

**Textos para
Discussão**

322

Abril
de 2013



CENTRALIZED ALLOCATION IN MULTIPLE MARKETS

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Centralized Allocation in Multiple Markets^{*†}

Daniel Monte[‡]

Norovsambuu Tumennasan[§]

April 8, 2013

Abstract

We study the problem of centralized allocation of indivisible objects in multiple markets. We show that the set of allocation rules that are group strategy-proof and Pareto-efficient are sequential dictatorships. Therefore, the solution of the joint allocation in multiple markets is significantly narrower than in the single-market case. Our result also applies to dynamic allocation problems. Finally, we provide conditions under which the solution of the single-market allocation coincides with the multiple-market case, and we apply this result to the study of the school choice problem with sibling priorities.

JEL classification: C78, D61, D78, I20.

Keywords: Matching, Group Strategy-Proofness, Pareto efficiency.

1 Introduction

A central planner often faces the task of distributing indivisible objects to the agents. For example, municipalities assign public houses to families, education departments allocate students to public schools, and firms allocate projects among workers. This class of assignment problems has been widely studied from many different perspectives. Pápai (2000), for example, shows that if the planner's goal is to implement a Pareto-efficient allocation in the single-market assignment problem, then one way of doing this with a rule that is both group

^{*}We thank the Center for Research in the Foundations of Electronic Markets for financial support.

[†]Tumennasan thanks the Department of Economics, Harvard University, for its hospitality and the Danish Council for Independent Research for travel grant #12-128681.

[‡]Sao Paulo School of Economics -FGV. *E-mail:* daniel.monte@fgv.br

[§]School of Economics and Business, Aarhus University, Denmark. *E-mail:* ntumennasan@econ.au.dk

strategy-proof and reallocation-proof is through the use of a hierarchical exchange rule.^{1,2} Most importantly, this is the *only* way when monetary transfers are not allowed. Not only is this result of theoretical importance, but it also provides important guidance for practitioners and policy makers.³

In reality, agents are typically involved in more than one assignment problem at one time; people who participate in the allocation of public housing, for example, might also have their children enrolled in public schools. In the US there are more than one thousand federally-funded benefit and assistance programs, many of which involve the assignment of indivisible objects. Moreover, a single family may be eligible for many of these programs at the same time.⁴ Additionally, many matching problems take place during multiple periods. One example previously studied in the literature is the allocation of new physicians in the United Kingdom, where each young doctor applies for two successive positions: a medical post and a surgical post (Roth, 1991; Irving, 1998). Another illustrative example is the allocation of courses among the faculty of a department in which each professor teaches one undergraduate and one graduate course.

We study the centralized allocation problem that takes place in multiple markets, where each market may be interpreted either as a different type of object or as a different period. There are n agents and two (or more) markets, and each agent must be assigned at most one object from each market. Agents have preferences over the different bundles, where a bundle is a vector consisting of one object per market. We restrict our attention to the cases in which markets are *independent*, by which we mean that the set of objects available in a particular market is exogenous and not affected by the other markets.

In environments with multiple markets, there might be scope for a mutually beneficial

¹A hierarchical exchange rule is a generalization of the top trading cycles allocation rule and can be described as follows. In the first stage, the planner distributes the objects to the agents; in particular, some agents might receive multiple objects while others might receive none. Then, the top trading cycles algorithm is applied, with each agent pointing to her preferred object, while the object points to its owner. Once all endowed agents receive their objects, the agents who did not participate in this first stage inherit the leftover objects and the top trading cycles algorithm is applied again. The procedure is repeated until all agents are assigned an object.

²Pycia and Ünver (2011) further show that the only rules that are group strategy-proof and Pareto-efficient are the trading cycles rules.

³For example, on April 16, 2012, it was announced that the New Orleans Recovery School District would utilize a version of the top trading cycles allocation rule as the allocation rule for the centralized enrollment of children in public schools (Vanacore, 2012).

⁴The website *benefits.gov* (formerly GovBenefits.gov) is a partnership of seventeen federal agencies as well as other governmental agencies that provides a centralized source of information for many of these assistance programs.

trade between agents even if the allocation is Pareto-efficient within each market. This raises the main question of our paper: is it possible to characterize the set of strategy-proof rules that can implement a Pareto-efficient outcome in the multiple-market problem?

In our main result (Theorem 2) we show that the set of rules that are group strategy-proof and that implement a Pareto-efficient allocation are the sequential dictatorships. These rules generalize the serial dictatorship rule in that the order of agents who choose the objects might be a function of the agents' previous choices.^{5,6}

We assume that agents have separable preferences, which we believe is the most restrictive assumption we can make while keeping with realism. Given the negative nature of our result, it also holds for more general preference domains, namely, for those domains that contain additively separable preferences.

From a technical perspective, one of the main contributions of our paper is to work with a special class of separable preferences, which we denote by the term *lexicographical preferences*. Given the tractability of this class of preferences, we believe that it will be useful in other studies that involve multiple-market allocations.

The results of this paper also have implications for the seemingly unrelated problem of dynamic allocation with overlapping generations of agents. Examples that have been recently studied include the assignment of teachers to public schools (Pereyra, 2013), the allocation of young children to public daycare centers (Kennes et al., 2012), and the housing allocation problem with overlapping generations (Bloch and Cantala, 2011; Kurino, 2013).⁷ In these problems, the set of objects in all periods is identical, which would seem to differentiate their problem from ours. However, in some periods some objects are not available for the younger generation because they are already allocated to the older generation, due to property rights. But in the following period the younger generation from the previous period is now the older generation; thus, the whole set of objects is available for them. If the objects that were not available in the previous period are superior to the ones that were available, then the dynamic allocation problem with overlapping generations is the same from the agents' perspective as

⁵In the single-market case, the sequential dictatorship rules are special cases of Pápai's (2000) hierarchical exchange rules.

⁶Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009) study the problem of multi-unit allocation. They also show that the sequential dictatorship is the only rule that is strategy-proof, Pareto-efficient, and nonbossy. While our result has a similar flavor, none of the proofs utilized in the above-mentioned papers can be applied to our setting because the problem that we study here is substantially different from the multi-unit allocation problem.

⁷Ünver (2010) studies a dynamic matching problem, but under a different setup, where the pool of agents evolves over time.

the allocation problem in multiple markets. Thus, the results of this paper shed some light on why strategy-proof and Pareto-efficient allocation rules are so elusive in dynamic settings with overlapping generations.

Finally, we present two classes of models in which the rules that are group strategy-proof and Pareto-efficient in the single-market case (not necessarily a sequential dictatorship), also satisfy the two properties in the multiple-market setting. The school choice model, for example, has been modeled as a static problem in the literature, which ignores the fact that children typically have a higher priority in the schools where their older siblings are enrolled at. This priority structure generates an intertemporal problem for the applicants. Pereyra (2013), Kennes et al. (2012) and Dur (2011) independently study different versions of the dynamic school choice problem and prove negative results concerning stability and strategy-proofness. In particular, Dur (2011) studies the problem where sibling priorities are considered, and he shows that there is no fair and strategy-proof allocation rule. In contrast to the negative results proved in these recent papers, we show that if the older siblings' preferences are lexicographic on their own allocation, then a rule that is group strategy-proof and Pareto-efficient in a single-market allocation problem, also satisfies these properties in the dynamic version. In particular, the top trading cycles (TTC) rule is group strategy-proof and Pareto-efficient even when sibling priorities are taken into account. The assumption that older children have lexicographic preferences on their own allocation means that at first they care for their own allocation and only then they care for that of their younger siblings. This assumption, we believe, is specially reasonable for middle-school and high-school students. Our result combined with Dur (2011)'s result that the Gale-Shapley deferred acceptance (DA) rule is not strategy-proof presents a setting in which the TTC outperforms the DA in terms of strategy-proofness.⁸

To the best of our knowledge, this is the first paper that provides a complete characterization of centralized allocation in multiple markets without an endowment structure. Konishi et al. (2001) considered the multi-type allocation problem,⁹ but in their work each agent is initially endowed with one object— as in the economy proposed by Shapley and Scarf (1974). Konishi et al. (2001) show that the core may be empty in these multi-type Shapley-Scarf economies and also that there are no Pareto-efficient, individually rational, and strategy-proof rules. Here, since we do not assume an initial endowment structure, we do not impose

⁸Dur (2011) further notes that the DA is strategy-proof if and only if priority structure is acyclic.

⁹See Klaus (2008) for further reference.

the individual rationality constraint, which plays a crucial role in the results obtained by Konishi et al. (2001).

This paper is organized as follows. In the following section we describe the model and state its main assumptions. In Section 3, we describe and define an allocation rule and its main properties. In Section 4, we describe the sequential dictatorship. We prove our main result (Theorem 2) in Section 5. In Section 6, we present two classes of dynamic problems in which the set of group strategy-proof and Pareto-efficient rules is larger than the set of sequential dictatorships. Finally, we conclude the paper in Section 7. The proof of Theorem 2 for the case in which the number of agents is greater than two is left for the Appendix.

