7595
β_{r_m}	0.9967	ρ_ϕ	0.7595
β_{r_l}	0.9958	ρ_B	0.8135
$100\zeta_{r_m}$	0.0903	ρ_ζ	0.8135
$100\zeta_{r_l}$	0.1806	ρ_g	0.7595
S''	3.9170	σ_z	0.3433
a''	0.1836	σ_{λ_f}	0.3433
h	0.6029	σ_μ	0.3433
σ_u	1.8360	σ_b	0.3433
σ_{r_m}	1.8360	σ_ϕ	0.3433
σ_{r_l}	1.8360	σ_B	0.3433
$100\zeta'_{r_m}$	1.2846	σ_T	0.3433
$100\zeta'_{r_l}$	1.2846	σ_m	0.1700
$\frac{C^{r_m}}{C^u}$	1.0883	σ_ζ	0.1700
$\frac{C^{r_l}}{C^u}$	1.0883	σ_g	0.3433
ω_u	0.7334	ϕ_y	0.3672
ω_{r_m}	0.1333	ϕ_π	1.7026
ω_{r_l}	0.1333	ρ_r	0.7068
$\chi\omega_u$	0.6143	ϕ_T	1.4448
$\chi\omega_{r_l}$	0.1929	ζ_p	0.7595
ν	1.9585	ζ_w	0.7595

Log-linear Approximation

Exogenous Processes

- Technology process:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z,t} \tag{5.1}$$

- Preference for leisure:

$$\hat{\varphi}_{i,t} = \rho_{\varphi,i} \hat{\varphi}_{i,t-1} + \epsilon_{\varphi,i,t} \tag{5.2}$$

  for $i = u, r_m, r_l$

- Price Mark-up shock:

$$\hat{\lambda}_{f,t} = \epsilon_{\lambda,t} \tag{5.3}$$

- Capital adjustment cost process:

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \epsilon_{\mu,t} \tag{5.4}$$

- Intertemporal preference shifter:

$$\hat{b}_{i,t} = \rho_b \hat{b}_{i,t-1} + \epsilon_{b,i,t} \tag{5.5}$$

  for $i = u, r_m, r_l$

- Government spending process:

$$\hat{G}_t = \rho_g \hat{G}_{t-1} + \epsilon_{g,t} \tag{5.6}$$

- Monetary Policy Shocks: $\epsilon_{m,t}$.

- Exogenous risk premium shock:

$$\epsilon_{\zeta,i,t} = \rho_{\zeta,i} \epsilon_{\zeta,i,t-1} + \varepsilon_{\zeta,i,t} \tag{5.7}$$

  for $i = r_m, r_l$

- Fiscal shock $\epsilon_{T,t}$

- Long-term bond supply shock $\epsilon_{B,i,t}$, for $i = r_m, r_l$

Log-linear Approximation

- Real marginal cost

$$\widehat{mc}_t = \alpha \hat{r}_t^k + (1-\alpha)\hat{w}_{z,t} \tag{5.8}$$

- Capital demand

$$\hat{K}_{z,t} = \hat{w}_{z,t} - \hat{r}_t^k + \hat{L}_t \tag{5.9}$$

- Technology

$$\hat{Y}_{z,t} = \alpha \hat{K}_{z,t} + (1-\alpha)\hat{L}_t \tag{5.10}$$

- Effective capital

$$\hat{K}_{z,t} = -z_t + \hat{u}_t + \hat{\bar{K}}_{z,t-1}. \tag{5.11}$$

- Law of motion of capital

$$\hat{\bar{K}}_{z,t} = (1-\delta)e^{-\gamma}(\hat{\bar{K}}_{z,t-1} - z_t) + [1-(1-\delta)e^{-\gamma}](\hat{\mu}_t + \hat{I}_{z,t}). \tag{5.12}$$

- Price setting

$$\hat{X}_t^{pn,i} = (1-\beta_i\zeta_p)(\hat{\Xi}_t^i + \hat{Y}_{z,t} + \hat{\lambda}_{f,t} + \widehat{mc}_t) + \beta_i\zeta_p E_t\left[\frac{1+\lambda_f}{\lambda_f}\pi_{t+1} + \hat{X}_{t+1}^{pn,i}\right] \tag{5.13}$$

$$\hat{X}_t^{pd,i} = (1-\beta_i\zeta_p)(\hat{\Xi}_t^i + \hat{Y}_{z,t}) + \beta_i\zeta_p E_t\left[\frac{1}{\lambda_f}\pi_{t+1} + \hat{X}_{t+1}^{pd,i}\right] \tag{5.14}$$

for $i = u, r_m, r_l$

- Law of motion of prices

$$\pi_t = \frac{1-\zeta_p}{\zeta_p}\left(\frac{1}{\sum_i \omega_i \left(\frac{\Xi^i}{\Xi^u}\frac{1-\beta_u\zeta_p}{1-\beta_i\zeta_p}\right)}\right)\left[\sum_i \omega_i \left(\frac{\Xi^i}{\Xi^u}\frac{1-\beta_u\zeta_p}{1-\beta_i\zeta_p}\right)(\hat{X}_t^{pn,i} - \hat{X}_t^{pd,i})\right] \tag{5.15}$$

for $i = u, r_m, r_l$

- Capital utilization

$$\hat{r}_t^k = \frac{a''(1)}{r^k}\hat{u}_t \tag{5.16}$$

- Law of motion of Q

$$\hat{q}_t = \bar{\beta}e^{-\gamma}E_t\left[r^k\hat{r}^k_{t+1} + (1-\delta)\hat{q}_{t+1}\right] - E_t\hat{z}_{t+1} + \tag{5.17}$$

$$+E_t\left[\sum_i q_i\left(\frac{\beta_i}{\bar{\beta}}\hat{\Xi}^i_{t+1} - \hat{\Xi}^i_t\right)\right]$$

for $i = u, r_m, r_l$

where

$$q_i = \frac{\omega_i(\Xi^i \setminus \Xi^u)}{\sum_j \omega_j(\Xi^j \setminus \Xi^u)},$$

for $j = u, r_m, r_l$

- Law of motion of Q (Rearranged)

$$\begin{aligned}\hat{q}_t = \;& \bar{\beta}e^{-\gamma}E_t\left[r^k\hat{r}^k_{t+1} + (1-\delta)\hat{q}_{t+1}\right] - E_t\hat{z}_{t+1} \\ & +E_t\left[\left(\tfrac{1}{\sum_i \omega_i \Xi_i/\Xi_u}\right)\left(\sum_i \omega_i \tfrac{\Xi_i}{\Xi_u}\left(\tfrac{\beta_i}{\bar{\beta}}\hat{\Xi}^i_{t+1} - \hat{\Xi}^i_t\right)\right)\right]\end{aligned}$$

for $i = u, r_m, r_l$

- Investment decisions

$$0 = \hat{q}_t + \hat{\mu}_t - e^{2\gamma}S''(\hat{z}_t + \hat{I}_{z,t} - \hat{I}_{z,t-1}) + \bar{\beta}e^{2\gamma}S''\mathbb{E}_t\left[z_{t+1} + \hat{I}_{z,t+1} - \hat{I}_{z,t}\right] \tag{5.18}$$

- Marginal utility for each type

$$\hat{\Xi}^j_t = \frac{1}{1-\beta_j h}\left[\left(\hat{b}^j_t - \beta_j h E_t \hat{b}^j_{t+1}\right) - \frac{\sigma_j}{1-h}\left((1+\beta_j h^2)\hat{C}^j_{z,t} - \beta_j h E_t \hat{C}^j_{z,t+1} - h\hat{C}^j_{z,t-1}\right)\right] \tag{5.19}$$

for $j = u, r_m, r_l$.

- Euler equation: Unconstrained, short-term bonds

$$\hat{\Xi}^u_t = r_t + \mathbb{E}_t\left(\hat{\Xi}^u_{t+1} - z_{t+1} - \pi_{t+1}\right) \tag{5.20}$$

- Euler equation: Unconstrained, medium and long-term bonds

$$\hat{\zeta}_{i,t} + \hat{\Xi}^u_t = \frac{R_i}{R_i - \kappa}r_{i,t} + \mathbb{E}_t\left(\hat{\Xi}^u_{t+1} - z_{t+1} - \pi_{t+1} - \frac{\kappa_i}{R_i - \kappa_i}r_{i,t+1}\right) \tag{5.21}$$

for $i = r_m, r_l$

- Euler equation: Constrained, medium and long-term bonds

$$\hat{\Xi}^i_t = \frac{R_i}{R_i-\kappa_i} r_{i,t} + \mathbb{E}_t\left(\hat{\Xi}^i_{t+1} - z_{t+1} - \pi_{t+1} - \frac{\kappa_i}{R_i-\kappa_i} r_{i,t+1}\right) \tag{5.22}$$

for $i = r_m, r_l$

- Wage setting

$$\hat{X}^{wn,j}_t = (1-\zeta_w\beta_j)\left[\hat{b}^j_t + \hat{\varphi}^j_t + (1+\nu)\hat{L}_t + \left(\frac{1+\lambda_w}{\lambda_w}\right)(1+\nu)\hat{w}_{z,t}\right] + \tag{5.23}$$

$$+\zeta_w\beta_j\mathbb{E}_t\left[\frac{1+\lambda_w}{\lambda_w}(1+\nu)(\pi_{t+1}+z_{t+1}) + \hat{X}^{wn,j}_{t+1}\right]$$