2 Model

Let $N = \{1, \dots, n\}$, where $n < \infty$, be the set of agents. There are two types of indivisible objects, A and B , which also stand for the respective sets of objects types. We refer to a pair $(a, b) \in A \times B$ as a bundle. For convenience, we assume that an artificial null object, 0 , is in both sets A and B . Throughout the paper, we assume that $|A \setminus \{0\}| \geq n$ and $|B \setminus \{0\}| \geq n$, i.e., there are enough A - and B -objects to distribute to the agents. An allocation $x = (x_1, \dots, x_n)$ is a list of the assignments for the n agents, where $x_i \in A \times B$. If $x_i = (a, b)$, then agent i is assigned the bundle (a, b) . We write x_i^A (x_i^B) to denote the A -object (B -object) that agent i obtains under allocation x . We refer to $x^A = (x_i^A)_{i \in N}$ and $x^B = (x_i^B)_{i \in N}$ as the A - and B -allocation, respectively. An allocation x is feasible if no object (except the null object) is assigned to more than one agent. Similarly, we define feasible A - and B -allocations. Let X stand for the set of all feasible allocations. The notations X^A and X^B stand for the sets of feasible A - and B -allocations, respectively.

Each agent i has a preference relation R_i over $A \times B$, and $R = (R_i)_{i \in N}$ is the preference profile of the agents. We use the following conventional notations: $R_{-i} := (R_j)_{j \neq i}$; $R_S := (R_j)_{j \in S}$ for all $S \subseteq N$; and $R_{-S} := (R_j)_{j \notin S}$. Throughout the paper we will maintain the following three assumptions on preferences:

Assumption 1 (Strictness). *Each agent's preference relation R_i is strict, i.e., the conditions $(a, b)R_i(\hat{a}, \hat{b})$ and $(\hat{a}, \hat{b})R_i(a, b)$ together imply that $(a, b) = (\hat{a}, \hat{b})$.*

Assumption 2 (Desirability). *For any $(a, b) \neq (0, 0)$ and $i \in N$, $(a, b)R_i(0, 0)$.*

Assumption 3 ((Additive) Separability). *For each agent i , there exists $u_i : A \cup B \rightarrow \mathbb{R}$ such that*

$$(a, b) R_i(\hat{a}, \hat{b}) \text{ for some } a, \hat{a} \in A \text{ and } b, \hat{b} \in B \text{ if and only if } u_i(a) + u_i(b) \geq u_i(\hat{a}) + u_i(\hat{b}).$$

The separability assumption rules out complementarity between A - and B -objects, but note that our main characterization result holds in all preference domains that contain all additively separable preferences. The domain of the additively separable preferences could be too general for some applications,¹⁰ but it is perhaps the most natural domain in a general model such as ours.

We use the notation $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$, where \mathcal{R}_i stands for the set of all preference relations for agent i that satisfy Assumptions 1, 2, and 3.¹¹ For any separable preference relation $R_i \in \mathcal{R}_i$, we define two (strict) preference relations, R_i^A and R_i^B . The A -preference relation R_i^A is defined over A , and $a R_i^A a'$ holds only if $(a, b) R_i(a', b)$, for $\forall b \in B$. The B -preference relation R_i^B is defined in a similar manner. Let $\mathcal{R}^A = \Pi_{i \in N} \mathcal{R}_i^A$ and $\mathcal{R}^B = \Pi_{i \in N} \mathcal{R}_i^B$, where \mathcal{R}_i^A and \mathcal{R}_i^B are the sets of agent i 's A - and B -preferences, respectively.

Definition 1 (Pareto Dominance). *An allocation x (weakly) Pareto-dominates $y \neq x$ if $x_i R_i y_i$ for all i . An allocation x is (strongly) Pareto-efficient if it is feasible and if there exists no feasible $y \neq x$ which Pareto-dominates x . In a similar manner we define A - and B -Pareto-efficient allocations.*

3 Allocation Rule and Its Properties

An allocation rule (or a direct mechanism) $\varphi = (\varphi^A, \varphi^B)$ is a mapping from the set of preferences \mathcal{R} to the set of feasible allocations X . For a given allocation rule, the agents play a revelation game in which each agent's strategy set is \mathcal{R}_i .

We now turn our attention to the properties of the allocation rules that we will consider in this paper. First, we say an allocation rule is efficient if it returns an efficient allocation for each preference profile.

Definition 2 (Pareto Efficiency). *An allocation rule φ is Pareto-efficient if for all $R \in \mathcal{R}$,*

¹⁰See Section 6.

¹¹Although \mathcal{R}_i is common for all the agents, we are using the notations \mathcal{R}_i and \mathcal{R} to keep them consistent with the notations R_i and R .

the allocation $\varphi(R)$ is Pareto-efficient under R . Similarly, φ is A -Pareto-efficient (or B -Pareto-efficient) if φ^A (φ^B) is A -Pareto (B -Pareto) efficient.

An allocation rule is group strategy-proof if in its associated revelation game no group of agents exists so that by misreporting their preferences every agent in the group could obtain a weakly better allocation and at least one agent in the group could obtain a strictly better allocation. Below we present the formal definition of group strategy-proofness.

Definition 3 (Group Strategy-Proofness). *An allocation rule φ is group strategy-proof if, for all $R \in \mathcal{R}$, there exists no $S \subseteq N$ and $\hat{R}_S = (\hat{R}_i)_{i \in S}$ such that*

$$\varphi_i(\hat{R}_S, R_{-S}) R_i \varphi_i(R) \text{ for all } i \in S$$

and

$$\varphi_j(\hat{R}_S, R_{-S}) \neq \varphi_j(R) \text{ for at least one } j \in S.$$

Although the concept of group strategy-proofness is intuitively simple, checking whether a given allocation rule is group strategy-proof, requires that one consider all possible group deviations, which can be a daunting task. Pápai (2000) shows that group strategy-proofness is equivalent to the combination of two properties—individual strategy-proofness and nonbossiness—which are simpler to check.

An allocation rule is strategy-proof if, in its associated revelation game, reporting one's true preferences is a weakly dominant strategy for every agent.

Definition 4 (Strategy-Proofness). *An allocation rule φ is (individually) strategy-proof if, for all $i \in N$, all $R \in \mathcal{R}$, and all $\hat{R}_i \in \mathcal{R}_i$,*

$$\varphi_i(R) R_i \varphi_i(\hat{R}_i, R_{-i})$$

where $R_{-i} = (R_j)_{j \neq i}$.

Finally, an allocation rule is nonbossy if no agent can change the others' allocations without changing her own allocation.

Definition 5 (Nonbossiness). *An allocation rule φ is nonbossy if, for all $R \in \mathcal{R}$, all $i \in N$, and all $\hat{R}_i \in \mathcal{R}_i$,*

$$\varphi_i(R_i, R_{-i}) = \varphi_i(\hat{R}_i, R_{-i}) \implies \varphi(R_i, R_{-i}) = \varphi(\hat{R}_i, R_{-i}).$$

Although Pápai (2000)'s result – that an allocation rule is group strategy-proof if and only if it is also strategy-proof and nonbossy – was proven in that paper for single-market allocation problems, the proof extends to our domain.

Lemma 1 (Lemma 1 of Pápai (2000)). *Any allocation rule φ is group strategy-proof if and only if it is individually strategy-proof and nonbossy.*

We now introduce another useful notion which is closely related to Maskin Monotonicity as used in the implementation literature.

Definition 6 (Monotonicity). *For a given allocation rule φ , we say that a preference profile R^1 is a φ -monotonic change of R if, for each agent i , the relative ranking of the allocation $\varphi_i(R)$ weakly improves under R^1 , specifically,*

$$\{(a, b) \in A \times B : \varphi_i(R) R_i(a, b)\} \subseteq \{(a, b) \in A \times B : \varphi_i(R) R_i^1(a, b)\}.$$

An allocation rule φ is monotonic if, for each R and for any of its φ -monotonic changes R^1 , φ yields the same allocation for both R and R^1 , i.e., $\varphi(R^1) = \varphi(R)$.

In words, a φ -monotonic allocation rule must have the following property: if each agent's lower contour set of $\varphi(R)$ expands (weakly) when going from preference profile R to R^1 , then φ prescribes the same allocations for both R and R^1 . For a given allocation rule φ , checking whether a preference profile is a φ -monotonic change of another is simple. The next lemma, which is from Svensson (1999), establishes that each nonbossy and strategy-proof allocation rule φ is monotonic.

Lemma 2 (Lemma 1 of Svensson (1999)). *If an allocation rule φ is nonbossy and strategy-proof, then φ is monotonic.*

Lemmas 1 and 2 establish the equivalency of group strategy-proofness and monotonicity.

Below we present a simple example that demonstrates that an allocation that is Pareto-efficient in each single market might fail efficiency when we consider the joint-allocation problem. We will return to this example in the following section.

Example 1 (Failure of Pareto Efficiency). *Let $n = 2$, $A = \{a_1, a_2\}$, and $B = \{b_1, b_2\}$. Consider the preference profile \tilde{R} in which $a_1 \tilde{R}_1^A a_2$ and $b_2 \tilde{R}_1^B b_1$ for both $i = 1, 2$. Also assume that $(a_2, b_2) \tilde{R}_1(a_1, b_1)$ and $(a_1, b_1) \tilde{R}_2(a_2, b_2)$. The allocation $((a_1, b_1), (a_2, b_2))$ is Pareto-efficient within each market, but it is clearly Pareto-dominated by $((a_2, b_2), (a_1, b_1))$.*

Before we move on, let us introduce a special class of allocation rules: an allocation rule $\varphi = (\varphi^A, \varphi^B)$ is *market-independent* if the A - and B -allocations depend only on A - and B -preference profiles, i.e., if there exist A - and B -allocation rules $\phi^A : \mathcal{R}^A \rightarrow X^A$ and $\phi^B : \mathcal{R}^B \rightarrow X^B$ such that $\varphi^A(R) = \phi^A(R^A)$ and $\varphi^B(R) = \phi^B(R^B)$. For any market specific allocation rule (i.e., ϕ^A or ϕ^B) we can define the notions of market specific efficiency, group strategy-proofness, strategy-proofness, and nonbossiness as these notions are defined for allocation rules in single markets.

We note here that because the agents have separable preferences in our setting, any market-independent allocation rule that consist of two group strategy-proof rules is also group strategy-proof in the multi-market setting.¹² In this sense, as long as the preferences are separable, achieving group strategy-proofness in multiple-market settings is no more difficult than achieving them in single-market settings. On the other hand, Pareto efficiency is much harder to achieve in a multiple-market setting than in a single-market setting, as noted in the example above. Therefore, we conclude that efficiency is the driving force for why the set of allocation rules that are group strategy-proof and Pareto-efficient narrows in multiple-market settings.

4 Sequential Dictatorship

In this section, we define a sequential-dictatorship allocation rule which is group strategy-proof and Pareto-efficient. In this allocation rule, the determination of who is the first agent to choose is exogenous and this agent is free to choose any bundle from the set of all bundles. The first agent's choice determines who will be the second agent to choose, and this agent will choose any bundle from the set of available bundles, which excludes the bundle chosen by the first agent. Then the second agent's choice determines who will be the third agent to choose. This agent will choose from the set of bundles available, which excludes the bundles chosen by the first and second agents. The process continues until all agents have made their

¹²This differs from the problems with multi-unit goods (see, for example, Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009)). In that literature, it is shown that natural adaptations of allocation rules that are strategy-proof in the single-object case might not be strategy-proof in the multi-unit case. For concreteness, consider the *HBS draft allocation rule* described by Budish and Cantillon (2012), in which the choosing order of the agents is reversed at every round. That allocation rule is not strategy-proof. Now consider a version of that allocation rule in our setting: Agents choose in market A according to some exogenous ordering, but in market B they choose according to the exact opposite ordering of market A. This allocation rule is strategy-proof, but fails efficiency. The same intuition applies to the setting of Manea (2007), in which he shows that the serial dictatorship is manipulable.

choices. Below we define the sequential-dictatorship algorithm formally.

For any nonempty subsets $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$ and any preferences of agent i , R_i , we define $\tau(R_i, \bar{A} \times \bar{B})$ as the most preferred bundle of agent i (under preferences R_i) in the set $\bar{A} \times \bar{B}$. Because preferences are separable, it must be the case that

$$\tau(R_i, \bar{A} \times \bar{B}) = (\tau^A(R_i^A, \bar{A}), \tau^B(R_i^B, \bar{B})),$$

where $\tau^A(R_i^A, \bar{A})$ and $\tau^B(R_i^B, \bar{B})$ are the most preferred A - and B - objects of agent i (under R_i^A and R_i^B) in sets \bar{A} and \bar{B} , respectively.

Let $\pi : N \times \mathcal{R} \rightarrow \{1, \dots, n\}$ be a permutation of N that depends on the preference profiles of the agents. In addition, let $i_j(R, \pi)$ be the agent for whom $\pi(i_j(R, \pi), R) = j$; that is, if the preference profile is R , then $i_j(R, \pi)$ is the individual who will make the j th choice under the specific permutation $\pi(\cdot, R)$.

When the order of choice is given according to $\pi(\cdot, R)$, agent $i_1(R, \pi)$ chooses her most preferred bundle from $A \times B$ and then agent $i_2(R, \pi)$ chooses her most preferred bundle from the remaining set of bundles, and so on. The set of A -type objects that are available for agent $i_j(R, \pi)$, namely, the set $A_j(R, \pi)$, is determined as follows:

$$\begin{aligned} A_1(R, \pi) &= A \\ A_2(R, \pi) &= A_1(R, \pi) \setminus \left\{ \tau^A(R_{i_1(R, \pi)}^A, A_1(R, \pi)) \right\} \\ &\vdots \\ A_n(R, \pi) &= A_{n-1}(R, \pi) \setminus \left\{ \tau^A(R_{i_{n-1}(R, \pi)}^A, A_{n-1}(R, \pi)) \right\}. \end{aligned}$$

We define $B_j(R, \pi)$ in a similar manner.

With the above notations, the choice made by agent $i_j(R, \pi)$ is $\tau(R_{i_j(R, \pi)}, A_j(R, \pi) \times B_j(R, \pi))$ when the order of the agents is given by $\pi(\cdot, R)$. Now we are ready to define the sequential-dictatorship allocation rule.

Definition 7 (Sequential Dictatorship). *An allocation rule φ is a sequential-dictatorship allocation rule if there is a permutation $\pi : N \times \mathcal{R} \rightarrow \{1, \dots, n\}$ such that all of the following hold:*

1. *For all $R \in \mathcal{R}$ and $j = 1, \dots, n$, it must be that $\varphi_{i_j(R, \pi)}(R) = \tau(R_{i_j(R, \pi)}, A_j(R, \pi) \times B_j(R, \pi))$.*
2. *If $i = i_1(R, \pi)$ for some R , then $i = i_1(R', \pi)$ for all $R' \in \mathcal{R}$.*

3. If $i = i_j(R, \pi)$ for some R and $j = 2, \dots, n$, then $i = i_j(R', \pi)$ for all R' in which $\varphi_{i_k(R, \pi)}(R) = \varphi_{i_k(R', \pi)}(R')$ for all $k \leq j - 1$.

The first item in the definition above means that each agent must be assigned to her most preferred bundle among the available bundles. The second item requires that there be only one agent who makes the first choice. The third item requires that if the first $j - 1$ agents make the same choices under two different preference profiles, then the j th agent who makes a choice under the two different profiles must be the same individual.

The standard serial-dictatorship allocation rule is the one in which $\pi(\cdot, R)$ is constant for all $R \in \mathcal{R}$. That is, the order in which the agents make their choices is the same, regardless of the preference profile.

Example 2 (Example 1 revisited). *For preference profile \tilde{R} defined in example 1, the sequential dictatorship allocation rule yields the allocation $((a_1, b_2), (a_2, b_1))$ if agent 1 is the first to choose, and $((a_2, b_1), (a_1, b_2))$ if agent 1 is the second to choose. Clearly, in both cases the final allocation is efficient.*

Remark 1. *In the example above, observe that for preference profile \tilde{R} , the sequential-dictatorship allocation rule never yields the allocation $((a_2, b_2), (a_1, b_1))$ which is also Pareto-efficient. This result contrasts with the result in the allocation problem in single markets, in which all Pareto-efficient allocations are reached through some serial dictatorship (Abdulkadiroğlu and Sönmez, 1999).*

The sequential-dictatorship rule is group strategy-proof and Pareto-efficient in the multiple market allocation setting, as will be discussed in the next section. Moreover, from example 1 above we conclude that the sequential dictatorship does not span the entire set of Pareto-efficient allocations in the joint problem. Therefore, a natural question is whether there is any other rule that is group strategy-proof and also Pareto-efficient. The main contribution of our paper is to show that sequential dictatorships are the *only* rules that satisfy group strategy-proofness and Pareto efficiency.

5 Efficiency and Strategy-Proofness

In this section we characterize the allocation rules that are group strategy-proof and Pareto-efficient. First, let us note that any sequential-dictatorship allocation rule is group strategy-proof and Pareto-efficient, which we state as a theorem below.

Theorem 1. *The sequential dictatorship allocation rules are group strategy-proof and Pareto-efficient.*

Now we turn our attention to the main result of the paper: *only* the sequential-dictatorship allocation rules satisfy nonbossiness, strategy-proofness and Pareto-efficiency.

One of the challenges that we face in our main proof is that all the preference profiles considered in it must be separable. Specifically, if R_i and R_i^1 differ in the relative rankings of some (a, b) and (a, \hat{b}) , then they must also differ in the ranking of any two alternatives (\bar{a}, b) and (\bar{a}, \hat{b}) . Therefore, we do not have the luxury of considering two profiles that differ in the rankings of only two bundles. Nevertheless, it turns out that there is a class of preferences that are rather tractable. We define this class of preferences as follows:

Definition 8 (Generalized lexicographical preference). *A preference relation of agent i , R_i , is a (generalized) lexicographical preference if there exists a bijection $\eta_i : A \cup B \rightarrow \{1, 2, \dots, |A \cup B|\}$, which we call a lexicographical ordering, such that whenever $(a, b) R_i (\bar{a}, \bar{b})$ for some $(a, b), (\bar{a}, \bar{b}) \in A \times B$, one of the following conditions is satisfied:*

- (i) $\min\{\eta_i(a), \eta_i(b)\} < \min\{\eta_i(\bar{a}), \eta_i(\bar{b})\}$;
- (ii) $\min\{\eta_i(a), \eta_i(b)\} = \min\{\eta_i(\bar{a}), \eta_i(\bar{b})\} \& \max\{\eta_i(a), \eta_i(b)\} \leq \max\{\eta_i(\bar{a}), \eta_i(\bar{b})\}$.

If the preference relation of agent i is lexicographical, then we use the notation L_i to denote i 's preference relation. The notation $(a, b) L_i (\bar{a}, \bar{b})$ means that either agent i (strictly) prefers (a, b) to (\bar{a}, \bar{b}) or $(a, b) = (\bar{a}, \bar{b})$. When L_i is a lexicographical preference relation associated with ordering η_i , we usually write $L_i : \eta_i^{-1}(1), \eta_i^{-1}(2), \dots, \eta_i^{-1}(|A \cup B|)$. Each lexicographical preference relation can therefore be represented by only one list of objects in $A \cup B$, which is itself a huge simplification in that even separable preferences have to be represented by a list of pairs in $A \times B$.