for $j = u, r_m, r_l$

$$\hat{X}^{wd,j}_t = (1-\zeta_w\beta_j)\left[\hat{\Xi}^j_t + \hat{L}_t + \left(\frac{1+\lambda_w}{\lambda_w}\right)\hat{w}_{z,t}\right] + \zeta_w\beta_j\mathbb{E}_t\left[\frac{1}{\lambda_w}(\pi_{t+1}+z_{t+1}) + \hat{X}^{wd,j}_{t+1}\right] \tag{5.24}$$

for $j = u, r_m, r_l$

- Law of motion of real wages

$$\hat{w}_{z,t} = \zeta_w(\hat{w}_{z,t-1} - \pi_t - z_t) + (1-\zeta_w)\frac{1}{1+\frac{1+\lambda_w}{\lambda_w}\nu}\left[\sum_i \frac{\omega_i}{\sum_j \omega_j\left(\frac{b_i}{b_j}\frac{\varphi^i}{\varphi^j}\frac{\Xi^j}{\Xi^i}\right)^{\frac{1}{\lambda_w+(1+\lambda_w)\nu}}}[\hat{X}^{wn,i} - \hat{X}^{wd,i}]\right] \tag{5.25}$$

for $i = u, r_m, r_l$ and $j = u, r_m, r_l$

$$\hat{w}_t \equiv (1-\zeta_w)\frac{1}{1+\frac{1+\lambda_w}{\lambda_w}\nu}\left[\sum_i \chi_{\omega_i}[\hat{X}^{wn,i} - \hat{X}^{wd,i}]\right] + \zeta_w(\hat{w}_{z,t-1} - \pi_t - z_t)$$

where $\sum_i \chi_{\omega_i} = 1$ and $i = u, r_m, r_l$

- Government budget constraint

$$\begin{aligned}\hat{B}_{z,t} + \textstyle\sum_i \frac{B^i_z/B_z}{R_i-\kappa_i}\hat{B}^i_{z,t} = \quad & \beta_u^{-1}\left(\hat{B}_{z,t-1} + r_{t-1}\right) + \textstyle\sum_i \frac{B^i_z/B_z}{R_i-\kappa_i}\beta_i^{-1}\hat{B}^i_{z,t-1} \\ & + \textstyle\sum_i \frac{B^i_z/B_z}{R_i-\kappa_i}\frac{(1-e^{-\gamma}\Pi^{-1}\kappa_i)R_i}{R_i-\kappa_i} r_{i,t} + \frac{G_z}{B_z}\hat{G}_{z,t} \\ & - \frac{\hat{Y}_z}{B_z}\hat{T}_{z,t} - \left(\beta_u^{-1} + \textstyle\sum_i \frac{B^i_z/B_z}{R_i-\kappa_i}\beta_i^{-1}\right)(z_t + \pi_t).\end{aligned}$$

with

$$T_{z,t} \equiv T_z + Y_z\hat{T}_{z,t} \Rightarrow \hat{T}_{z,t} = \frac{T_{z,t}}{Y_z} - \frac{T_z}{Y_z}$$

and $i = Lm, Ll$

- Long term bond policy

$$\hat{B}^i_{z,t} - \frac{R_i}{R_i - \kappa_i} r_{i,t} = \rho_{B,i} \left(\hat{B}_{z,t-1} + \sum_j \left(\hat{B}^j_{z,t-1} - \frac{R_j}{R_j - \kappa_j} r_{j,t-1} \right) \right) + \epsilon_{B,i,t} \quad (5.26)$$

for $i = Lm, Ll$ and $j = Lm, Ll$

- Tax rule

$$\frac{\hat{T}_{z,t} - G_z \hat{G}_{z,t}}{T_z - G_z} = \phi_T \left(\hat{B}_{z,t-1} + \sum_j \left(\hat{B}^j_{z,t-1} - \frac{R_j}{R_j - \kappa_j} r_{j,t-1} \right) \right) + \epsilon_{T,t} \quad (5.27)$$

where $j = Lm, Ll$

- Monetary policy

$$r_t = \rho_r r_{t-1} + (1 - \rho_r) \left[\phi_\pi \pi_t + \phi_y \left(\hat{Y}_{z,t} - \hat{Y}_{z,t-4} + \sum_{i=0}^{3} z_{t-i} \right) \right] + \epsilon_{m,t} \quad (5.28)$$

- Term premium

$$\hat{\zeta}_{i,t} = \zeta' \hat{B}^i_{z,t} + \epsilon_{\zeta,i,t} \quad (5.29)$$

for $i = Lm, Ll$

- Aggregate resources constraint

$$\hat{Y}_{z,t} = \sum_i \frac{\omega_i C^i_z}{Y_z} \hat{C}^i_{z,t} + \frac{I_z}{Y_z} \hat{I}_{z,t} + \frac{G_z}{Y_z} \hat{G}_{z,t} + e^{-\gamma} r^k \frac{\bar{K}_z}{Y_z} \hat{u}_t \quad (5.30)$$

where $i = u, r_m, r_l$