Before we move on, let us consider an example of lexicographical preferences. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$. Consider $L_i : a_1, b_1, a_2, b_2, \dots, a_m, b_m$. Then

$$\begin{aligned}
& (a_1, b_1) L_i (a_1, b_2) L_i \dots L_i (a_1, b_m) L_i \\
& (a_2, b_1) L_i (a_3, b_1) L_i \dots L_i (a_m, b_1) L_i \\
& (a_2, b_2) L_i (a_2, b_3) L_i \dots L_i (a_2, b_m) L_i \\
& (a_3, b_2) L_i (a_4, b_2) L_i \dots L_i (a_m, b_2) L_i \\
& \vdots
\end{aligned}$$

We will use the lexicographical preferences extensively in the proof of our main theorem because they are separable and are tractable. First, let us show the separability of the lexicographical preferences in the following lemma.

Lemma 3 (Separability). *Any lexicographical preferences L_i is separable.*

Proof. Let η_i be an ordering associated with L_i . For all $c \in A \cup B$, set $u_i(c) = 2^{-\eta_i(c)}$. Consider any $a, \hat{a} \in A$ and $b, \hat{b} \in B$. One can easily verify that $(a, b)L_i(\hat{a}, \hat{b})$ if and only if $u_i(a) + u_i(b) \geq u_i(\hat{a}) + u_i(\hat{b})$. \square

If one has lexicographical preferences, then it is easy to identify one's most preferred bundle in any given nonempty subset $\bar{A} \times \bar{B} \subseteq A \times B$.

Lemma 4. *Let L_i be a lexicographical preference relation associated with ordering η . Then, for any nonempty subsets $\bar{A} \subseteq A$ and $\bar{B} \subseteq B$,*

$$\tau(L_i, \bar{A} \times \bar{B}) = \left(\arg \min_{a \in \bar{A}} \eta_i(a), \arg \min_{b \in \bar{B}} \eta_i(b) \right).$$

Proof. The proof follows directly from the definition of lexicographical preferences. \square

With lexicographical preferences it is also easy to determine whether a preference profile is a φ -monotonic change of another.

Lemma 5. *Consider an allocation rule φ and a lexicographical preference profile $L = (L_i)_{i \in N}$ associated with $\eta = (\eta_i)_{i \in N}$. Let \bar{L} be a lexicographical preference profile associated with $\bar{\eta} = (\bar{\eta}_i)_{i \in N}$ that satisfies the following three conditions:*

- (i) *If $\bar{\eta}_i(a) < \bar{\eta}_i(\varphi_i^A(L))$ for any $a \in A$ and $i \in N$, then $\eta_i(a) < \eta_i(\varphi_i^A(L))$.*
- (ii) *If $\bar{\eta}_i(b) < \bar{\eta}_i(\varphi_i^B(L))$ for any $b \in B$ and $i \in N$, then $\eta_i(b) < \eta_i(\varphi_i^B(L))$.*
- (iii) *If $\eta(\varphi_i^A(L)) < \eta(\varphi_i^B(L))$ for any i , then $\bar{\eta}(\varphi_i^A(L)) \leq \bar{\eta}(\varphi_i^B(L)) + 1$. Similarly, if $\eta(\varphi_i^B(L)) < \eta(\varphi_i^A(L))$ for any i , then $\bar{\eta}(\varphi_i^B(L)) \leq \bar{\eta}(\varphi_i^A(L)) + 1$.*

Then \bar{L} is an φ -monotonic change of L .

Proof. The proof follows directly from the definitions of lexicographical preferences and φ -monotonic change. \square

We are now ready to present the main result of our paper.

Theorem 2. *Any allocation rule that is group strategy-proof and Pareto-efficient is a sequential dictatorship.*¹³

Proof. Assume that $n = 2$ and fix an efficient, nonbossy, and strategy-proof allocation rule φ .

Claim 1. *For any $(a, b) \in A \times B$, there exists $i \in N$ such that $\varphi_i(R) = (a, b)$ for all R in which $(a, b) = \tau(R_i, A \times B)$.*

Proof of Claim 1. Without loss of generality let $a = a_1$ and $b = b_1$. Consider two lexicographical preferences, L_1^1 and L_2^1 , in which the first four objects in their respective lexicographical orderings are as follows:

$$\begin{aligned} L_1^1 &: b_1, a_1, b_2, a_2; \\ L_2^1 &: b_2, a_1, b_1, a_2. \end{aligned}$$

Because φ is efficient, it must be true that either (1) $\varphi(L_1^1, L_1^2) = ((a_1, b_1), (a_2, b_2))$, or (2) $\varphi(L_1^1, L_2^1) = ((a_2, b_1), (a_1, b_2))$.

Suppose that Case (1) occurs. We claim that if $(a_1, b_1) = \tau(R_1, A \times B)$ for some R_1 , then $\varphi_1(R_1, R_2) = (a_1, b_1)$ for any R_2 .

Consider two more lexicographical preferences, L_1^2 and L_2^2 , in which the first four objects in their respective lexicographical orderings are follows:

$$\begin{aligned} L_1^2 &: a_1, b_1, b_2, a_2; \\ L_2^2 &: a_1, b_2, b_1, a_2. \end{aligned}$$

We now show that

$$\varphi(L_1^1, L_2^1) = (L_1^2, L_2^1) = \varphi(L_1^1, L_2^2) = \varphi(L_1^2, L_2^2). \quad (1)$$

Because (L_1^2, L_2^1) is a φ -monotonic change of (L_1^1, L_2^1) (Lemma 5), we obtain the first equality above due to Lemma 2. Consider now $\varphi(L_1^1, L_2^2)$. Observe that $\varphi_2(L_1^1, L_2^2) \neq (a_1, b_2)$;

¹³For the special case of two markets with two goods in each market and two agents, there is an alternative proof that makes use of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975). Proof upon request.

otherwise, $\varphi_2(L_1^1, L_2^2)L_2^1\varphi_2(L_1^1, L_2^1)$, which is a contradiction with the fact that φ is strategy-proof. This and the efficiency of φ imply that either $\varphi_2(L_1^1, L_2^2) = (a_2, b_2)$ or $\varphi_2(L_1^1, L_2^2) = (a_1, b_1)$. If the latter case occurs, then by the efficiency of φ , it must be the case that $\varphi_1 = (a_2, b_2)$. Then $\varphi(L_1^1, L_2^2)$ is Pareto-dominated by $((a_2, b_1), (a_1, b_2))$ under (L_1^1, L_2^2) , a contradiction. Thus, $\varphi_2(L_1^1, L_2^2) = (a_2, b_2)$. Then the nonbossiness of φ implies that $\varphi(L_1^1, L_2^2) = \varphi(L_1^1, L_2^1)$, the second equality in (1). Consider now $\varphi(L_1^2, L_2^2)$, which is a φ -monotonic change of (L_1^1, L_2^2) . Thus, $\varphi(L_1^2, L_2^2) = \varphi(L_1^1, L_2^1)$, due to Lemma 2, the third equality in (1).

Now consider lexicographical preference relation L_2^3 in which the first four objects in its lexicographical ordering are as follows:

$$L_2^3 : a_1, b_1, b_2, a_2.$$

We now show that $\varphi(L_1^1, L_2^3) = \varphi(L_1^1, L_2^2)$. Observe here that $\varphi_2^A(L_1^1, L_2^3) \neq a_1$; otherwise, $\varphi_2(L_1^1, L_2^3)L_2^2\varphi_2(L_1^1, L_2^2)$, which is a contradiction with the strategy-proofness of φ . Then, due to the efficiency of φ , it must be the case that $\varphi_2(L_1^1, L_2^3)$ is either (a_2, b_1) or (a_2, b_2) . If $\varphi_2(L_1^1, L_2^3) = (a_2, b_1)$, then $((a_2, b_1), (a_1, b_2))$ Pareto-dominates $\varphi(L_1^1, L_2^3)$ under (L_1^1, L_2^3) , which is a contradiction. Hence, $\varphi_2(L_1^1, L_2^3) = (a_2, b_2)$. Then the nonbossiness of φ implies that $\varphi(L_1^1, L_2^3) = \varphi(L_1^1, L_2^1)$.

Consider any R_2 . We claim that $\varphi_2^A(L_1^1, R_2) \neq a_1$. Suppose otherwise, i.e., suppose that $\varphi_2(L_1^1, R_2) = (a_1, b)$ for some $b \in B$. Then, $\varphi_2(L_1^1, R_2)L_2^3\varphi_2(L_1^1, L_2^3)$, which is a contradiction with the fact that φ is strategy-proof. Thus, $\varphi_2^A(L_1^1, R_2) \neq a_1$. Similarly, we can show that $\varphi_2^B(L_1^1, R_2) \neq b_1$. Thus, by efficiency, $\varphi_1(L_1^1, R_2) = (a_1, b_1)$.

Finally, consider any R_1 in which $(a_1, b_1) = \tau(R_1, A \times B)$. Clearly, (R_1, R_2) is a φ -monotonic change of (L_1^1, R_2) . Thus, it must be true that $\varphi_1(R_1, R_2) = (a_1, b_1)$, thanks to Lemma 2. This proves that if case (1) occurs, then $\varphi_1(R) = (a_1, b_1)$ for all R in which $(a_1, b_1) = \tau(R_1, A \times B)$.

Suppose now that Case (2) occurs. We claim that if $(a_1, b_1) = \tau(R_2, A \times B)$ for some R_2 , then $\varphi_1(R_1, R_2) = (a_1, b_1)$ for any R_1 .

For this case a proof similar to that used in Case (1) implies that for all R_2 in which $(a_1, b_2) = \tau(R_2, A \times B)$, it must be the case that $\varphi_2(R_1, R_2) = (a_1, b_2)$ for all R_1 . Fix \bar{R}_2 in which $(a_1, b_2) = \tau(\bar{R}_2, A \times B)$.

Now consider two lexicographical preferences, \bar{L}_1^1 and \bar{L}_2^1 , in which the first four objects

in their respective lexicographical orderings are follows:

$$\begin{aligned}\bar{L}_1^1 &: b_2, a_1, b_1, a_2; \\ \bar{L}_2^1 &: b_1, a_1, b_2, a_2.\end{aligned}$$

By efficiency, it must be the case that either (i) $\varphi(\bar{L}_1^1, \bar{L}_2^1) = ((a_1, b_2), (a_2, b_1))$ or (ii) $\varphi(\bar{L}_1^1, \bar{L}_2^1) = ((a_2, b_2), (a_1, b_1))$. In case (i), again using the logic from case (1), we obtain the result that for all R_1 in which $(a_1, b_2) = \tau(R_1, A \times B)$, it must be true that $\varphi_1(R_1, R_2) = (a_1, b_2)$. Fix \bar{R}_1 such that $(a_1, b_2) = \tau(\bar{R}_1, A \times B)$, and that (\bar{R}_1, \bar{R}_2) . But the agents cannot both obtain (a_1, b_2) which means that case (i) cannot occur. In case (ii), using a similar logic as in case (1), we obtain that $\varphi_2(R) = (a_1, b_1)$ for all R in which $(a_1, b_1) = \tau(\bar{R}_2, A \times B)$. This completes the proof that if case 2 occurs, then $\varphi_2(R) = (a_1, b_1)$ for all R in which $(a_1, b_1) = \tau(\bar{R}_2, A \times B)$.

This completes the proof of Claim 1.

To complete the proof of the theorem for the $n = 2$ case, fix any (a, b) . By claim 1 there exists an agent i such that $\varphi_i(R) = (a, b)$ for all R with $\tau(R_i, A \times B) = (a, b)$. We need to show that $\varphi_i(R) = \tau(R_i, A \times B)$ for all $R \in \mathcal{R}$. Fix any (a, \bar{b}) . Then by claim 1 there must exist j such that $\varphi_j(R) = (a, \bar{b})$ for all R with $\tau(R_j, A \times B) = (a, \bar{b})$. If $(a, \bar{b}) = (a, b)$, then clearly $i = j$. If $(a, \bar{b}) \neq (a, b)$, we need to show that $i = j$. If $i \neq j$, consider \bar{R} such that $(a, b) = \tau(\bar{R}_i, A \times B)$ and $(a, \bar{b}) = \tau(\bar{R}_j, A \times B)$. Then it must be the case that $\varphi_i^A(\bar{R}) = \varphi_j^A(\bar{R}) = a$, which is a contradiction. Thus, $i = j$.

A similar proof shows that for any (\bar{a}, \bar{b}) , it must be true that $\varphi_i(R) = (\bar{a}, \bar{b})$ for all R with $\tau(R_i, A \times B) = (\bar{a}, \bar{b})$. Given that we picked arbitrary (a, b) and (\bar{a}, \bar{b}) , it must be the case that $\varphi_i(R) = \tau(R_i, A \times B)$ for all R . Combining this result with the efficiency of φ completes the proof of the theorem for the $n = 2$ cases. The proof for the $n \geq 3$ cases are in the appendix. \square

We conclude this section by showing that each of nonbossiness, strategy-proofness, and Pareto efficiency plays an indispensable role for Theorem 2. Below we present three examples in which a non-sequential dictatorship rule satisfies two of the three properties.

Example 3 (Allocation Rule that is Strategy-proof and Nonbossy but not Efficient). *Consider a constant allocation rule, (i.e., a rule that does not depend on the preference profiles*

of the agents). Clearly, this rule is both strategy-proof and nonbossy, but not necessarily Pareto-efficient.

Example 4 (Allocation Rule that is Efficient and Nonbossy but not Strategy-proof). Recall Example 1 and consider an allocation rule φ which differs from the serial dictatorship rule in which agent 1 is the first agent to select only in that $\varphi(\tilde{R}) = ((a_2, b_2), (a_1, b_1))$. One can easily check that φ is both Pareto-efficient and nonbossy, but it is not a serial dictatorship rule. Thus, Theorem 2 implies that φ is not strategy-proof.

Example 5 (Allocation Rule that is Efficient and Strategy-proof but not Nonbossy). Let $n \geq 3$ and consider the following allocation rule, φ , which is a slight modification of a sequential dictatorship rule: agent 1 is the first agent to select, and agent 2 (agent 3) is the second agent to make a selection only if agent 1's second most preferred bundle contains a_1 (a_2 or a_3). One can easily check that φ is both Pareto-efficient and strategy-proof but not nonbossy.

6 Dynamic Matching Problems

In this section, we present positive results with respect to two classes of dynamic matching problems. First, we consider the class of dynamic problems without overlapping generations and in which the set of objects to be allocated is the same in all periods. Note that throughout the paper and, in particular, in Theorem 2, we have assumed that markets are independent, in the sense that the sets of objects in both markets are exogenously and independently given. However, in the dynamic-allocation problem in which the two sides of the market remain constant across periods—in particular, when there are no overlapping generations—the independence assumption is not satisfied. The set of objects available to the agents in a particular period is the same as the set of objects available in the other period. That is, $A = B$. In this special case, rules other than the sequential dictatorship might also be group strategy-proof and Pareto-efficient. In particular, any trading cycle rule adapted to this setting is group strategy-proof and Pareto-efficient, as we argue in the example below.

Example 6 (Dynamic Matching without OLG). Suppose that the set of objects is the same in both markets, i.e., $A = B$; and also that the preference domain is such that if $(a, b) R_i (c, d) \Rightarrow (b, a) R_i (c, d)$, for all $a, b, c, d \in A = B$ and for all $i = 1, \dots, n$. Moreover, let us restrict our attention to allocation rules in which each agent reports a list that represents

her preference profile over bundles consisting of repetition of the objects; for example, the list a, b, c, d for agent i , represents $(a, a) R_i (b, b) R_i (c, c) R_i (d, d)$. Now, consider a trading cycle rule in which, given the reported lists (respecting the restriction above), the allocation rule assigns the pairs of objects to the agents using the same procedure as the trading cycle rules in the single-market case. With this allocation rule each agent is assigned the same object in both markets. Thus, within this special class of problems, this allocation rule is group strategy-proof and Pareto-efficient. This special class of allocation problems might fit dynamic-matching problems without overlapping generations. When there are overlapping generations, however, the set of objects available to a cohort might differ in periods t and $t + 1$ if some of the objects had been previously allocated to the older generations.

We now consider a second class of dynamic problems that fit our multiple-market framework. Consider the well-known school choice problem (Abdulkadiroğlu and Sönmez (2003)). This problem has been modeled as a static matching problem, and two allocation rules have been proposed for this class of problems: the Gale-Shapley deferred acceptance allocation rule (which has been adopted by New York and Boston school systems), and the top trading cycles allocation rule (recently adopted by the city of New Orleans). One feature of the school choice problem that has been largely ignored is the fact that at the time of her application, an older child might have dynamic incentives if she has younger siblings.¹⁴ Namely, a school usually gives high priority to children that have siblings already enrolled in that school, which generates an intertemporal problem at the time of the assignment: the school where the older sibling is enrolled affects the priority structure that her younger siblings face. The results in the dynamic school choice problem are mostly negative. For example, Kennes et al. (2012) show that there are no strategy-proof and stable allocation rules and also that the top trading cycles is neither Pareto-efficient nor strategy-proof. Dur (2011) shows that there are no fair and stable allocation rules in the dynamic school choice problem. Here we prove a positive result for a particular preference domain, which we denote by *initial-period lexicographic preferences*—a subset of the domain of separable preferences.

We interpret the initial period as being market A and the subsequent period as market B . For the expositional simplicity we assume that there are n families with two children—one in A and one in B .

Definition 9 (Initial-period lexicographical preference). *We say that a preference ranking*

¹⁴See Dur (2011) for a recent working paper on this topic.

of agent i , R_i , is initial-period lexicographic if, for $\forall a, c \in A$ and $\forall b, d \in B$, we have that $(a, b) R_i (c, d)$ if and only if one of the following two conditions hold:

- (i) $(a, b') R_i (c, d')$ for all $b', d' \in B$, if $a \neq c$; or
- (ii) $(\hat{a}, b) R_i (\hat{a}, d)$ for all $\hat{a} \in A$, if $a = c$.

Assume that preferences are initial-period lexicographic, where A is the initial period, in which a family's oldest child is entering the allocation rule. Observe here that initial-period lexicographical preferences are additively separable. In addition, R_i^A is allowed to be different from R_i^B , which we believe is a reasonable assumption, because R_i^A is the older sibling's preference while R_i^B is the younger sibling's preference.

We consider allocation rules in which the older siblings' allocation is only dependent on the reported preferences of their generation, R^A , but the younger siblings' allocation is dependent on their older siblings' allocations and on the reported preferences of their generation, R^B . This formulation is consistent with the real life sibling priorities because the younger siblings' priorities at schools, which are dependent on the older sibling's allocations, and their preferences are inputs to any allocation rule for the younger siblings. To formalize this, fix functions $f^A : \mathcal{R}^A \rightarrow X^A$ and $f^B : X^A \times \mathcal{R}^B \rightarrow X^B$. Here observe that f^A is an A -allocation rule (or an allocation rule for the older siblings), and more importantly, $f^B(x^A, \cdot)$ is a B -allocation rule (or allocation rule for the younger siblings) when x^A is fixed. Let Φ^f be the set of allocation rules such that if $\varphi \in \Phi^f$ then

- $\varphi^A(R) = f^A(R^A)$, for all $R \in \mathcal{R}$ and
- $\varphi^B(R) = f^B(\varphi^A(R^A), R^B)$ for all $R \in \mathcal{R}$.

Below we show that if the preferences are initial-period lexicographic and if allocation rules for both markets A and B are group strategy-proof and efficient, then this allocation rule is both group strategy-proof and Pareto-efficient.

Theorem 3. Fix functions $f^A : \mathcal{R}^A \rightarrow X^A$ and $f^B : X^A \times \mathcal{R}^B \rightarrow X^B$. In addition, suppose that f^A is Pareto-efficient and group strategy-proof in market A , and $f^B(x^A, \cdot)$ is Pareto-efficient and group strategy-proof in market B for all x^A . If preferences are initial-period lexicographic, then any $\varphi = (\varphi^A, \varphi^B) \in \Phi^f$ is group strategy-proof and efficient in the multiple-market problem.

Proof. First, let us argue that for all R , $\varphi(R)$ is Pareto-efficient. Let R be the preference profile of the agents. Let $\varphi(R) = y$. Suppose that there exists an allocation z , such that z Pareto-dominates y . Then, for all agents, it must be true that $(z_i^A, z_i^B) R_i (y_i^A, y_i^B)$, where for at least one agent it must also be true that $(y_i^A, y_i^A) \neq (z_i^A, z_i^B)$. Given that preferences are initial-period lexicographic, it must be that $z_i^A R_i^A y_i^A$ for all agents. However, given that y^A is Pareto-efficient in market A , it must be that $y_i^A = z_i^A$. Given that z Pareto-dominates y , it must be that $z_j^B R_j^B y_j^B$, for all j and $z_i^B \neq y_i^B$ for at least one i . However, this contradicts the assumption that $f^B(y^A, R^B)$ is Pareto-efficient in B .

Now suppose that φ is not strategy-proof. There must exist R and \bar{R}_i such that $\varphi(\bar{R}_i, R_{-i}) R_i \varphi(R)$ and $\varphi(\bar{R}_i, R_{-i}) \neq \varphi(R)$. Denote $y_i = \varphi_i(R_i, R_{-i})$ and $\hat{y}_i = \varphi_i(\hat{R}_i, R_{-i})$. Given that preferences are initial-period lexicographic, it must be true that either (i) $\hat{y}_i^A R_i^A y_i^A$ and $\hat{y}_i^A \neq y_i^A$, or (ii) $\hat{y}_i^A = y_i^A$ and $\hat{y}_i^B R_i^B y_i^B$.

Suppose that (i) is true, that is, $\hat{y}_i^A R_i^A y_i^A$ and $\hat{y}_i^A \neq y_i^A$. Given that f^A is group strategy-proof, (i) cannot hold.

Suppose that (ii) holds, that is $\hat{y}_i^A = y_i^A$ and $\hat{y}_i^B R_i^B y_i^B$. Because f^A is nonbossy, it must be true that $\hat{y}^A = y^A$. Thus, $y_i^B = f_i^B(y^A, R^B)$ and $\hat{y}_i^B = f_i^B(y^A, (\bar{R}_i^B, R_{-i}^B))$. Now recall that $f^B(y^A, \cdot)$ is strategy-proof in market B . Thus, $y_i^B R_i^B \bar{y}_i^B$ which contradicts (ii).

Finally, suppose that φ is not nonbossy. Then there must exist R and \bar{R}_i such that $\varphi_i(\bar{R}_i, R_{-i}) = \varphi_i(R)$ and $\varphi(\bar{R}_i, R_{-i}) \neq \varphi(R)$. Denote $y_i = \varphi_i(R_i, R_{-i})$ and $\hat{y}_i = \varphi_i(\hat{R}_i, R_{-i})$. Given that f^A only depends on R^A and it is nonbossy in market A , it must that $y^A = \hat{y}^A$. Then $f^B(y^A, \cdot)$ is nonbossy in market B , thus $y^B = \hat{y}^B$, which is a contradiction. Thus, φ is nonbossy. \square

The theorem above implies directly that the TTC allocation rule achieves both strategy-proofness and Pareto efficiency in the school choice problem with sibling priorities if the preferences are initial-period lexicographical. In the exact same setting Dur (2011) shows that the deferred acceptance allocation rule is not strategy-proof. The main reason why the TTC is strategy-proof while the DA is manipulable is that the TTC is nonbossy while the DA is not. Thus, TTC has an edge over DA in terms of non-manipulability.

7 Conclusion

We have studied the problem of centralized assignment in multiple markets, which includes the class of dynamic matching problems. In our main result, we showed that the set of rules that are group strategy-proof and implement a Pareto-efficient allocation is the set of sequential dictatorship rules. This result contrasts sharply with centralized allocation in a single market, and with single-object allocation in static environments. In those problems, the trading cycles allocation rule satisfies the above-mentioned criteria.

Our result provides further support for the use of sequential dictatorships in some dynamic matching problems. While these rules have the shortcoming that some agents might have a larger choice set than others, in some applications this shortcoming is less severe (see, for example, Kennes et al. (2012)).

Finally, there are classes of problems for which rules other than the sequential dictatorship are strategy-proof and Pareto-efficient within specific preference domains. In one of our applications, we showed that in the school choice problem where sibling priorities are taken seriously, the top trading cycle allocation rule is group strategy-proof and Pareto-efficient under a realistic, albeit restrictive, preference domain.

One possible direction for future research is to work with a solution concept other than Pareto efficiency. As we have argued in the text, many well-known allocation rules that are strategy-proof in single markets remain strategy-proof in the multiple-markets case if applied separately and independently to each different market. In this sense, Pareto efficiency seems to be a very demanding concept for the class of multiple-market problems.

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8 Appendix

Proof of Theorem 2. Let $n \geq 2$, $|A| \geq n$ and $|B| \geq n$. Without loss of generality we assume that $|B| \geq |A|$. First we will prove that for each efficient, nonbossy and strategy-proof allocation rule φ there exists an agent i such that $\varphi_i(R) = \tau(R_i, A \times B)$ for all $R \in \mathcal{R}$. Because we have proved this for the $n = 2$ case (in the main text of the paper), our proof will be by induction:

Induction Assumption: For each multi-market allocation problem in which $2 \leq n \leq m - 1$, $|A| \geq n$, and $|B| \geq n$ and for each efficient, strategy-proof, and nonbossy allocation rule of this market, there exists an agent who is assigned her most preferred bundle for each preference profile.

Fix any multi-market allocation problem in which $n = m$ and $|B| \geq |A| \geq n$. Fix any efficient, strategy-proof, and nonbossy allocation rule φ for this market. We now will show that there exists an agent i such that $\varphi_i(R) = \tau(R_i, A \times B)$ for all $R \in \mathcal{R}$.

For this proof we will need several steps.

Claim 2. Consider any nonempty and (strict) subset $S \subset N$ and any preference profile $R \in \mathcal{R}$. Let $A_S = \{a \in A : \varphi_i^A(R) = a \text{ for some } i \in S\}$. Similarly, define B_S . Then there must exist an agent $i \notin S$ such that $\varphi_i(R) = \tau(R_i, A \setminus A_S \times B \setminus B_S)$.

Proof of Claim 2. To the contrary of Claim 2, suppose that $\varphi_i(R) \neq \tau(R_i, A \setminus A_S \times B \setminus B_S)$ for all $i \notin S$. Let R^1 be an φ -monotonic change of R satisfying the following conditions:

1. if $j \in S$, then $\varphi_j(R)$ is the most preferred bundle of j in $A \times B$ (under R_j^1), i.e., $\varphi_j(R) = \tau(R_j^1, A \times B)$ and
2. if $i \notin S$, then i 's preferences satisfy that
 - (a) whenever $(a, b) \in A \setminus A_S \times B \setminus B_S$ and $(\bar{a}, \bar{b}) \notin A \setminus A_S \times B \setminus B_S$, then $(a, b) R_i^1 (\bar{a}, \bar{b})$
 - (b) whenever $(a, b) \in A \setminus A_S \times B \setminus B_S$ and $(\bar{a}, \bar{b}) \in A \setminus A_S \times B \setminus B_S$, then $(a, b) R_i^1 (\bar{a}, \bar{b})$ if and only if $(a, b) R_i (\bar{a}, \bar{b})$

By Lemma 2, $\varphi(R^1) = \varphi(R)$. Here observe that we have constructed R^1 so that we reach a desired contradiction once we show that there is an agent $i \notin S$ such that $\varphi(R^1) = \tau(R_i^1, A \setminus A_S \times B \setminus B_S)$.

To show this consider the class of preferences $\mathcal{R}^S(R)$ such that each $R' \in \mathcal{R}^S(R)$ satisfies the following conditions:

1. if $j \in S$, then $\varphi_j(R)$ is the most preferred bundle of j in $A \times B$ (under R'_j), i.e.,
 $\varphi_j(R) = \tau(R'_j, A \times B)$
2. if $i \notin S$, then i 's preferences satisfy that

$$(a) \text{ whenever } (a, b) \in A \setminus A_S \times B \setminus B_S \text{ and } (\bar{a}, \bar{b}) \notin A \setminus A_S \times B \setminus B_S, \text{ then } (a, b) R'_i (\bar{a}, \bar{b})$$

Observe that $R^1 \in \mathcal{R}^S(R)$. For each preference profile in $\mathcal{R}^S(R)$, each agent $j \in S$ must obtain $\varphi_i(R)$ due to the efficiency of φ . Consequently, for $\mathcal{R}^S(R)$, we can treat φ as the allocation rule that allocates $A \setminus A_S \times B \setminus B_S$ among the agents in $N \setminus S$. Then by the induction assumption, there must exist an agent $i \in N \setminus S$ such that $\varphi_i(\bar{R}) = \tau(\bar{R}_i, A \setminus A_S \times B \setminus B_S)$ for all $\bar{R} \in \mathcal{R}^S(R)$. Consequently, because $R^1 \in \mathcal{R}^S(R)$, it must be that $\varphi_i(R^1) = \tau(R^1_i, A \setminus A_S \times B \setminus B_S)$, reaching a contradiction.

In fact, we can strengthen Claim 2 as follows:

Claim 3. Consider any nonempty and (strict) subset $S \subset N$ and consider the set of preference profiles \mathcal{R}^S such that $\varphi_i(R) = \varphi_i(\bar{R})$ for all $i \in S$ and $R, \bar{R} \in \mathcal{R}^S$. Then there must exist an agent $j \notin S$ such that $\varphi_j(R) = \tau(R_j, A \setminus A_S \times B \setminus B_S)$ for all $R \in \mathcal{R}^S$.

Proof of Claim 3. Recall that how $\mathcal{R}^S(R)$ is defined in the proof of Claim 2. Observe here that $\mathcal{R}^S(R) = \mathcal{R}^S(\bar{R})$ for all $R, \bar{R} \in \mathcal{R}^S$. This and proof of Claim 2 complete the proof of claim 3.

In the next 3 claims (4-6), we prove that for any $(a, b) \in A \times B$, there exists an agent i such that $\varphi_i(R) = (a, b)$ whenever $\tau(R_i, A \times B) = (a, b)$. Without loss of generality, let us set $a = a_1$ and $b = b_1$.

Claim 4. Let L be a lexicographical preference profile in which each agent's lexicographical ordering of the objects is the same and as follows:

$$L_i : a_1, b_1, a_2, b_2, \dots, a_{|A|}, b_{|A|}, b_{|A|+1}, \dots, b_{|B|}.$$

Then φ allocates each (a_k, b_k) where $k \leq n$ to some agent under L .

Proof of Claim 4. Since $n \geq 3$ there must exist an agent i for whom $\varphi_i^A(L) \neq a_1$ and $\varphi_i^B(L) \neq b_1$. Set $S = \{i\}$, and observe that $(a_1, b_1) \in A \setminus A_S \times B \setminus B_S$. Then by Claim 2, there must exist an agent $i_1 \notin S$ for whom $\varphi_{i_1}(L) = (a_1, b_1)$ because $(a_1, b_1) = \tau(L_j, A \setminus A_S \times B \setminus B_S)$ for all $j \in N \setminus S$, due to Lemma 4. Now set $S_1 = \{i_1\}$ and observe that $(a_2, b_2) = \tau(L_j, A \setminus A_{S_1} \times B \setminus B_{S_1})$ for all $j \in N \setminus S_1$, due to Lemma 4. Then by Claim 2, there exists an agent $i_2 \notin S_1$ for whom $\varphi_{i_2}(L) = (a_2, b_2)$. Next set $S_2 = \{i_1, i_2\}$. Using a similar argument as before we obtain that there exists an agent $i_3 \notin S_2$ for whom $\varphi_{i_3}(L) = (a_3, b_3)$. We complete the proof of this claim by applying the same argument repeatedly.

Without loss of generality, let us assume $\varphi_i(L) = (a_i, b_i)$. We now show that $\varphi_1(R) = (a_1, b_1)$ for all R in which $(a_1, b_1) = \tau(R_1, A \times B)$.

Claim 5. Consider any lexicographical preference profile in which all agents' lexicographical ordering of the objects is the same and starts listing a_1 and b_1 and then alternates the remaining elements of A and B . Then for this lexicographical preference profile φ must assign (a_1, b_1) to agent 1.

Proof of Claim 5. To prove this claim it suffices to prove the following claim.

“Consider a lexicographical preference profile \hat{L} in which each agent's lexicographical ordering of the objects is the same and that is obtained from L (considered in Claim 4) by reversing the lexicographical orderings of only two neighboring B -objects (except b_1), i.e., each agent's lexicographical ordering of the objects is:

$$\hat{L}_i : a_1, b_1, a_2, b_2, \dots, a_{j-1}, b_{j-1}, a_j, \mathbf{b}_{j+1}, a_{j+1}, \mathbf{b}_j, a_{j+2}, b_{j+2}, a_{j+3}, b_{j+3}, \dots, b_{|B|}$$

where $j \geq 2$. Then for each $i < j$, $\varphi_i(\hat{L}) = \varphi_i(L)$ and $\varphi_j(\hat{L}) = (a_j, b_{j+1})$.”

If $j > n$, then \hat{L} is an φ -monotonic change of L (Lemma 5). Hence, Lemma 2 yields the statement above. Thus, let us concentrate on the $j \leq n$ cases.

Fix any j such that $j \leq n$, and consider a lexicographical preference profile L^1 such that

$$\begin{aligned} L_i^1 : & a_1, b_1, \dots, a_i, b_i, & \mathbf{a}_j, \mathbf{b}_j, & a_{i+1}, b_{i+1}, \dots, a_{|A|}, b_{|A|}, \dots, b_{|B|} \text{ if } i < j \\ L_j^1 : & a_1, b_1, \dots, a_j, b_j, & \mathbf{b}_{j+1}, \mathbf{a}_{j+1}, & a_{j+2}, b_{j+2}, \dots, a_{|A|}, b_{|A|}, \dots, b_{|B|} \\ L_i^1 : & \mathbf{a}_j, a_1, b_1, \dots, a_{j-1}, & b_{j-1}, b_j, & a_{j+1}, b_{j+1}, \dots, a_{|A|}, b_{|A|}, \dots, b_{|B|} \text{ if } i > j. \end{aligned}$$

Clearly L^1 is an φ -monotonic change of L . Hence, $\varphi(L) = \varphi(L^1)$. Now let L^2 be

the lexicographical preference obtained from L^1 by changing only agent j 's lexicographical ordering of the objects as follows:

$$L_j^2 : a_1, b_1, \dots, a_j, \mathbf{b}_{j+1}, \mathbf{b}_j, a_{j+1}, a_{j+2}, b_{j+2}, \dots, a_{|A|}, b_{|A|}, \dots, b_{|B|}$$

Observe here that there is only one bundle, (a_j, b_{j+1}) , such that j prefers it to $\varphi_j(L^1) = (a_j, b_j)$ under L_j^2 but not under L_j^1 . As φ is strategy-proof, $\varphi_j(L^2)$ is either (a_j, b_j) or (a_j, b_{j+1}) . In the former case, thanks to nonbossiness, $\varphi(L^2) = \varphi(L^1)$. Then by Claim 3, it must be that $\varphi_j(L^2) = \tau(L_j^2, A \setminus \{a_1, \dots, a_{j-1}\} \times B \setminus \{b_1, \dots, b_{j-1}\}) = (a_j, b_{j+1})$, a contradiction. Hence, $\varphi_j(L^2) = (a_j, b_{j+1})$. Because $(a_1, b_1) = \tau(L_i^2, A \setminus \{a_j\} \times B \setminus \{b_{j+1}\})$ for all $i \neq j$, some agent other than j must obtain (a_1, b_1) under $\varphi(L^2)$ by Claim 2. In addition, when $j > 2$, because $(a_2, b_2) = \tau(L_i^2, A \setminus \{a_j, a_1\} \times B \setminus \{b_{j+1}, b_1\})$, some agent other than j must obtain (a_2, b_2) under $\varphi(L^2)$ by Claim 2. A similar logic yields that each of the $\{(a_1, b_1), \dots, (a_{j-1}, b_{j-1})\}$ is allocated to some agent under $\varphi(L^2)$. However, observe that (a_1, b_1) cannot be allocated to any agent $i > j$ (if such i exists) under $\varphi(L^2)$. Otherwise, by swapping their allocations agents j and i Pareto improve. Similarly, we obtain that none of the $\{(a_1, b_1), \dots, (a_{j-1}, b_{j-1})\}$ are allocated to agents $\{j+1, \dots, n\}$ under $\varphi(L^2)$. Now let us show that agent 1 obtains (a_1, b_1) under $\varphi(L^2)$. Otherwise, she obtains one of the $\{(a_2, b_2), \dots, (a_{j-1}, b_{j-1})\}$. But then agents 1 and j can swap their allocations and Pareto improve. Then agent 2 must obtain (a_2, b_2) under $\varphi(L^2)$; otherwise agents 2 and j can swap their allocations and Pareto improve. A similar logic yields that all agents $i \leq j-1$, $\varphi_i(L^2) = (a_i, b_i)$ and $\varphi_j(L^2) = (a_j, b_{j+1})$.

If $j = n$, then observe that \hat{L} is an φ -monotonic change of L^1 . Thus, we obtain the desired result, due to Lemma 2. Let $j < n$. We now need to show that $\varphi_i(L^1) = \varphi_i(\hat{L})$ for all $i = 1, \dots, j$. To prove this, we need some extra steps. First, observe that by Claim 2 there exists be an agent $k > j$ for whom $\varphi_k(L^2) = \tau(L_k^2, A \setminus \{a_1, \dots, a_j\} \times B \setminus \{b_1, \dots, b_{j-1}, b_{j+1}\}) = (a_{j+1}, b_j)$. Also, due to Claim 2, each of the $\{(a_{j+2}, b_{j+2}), (a_{j+3}, b_{j+3}), \dots, (a_{|A|}, b_{|B|})\}$ is allocated to some agent $i \neq k$ ($i > j$). Now consider a lexicographical preference L^3 such that $L_k^3 = L_k^2$ and $L_i^3 = \hat{L}_i$, for all $i \neq k$. Observe that L^3 is an φ -monotonic change of L^2 , hence $\varphi(L^3) = \varphi(L^2)$.

Consider a lexicographical preference profile L^4 in which

$$\begin{aligned}
L_i^4 &: a_1, b_1, \dots, a_i, b_i, & \mathbf{a}_{j+1}, \mathbf{b}_{j+1}, & a_{i+1}, b_{i+1}, & \dots, & a_{|A|}, b_{|A|}, \dots, b_{|B|} \text{ if } i < j \\
L_j^4 &: a_1, b_1, \dots, a_j, \mathbf{b}_{j+1}, & \mathbf{a}_{j+1}, \mathbf{b}_j, & a_{j+2}, b_{j+2}, & \dots, & a_{|A|}, b_{|A|}, \dots, b_{|B|} \\
L_k^4 &: a_1, b_1, \dots, a_j, b_j, & \mathbf{b}_{j+1}, \mathbf{a}_{j+1}, & a_{j+2}, b_{j+2}, & \dots, & a_{|A|}, b_{|A|}, \dots, b_{|B|} \\
L_i^4 &: \mathbf{a}_{j+1}, a_1, b_1, \dots, & a_j, b_j, b_{j+1}, & a_{j+2}, b_{j+2}, & \dots, & a_{|A|}, b_{|A|}, \dots, b_{|B|} \text{ if } i \neq k \text{ \& } i > j
\end{aligned}$$

Clearly, L^4 is an φ -monotonic change of L^3 . Hence, $\varphi(L^4) = \varphi(L^3)$. Now let L^5 be a lexicographical preference obtained from L^4 by changing agent k 's order of the objects as follows:

$$L_k^5 : a_1, b_1, \dots, a_j, \mathbf{b}_{j+1}, \mathbf{b}_j, a_{j+1}, a_{j+2}, b_{j+2}, \dots, a_{|A|}, b_{|A|}, \dots, b_{|B|}.$$

Going from L^4 to L^5 only the relative ranking of (a_{j+1}, b_{j+1}) improves with respect to $\varphi_k(L^4) = (a_{j+1}, b_j)$ for agent k . As φ is strategy-proof, $\varphi_j(L^5)$ is either (a_{j+1}, b_j) or (a_{j+1}, b_{j+1}) . Now we rule out the latter case. Suppose the latter case occurs. Using the same steps as we used to prove that $\{(a_1, b_1), \dots, (a_{j-1}, b_{j-1})\}$ is allocated among the agents $\{1, \dots, j-1\}$ under L^2 , we obtain that $\{(a_1, b_1), \dots, (a_j, b_j)\}$ is allocated among the agents $\{1, \dots, j\}$. If agent 1 does not obtain (a_1, b_1) , by swapping the allocations of 1 and k , we can Pareto improve. Similarly, each agent $i \leq j-1$ must obtain (a_i, b_i) . Therefore, agent j obtains (a_j, b_j) . But this contradicts Claim 3 because j is not obtaining her most preferred bundle in $A \setminus \{a_1, \dots, a_{j-1}\} \times B \setminus \{b_1, \dots, b_{j-1}\}$ under L^5 . Hence, $\varphi_k(L^5) = \varphi_k(L^4) = (a_{j+1}, b_j)$. Now the nonbossiness of φ yields that $\varphi(L^5) = \varphi(L^4)$. Finally, observe that \hat{L} is a φ -monotonic change of L^5 . Thus, $\varphi(\hat{L}) = \varphi(L^5)$. This completes the proof of Claim 5.

Claim 6. For any preference profile R in which $(a_1, b_1) = \tau(R_1, A \times B)$ it must be that $\varphi_1(R) = (a_1, b_1)$.

Proof of Claim 6. Pick any preference profile R such that $\tau(R_1, A \times B) = (a_1, b_1)$. Now let us construct a lexicographical preference L^n in n iterative rounds.

Round 1. Set $i_1 = 1$. Pick any lexicographical preference L^1 in which everyone's order of the objects is the same and starts with (a_1, b_1) and alternates the remaining A and B -objects. Set $I_1 = \{i_1\}$ and $A_1 = A \setminus \{a_1\}$ and $B_1 = B \setminus \{b_1\}$. Observe that $\varphi_{i_1}(L^1) = (a_1, b_1)$ by Claim 5.

Round 2. Let $i_2 \in N \setminus I_1$ be the agent for whom $\varphi_{i_2}(L_1) = \tau(L_{i_2}, A_1 \times B_1)$. This is always feasible thanks to Claim 2.¹⁵ Set $I_2 = I_1 \cup \{i_2\}$. Let $\tau(R_{i_2}, A_1 \times B_1) := (\hat{a}_2, \hat{b}_2)$. Set $A_2 = A_1 \setminus \{\hat{a}_2\}$ and $B_2 = B_1 \setminus \{\hat{b}_2\}$. Pick a lexicographical preference L^2 in which the

¹⁵In fact, this agent is the second agent.

order of the objects is the same for everyone, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2)$, and alternates the remaining A and B -objects. Observe that $\varphi_{i_1}(L^2) = (a_1, b_1)$ and $\varphi_{i_2}(L^2) = (\hat{a}_2, \hat{b}_2)$.

Round $k \leq n$. Let $i_k \in N \setminus I_{k-1}$ be the agent for whom $\varphi_{i_k}(L^{k-1}) = \tau(L_{i_k}^{k-1}, A_{k-1} \times B_{k-1})$. Set $I_k = I_{k-1} \cup \{i_k\}$. Let $\tau(R_{i_k}, A_{k-1} \times B_{k-1}) := (\hat{a}_k, \hat{b}_k)$. Set $A_k = A_{k-1} \setminus \{\hat{a}_k\}$ and $B_k = B_{k-1} \setminus \{\hat{b}_k\}$. Pick a lexicographical preference L^k in which the order of the objects is the same, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2, \dots, \hat{a}^k, \hat{b}^k)$, and then alternates the remaining A and B -objects. Observe that $\varphi_{i_1}(L^k) = (a_1, b_1)$ and $\varphi_{i_j}(L^k) = (\hat{a}_j, \hat{b}_j)$ where $j \leq k$.

Consider L^n and we now show that R is an φ -monotonic change of L^n . In other words, we need to show that $\varphi_i(L^n)R_i(a, b)$ for each i and $(a, b) \in A \times B$ satisfying $\varphi_i(L^n)L_i^n(a, b)$. Clearly, this is true for $i = i_1$ because $\varphi_{i_1}(L^n) = (a_1, b_1) = \tau(R_{i_1}, A \times B) = \tau(L_{i_1}^n, A \times B)$. Consider now the $i = i_2$ case. Fix any $(a, b) \in A \times B$ such that $\varphi_{i_2}(L^n) = (\hat{a}_2, \hat{b}_2)L_{i_2}^n(a, b)$. Then the definition of lexicographical preferences and the construction of L^n yield that $a \neq a_1$ and $b \neq b_1$. Recall that $(\hat{a}_2, \hat{b}_2) = \tau(R_{i_2}, A \setminus \{a_1\} \times B \setminus \{b_1\})$. Thus, it must be that $\hat{a}_2 R_{i_2}^A a$ and $\hat{b}_2 R_{i_2}^B b$. This and the separability of R yield that $(\hat{a}_2, \hat{b}_2)R_{i_2}(a, b)$, our desired result. A similar proof applies for all $i = i_k$ where $k \leq n$. Thus, R is a φ -monotonic change of L^n . Consequently, by Lemma 2, $\varphi(R) = \varphi(L^n)$ and $\varphi_{i_1}(R) = (a_1, b_1)$.

Claim 7. There exists an agent such that $\varphi_i(R) = \tau(R_i, A \times B)$ for all R .

Proof of Claim 7. Fix any (a, b) . By Claim 6 there exists an agent i such that $\varphi_i(R) = (a, b)$ for all R with $\tau(R_i, A \times B) = (a, b)$. We need to show that $\varphi_i(R) = \tau(R_i, A \times B)$ for all $R \in \mathcal{R}$. Fix any (a, \bar{b}) . Then by Claim 6 there must exist j such that $\varphi_j(R) = (a, \hat{b})$ for all R with $\tau(R_j, A \times B) = (a, \bar{b})$. If $(a, \bar{b}) = (a, b)$, then clearly $i = j$. If $(a, \bar{b}) \neq (a, b)$, we need to show that $i = j$. If $i \neq j$, consider \bar{R} such that $(a, b) = \tau(\bar{R}_i, A \times B)$ and $(a, \bar{b}) = \tau(\bar{R}_j, A \times B)$. Then it must be that $\varphi_i^A(\bar{R}) = \varphi_j^A(\bar{R}) = a$, a contradiction. Thus, $i = j$. A similar proof shows that for any (\bar{a}, \bar{b}) , it must be that $\varphi_i(R) = (\bar{a}, \bar{b})$ for all R with $\tau(R_i, A \times B) = (\bar{a}, \bar{b})$. Given that we picked arbitrary (a, b) and (\bar{a}, \bar{b}) , it must be that $\varphi_i(R) = \tau(R_i, A \times B)$ for all R .

Claim 8. Any strategy-proof, Pareto-efficient allocation rule φ is a sequential serial dictatorship.

Proof of Claim 8. This claim is a consequence of Claims 3 and 7. □